

Closed formula for the matrix elements of the volume operator in canonical quantum gravity

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Abstract

We derive a closed formula for the matrix elements of the volume operator for canonical Lorentzian quantum gravity in four spacetime dimensions in the continuum in a spin-network basis.

We also display a new technique of regularization which is state dependent but we are forced to it in order to maintain diffeomorphism covariance and in that sense it is natural.

We arrive naturally at the expression for the volume operator as defined by Ashtekar and Lewandowski up to a state independent factor.

1 Introduction

The volume functional V for Lorentzian canonical quantum gravity plays a quite important role in the quantization process : if one follows the approach advertized in [1] and chooses a representation in which traces of the holonomy along closed loops for a certain $SU(2)$ A_a^i connection can be promoted to basic configuration operators, then it becomes technically very hard to define physically interesting operators such as the Euclidean and Lorentzian Hamiltonian constraint operators, an operator corresponding to the length of a curve, an operator corresponding to the ADM energy, matter Hamiltonian operators and so forth. The common reason is that all these operators involve only the co-triad e_a^i of the intrinsic metric of the initial data hypersurface Σ polynomially which, however, cannot be written as a polynomial in the momentum conjugate to A_a^i and therefore does not seem to make any sense as a quantum operator.

Fortunately, there is a trick available : as proved in [2, 3], the total volume of Σ is the generating functional of the co-triad meaning that one gets e_a^i from V essentially by functional derivation. Therefore, if we write $\delta t s^a e_a^i \propto \{h_s, V\} + o(\delta t^2)$ where s is an edge with parameter length δt , h_s is the holonomy along s and $\{.,.\}$ denotes Poisson brackets then it would be sufficient to quantize V in such a way that (gauge-invariant) functions of a finite number of holonomies form a dense domain. Such a satisfactory quantization of V which has precisely this property was indeed given already in the literature [4, 5].

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Therefore, the strategy outlined above and in [2, 3] can be implemented and, in particular, was successfully employed to construct various operators listed above [2, 3, 6, 7, 8]. However, the explicit computation of the spectrum of most of those operators was not performed so far. The reason is that the spectra of all these operators are clearly largely determined by the spectrum of the volume operator whose precise spectrum is fairly unknown except for the most simple situations [9, 10].

The present paper is devoted to the derivation of the complete set of matrix elements of this operator in closed form from which the spectrum can be obtained straightforwardly.

In section 3, after introducing the notation in section 2, we develop a novel technique to regularize the volume operator which is state dependent very much in the same way as the regularization displayed in the second reference of [5]. This part of the paper is a continuation of the work on the point splitting regularizations of the volume and area operators as displayed in [1, 5]. The derivation is rather short and gives a diffeomorphism covariant end result. We are unambiguously led to the Ashtekar-Lewandowski volume operator defined in [18] up to a factor of $\sqrt{27/8}$.

In section 4 we repeat the argument given in [5, 9] which shows that the spectrum is discrete and that its computation reduces to linear algebra, in the sense that we just need to determine separately the eigenvalues of an infinite number of finite-dimensional, positive semi-definite and symmetric real-valued matrices.

Finally, in section 5 we derive the explicit formula for an arbitrary matrix element in a spin-network basis and compute the eigenvalues of the associated matrices in a number of simple cases. It should be noted that the formulae obtained, albeit quite complicated, are the starting point for a numerical evaluation of the spectrum and are in particular useful to determine the large spin behaviour of the volume operator.

2 Notation

Let us first introduce the notation. We consider the Hamiltonian formulation of the Palatini action of four-dimensional Lorentzian vacuum gravity.

Denote by q_{ab} the intrinsic metric on the initial data hypersurface Σ and let e_a^i be its co-triad where i, j, k, \dots denote $su(2)$ indices. The densitized triad $E_i^a := \det((e_b^j))e_i^a$ multiplied by $1/\kappa$, κ being Newton's constant, is the momentum conjugate to $K_a^i := \text{sgn}(\det((e_c^j)))K_{ab}e_i^b$ where K_{ab} is the extrinsic curvature of Σ . Upon performing a canonical point transformation [11, 12] on the phase space coordinatized by $(K_a^i, E_i^a/\kappa)$ we arrive at the chart $(A_a^i/\kappa := (\Gamma_a^i + K_a^i)/\kappa, E_i^a)$ where Γ_a^i is the spin-connection of e_a^i . It is easy to see that the configuration variable A_a^i is a $su(2)$ connection.

The virtue of casting general relativity into a connection dynamics formulation is as follows.

- 1) First of all, the phase space of general relativity is now embedded into that of an $su(2)$ gauge theory, thereby opening access to a wealth of techniques that have been proving useful in quantizing those theories. Probably the most important technique is the use of traces of the holonomy along piecewise analytic loops in Σ , so-called Wilson-loop functionals, to coordinatize the space of smooth connections modulo gauge transformations \mathcal{A}/\mathcal{G} [13, 14].

- 2) Using those Wilson loop functionals one can construct [15, 16, 17] a quantum configuration space of generalized connection modulo gauge transformations $\overline{\mathcal{A}/\mathcal{G}}$ and find natural, σ -additive, diffeomorphism invariant, regular Borel probability measures μ_0 thereon which equips us with a Hilbert space structure $\mathcal{H} := L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$. Of course, for this to make sense we have to make the following assumption : Wilson loop functionals continue to make sense as operators in the quantum theory. We will adopt this viewpoint in the sequel in the hope to capture one of the physically interesting phases of quantum general relativity. The interested reader is referred to [1] and references therein for further details.

The following paragraph is supposed to equip the reader with a working knowledge of the tools associated with $\overline{\mathcal{A}/\mathcal{G}}$.

We begin by defining cylindrical functions. Since the gauge invariant information contained in a connection is captured by finite linear sums of products of Wilson loop functionals we may label gauge invariant functions by piecewise analytic graphs γ which are just the union of the loops involved. Such a graph consists of maximally analytic pieces, called the edges e of γ and the edges meet in the vertices v of γ . A function f on $\overline{\mathcal{A}/\mathcal{G}}$ is said to be cylindrical with respect to a graph γ iff it can be written as $f = f_\gamma \circ p_\gamma$ where $p_\gamma(A) = (h_{e_1}(A), \dots, h_{e_n}(A))$ and where e_1, \dots, e_n are the edges of γ . Here $h_e(A)$ is the holonomy along e evaluated at $A \in \overline{\mathcal{A}/\mathcal{G}}$ and f_γ is a complex valued function on $SU(2)^n$. Since a function cylindrical with respect to a graph γ is automatically cylindrical with respect to any graph bigger than γ , a cylindrical function is actually given by a whole equivalence class of functions f_γ . In the sequel we will not distinguish between this equivalence class and one of its representants.

We say that a function cylindrical with respect to a graph γ is of class C^n if it is of class C^n with respect to the standard differentiable structure of $SU(2)^n$ and denote this class of functions by $\text{Cyl}_\gamma^n(\overline{\mathcal{A}/\mathcal{G}})$, $\text{Cyl}^n(\overline{\mathcal{A}/\mathcal{G}}) := \cup_{\gamma \in \Gamma} \text{Cyl}_\gamma^n(\overline{\mathcal{A}/\mathcal{G}})$ where Γ is the set of piecewise analytic graphs in Σ . Thus we have a differential calculus [18] on $\overline{\mathcal{A}/\mathcal{G}}$.

Finally, the measure μ_0 is the faithful projective limit [19, 20] of the self-consistent projective family $(\mu_{0,\gamma})_{\gamma \in \Gamma}$ defined by

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0(A) f(A) := \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_{0,\gamma}(A) f_\gamma(p_\gamma(A)) := \int_{SU(2)^n} d\mu_H(g_1) \dots d\mu_H(g_n) f_\gamma(g_1, \dots, g_n)$$

where $g_I := h_{e_I}(A)$. This equips us with an integral calculus on $\overline{\mathcal{A}/\mathcal{G}}$.

3 Regularization of the volume operator

In this section we present a new regularization of the volume operator which uses the methods derived by Abhay Ashtekar and Jurek Lewandowski in [18, 5, 9]. Most of the issues involved in the regularization as presented here are borrowed from their work.

The advantage of our procedure is that it is rather short and compact.

Let $R \subset \Sigma$ be an open, connected region of Σ . We have the identity

$$\frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} E_i^a E_j^b E_k^c = \det((E_i^a)) = \det((q_{ab})) = [\det((e_a^i))]^2 =: \det(q) \geq 0 \quad (3.1)$$

and can thus write the volume of the region R as measured by the metric q_{ab} as follows

$$V(R) := \int_R d^3x \sqrt{\det(q)} = \sqrt{\frac{1}{3!} \epsilon_{ijk} \epsilon^{abc} E_i^a E_j^b E_k^c}. \quad (3.2)$$

The next step is to smear the fields E_i^a . Let $\chi_\Delta(p, x)$ be the characteristic function in the coordinate x of a cube with center p spanned by the three vectors $\vec{\Delta}_i = \Delta_i \vec{n}_i(\Delta)$ where \vec{n}_i is a normal vector in the frame under consideration and which has coordinate volume $\text{vol} = \Delta_1 \Delta_2 \Delta_3 \det(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ (we assume the three normal vectors to be right oriented). In other words, $\chi_\Delta(p, x) = \prod_{i=1}^3 \theta(\Delta_i - | \langle \vec{n}_i, x - p \rangle |)$ where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product and $\theta(y) = 1$ for $y > 0$ and zero otherwise.

We consider the smeared quantity

$$E(p, \Delta, \Delta', \Delta'') := \frac{1}{\text{vol}(\Delta) \text{vol}(\Delta') \text{vol}(\Delta'')} \int_\Sigma d^3x \int_\Sigma d^3y \int_\Sigma d^3z \chi_\Delta(p, x) \chi_{\Delta'}(p, \frac{x+y}{2}) \times \\ \times \chi_{\Delta''}(p, \frac{x+y+z}{3}) \frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} E_i^a(x) E_j^b(y) E_k^c(z). \quad (3.3)$$

Notice that if we take the limits $\Delta_i, \Delta'_i, \Delta''_i \rightarrow 0$ in any combination and in any rate with respect to each other then we get back to ([1]) evaluated at the point p . This holds for any choice of linearly independent normal vectors $\vec{n}_i, \vec{n}'_i, \vec{n}''_i$. The strange argument $(x+y+z)/3$ will turn out to be very crucial in obtaining a manifestly diffeomorphism covariant result. We will see this in a moment.

Then it is easy to see that the classical identity

$$V(R) = \lim_{\Delta \rightarrow 0} \lim_{\Delta' \rightarrow 0} \lim_{\Delta'' \rightarrow 0} \int_R d^3p \sqrt{|E(p, \Delta, \Delta', \Delta'')|} \quad (3.4)$$

holds. We could introduce absolute values because we have the classical identity $\det(q) = |\det(q)|$ and the modulus is necessary for the regulated quantity (3.3) is not non-negative anymore.

The virtue of introducing the quantities (3.3) is that they can be promoted to quantum operators which have the dense domain $\text{Cyl}^1(\overline{\mathcal{A}/\mathcal{G}})$. To see this, notice that due to the canonical brackets $\{A_a^i(x), E_j^b(y)\} = \kappa \delta_a^b \delta_j^i \delta^{(3)}(x, y)$ we are naturally led to represent the operator corresponding to E_i^a by $\hat{E}_i^a(x) = -i \ell_p^2 \delta / \delta A_a^i(x)$ where $\ell_p = \sqrt{\hbar \kappa}$ is the Planck length.

Now let be given a graph γ . In order to simplify the notation, we subdivide each edge e with endpoints v, v' which are vertices of γ into two segments s, s' where $e = s \circ (s')^{-1}$ and s has an orientation such that it is *outgoing* at v while s' has an orientation such that it is *outgoing* at v' . This introduces new vertices $s \cap s'$ which we will call pseudo-vertices because they are not points of non-analyticity of the graph. Let $E(\gamma)$ be the set of these segments of γ but $V(\gamma)$ the set of true (as opposed to pseudo) vertices of γ . Let us now evaluate the action of $\hat{E}_i^a(p, \Delta) := 1/\text{vol}(\Delta) \int_\Sigma d^3x \chi_\Delta(p, x) \hat{E}_i^a(x)$ on a function $f = p_\gamma^* f_\gamma$ cylindrical with respect to γ . We find $(e : [0, 1] \rightarrow \Sigma; t \mapsto e(t))$ being a parametrization of the edge e)

$$\hat{E}_i^a(p, \Delta) f = -\frac{i \ell_p^2}{\text{vol}(\Delta)} \sum_{e \in E(\gamma)} \int_0^1 dt \chi_\Delta(p, e(t)) \dot{e}^a(t) \times \\ \times \frac{1}{2} \text{tr}(h_e(0, t) \tau_i h_e(t, 1) \frac{\partial}{\partial h_e(0, 1)}) f_\gamma. \quad (3.5)$$

Here we have used 1) the fact that a cylindrical function is already determined by its values on \mathcal{A}/\mathcal{G} rather than $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ so that it makes sense to take the functional derivative, 2) the definition of the holonomy as the path ordered exponential of $\int_e A$ with the smallest parameter value to the left, 3) $A = dx^a A_a^i \tau_i / 2$ where $SU(2) \ni \tau_j = -i\sigma_j$, σ_i being the usual Pauli matrices, so that $[\tau_i/2, \tau_j/2] = \epsilon_{ijk} \tau_k / 2$ and 4) we have defined $\text{tr}(h\partial/\partial g) = h_{AB} \partial/\partial g_{AB}$, A, B, C, \dots being $SU(2)$ indices.

We now wish to evaluate the whole operator $\hat{E}(p, \Delta, \Delta', \Delta'')$ on f . It is clear that we obtain three types of terms, the first type comes from all three functional derivatives acting on f only, the second type comes from two functional derivatives acting on f and the remaining one acting on the trace appearing in (3.5) and finally the third type comes from only one derivative acting on f_γ and the remaining two acting on the trace. Explicitly we find (we mean by $\theta(t, t')$ the theta function which is unity if $0 < t < t' < 1$, $1/2$ if $t = 0 < t' < 1, 0 < t < t' = 0$ and $1/4$ if $t \leq t' \in \{0, 1\}$ and zero otherwise. Likewise $\theta(t, t', t'')$ is 1 if $t < t' < t''$, $1/2$ if $t = 0 < t' < t'', 0 < t < t' < t'' = 1$, $1/4$ if $0 = t = t' < t'', 0 < t < t' = t''$ and $1/8$ if $t \leq t' \leq t'' \in \{0, 1\}$. Finally $m(t, t', t'') = 1/2^k$ where $k = 0, 1, 2, 3$ is the possible number of arguments that equal 0, 1)

$$\begin{aligned}
\hat{E}(p, \Delta, \Delta', \Delta'')f &= \frac{i\ell_p^6}{8 \cdot 3! \text{vol}(\Delta) \text{vol}(\Delta') \text{vol}(\Delta'')} \epsilon_{abc} \epsilon^{ijk} \int_{[0,1]^3} dt dt' dt'' \times \\
&\times \left\{ \sum_{e, e', e'' \in E(\gamma)} \dot{e}(t)^a \dot{e}'(t')^b \dot{e}''(t'')^c \chi_\Delta(p, e(t)) \chi_{\Delta'}(p, \frac{e(t) + e'(t')}{2}) \times \right. \\
&\times \chi_{\Delta''}(p, \frac{e(t) + e'(t') + e''(t'')}{3}) m(t, t', t'') \text{tr}(h_{e''}(0, t'') \tau_k h_{e''}(t'', 1) \frac{\partial}{\partial h_{e''}(0, 1)}) \times \\
&\times \text{tr}(h_{e'}(0, t') \tau_j h_{e'}(t', 1) \frac{\partial}{\partial h_{e'}(0, 1)}) \text{tr}(h_e(0, t) \tau_i h_e(t, 1) \frac{\partial}{\partial h_e(0, 1)}) \\
&+ \sum_{e', e'' \in E(\gamma)} \dot{e}''(t)^a \dot{e}'(t')^b \dot{e}''(t'')^c \chi_\Delta(p, e''(t)) \chi_{\Delta'}(p, \frac{e''(t) + e'(t')}{2}) \times \\
&\times \chi_{\Delta''}(p, \frac{e''(t) + e'(t') + e''(t'')}{3}) [\theta(t, t'') \text{tr}(h_{e''}(0, t) \tau_i h_{e''}(t, t'') \tau_k h_{e''}(t'', 1) \frac{\partial}{\partial h_{e''}(0, 1)}) \\
&+ \theta(t'', t) \text{tr}(h_{e''}(0, t'') \tau_k h_{e''}(t'', t) \tau_i h_{e''}(t, 1) \frac{\partial}{\partial h_{e''}(0, 1)})] \text{tr}(h_{e'}(0, t') \tau_j h_{e'}(t', 1) \frac{\partial}{\partial h_{e'}(0, 1)}) \\
&+ \sum_{e', e'' \in E(\gamma)} \dot{e}'(t)^a \dot{e}'(t')^b \dot{e}''(t'')^c \chi_\Delta(p, e'(t)) \chi_{\Delta'}(p, \frac{e'(t) + e'(t')}{2}) \times \\
&\times \chi_{\Delta''}(p, \frac{e'(t) + e'(t') + e''(t'')}{3}) \text{tr}(h_{e''}(0, t'') \tau_k h_{e''}(t'', 1) \frac{\partial}{\partial h_{e''}(0, 1)}) \times \\
&\times [\theta(t, t') \text{tr}(h_{e'}(0, t) \tau_i h_{e'}(t, t') \tau_j h_{e'}(t', 1) \frac{\partial}{\partial h_{e'}(0, 1)}) \\
&+ \theta(t', t) \text{tr}(h_{e'}(0, t') \tau_j h_{e'}(t', t) \tau_i h_{e'}(t, 1) \frac{\partial}{\partial h_{e'}(0, 1)})] \\
&+ \sum_{e, e'' \in E(\gamma)} \dot{e}(t)^a \dot{e}''(t')^b \dot{e}''(t'')^c \chi_\Delta(p, e(t)) \chi_{\Delta'}(p, \frac{e(t) + e''(t')}{2}) \times \\
&\times \chi_{\Delta''}(p, \frac{e(t) + e''(t') + e''(t'')}{3}) [\theta(t', t'') \text{tr}(h_{e''}(0, t') \tau_j h_{e''}(t', t'') \tau_k h_{e''}(t'', 1) \frac{\partial}{\partial h_{e''}(0, 1)})
\end{aligned}$$

$$\begin{aligned}
& +\theta(t'', t')\text{tr}(h_{e''}(0, t'')\tau_k h_{e''}(t'', t')\tau_j h_{e''}(t', 1)\frac{\partial}{\partial h_{e''}(0, 1)})]\text{tr}(h_e(0, t)\tau_i h_e(t, 1)\frac{\partial}{\partial h_e(0, 1)}) \\
& + \sum_{e'' \in E(\gamma)} \dot{e}''(t)^a \dot{e}''(t')^b \dot{e}''(t'')^c \chi_{\Delta}(p, e''(t)) \chi_{\Delta'}(p, \frac{e''(t) + e''(t')}{2}) \times \\
& \times \chi_{\Delta''}(p, \frac{e''(t) + e''(t') + e''(t'')}{3}) \times \\
& \times [\theta(t, t', t'')\text{tr}(h_{e''}(0, t)\tau_i h_{e''}(t, t')\tau_j h_{e''}(t', t'')\tau_k h_{e''}(t'', 1)\frac{\partial}{\partial h_{e''}(0, 1)}) \\
& +\theta(t, t'', t')\text{tr}(h_{e''}(0, t)\tau_i h_{e''}(t, t'')\tau_k h_{e''}(t'', t')\tau_j h_{e''}(t', 1)\frac{\partial}{\partial h_{e''}(0, 1)}) \\
& +\theta(t', t'', t)\text{tr}(h_{e''}(0, t')\tau_j h_{e''}(t', t'')\tau_k h_{e''}(t'', t)\tau_i h_{e''}(t, 1)\frac{\partial}{\partial h_{e''}(0, 1)}) \\
& +\theta(t', t, t'')\text{tr}(h_{e''}(0, t')\tau_j h_{e''}(t', t)\tau_i h_{e''}(t, t'')\tau_k h_{e''}(t'', 1)\frac{\partial}{\partial h_{e''}(0, 1)}) \\
& +\theta(t'', t, t')\text{tr}(h_{e''}(0, t'')\tau_k h_{e''}(t'', t)\tau_i h_{e''}(t, t')\tau_j h_{e''}(t', 1)\frac{\partial}{\partial h_{e''}(0, 1)}) \\
& + \theta(t'', t', t)\text{tr}(h_{e''}(0, t'')\tau_k h_{e''}(t'', t')\tau_j h_{e''}(t', t)\tau_i h_{e''}(t, 1)\frac{\partial}{\partial h_{e''}(0, 1)})] \Big\} f_{\gamma} \\
& =: [\hat{O}_{1,2,3} + \hat{O}_{2,3,1} + \hat{O}_{12,3} + \hat{O}_{1,23} + \hat{O}_{123}] f_{\gamma} .
\end{aligned} \tag{3.6}$$

The fact that the integrand of the terms involved in $\hat{O}_{12,3}, \hat{O}_{1,23}, \hat{O}_{2,3,1}, \hat{O}_{123}$ vanishes if either of the cases $0 < t = t' < 1, 0 < t' = t'' < 1, 0 < t = t' = t'' < 1$ occurs is due to the fact that in this case in $\hat{O}_{12,3}, \hat{O}_{1,23}, \hat{O}_{2,3,1}, \hat{O}_{123}$ we get a trace which contains $\tau_{(i}\tau_{j)}, \tau_{(j}\tau_{k)}, \tau_{(k}\tau_{i)}$ contracted with ϵ^{ijk} which vanishes (to see this recall that the functional derivative is

$$\begin{aligned}
\delta h_e(A)/\delta A_a^i(x) &= \frac{1}{2} \int_0^1 dt [\frac{1}{2} \delta^{(3)}(e(t+), x) \dot{e}(t+)^a h_e(0, t) \tau_i h_e(t, 1) \\
& + \frac{1}{2} \delta^{(3)}(e(t-), x) \dot{e}(t-)^a h_e(0, t) \tau_i h_e(t, 1)]
\end{aligned}$$

(one sided derivatives and δ distributions). This expression is also correct if x is an endpoint of e (in which case there is only one term which survives in (3.7)). which results in the case that we consider $h_{e_1} \tau_j h_{e_2}$ instead of $h_e, e = e_1 \circ e_2, x = e_1 \cap e_2$ a point of analyticity, in a term involving $\tau_{(i}\tau_{j)}$).

Given a triple e, e', e'' of (not necessarily distinct) edges of γ , consider the functions

$$x_{ee'e''}(t, t', t'') := \frac{e(t) + e'(t') + e''(t'')}{3} . \tag{3.7}$$

This function has the interesting property that the Jacobian is given by

$$3^3 \det\left(\frac{\partial(x_{ee'e''}^1, x_{ee'e''}^2, x_{ee'e''}^3)(t, t', t'')}{\partial(t, t', t'')}\right) = \epsilon_{abc} \dot{e}(t)^a \dot{e}'(t')^b \dot{e}''(t'')^a \tag{3.8}$$

which is precisely the form of the factor which enters all the integrals in (3.6). This is why we have introduced the strange argument $(x + y + z)/3$.

We now consider the limit $\Delta_i, \Delta'_i, \Delta''_i \rightarrow 0$.

Lemma 3.1 *For each triple of edges e, e', e'' there exists a choice of vectors $\vec{n}_i, \vec{n}'_i, \vec{n}''_i$ and a way to guide the limit $\Delta_i, \Delta'_i, \Delta''_i \rightarrow 0$ such that*

$$\int_{[0,1]^3} \det\left(\frac{\partial(x_{ee'e''}^a)}{\partial(t, t', t'')}\right) \chi_\Delta(p, e) \chi_{\Delta'}(p, (e + e')/2) \chi_{\Delta''}(p, (e + e' + e'')/3) \hat{O}_{ee'e''} \quad (3.9)$$

vanishes

a) *if e, e', e'' do not all intersect p or*

b) $\det\left(\frac{\partial(x_{ee'e''}^a)}{\partial(t, t', t'')}\right)_p = 0$ *(which is a diffeomorphism invariant statement).*

Otherwise it tends to

$1/8 \text{sgn}(\det(\frac{\partial(x_{ee'e''}^a)}{\partial(t, t', t'')}))_p \hat{O}_{e, e', e''}(p) \prod_{i=1}^3 \Delta''_i$. *Here we have denoted by $\hat{O}_{ee'e''}(t, t', t'')$ the trace(s) involved in the various terms of (3.6).*

Proof¹ :

If at least one of e, e', e'' does not intersect p then, if we choose Δ_i etc. smaller than some finite number Δ_0 , (3.9) vanishes identically since the support of the characteristic functions is in a neighbourhood around p which shrinks to zero with the Δ_i etc.

So let us assume that all of e, e', e'' intersect p at parameter value t_0, t'_0, t''_0 (this value is unique because the edges are not self-intersecting). Then we can write $e(t) = p + c(t - t_0)$ where c is analytic and vanishes at $\tau = t - t_0 = 0$. We have the case subdivision :

Case I : $\det(\frac{\partial(x_{ee'e''}^a)}{\partial(t, t', t'')})_{t_0} = 0$.

Case Ia) : All of $\dot{c}(0), \dot{c}'(0), \dot{c}''(0)$ are co-linear.

Case Ib) : Two of $\dot{c}(0), \dot{c}'(0), \dot{c}''(0)$ are co-linear and the third is linearly independent of them.

Case Ic) : No two of $\dot{c}(0), \dot{c}'(0), \dot{c}''(0)$ are co-linear.

Case II : $\det(\frac{\partial(x_{ee'e''}^a)}{\partial(t, t', t'')})_{t_0} \neq 0$.

Notice that all vectors $\dot{c}(0), \dot{c}'(0), \dot{c}''(0)$ are non-vanishing by the definition of a curve.

We consider first case I). We exclude the trivial case that all three curves lie in a coordinate plane or line such that the determinant already vanishes for all finite values of the Δ 's. Therefore there exist linearly independent unit vectors u, v, w (not necessarily orthogonal) in terms of which we may express c, c', c'' .

In case Ia) we have an expansion of the form

$$\begin{aligned} c(t) &= au(t + o(t^2)) + bv(t^m + o(t^{m+1})) + cw(t^n + o(t^{n+1})) \\ c'(t) &= au(t + o(t^2)) + b'v(t^{m'} + o(t^{m'+1})) + c'w(t^{n'} + o(t^{n'+1})) \\ c''(t) &= a''u(t + o(t^2)) + b''v(t^{m''} + o(t^{m''+1})) + c''w(t^{n''} + o(t^{n''+1})) \end{aligned} \quad (3.10)$$

where $a, b, c, a', b', c', a'', b'', c''$ are real numbers with $aa'a'' \neq 0$ and at least one of the b 's and c 's being different from zero (also not for instance $b = c = b' = c' = 0$). Furthermore $m, m', m'', n, n', n'' \geq 2$. The characteristic functions have support in coordinate cubes spanned by the vectors $\vec{n}_i, \vec{n}'_i, \vec{n}''_i$. Now, since u, v, w are linearly independent we may

¹The idea that one should adapt the regularization to each triple of edges was first communicated to the author by Jurek Lewandowski. Its justification relies on the fact that the classical expression does not depend on the way we regularize. What is new here is to introduce even more parameters than in [9] to be adapted to a triple which simplifies the proofs. What is also new is that we introduced the $(x + y + z)/3$ argument which makes the proof especially clear.

simply *choose*, for instance, $\vec{n}_i := u, \vec{n}'_i := v, \vec{n}''_i := w$. It follows then and from the fact that $0 \leq \chi \leq 1$ that

$$\begin{aligned} & \chi_{\Delta}(p, e) \chi_{\Delta'}(p, (e + e')/2) \chi_{\Delta''}(p, (e + e' + e'')/3) \\ &= \tilde{\chi}_{\Delta}(0, c) \tilde{\chi}_{\Delta'}(0, (c + c')/2) \tilde{\chi}_{\Delta''}(0, (c + c' + c'')/3) \\ &\leq \theta_{\Delta_1}(< c, u >) \theta_{\Delta'_1}(< (c + c')/2, v >) \theta_{\Delta''_1}(< (c + c' + c'')/3, w >) . \end{aligned} \quad (3.11)$$

From the explicit expansions of c, c', c'' we conclude that (3.11) has the bound

$$\theta_{\delta_1 \Delta_1}(t) \theta_{\delta'_1 \Delta'_1}(t') \theta_{\delta''_1 \Delta''_1}(t'') \quad (3.12)$$

for some sufficiently large numbers $\delta_1, \delta'_1, \delta''_1$. On the other hand we also see from the explicit expansion of $|\det(\frac{\partial(x_{ee'e''}^a)}{\partial(t, t', t'')})|$ around t_0 that it is bounded by $M(|t|^k + |t'|^k + |t''|^k)$ where M is a positive number and where $k = \min(m + n', m + n'', m' + n, m' + n'', m'' + n, m'' + n') - 2 \geq 2$.

The prescription of how to guide the limit in case Ia) is then to synchronize $\Delta_1 = \Delta'_1 = \Delta''_1 = \Delta$ and to take the limit $\Delta \rightarrow 0$ first. The integral is at least of order Δ^5 while we divide only by an order of Δ^3 so that the result vanishes.

In case Ib) we have an expansion of the form (let w.l.g. c, c' have co-linear tangents)

$$\begin{aligned} c(t) &= au(t + o(t^2)) + bv(t^m + o(t^{m+1})) + cw(t^n + o(t^{n+1})) \\ c'(t) &= au(t + o(t^2)) + b'v(t^{m'} + o(t^{m'+1})) + c'w(t^{n'} + o(t^{n'+1})) \\ c''(t) &= a''v(t + o(t^2)) + b''u(t^{m''} + o(t^{m''+1})) + c''w(t^{n''} + o(t^{n''+1})). \end{aligned} \quad (3.13)$$

We now argue as above and find that the product of the characteristic functions can be estimated by

$$\theta_{\delta_1 \Delta_1}(t) \theta_{\delta'_1 \Delta'_1}(t') \theta_{\delta''_2 \Delta''_2}(t'')$$

while the determinant can be estimated as above just that k is now given by $k = \min(m, m', m'', n, n', n'') - 1 \geq 1$.

The prescription is now $\Delta_1 = \Delta'_1 = \Delta''_2 =: \Delta \rightarrow 0$ first and we conclude that the integral is at least of order Δ^4 while we divide again only by Δ^3 such that the limit vanishes.

In case Ic) finally we have an expansion of the form

$$\begin{aligned} c(t) &= au(t + o(t^2)) + bv(t^m + o(t^{m+1})) + cw(t^n + o(t^{n+1})) \\ c'(t) &= av(t + o(t^2)) + b'v(t^{m'} + o(t^{m'+1})) + c'w(t^{n'} + o(t^{n'+1})) \\ c''(t) &= a''u(t + o(t^2)) + b''v(t + o(t^2)) + c''w(t^{n''} + o(t^{n''+1})). \end{aligned} \quad (3.14)$$

This time we estimate the product of the characteristic functions for instance by

$$\theta_{\delta_1 \Delta_1}(t) \theta_{\delta'_2 \Delta'_2}(t') \theta_{\delta''_2 \Delta''_2}(t'')$$

while the determinant can be estimated as above and k is given by $k = \min(m, m', n, n', n'') - 1 \geq 1$ so that we have actually the same situation as in case Ib) upon synchronizing this time $\Delta_1 = \Delta'_2 = \Delta''_2 =: \Delta \rightarrow 0$.

As for case II) we observe that the nonvanishing of the functional determinant at p implies that the map $x_{ee'e''}$ is actually invertible in a neighbourhood of p by the inverse function theorem. In other words, there is only one point (t_0, t'_0, t''_0) such that $x_{ee'e''}(t_0, t'_0, t''_0) = p$. Moreover, since the determinant is non-vanishing at p , all three edges must be distinct from each other. It follows now from our choice of edges that p must be a vertex $v = e \cap e' \cap e''$ of γ in order that the result is non-vanishing and thus from the choice of parametrization $t_0 = t'_0 = t''_0 = 0$.

Therefore, if we take the limit $\Delta''_i \rightarrow 0$ first in any order then the condition $\chi_{\Delta''}(p, x_{ee'e''}) = 1$ will actually imply $\chi_{\Delta}(p, e) = \chi_{\Delta'}(p, (e + e')/2) = 1$ for small enough Δ''_i so that we can take these characteristic functions out of the integral and replace them by 1 if p is a common vertex of all three edges. Also we can replace the operator $\hat{O}_{ee'e''}(t, t', t'')$ by $\hat{O}_{ee'e''}(v)$. This holds only if the triple intersects in p .

If not all of e, e', e'' intersect in p then the limit will vanish anyway if we take a suitable limit of the Δ_i as we have shown before. We can account for that case by replacing $\chi_{\Delta}(p, e), \chi_{\Delta'}(p, (e + e')/2)$ by $\chi_{\Delta}(p, v)\chi_{\Delta'}(p, v)$. Here v is the common vertex at which the distinct e, e', e'' must be incident otherwise they could not even pass through a small enough neighbourhood of p . We can also assume that all three edges have linearly independent tangents at v and expand still around $t = 0$. The remaining integral divided by $\Delta''_1\Delta''_2\Delta''_3$ then tends to

$$\int_{[0,1]^3} d^3t \det\left(\frac{\partial x_{ee'e''}}{\partial t}\right) \delta^{(3)}(p, x_{ee'e''}) = s(e, e', e'') \int_{C_{ee'e''}} d^3x \delta^{(3)}(p, x) = \frac{1}{8} s(e, e', e'') \quad (3.15)$$

where

$$s(e, e', e'')_v := \text{sgn}(\det(\dot{e}(0), \dot{e}'(0), \dot{e}''(0))). \quad (3.16)$$

The factor $1/8$ is due to the fact that in the limit $\Delta'' \rightarrow 0$ we obtain an integral over $C(e, e', e'')$, the cone based at p and spanned by $\dot{e}(0), \dot{e}'(0), \dot{e}''(0)$ where the orientation is taken to be positive. This integral just equals $\int_{\mathbb{R}_+^3} d^3t \delta(0, t) \delta(0, t') \delta(0, t'') = 1/8$ as one can easily check. This furnishes the proof.

□

We conclude that (3.6) reduces to (in particular, the operators $\hat{O}_{12,3}\hat{O}_{1,23}\hat{O}_{2,31}\hat{O}_{123}$ drop out)

$$\lim_{\Delta'' \rightarrow 0} \hat{E}(p, \Delta, \Delta', \Delta'') f = \sum_{e, e', e''} \frac{i \ell_p^6 s(e, e', e'')_v}{8 \cdot 3! \text{vol}(\Delta) \text{vol}(\Delta')} \chi_{\Delta}(p, v) \chi_{\Delta'}(p, v) \frac{3^3}{8} \hat{O}_{e, e', e''}(0, 0, 0)$$

where v on the right hand side is the intersection point of the triple of edges and it is understood that we only sum over such triples of edges which are incident at a common vertex. Moreover,

$$\hat{O}_{e, e', e''}(0, 0, 0) = \frac{1}{8} \epsilon_{ijk} X_{e''}^i X_{e'}^j X_e^k \text{ and } X_e^i := X^i(h_e(0, 1)) := \text{tr}(\tau_i h_e(0, 1)) \frac{\partial}{\partial h_e(0, 1)} \quad (3.17)$$

is a right invariant vector field² in the τ_i direction of $su(2)$, that is, $X(hg) = X(h)$. The factor of $1/8$ in (3.17) comes from the fact that at $t = t' = t'' = 0$ we get only one-sided functional derivatives of the various edges involved ($m(0, 0, 0) = 1/8$, see above). We

²This important observation, which is fundamental for all that follows, is due to Abhay Ashtekar and Jurek Lewandowski

also have extended the values of the sign function to include 0 which takes care of the possibility that one has triples of edges with linearly dependent tangents.

The final step consists in choosing $\Delta = \Delta'$ and taking the square root of the modulus. We replace the sum over all triples incident a common vertex $\sum_{e,e',e''}$ by a sum over all vertices followed by a sum over all triples incident at the same vertex $\sum_{v \in V(\gamma)} \sum_{e \cap e' \cap e'' = v}$. Now, for small enough Δ and given p , at most one vertex contributes, that is, at most one of $\chi_\Delta(v, p) \neq 0$ because all vertices have finite separation. Then we can take the relevant $\chi_\Delta(p, v) = \chi_\Delta(p, v)^2$ out of the square root and take the limit which results in

$$\begin{aligned}
\hat{V}(R)_\gamma &= \int_R d^3p \sqrt{\widehat{\det(q)(p)}}_\gamma = \int_R d^3p \hat{V}(p)_\gamma \\
\hat{V}(p)_\gamma &= \ell_p^3 \sum_{v \in V(\gamma)} \delta^{(3)}(p, v) \hat{V}_{v,\gamma} \\
\hat{V}_{v,\gamma} &= \sqrt{\left| \frac{i}{3! \cdot 8} \left(\frac{3}{4}\right)^3 \sum_{e,e',e'' \in E(\gamma), e \cap e' \cap e'' = v} s(e, e', e'') q_{ee'e''} \right|} \\
q_{ee'e''} &= \epsilon_{ijk} X_e^i X_{e'}^j X_{e''}^k
\end{aligned} \tag{3.18}$$

where we could switch the order of the X 's because a triple contributes only if the corresponding edges are distinct and so the X 's commute.

Expression (3.18) is the final expression for the volume operator and coincides up to a factor $\sqrt{27/8}$ with the expression found in [5, 9] while it is genuinely different from the one found in [4] as pointed out in [9]. Note that the final expression is manifestly diffeomorphism covariant. Although the procedure of adapting the limiting to a given triple of edges is somewhat non-standard there is an argument in favour of such a procedure : the discussion in lemma (3.1) reveals that any other regularization which would result in a finite contribution for the case where $s(e, e'e'')$ is zero would necessarily depend on the higher order intersection characteristics of a triple of edges. However, since such a quantity is not diffeomorphism covariant which is unacceptable, the dependence must be trivial, that is, a constant (this is precisely what happens in (3.4) according to (3.11). While there is no a priori *kinematical* reason to prefer one operator over the other, there is a *dynamical* reason : the volume operator can be seen as an essential ingredient in the regularization of the Lorentzian Wheeler-DeWitt constraint operator [2, 3]. The fact that (3.18) contains $s(e, e', e'')$ is the precise reason for this operator to be anomaly-free ! Therefore, the regularization leading to (3.18) is the one singled out by the dynamics of the theory !

4 Spectral Analysis

In this section we summarize a number of results due to Abhay Ashtekar and Jurek Lewandowski obtained in the series of papers [18, 5, 9]. We have included them here for the sake of self-containedness of the present paper. More specifically, the cylindrical consistency of the family of volume operators \hat{V}_γ as defined in [18] and in the present paper, the self-adjointness of the corresponding projective limit and the fact that the spectrum of the volume operator is discrete was first observed in [9]. The proof of these properties given in [9] uses methods developed in [18, 1, 5, 9].

This section is subdivided into three parts. First we prove that the family of operators derived in (3.18) defines a linear unbounded operator on \mathcal{H} . Next we show that the operator is symmetric, positive semi-definite and admits self-adjoint extensions and finally we show that its spectrum is discrete and that the operator so defined is anomaly-free.

4.1 Cylindrical Consistency

What we have obtained in (3.18) is a family of operators $(\hat{V}(R)_\gamma, D_\gamma)_{\gamma \in \Gamma}$. That is not enough to show that this family of cylindrical projections “comes from” a linear operator on \mathcal{H} . As proved in [18], for this to be the case we need to check that whenever $\gamma \subset \gamma'$ then

- 1) $p_{\gamma\gamma'}^* D_\gamma \subset D_{\gamma'}$ where $p_{\gamma\gamma'}$ is the restriction from γ' to γ . This condition makes sure that the operator defined on bigger graphs can be applied to functions defined on smaller graphs.
- 2) $(\hat{V}(R)_{\gamma'})|_\gamma = \hat{V}(R)_\gamma$, this is the condition of cylindrical consistency and says that the operator on bigger graphs equals the operator on smaller graphs when restricted to functions thereon.

A graph $\gamma \subset \gamma'$ can be obtained from a bigger graph γ' by a finite series of steps consisting of the following basic ones :

- i) remove an edge from γ' ,
- ii) join two edges e', e'' , such that $e' \cap e''$ is a point of analyticity, to a new edge $e = e' \circ (e'')^{-1}$
- iii) reverse the orientation of an edge.

Clearly, a dense domain for $\hat{V}(R)_\gamma$ is given by $D_\gamma := \text{Cyl}_\gamma^3(\overline{\mathcal{A}/\mathcal{G}})$. This choice trivially satisfies requirement 1) since functions which just do not depend on some arguments or only on special combinations $h_e = h_{e'} h_{e''}, h_{e'} = h_e^{-1}$ are still thrice continuously differentiable if the original function was (here we have used the fact that $SU(2)$ is a Lie group, that is, group multiplication and taking inverses is an analytic map).

Next, let us check cylindrical consistency. Consider first the case i) that γ does not depend on an edge e on which γ' does. Then clearly $X_e^i f_\gamma = 0$ for any function cylindrical with respect to γ and so in the sum over triples over vertices in (3.18) the terms involving e drop out.

Next consider the case ii). If $e = e' \circ (e'')^{-1}$ is an edge of γ and e', e'' are edges of γ' where $v := e' \cap e''$ is a point of analyticity for γ while for γ' it is not, then while v is a vertex for γ' it is only a pseudo-vertex for γ and so in $\hat{V}(D)_\gamma$ there is no term corresponding to v . On the other hand, since the vertex v is a pseudo vertex for γ it is in particular only two-valent and so the corresponding term in $\hat{V}(D)_{\gamma'}$ drops out. Likewise, if v is a vertex for γ at which the outgoing edge e is incident, then from right invariance of the vector field we have $X_e = X_{e' \circ (e'')^{-1}} = X_{e'}$ and so at vertices that belong to both γ and γ' the corresponding vertex operators coincide.

Finally, case iii) is actually excluded by our unambiguous choice of orientation.

We conclude that there exists an operator $(\hat{V}(R), D)$ on \mathcal{H} which is densely defined on $D = \text{Cyl}^3(\overline{\mathcal{A}/\mathcal{G}})$.

4.2 Symmetry, Positivity and Self-Adjointness

Notice that the vector field iX_e is symmetric on \mathcal{H}_γ , the completion of $\text{Cyl}_\gamma^1(\overline{\mathcal{A}/\mathcal{G}})$ with respect to $\mu_{0,\gamma}$, e an edge of γ , because the Haar measure is right invariant. It follows

from the explicit expression (3.18) in terms of the iX_e that all the projections $\hat{V}(R)_\gamma$ are symmetric. In this special case (namely, the volume operator leaves the space D_γ invariant) this is enough to show that $\hat{V}(R)$ is symmetric on D . Furthermore, all $\hat{V}(R)_\gamma$ are positive semi-definite by inspection so that $\hat{V}(R), D$ is a densely defined, positive semidefinite and symmetric operator. It follows that it has self-adjoint extensions, for instance its Friedrich extension.

4.3 Discreteness and Anomaly-freeness

The operator $\hat{V}(D)$ has the important property that it leaves the dense subset $\text{Cyl}_\gamma^\infty(\overline{\mathcal{A}/\mathcal{G}}) \subset \mathcal{H}$ invariant, separately for each $\gamma \in \Gamma$. Moreover, recall that \mathcal{H} possesses a convenient basis, called the spin-network basis in the sequel [21, 22, 23].

Let us recall the most important features of that basis. Given a graph γ , let us associate with each edge $e \in E(\gamma)$ an spin quantum number $j_e > 0$. Also, with each vertex $v \in V(\gamma)$ we associate a certain contraction matrix c_v as follows : if π_j is the irreducible representation of $SU(2)$ corresponding to spin j then form the tensor product $\otimes_{e \in E(\gamma)} \pi_{j_e}(h_e)$ and for each vertex v contract the indices A of the matrices $\pi_{j_e}(h_e)_{AB}$ for all e incident at v in a gauge invariant fashion. We obtain a gauge invariant function $T_{\gamma, \vec{j}, \vec{c}}(A)$, called a spin-network function. It turns out that given γ, \vec{j} there are only a finite number of linearly independent \vec{c} compatible with γ, \vec{j} . Moreover, one can show that spin-network functions defined on different graphs are orthogonal and even on the same graph they are orthogonal whenever the vectors \vec{j} don't equal each other. It follows that there is an orthonormal basis

$$\langle T_{\gamma, \vec{j}, \vec{c}}, T_{\gamma', \vec{j}', \vec{c}'} \rangle = \delta_{\gamma, \gamma'} \delta_{\vec{j}, \vec{j}'} \delta_{\vec{c}, \vec{c}'} .$$

Now it is obvious that the operator $\hat{V}(R)$ leaves the finite dimensional vector space $U_{\gamma, \vec{j}}$ spanned by spin-network states compatible with γ, \vec{j} invariant. The matrix

$$(V(R)_{\gamma \vec{j}})_{\vec{c}, \vec{c}'} := \langle T_{\gamma, \vec{j}, \vec{c}} | \hat{V}(R) | T_{\gamma, \vec{j}, \vec{c}'} \rangle \quad (4.1)$$

is therefore finite-dimensional, positive semi-definite and symmetric. The task of computing its eigenvalues therefore becomes a problem in linear algebra !

Next, since from (3.18)

$$\hat{V}(R)_\gamma = \ell_p^3 \sum_{v \in V(\gamma) \cap R} \hat{V}_{v, \gamma} \quad (4.2)$$

and since $\hat{V}_{v, \gamma}$ involves only those $e \in E(\gamma)$ with $v \in e$, we find that $\hat{V}_{v, \gamma}$ can only change the entry c_v in \vec{c} . In other words, $[\hat{V}_{v, \gamma}, \hat{V}_{v', \gamma}] = 0$ and each $\hat{V}_{v, \gamma}$ can be diagonalized separately.

Finally, since the spins j_e only take discrete values it follows that \mathcal{H}_γ has a countable basis and the spectrum that $\hat{V}(R)$ attains on D_γ is therefore pure point. Let us check whether this is the complete spectrum. Assume it were not and let \hat{P} the spectral projection on the rest of the spectrum (the existence of the spectral projections relies on the fact that $\hat{V}(R)$ is self-adjoint and not only symmetric). It follows that $u = \hat{P}v$ is orthogonal to D_γ where v is any vector in \mathcal{H}_γ . But D_γ is dense in \mathcal{H}_γ and so we find for every $\epsilon > 0$ a $\phi \in D_\gamma$ with $\|u - \phi\| < \epsilon$. Now we have from orthogonality $\epsilon^2 > \|u - \phi\|^2 = \|u\|^2 + \|\phi\|^2 > \|u\|^2$ and so $u = 0$. This shows that the complete spectrum is already attained on D_γ . It is

purely discrete as well³.

Last, we wish to show that the volume operators are anomaly free (given the fact that we have largely adapted our regularization to a graph, this statement is far from trivial). By this we mean the following : given any two open sets $R_1, R_2 \subset \Sigma$ we have vanishing Poisson brackets $\{V(R_1), V(R_2)\} = 0$ because the functionals $V(R)$ depend on the momentum variable $E_i^a(x)$ only. Now, given a function f cylindrical with respect to a graph γ , it is not at all obvious any more that $[\hat{V}(R_1), \hat{V}(R_2)]f = 0$ for any such f . Fortunately, given the above characterization of the spectrum, the commutator can be easily proved to vanish on cylindrical functions. To see this, note that the above results imply that if we choose any region $R(\gamma)$ such that $\gamma \subset R(\gamma)$ then there exists an eigenbasis of \mathcal{H}_γ of $\hat{V}(R(\gamma))$. Now consider any region R . Since all regions are open by construction, all regions fall into equivalence classes with respect to γ : R, R' are equivalent if they contain the same vertices of γ (any vertex either has a neighbourhood which lies completely inside R or it lies outside). Therefore any two $\hat{V}(R), \hat{V}(R')$ differ at most by some of the $\hat{V}_{v,\gamma}$ all of which are contained in the expression for $\hat{V}(R(\gamma))$. Since the $\hat{V}_{v,\gamma}$ commute the eigenbasis of $\hat{V}(R(\gamma))$ is a simultaneous eigenbasis of all $\hat{V}_{v,\gamma}$ for all $v \in V(\gamma)$ and so this eigenbasis is a simultaneous eigenbasis of all $\hat{V}(R)_\gamma$. Since all \mathcal{H}_γ are orthogonal, we have a simultaneous eigenbasis for all $\hat{V}(R)$.

While it is in general not enough to verify that two self-adjoint, unbounded operators commute on a dense domain (rather, by definition, we have to check that the associated spectral projections commute) in our case we are done because the spectral projections *are* the projections on the various D_γ because the point spectrum is already the complete spectrum. Thus we have verified that the commutator algebra mirrors the classical Poisson algebra.

5 The complete set of matrix elements

In this final section we are going to compute (4.1) in a spin-network basis for each $\gamma, \vec{j}, \vec{e}, \vec{e}', R$. The advantage of our approach as compared to the ones proposed already in the literature [10] will be obvious : in [10] one works with an overcomplete and non-orthogonal set of states. This prevents one from giving a unified treatment as it is done here and only allowed one to find a formula for the eigenvalues in a few simple cases. More generally one has to check all the time the linear independence of the states and can compute the matrix elements only in a case by case analysis.

In the next subsection we show in more detail a result which follows immediately from [1, 5, 9] : namely, that the problem of computing the matrix elements of the volume operator reduces to that of computing the matrix elements of a homogenous polynomial of degree three of spin operators for a spin system.

This opens access to powerful techniques well-known from the quantum theory of angular momentum for spin systems. Then we will show how the actual computation

³in the physical sense that it is attained on a countable basis so that the eigenvalues only comprise a countable set. In a mathematical sense one would need to check that there are no accumulation points and no eigenvalues of infinite multiplicity. To settle this question it would be enough to show that the formula for the eigenvalue is an unbounded function of $j := \sum_{e \in E(\gamma)} j_e$ because every countably infinite set of j 's corresponding to different choices of \vec{j} diverges. This is one possible future application of the explicit matrix element formulae which we derive in the next section.

can be done by viewing the problem as a task in spin recoupling theory and that all the matrix elements can be written as algebraic functions of $6j$ symbols for which a closed formula exists (Racah formula). Finally we will diagonalize the volume operator for an infinite number of (very special) graphs.

5.1 The left regular representation and the spin-network representation

Consider a spin-network state $T_{\gamma, \vec{j}, \vec{c}}(A)$. This is a state for the connection representation $L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ in the sense that the (equivalence classes modulo gauge transformations of) connection operators are diagonal in this representation. Notice that a spin-network state is gauge invariant so that it can be viewed alternatively as a function of $[A]$, the gauge equivalence class of A . Here we will use the fact that $d\mu_0$ extends to $\overline{\mathcal{A}}$ [20] and reduces to the measure indicated in the introduction when integrating gauge invariant functions. Consider the usual Dirac (generalized) states δ_A defined by $\delta_A(A') := \delta_{\mu_0}(A, A')$ where δ_{μ_0} is the δ distribution with respect to μ_0 [23] and $A, A' \in \overline{\mathcal{A}}$. We define the *spin-network representation* by

$$\delta_A(T_{\gamma, \vec{j}, \vec{c}}) := \int_{\overline{\mathcal{A}}} d\mu_0(A') \overline{\delta_A(A')} T_{\gamma, \vec{j}, \vec{c}}(A') = T_{\gamma, \vec{j}, \vec{c}}(A) =: \langle A | \gamma, \vec{j}, \vec{c} \rangle . \quad (5.1)$$

So far this is only notation. The advantage of this notion will become transparent upon proving that the states $|\gamma, \vec{j}, \vec{c}\rangle$ are nothing else than the usual angular momentum states associated with an abstract spin system.

To see this, let v be a vertex of γ and e_1, \dots, e_n the edges of γ incident at v and we have chosen orientations such that all of them are outgoing at v . The spin-network state $T_{\gamma, \vec{j}, \vec{c}}(A)$ can be written

$$\begin{aligned} T_{\gamma, \vec{j}, \vec{c}}(A) &= (c_v)_{m_1 \dots m_n} \prod_{i=1}^n (\pi_{j_{e_i}}(h_{e_i}(A)))_{m_i m'_i} (M_v)_{m'_1 \dots m'_n} \\ &= \text{tr}(c_v \cdot \otimes_{i=1}^n \pi_{j_{e_i}}(h_{e_i}(A)) \cdot M_v) \end{aligned} \quad (5.2)$$

where c_v is the vertex contractor corresponding to v which contracts the indices corresponding to the starting points of the e_i and M_v contracts the indices corresponding to the endpoints (recall that the holonomy is a path ordered exponential with the smallest parameter value to the left). Note that c_v is \mathbb{C} -valued while M_v still depends on the rest of the graph.

Consider a local gauge transformation at v , that is, $h_{e_i} \rightarrow gh_{e_i}$ for some $g \in SU(2)$. Then (5.2) becomes $\text{tr}(c_v \cdot \otimes_{i=1}^n \pi_{j_{e_i}}(g) \cdot \otimes_{i=1}^n \pi_{j_{e_i}}(h_{e_i}(A)) \cdot M_v)$. In order for this to be gauge invariant, notice that the tensor product representation $\otimes_i \pi_{j_{e_i}}(g)$ can be written as a direct sum of orthogonal irreducible representations $\pi_j(g)$, and so we just need to choose c_v to be proportional to one of the various equivalent (but orthogonal) trivial representations $\pi_0(g) = \pi_0(1)$ contained in that decomposition (see [23] for more details) in order for (5.2) to be gauge invariant. This is precisely what a spin-network state is.

The point is now that

$$\prod_{i=1}^n (\pi_{j_{e_i}}(h_{e_i}(A)))_{m_i m'_i} =: \langle A | (m_1, j_1), \dots, (m_n, j_n); m'_1, \dots, m'_n \rangle \quad (5.3)$$

transforms *precisely* as the usual angular momentum eigenstates

$$|(m_1, j_1), \dots, (m_n, j_n); k_1, \dots, k_m \rangle := \prod_{i=1}^n |m_i, j_i; k_1, \dots, k_m \rangle$$

of an abstract spin system of n spins j_1, \dots, j_n under rotations (see, e.g., [24]). This is because $|(m, j); k \rangle$ transforms as $\hat{U}(g)|m, j; k \rangle = \pi_j(g)_{m, m'}|(m', j); k \rangle$ by definition, where $\hat{U}(g)$ is a unitary representation of the rotation group. Here k_1, \dots, k_m are some additional quantum numbers of observables commuting with the operators \hat{J}_i which in our case coincide with the m'_1, \dots, m'_n . In other words, the left regular representation $L_g f(h_e) = f(gh_e)$ that we are dealing with in the connection representation can be seen to be equivalent with the representation $\hat{U}(g)$ of the spin-network representation.

Let us now couple the spins j_1, \dots, j_n to resulting spin $j = 0$ and denote a particular state with zero angular momentum by $|0 \rangle$. We claim that up to a constant

$$(c_v)_{m_1 \dots m_n} = \langle (m_1, j_1), \dots, (m_n, j_n); k_1, \dots, k_m | 0 \rangle \quad (5.4)$$

where the inner product is now to be understood in the angular momentum Hilbert space of the abstract spin system. In other words, the c_v are just usual Clebsh-Gordan coefficients. The proof is easy, we have

$$\begin{aligned} & (c_v)_{m_1 \dots m_n} \prod_{i=1}^n \pi_{j_i}(g)_{m_i m'_i}(g) \\ &= \langle \hat{U}(g)((m'_1, j_1), \dots, (m'_n, j_n); k_1, \dots, k_m) | 0 \rangle \\ &= \langle (m'_1, j_1), \dots, (m'_n, j_n); k_1, \dots, k_m | \hat{U}(g^{-1})0 \rangle \\ &= (c_v)_{m'_1 \dots m'_n} \end{aligned}$$

where we have used unitarity of the group and of the representation and that $\hat{U}(g)|0 \rangle = |0 \rangle$ is rotation invariant. The factor of proportionality is seen to equal unity because $c_v^2 = 1$ and since $\sum_{m_1 \dots m_n} |(m_1, j_1) \dots (m_n, j_n); k_1 \dots k_m \rangle \langle (m_1, j_1) \dots (m_n, j_n); k_1 \dots k_m| = 1$ due to completeness and Wigner-Eckart theorem [24].

Next, consider the self-adjoint right invariant vector fields $Y_e := -i/2 X_e$. We have $[X^i, X^j] = -2\epsilon^{ijk} X^k$ and so $[Y^i, Y^j] = i\epsilon^{ijk} Y^k$ which is the usual angular momentum algebra. It follows that (we use a dual interpretation of the $SU(2)$ indices A, B, C, \dots which we may choose to take values $\pm 1/2$ and so can be interpreted to be the eigenvalues of the 3-component of angular momentum)

$$\begin{aligned} & |j = 1/2, C \rangle Y_e^j h_e(A)_{CD} = \langle j = 1/2, C | (-i/2)(\tau_j h_e(A))_{CD} \\ &= (i/2) \langle (\tau_j)_{EC} (1/2, C) | h_e(A) \rangle_{ED} \\ &= (i/2) \frac{d}{dt}_{t=0} \langle \hat{U}(\exp(t\tau_j)(1/2, E) | h_e(A) \rangle_{ED} = \langle 1/2, E | J_j h_e(A) \rangle_{ED} \quad (5.5) \end{aligned}$$

and so by multilinearity of the tensor product representation we find due to self-adjointness of J_I (set $Y_I := Y_{e_I}$)

$$\begin{aligned} & Y_I^j T_{\gamma, \vec{j}, \vec{c}}(A) \\ &= \frac{i}{2} \frac{d}{dt}_{t=0} \langle \vec{j}, \vec{m} \rangle \langle A | \gamma, \vec{j}, \vec{m}; \vec{m}' | \hat{U}(-\exp(t\tau_j))_I 0 \rangle (M_v)_{m'_1 \dots m'_n} \\ &= \langle \vec{j}_v, \vec{m} \rangle \langle A | \gamma, \vec{j}_v, \vec{m}; \vec{m}' | J_I^j 0 \rangle (M_v)_{m'_1 \dots m'_n} \quad (5.6) \end{aligned}$$

where \vec{j}_v comprises the components of \vec{j} which correspond to the edges incident at v . Since spin-network states are thus characterized by the various zero-angular momentum

eigenstates $|0\rangle$ of the abstract spin system, there is a one-to-one correspondence between the possible choices of c_v and $|0\rangle$. So let $|0\rangle$ correspond to c_v and $|0'\rangle$ to c'_v . Let $p(\vec{Y})$ be any polynomial of the Y_I^i corresponding to the vertex v then we have for its matrix elements

$$\begin{aligned}
& \langle T_{\gamma, \vec{j}, \vec{c}} | p(\vec{Y}) | T_{\gamma, \vec{j}, \vec{c}'} \rangle \\
&= \int_{\mathcal{A}/\mathcal{G}} d\mu_0(A) \langle \vec{j}_v, \vec{m} | 0 \rangle \langle A | \gamma, \vec{j}_v, \vec{m}; \vec{m}' \rangle (M_v)_{m'_1, \dots, m'_n} \langle \vec{j}_v, \vec{m} | p(\vec{J}) 0' \rangle \times \\
&\quad \times \langle A | \gamma, \vec{j}_v, \vec{m}; \vec{m}' \rangle (M_v)_{\vec{m}'_1, \dots, \vec{m}'_n} \\
&= \langle \vec{j}_v, \vec{m} | 0 \rangle \langle \vec{j}_v, \vec{m} | p(\vec{J})^\dagger 0' \rangle \prod_{v' \neq v} \delta_{c_v, c'_{v'}} \\
&= \langle 0 | p(\vec{J})^\dagger 0' \rangle \prod_{v' \neq v} \delta_{c_v, c'_{v'}} = \langle p(\vec{J}) 0 | 0' \rangle \prod_{v' \neq v} \delta_{c_v, c'_{v'}}. \tag{5.7}
\end{aligned}$$

Here we have used (5.6) in the first step, the orthonormality of the spin-network states in the second step and in the last step we have again recalled that $\sum_{\vec{m}} |\vec{j}_v, \vec{m}\rangle \langle \vec{j}_v, \vec{m}| = 1$ is the identity on the abstract angular momentum Hilbert space labelled by the spins \vec{j}_v (completeness, notice that we sum over \vec{m} in (5.7)).

Expression (5.7) is a huge simplification : in order to compute the matrix elements of the volume operator it is enough to compute the matrix elements of the angular momentum operator polynomial $\epsilon_{ijk} J_e^i J_{e'}^j J_{e''}^k$ between states that have zero angular momentum coming from \vec{j}_v . This task is devoted to the next subsection.

5.2 Recoupling Theory for n angular momenta

The first task is to label all the linearly independent states $|0\rangle$ which correspond to the coupling of n angular momenta j_1, \dots, j_n to zero angular momentum. This is a well known problem in the quantum theory of angular momentum and runs under the name recoupling theory. We review here what we need for our computation.

The states $|\vec{j}, \vec{m}\rangle$ are a complete basis for a spin system consisting of n angular momenta. In this basis the $2n$ mutually commuting operators $(J_I^i)^2, J_I^3$ are diagonal with eigenvalues $j_I(j_I + 1), m_I$.

We are obviously interested in a basis in which the square of the operator of total angular momentum $J := J_1 + \dots + J_n$ and its 3- component are diagonal with eigenvalues $j(j + 1), m$. In order to label that basis we need $2(n - 1)$ more quantum numbers of mutually commuting operators and commuting with J^2, J_3 . The n operators J_I^2 are readily verified to satisfy these conditions but we need $n - 2$ more. In order to motivate our choice notice that $\hat{V}_{v, \gamma}$ is the square root of the modulus of $i(3/2)^3 \frac{1}{4} \sum_{I < J < K} \epsilon(e_I, e_J, e_K) q_{IJK}$ where

$$\frac{1}{4} q_{IJK} := \frac{1}{8} \epsilon_{ijk} X_I^i X_J^j X_K^k = [Y_I^i Y_J^j, Y_J^j Y_K^k] = \frac{1}{4} [(Y_{IJ}^i)^2, (Y_{JK}^j)^2]$$

where $Y_{IJ} := Y_I + Y_J$. Here we have used total antisymmetry of $\epsilon(e_I, e_J, e_K), \hat{q}_{IJK}$ and the fact that $[X_I, X_J] = 0, I \neq J$ in order to get rid of $3!$ and in order to sum only over $I < J < K$. This observation⁴ urges us to study the operator

$$\hat{q}_{IJK} := [J_{IJ}^2, J_{JK}^2] \text{ where } J_{IJ} = J_I + J_J \tag{5.8}$$

⁴This fact that the third order polynomial $\text{tr}(X_I[X_J, X_K])$ in the X can be written as a commutator of two second order polynomials $\text{tr}(X_I X_J)$ which are gauge invariant was made independently also by Abhay Ashtekar and Jurek Lewandowski

and thus it would be convenient if we could work in a basis in which the J_{IJ} also were diagonal. Clearly we cannot achieve this for all $n(n-1)/2$ operators J_{IJ} but we can work in different bases adapted to \hat{q}_{IJK} for a particular triple $I < J < K$ and expand them in terms of each other.

We will develop a theory of recoupling schemes here which is somewhat different from the one encountered in the standard literature and we also consider arbitrary n . The following paragraphs will therefore be somewhat detailed.

Definition 5.1 *A coupling scheme based on a pair $1 \leq I < J \leq n$ is an orthonormal basis $|\vec{g}(IJ), \vec{j}, j, m\rangle$, diagonalizing besides $J^2, J_3, J_I^2, I = 1, \dots, n$ the squares of the additional $n-2$ operators G_2, \dots, G_{n-1} where :*

$$G_1 := J_I, G_2 := G_1 + J_J, G_3 := G_2 + J_1, G_4 := G_3 + J_2, \dots, G_{I+1} := G_I + J_{I-1}, G_{I+2} := G_{I+1} + J_{I+1}, G_{I+3} := G_{I+2} + J_{I+2}, \dots, G_J := G_{J-1} + J_{J-1}, G_{J+1} := G_J + J_{J+1}, G_{J+2} := G_{J+1} + J_{J+2}, \dots, G_n := G_{n-1} + J_n = J.$$

The vector $\vec{g}(IJ) := (g_2(IJ), \dots, g_{n-1}(IJ))$ denotes the eigenvalues of the squares of G_2, \dots, G_{n-1} . We will call the coupling scheme based on the pair (12) the standard basis.

Let us check that the G_I satisfy the angular momentum algebra and that the squares of the operators J, J_I, G_I and the operator J^3 are mutually commuting : for this it will be enough to check that the operators for the standard basis are mutually commuting since there is a relabelling of indices such that any coupling scheme becomes the standard basis. Since $[J_I, J_J] = 0, I \neq J$ we find $[G_I^i, G_I^j] = \sum_{J=1}^I [J_J^i, J_J^j] = i\epsilon_{ijk} \sum_{J=1}^I J_J = i\epsilon_{ijk} G_I^k$ which implies that G_I^2 has spectrum $g_I(g_I + 1)$ with g_I integral or half integral.

Let $I < J$, then we can write $G_J = G_I + (J_{I+1} + \dots + J_J) =: G_I + G'_I$ so that $[G_I, G'_I] = 0$ because G_I, G'_I involve disjoint sets of J_K 's and so $[G_J^2, G_I^2] = [G_I^2 + (G'_I)^2, G_I^2] + 2(G'_I)^i [G_I^j, G_I^j] + [G_I^i, G_I^j] G_I^j = 2i(G'_I)^i \epsilon_{ijk} \{G_I^j G_I^k + G_I^k G_I^j\} = 0$. The same argument can be used for the other commutators.

The virtue of this definition is the following : we have now a neat labelling of the vectors $|0\rangle$ referred to in the previous subsection : we can choose them to be the vectors of the standard basis corresponding to $j = 0$, that is, they are just the vectors $|\vec{g}(12), \vec{j}, j = 0, m = 0\rangle$ ($m = 0$ is forced by $j = 0$) and the different vertex contractors just correspond to the various possible values of the intermediate (or so-called re-coupling) vectors G_2, \dots, G_{n-1} . We will therefore compute the matrix elements of \hat{q}_{IJK} in this basis. In the sequel we will drop the indices $\vec{j}, j = 0, m = 0$ since they are the same in every coupling scheme and since \hat{q}_{IJK} commutes with $J^2, J_I^2, (J)^3$ matrix elements between vectors with different values of these operators therefore vanish anyway. We will also write \vec{g} instead of $\vec{g}(12)$.

After these long preparations we are now in the position to complete the derivation. We use completeness of the bases $|\vec{g}(IJ), \vec{j}, j, m\rangle$ and have

$$\begin{aligned} & \langle \vec{g} | \hat{q}_{IJK} | \vec{g}' \rangle \\ &= \sum_{\vec{g}(IJ)} g_2(IJ)(g_2(IJ) + 1) [\langle \vec{g} | \vec{g}(IJ) \rangle \langle \vec{g}(IJ) | J_{JK}^2 | \vec{g}' \rangle - \langle \vec{g} | J_{JK}^2 | \vec{g}(IJ) \rangle \langle \vec{g}(IJ) | \vec{g}' \rangle] \\ &= \sum_{\vec{g}(IJ), \vec{g}(JK), \vec{g}''} g_2(IJ)(g_2(IJ) + 1) g_2(JK)(g_2(JK) + 1) \langle \vec{g}(IJ) | \vec{g}'' \rangle \langle \vec{g}(JK) | \vec{g}'' \rangle \times \\ & \quad \times [\langle \vec{g}(IJ) | \vec{g} \rangle \langle \vec{g}(JK) | \vec{g}' \rangle - \langle \vec{g}(JK) | \vec{g} \rangle \langle \vec{g}(IJ) | \vec{g}' \rangle] \end{aligned} \quad (5.9)$$

where we have used that the so-called $3nj$ -symbols $\langle \vec{g}(IJ) | \vec{g}(JK) \rangle$ are real (also $\langle \vec{g}(IJ) | \vec{g}(KL) \rangle$ for any $I < J, K < L$ are real). To see this, insert a unit in terms of

the basis $|\vec{j}, \vec{m}\rangle$ which allows us to write the $3nj$ -symbol in terms of Clebsh-Gordan coefficients which are real [24]. It is understood that in (5.9) and in what follows the summation extends only over the allowed values of \vec{g} etc. which are purely fixed by the values of j_1, \dots, j_n .

The task left is to write the $3nj$ -symbols of the form $\langle \vec{g}(IJ) | \vec{g}' \rangle$ in terms of known quantities. We will derive below a formula which expresses it in terms of the $6j$ symbol for which a closed formula (the Racah formula [24]) exists.

We will denote $\vec{g}(IJ) = (g_2(j_I, j_J), g_3(g_2, j_1), \dots, g_{I+1}(g_I, j_{I-1}), g_{I+2}(g_{I+1}, j_{I+1}), \dots, g_J(g_{J-1}, j_{J-1}), g_{J+1}(g_J, j_{J+1}), \dots, g_{n-1}(g_{n-2}, j_{n-1}))$ where the notation $j''(j, j')$ means that we couple J, J' to resulting spin j'' of J'' thereby keeping explicit track of the coupling scheme. The following two elementary lemmata will be crucial in computing the $3nj$ -symbols.

Lemma 5.1 $\langle \vec{g}(IJ) | \vec{g}' \rangle$

$= \langle g_2(j_I, j_J), g_3(g_2, j_1), \dots, g_{I+1}(g_I, j_{I-1}), g_{I+2}(g_{I+1}, j_{I+1}), \dots, g_J(g_{J-1}, j_{J-1}) | g'_2(j_1, j_2), g'_3(g'_2, j_3), \dots, g'_J(g'_{J-1}, j_J) \rangle \delta_{g_J, g'_J} \dots \delta_{g_{n-1}, g'_{n-1}}$ where on the right hand side we consider a spin system consisting of spins J_1, \dots, J_J and total spin quantum number g_J .

Proof :

Since $G_J(12)^2, \dots, G_{n-1}(12)^2$ are diagonal on both vectors it is immediately clear that the $3nj$ -symbol must be proportional to the Kronecker delta δ_{g_K, g'_K} for $K = J, \dots, n-1$. To get the factor of proportionality we expand both vectors in terms of the basis $|\vec{j}, \vec{m}\rangle$ and Clebsh-Gordan coefficients. To do this, note that by definition

$\langle g_2(j_1, j_2), \dots, g_{n-2}(g_{n-3}, j_{n-2}); g_{n-1}m - m_n, j_n m_n | g_2(j_1, j_2), \dots, g_{n-2}(g_{n-3}, j_{n-2}), g_{n-1}(g_{n-2}, j_{n-1}), j(g_{n-1}, j_n), m \rangle = \langle g_{n-1}(g_{n-2}, j_{n-1})m - m_n, j_n m_n | j, m; g_{n-1}, j_n \rangle$ where on the right hand side we have the standard Clebsh-Gordan coefficient. Here

$|g_2(j_1, j_2), \dots, g_{n-2}(g_{n-3}, j_{n-2}); g_{n-1}m - m_n, j_n m_n \rangle := |g_2(j_1, j_2), \dots, g_{n-2}(g_{n-3}, j_{n-2}), g_{n-1}(g_{n-2}, j_{n-1}), m - m_n \rangle \otimes |j_n m_n \rangle$. We can now iterate this procedure and obtain

$|g'_2(j_1, j_2), \dots, g'_{n-2}(g'_{n-3}, j_{n-2}), g'_{n-1}(g'_{n-2}, j_{n-1}), j(g'_{n-1}, j_n), m \rangle = \sum_{m_J + \dots + m_n = m} \langle g'_{n-1}m - m_n, j_n m_n | jm \rangle \langle g'_{n-2}m - m_n - m_{n-1}, j_{n-1}m_{n-1} | g'_{n-1}m - m_n \rangle \dots$

$\dots \langle g'_J m_J, j_{J+1} m_{J+1} | g'_{J+1}, m_J + m_{J+1} \rangle |g'_2(j_1, j_2), \dots, g'_J(g'_{J-1}, j_J), m_J \rangle \otimes |j_{J+1}, m_{J+1} \rangle \otimes \dots \otimes |j_n, m_n \rangle$. Similarly we have $|g_2(j_I, j_J), \dots, g_{n-2}(g_{n-3}, j_{n-2}), g_{n-1}(g_{n-2}, j_{n-1}), j(g_{n-1}, j_n), m \rangle = \sum_{m_J + \dots + m_n = m} \langle g_{n-1}m - m_n, j_n m_n | jm \rangle$

$\langle g_{n-2}m - m_n - m_{n-1}, j_{n-1}m_{n-1} | g_{n-1}m - m_n \rangle \dots \langle g_J m_J, j_{J+1} m_{J+1} | g_{J+1}, m_J + m_{J+1} \rangle |g_2(j_I, j_J), \dots, g_J(g_{J-1}, j_{J-1}), m_J \rangle \otimes |j_{J+1}, m_{J+1} \rangle \otimes \dots \otimes |j_n, m_n \rangle$. Thus we have the inner product

$\langle \vec{g}(IJ) | \vec{g}' \rangle = \sum_{m_J + \dots + m_n = m} \langle g_2(j_I, j_J), \dots, g_J(g_{J-1}, j_{J-1}), m_J | g'_2(j_1, j_2), \dots, g'_J(g'_{J-1}, j_J), m_J \rangle \times \langle g'_{n-1}m - m_n, j_n m_n | jm \rangle \langle g'_{n-2}m - m_n - m_{n-1}, j_{n-1}m_{n-1} | g'_{n-1}m - m_n \rangle \dots \langle g'_J m_J, j_{J+1} m_{J+1} | g'_{J+1}, m_J + m_{J+1} \rangle \times \langle g_{n-1}m - m_n, j_n m_n | jm \rangle \langle g_{n-2}m - m_n - m_{n-1}, j_{n-1}m_{n-1} | g_{n-1}m - m_n \rangle \dots \langle g_J m_J, j_{J+1} m_{J+1} | g_{J+1}, m_J + m_{J+1} \rangle = \langle g_2(j_I, j_J), \dots, g_J(g_{J-1}, j_{J-1}), \tilde{m} | g'_2(j_1, j_2), \dots, g'_J(g'_{J-1}, j_J), \tilde{m} \rangle \times \sum_{m_J + \dots + m_n} \langle g'_{n-1}m - m_n, j_n m_n | jm \rangle \langle g'_{n-2}m - m_n - m_{n-1}, j_{n-1}m_{n-1} | g'_{n-1}m - m_n \rangle \dots \langle g'_J m_J, j_{J+1} m_{J+1} | g'_{J+1}, m_J + m_{J+1} \rangle \times$

$\times < g_{n-1}m - m_n, j_n m_n | jm > < g_{n-2}m - m_n - m_{n-1}, j_{n-1} m_{n-1} | g_{n-1}m - m_n > \dots < g_J m_J, j_{J+1} m_{J+1} | g_{J+1}, m_J + m_{J+1} >$
 $= < g_2(j_I, j_J), \dots, g_J(g_{J-1}, j_{J-1}), \tilde{m} | g'_2(j_1, j_2), \dots, g'_J(g'_{J-1}, j_J), \tilde{m} > \delta_{g_J, g'_J} \dots \delta_{g_{n-1}, g'_{n-1}}$ where in the second step we have used the fact that the $3nj$ -symbols are independent of m (Wigner-Eckart theorem⁵) so that \tilde{m} is arbitrary and in the last step we have used the sum rule for the Clebsh-Gordan coefficients, that is, $\sum_{m_1+m_2=m} (< j_1 m_1, j_2 m_2 | jm >)^2 = 1$.
 \square

Lemma 5.2 $< g_2(j_1, j_2), \dots, g_K(g_{K-1}, j_K), g_{K+1}(g_K, j_{K+1}), g_{K+2}(g_{K+1}, j_{K+2}); m |$
 $|g_2(j_1, j_2), \dots, g_K(g_{K-1}, j_K), g'_{K+1}(g_K, j_{K+2}), g'_{K+2}(g'_{K+1}, j_{K+1}); m >$
 $= < g_{K+1}(g_K, j_{K+1}), g_{K+2}(g_{K+1}, j_{K+2}); m | g'_{K+1}(g_K, j_{K+2}), g_{K+2}(g'_{K+1}, j_3); m >.$

Proof :

Again we expand

$$\begin{aligned} & |g_2, \dots, g_{K+2}; m > = \sum_{m_{K+1}, m_{K+2}} < g_K m - m_{K+1} - m_{K+2}, j_{K+1} m_{K+1} | g_{K+1} m - m_{K+2} > < g_{K+2} m - m_{K+2}, j_{K+2} m_{K+2} | g_{K+2} m - m_{K+2} > \times \\ & \times |g_2, \dots, g_K; m - m_{K+1} - m_{K+2} > \otimes |j_{K+1} m_{K+1}, j_{K+2} m_{K+2} > \text{ and} \\ & |g_2, \dots, g_K, g'_{K+1}, g_{K+2}; m > = \sum_{m_{K+1}, m_{K+2}} < g_K m - m_{K+1} - m_{K+2}, j_{K+2} m_{K+2} | g'_{K+1} m - m_{K+1} > < g_{K+2} m - m_{K+1}, j_{K+1} m_{K+1} | g_{K+2} m - m_{K+1} > \times \\ & \times |g_2, \dots, g_K; m - m_{K+1} - m_{K+2} > \otimes |j_{K+1} m_{K+1}, j_{K+2} m_{K+2} >. \text{ We find for the inner product of these states due to the normalization just} \\ & \sum_{m_{K+1}, m_{K+2}} < g_K m - m_{K+1} - m_{K+2}, j_{K+1} m_{K+1} | g_{K+1} m - m_{K+2} > \times \\ & \times < g_{K+2} m - m_{K+2}, j_{K+2} m_{K+2} | g_{K+2} m - m_{K+2} > \times \\ & \times < g_K m - m_{K+1} - m_{K+2}, j_{K+2} m_{K+2} | g'_{K+1} m - m_{K+1} > < g_{K+2} m - m_{K+1}, j_{K+1} m_{K+1} | g_{K+2} m - m_{K+1} > \\ & \equiv < g_{K+1}(g_K, j_{K+1}), g_{K+2}(g_{K+1}, j_{K+2}); m | g'_{K+1}(g_K, j_{K+2}), g_{K+2}(g'_{K+1}, j_3); m > \text{ as claimed.} \\ & \square \end{aligned}$$

We are now ready to reduce out the general $3nj$ -symbol. We assume that $g_K = g'_K, K = J, \dots, n-1$ as otherwise we get zero by lemma 5.1. Upon repeatedly using the completeness relations of the coupling scheme bases we first shift j_J and then j_I to the right in the matrix element

$$\begin{aligned} & < \vec{g}(IJ), \vec{g}' > \\ & \stackrel{=5.1}{=} < g_2(j_I, j_J), g_3(g_2, j_1), \dots, g_{I+1}(g_I, j_{I-1}), g_{I+2}(g_{I+1}, j_{I+1}), \dots, g_J(g_{J-1}, j_{J-1}) | \\ & |g'_2(j_1, j_2), g'_3(g'_2, j_3), \dots, g'_{J-1}(g'_{J-2}, j_{J-1}), g_J(g'_{J-1}, j_J) > \\ & \stackrel{=5.1}{=} \sum_{h_2} < g_2(j_I, j_J), g_3(g_2, j_1), g_4, \dots, g_J | h_2(j_I, j_1), g_3(h_2, j_J), g_4, \dots, g_J > \\ & < h_2(j_I, j_1), g_3(h_2, j_J), g_4, \dots, g_J | g'_2(j_1, j_2), g'_3(g'_2, j_3), \dots, g'_{J-1}, g_J > \\ & \stackrel{=5.1}{=} \sum_{h_2} < g_2(j_I, j_J), g_3(g_2, j_1) | h_2(j_I, j_1), g_3(h_2, j_J) > \times \end{aligned}$$

⁵Proof : Let us expand $|\vec{g}(IJ) > = \sum_{\vec{g}(KL)} < \vec{g}(KL); j, m | \vec{g}(IJ); j, m > |\vec{g}(KL); j, m >$ where we assume that $m < j$ w.l.g. Now apply the ladder operator J^\dagger to both sides of this equation. We find $J^\dagger |\vec{g}(MN); j, m > = c(j, m) |\vec{g}(MN); j, m+1 >$ for any $K < L$ where the coefficient only depends on j, m . Comparing coefficients we find $< \vec{g}(KL); j, m | \vec{g}(IJ); j, m > = < \vec{g}(KL); j, m+1 | \vec{g}(IJ); j, m+1 >$, that is, the $3nj$ -symbols are rotation invariant. \square

$$\begin{aligned}
& \times \sum_{h_3} \langle h_2(j_I, j_1), g_3(h_2, j_J), g_4(g_3, j_2), g_5, \dots, g_J | h_2(j_I, j_1), h_3(h_2, j_2), g_4(h_3, j_J), g_5, \dots, g_J \rangle \\
& \langle h_2(j_I, j_1), h_3(h_2, j_2), g_4(h_3, j_J), g_5, \dots, g_J | g'_2(j_1, j_2), g'_3(g'_2, j_3), \dots, g'_{J-1}, g_J \rangle \\
& \stackrel{=5.1,5.2}{=} \sum_{h_2} \langle g_2(j_I, j_J), g_3(g_2, j_1) | h_2(j_I, j_1), g_3(h_2, j_J) \rangle \times \\
& \times \sum_{h_3} \langle g_3(h_2, j_J), g_4(g_3, j_2) | h_3(h_2, j_2), g_4(h_3, j_J) \rangle \times \\
& \times \langle h_2(j_I, j_1), h_3(h_2, j_2), g_4(h_3, j_J), g_5, \dots, g_J | g'_2(j_1, j_2), g'_3(g'_2, j_3), \dots, g'_{J-1}, g_J \rangle \\
& = \dots \\
& = \sum_{h_2} \langle g_2(j_I, j_J), g_3(g_2, j_1) | h_2(j_I, j_1), g_3(h_2, j_J) \rangle \times \\
& \times \sum_{h_3} \langle g_3(h_2, j_J), g_4(g_3, j_2) | h_3(h_2, j_2), g_4(h_3, j_J) \rangle \dots \\
& \sum_{h_I} \langle g_I(h_{I-1}, j_J), g_{I+1}(g_I, j_{I-1}) | h_I(h_{I-1}, j_{I-1}), g_{I+1}(h_I, j_J) \rangle \times \\
& \times \sum_{h_{I+1}} \langle g_{I+1}(h_I, j_J), g_{I+2}(g_{I+1}, j_{I+1}) | h_{I+1}(h_I, j_{I+1}), g_{I+2}(h_{I+1}, j_J) \rangle \times \\
& \times \sum_{h_{I+2}} \langle g_{I+2}(h_{I+1}, j_J), g_{I+3}(g_{I+2}, j_{I+2}) | h_{I+2}(h_{I+1}, j_{I+2}), g_{I+3}(h_{I+2}, j_J) \rangle \dots \\
& \sum_{h_{J-1}} \langle g_{J-1}(h_{J-2}, j_J), g_J(g_{J-1}, j_{J-1}) | h_{J-1}(h_{J-2}, j_{J-1}), g_J(h_{J-1}, j_J) \rangle \times \\
& \times \langle h_2(j_I, j_1) h_3(h_2, j_2), \dots, h_I(h_{I-1}, j_{I-1}), h_{I+1}(h_I, j_{I+1}), \dots, h_{J-1}(h_{J-2}, j_{J-1}), g_J(h_{J-1}, j_J) | \\
& | g'_2(j_1, j_2), \dots, g'_I(g'_{I-1}, j_I), g'_{I+1}(g'_I, j_{I+1}), \dots, g'_J(g'_{J-1}, j_J) \rangle \dots \quad (5.10)
\end{aligned}$$

Now the last matrix element in the last line of (5.10) is different from zero only for $h_K = g'_K, K = I, \dots, J-1$ by lemma 5.1 (we have $g_J = g'_J$ already) and in that case reduces by the same reasoning to

$$\begin{aligned}
& \langle h_2(j_I, j_1) h_3(h_2, j_2), \dots, h_{I-1}(h_{I-2}, j_{I-2}), g'_I(h_{I-1}, j_{I-1}) | g'_2(j_1, j_2), \dots, g'_I(g'_{I-1}, j_I) \rangle \\
& = \sum_{k_2} \langle h_2(j_I, j_1), h_3(h_2, j_2) | k_2(j_1, j_2), h_3(k_2, j_I) \rangle \times \\
& \times \sum_{k_3} \langle h_3(k_2, j_I), h_4(h_3, j_3) | k_3(k_2, j_3), h_3(k_3, j_I) \rangle \dots \\
& \sum_{k_{I-1}} \langle h_{I-1}(k_{I-2}, j_I), h_I(h_{I-1}, j_{I-1}) | k_{I-1}(k_{I-2}, j_{I-1}), g'_I(k_{I-1}, j_I) \rangle \times \\
& \times \langle k_2(j_1, j_2), k_3(k_2, j_3), \dots, k_{I-1}(k_{I-2}, j_{I-1}), g'_I(k_{I-1}, j_I) | g'_2(j_1, j_2), g'_3(g'_2, j_3), \dots, g'_I(g'_{I-1}, j_I) \rangle \quad (5.11)
\end{aligned}$$

and the last matrix element in the last line of (5.11) is just given by $\prod_{K=2}^{I-1} \delta_{k_K, g'_K}$ so that the summation drops out. So, altogether, there are only the $I-3$ summation variables h_2, \dots, h_{I-1} .

Equations (5.10), (5.11) demonstrate that we manage to reduce the $3nj$ -symbol under investigation to a polynomial of matrix elements of the structure

$$\begin{aligned}
& \text{Type I : } \langle j_{12}(j_1, j_2), j(j_{12}, j_3) | j_{13}(j_1, j_3), j(j_{13}, j_2) \rangle \text{ or} \\
& \text{Type II : } \langle j_{12}(j_1, j_2), j(j_{12}, j_3) | j_{23}(j_2, j_3), j(j_{23}, j_1) \rangle \dots \quad (5.12)
\end{aligned}$$

In fact, all matrix elements in (5.10) are of type I and all matrix elements in (5.11) are of type I except for the first factor after the first equality which is of type II. The point is now that both matrix elements are related to the $6j$ -symbol [24]. The $6j$ -symbol is defined through the type II matrix element, namely

$$\begin{aligned} & \langle j_{12}(j_1, j_2), j(j_{12}, j_3) | j_{23}(j_2, j_3), j(j_{23}, j_1) \rangle \\ & =: \sqrt{(2j_{12} + 1)(2j_{23} + 1)}(-1)^{j_1 + j_2 + j_3 + j} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}. \end{aligned} \quad (5.13)$$

In order to relate the type I matrix element to the $6j$ -symbol we need to use the following identity for the CG-coefficients

$$\langle j_1 m_1, j_2 m_2 | j m \rangle = (-1)^{j_1 + j_2 - j} \langle j_2 m_2, j_1 m_1 | j m \rangle \quad (5.14)$$

which is a consequence of the choice of phases for the CG coefficients in order to have them real valued [24]. Let us now set $j'_1 := j_2, j'_2 := j_1, j'_3 := j_3, j'_{12} := j_{12}, j'_{23} := j_{13}, j' := j$ then we get

$$\langle j_{12}(j_1, j_2), j(j_{12}, j_3) | j_{13}(j_1, j_3), j(j_{13}, j_2) \rangle = \langle j'_{12}(j'_2, j'_1), j'(j'_{12}, j'_3) | j'_{23}(j'_2, j'_3), j'(j'_{23}, j'_1) \rangle$$

which is almost the standard form except for the wrong order of j'_1, j'_2 in the first entry. Using the expansion

$$\begin{aligned} & |j'_{12}(j'_2, j'_1), j'(j'_{12}, j'_3); m' \rangle \\ & \sum_{m_1 + m_2 + m_3 = m'} \langle j'_2 m_2, j'_1 m_1 | j'_{12} m_1 + m_2 \rangle \langle j'_{12} m_1 + m_2, j'_3 m_3 | j' m' \rangle |j'_1 m_1, \dots, j'_3 m'_3 \rangle \\ & \text{and (5.14) we therefore find} \\ & \langle j'_{12}(j'_2, j'_1), j'(j'_{12}, j'_3) | j'_{23}(j'_2, j'_3), j'(j'_{23}, j'_1) \rangle \\ & = (-1)^{j'_1 + j'_2 - j'_{12}} \langle j'_{12}(j'_1, j'_2), j'(j'_{12}, j'_3) | j'_{23}(j'_2, j'_3), j'(j'_{23}, j'_1) \rangle \text{ and so we have for the type} \\ & \text{I matrix element} \end{aligned}$$

$$\begin{aligned} & \langle j_{12}(j_1, j_2), j(j_{12}, j_3) | j_{13}(j_1, j_3), j(j_{13}, j_2) \rangle \\ & = (-1)^{j_1 + j_2 - j_{12}} \sqrt{(2j_{12} + 1)(2j_{13} + 1)}(-1)^{j_1 + j_2 + j_3 + j} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix}. \end{aligned} \quad (5.15)$$

Finally, for the benefit of the reader we note the Racah formula [24]

$$\begin{aligned} & \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \Delta(j_1, j_2, j_{12}) \Delta(j_1, j, j_{23}) \Delta(j_3, j_2, j_{23}) \Delta(j_3, j, j_{12}) w \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}, \\ & \Delta(a, b, c) = \sqrt{\frac{(a + b - c)!(a - b + c)!(-a + b + c)!}{(a + b + c + 1)!}} \\ & w \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \sum_n (-1)^n (n + 1)! \times \\ & \times [(n - j_1 - j_2 - j_{12})!(n - j_1 - j - j_{23})!(n - j_3 - j_2 - j_{23})!(n - j_3 - j - j_{12})!]^{-1} \times \\ & \times [(j_1 + j_2 + j_3 + j - n)!(j_2 + j_{12} + j + j_{23} - n)!(j_{12} + j_1 + j_{23} + j_3 - n)!]^{-1} \end{aligned} \quad (5.16)$$

and the sum extends over all positive integers such that no factorial in the denominator has a negative argument. In conclusion, formulae (5.9), (5.10), (5.11) provide us with

the general expression for the matrix elements of the operator \hat{q}_{IJK} in the standard basis where it is understood that all occurring matrix elements are known in terms of j_1, \dots, j_n via (5.13), (5.15), (5.16).

Let us call $M_{v,IJK}(\vec{g}, \vec{g}') := \langle \vec{g} | \hat{q}_{IJK} | \vec{g}' \rangle$ then we have the matrix

$$M_v(\vec{g}, \vec{g}') = \left(\frac{3}{2}\right)^3 \frac{i}{4} \sum_{I < J < K} \epsilon(I, J, K) M_{v,IJK}(\vec{g}, \vec{g}') \quad (5.17)$$

which by inspection of (5.9) is a Hermitean matrix of a special kind, namely, it is of the form of i times a real-valued skew matrix. It's eigenvalues therefore come in pairs $\pm\lambda, \lambda > 0$ or are zero. The eigenvalues of the positive semidefinite matrix $M_v^\dagger M_v$ are therefore $|\lambda|^2$ or 0 where λ is an eigenvalue of M_v . We conclude that the eigenvalues of $\hat{V}_{v,\gamma}$ are given by $\ell_p^3 \sqrt{|\lambda|}$ or 0 where both eigenvectors of M_v corresponding to eigenvalues $\pm\lambda$ produce the same eigenvalue $\ell_p^3 \sqrt{|\lambda|}$ of $\hat{V}_{v,\gamma}$. We see that the problem of diagonalizing the volume operator becomes equivalent to diagonalizing the matrices M_v which can be done for each $v \in V(\gamma)$ separately.

In the next subsection we will derive the spectrum in the most simple cases.

Remark :

We have nowhere made use of the fact that we are only interested in gauge invariant functions. Therefore, our results carry over, word by word, to the case of non-gauge invariant functions provided we use the notion of an extended spin-network state.

Definition 5.2 *An extended spin-network state is a state of the form $T_{\gamma, \vec{j}, \vec{c}}$ as before just that the vertex contractors now are Clebsh-Gordan coefficients of the form $\langle j_1 m_1, \dots, j_n m_n | j_1, \dots, j_n, g_2, \dots, g_{n-1}, j \rangle$ where j maybe different from zero.*

This is important for applications in quantum gravity [2, 3] and for the length operator [6] where we need the action of the volume operator on non-gauge invariant functions in an intermediate step.

5.3 Examples

Given a vertex v and a vector of spins \vec{j} colouring the edges incident at v , let us denote by $d(\vec{j}, v)$ the number of linearly independent vectors $|\vec{g}\rangle$ compatible with \vec{j} . This number is the dimension of a vector space of vertex contractors associated with v and it is given purely algebraically as the number of linearly independent trivial representations which appear in the decomposition into irreducibles of the tensor product representation $\pi_{j_1} \otimes \dots \otimes \pi_{j_n}$. One can determine this number for each case at hand by repeated application of the usual Clebsh-Gordan theorem $\pi_j \otimes \pi_{j'} = \pi_{j+j'} \oplus \pi_{j+j'-1} \oplus \dots \oplus \pi_{|j-j'|}$ and counting all the trivial representation of spin 0 that appear. Notice that the sum is direct so that all the representations are orthogonal and therefore linearly independent. What one would like to have is a closed formula for $d(\vec{j}, v)$ but that seems to be a complicated combinatorical problem. To our knowledge, such a dimension formula is only known for the case of an n -fold tensor product of fundamental representations $j_1 = \dots = j_n = 1/2$ for the groups $GL(N), U(N), SU(N)$ [25]. We will not deal with this problem in the present paper.

First of all, whenever $d(\vec{j}, v) = 1$ then 0 is the only possible eigenvalue (this observation was made first in [10] for the case of a trivalent graph in which the vertex contractors span only a one-dimensional vector space). More generally, whenever $d(\vec{j}, v)$ is odd, we

know that 0 is an eigenvalue of multiplicity at least one.

Since M_v is a $d \times d$ skew matrix its characteristic polynomial has the structure $t^k(t^2 - \lambda_1^2) \dots (t^2 - \lambda_{(d-k)/2}^2) = t^k p_{(d-k)/2}(t^2)$ where $0 \leq k \leq d$ for d even and $1 \leq k \leq d$ for d odd. It follows from Galois theory that we can determine the spectrum by quadratures exactly in general whenever $d(\vec{j}, v) \leq 9$ and in fortunate cases (whenever 0 has multiplicity at least $d - 8$) up to arbitrarily high dimension. In the most general case, however, a numerical evaluation is the only possible approach.

We consider the first non-trivial cases $d(\vec{j}, v) \leq 9$ for an arbitrary valence n of the vertex. We may label the spins in such a way that $0 < j_1 \leq \dots \leq j_n$. We may parametrize the situation by the non-negative number $k := j_1 + \dots + j_{n-1} - j_n$ (there is no constraint for $k < 0$). If k is low valued then it is clear from Clebsh-Gordan theory that $d(\vec{j}, v)$ will be low valued. Specifically, if $k = 0$ then the first $n - 1$ spins have to add up to maximum spin and thus $d = 1$, the spectrum is only the point 0 irrespective of the value of n . In the following we will assume that $2j_1 \geq k$, otherwise the discussion requires a case division.

If $k = 1$ then one sees readily that $d = n - 1$: the only irreducible representations contained in $j_1 \otimes j_2$ that can possibly add up with j_3, \dots, j_{n-1} to j_n are $j_1 + j_2, j_1 + j_2 - 1$. Repeating the argument, the assertion follows.

Let in general $c_k(n)$ be the number of representations with weight $j_1 + \dots + j_n - k$ contained in $j_1 \otimes \dots \otimes j_n$ for $j_{n+1} = j_1 + \dots + j_n - k$ to couple to angular momentum 0 where $j_1 \leq \dots \leq j_n$ and where we assume that $j_2 - j_1 \leq j_1 + j_2 - k$, that is $k \leq 2j_1$. We wish to show that $c_k(n) = (n + k - 2, k)$ where (n, k) is the usual binomial coefficient. Since $c_0(n) = 1$ as we showed above, the induction is started. Now suppose we know all coefficients up to k for each n and start deriving those for $k + 1$. We can assume that we know them already up to $n - 1$ since $c_k(2) = 1$ for all k is trivial to see since we have made the assumption $2j_1 \geq k$. Then we have the recursion $c_{k+1}(n - 1) + c_k(n) = c_{k+1}(n)$ which follows from the fact that if we have the tensor product $j_1 \otimes \dots \otimes j_{n-1}$ then there are $c_l(n - 1)$ representations with weight $j_1 + \dots + j_{n-1} - l, 0 \leq l \leq k + 1$. But if we now form the tensor product with all of them with j_n , then the number of times that $j_1 + \dots + j_n - k$ arises from those with weight $j_1 + \dots + j_{n-1} - l, 0 \leq l \leq k$, which is given by $c_k(n)$, is equal to the number of times that $j_1 + \dots + j_n - (k + 1)$ arises from them. In addition we have $c_{k+1}(n - 1)$ terms coming from the representation $j_1 + \dots + j_{n-1} - (k + 1)$ itself. It is now readily verified that $c_k(n)$ solves the recursion.

Notice that $n + 1$ is the valence of the vertex. So we get back the result that in the tri-valent case we have $d = c_k(2) = (k, k) = 1$. We can now ask for which combinations of k, n we have $c_k(n) \leq 9$. Notice that $c_k(3) = k + 1$:

Case $k = 0$: n arbitrary.

Case $k = 1$: $n - 1 \leq 9$ so $n \leq 10$.

Case $k = 2$: $n(n - 1)/2 \leq 9$ so $n \leq 4$.

Case $3 \leq k \leq 8$: $n \leq 3$.

Case $k > 8$: $n \leq 2$.

Reverse question (only $n \geq 2$ makes sense since we need valence at least 3 for the volume operator not to vanish trivially) :

Case $n = 2$: k arbitrary.

Case $n = 3$: $k + 1 \leq 9$ so $k \leq 8$.

Case $n = 4$: $(k + 1)(k + 2)/2 \leq 9$ so $k \leq 2$.

Case $5 \leq n \leq 10$: $k \leq 1$.

Case $n > 10$: $k = 0$.

So, for instance we can treat the general 4-valent case under the restriction that $j_4 = j_1 + j_2 + j_3 - k$, $0 \leq k \leq 8$ for $j_3 \geq j_2 \geq j_1 \geq k/2$ in which case $d = k + 1$. As the formulae already get quite involved, we will only discuss the cases $k = 1, 2, 3$ explicitly.

Notice first of all that due to gauge-invariance $J_1 + J_2 + J_3 + J_4|0\rangle = 0$ so that we can substitute for J_4 . Thus for instance $\hat{q}_{124}|0\rangle = -(\hat{q}_{121} + \hat{q}_{122} + \hat{q}_{123})|0\rangle$ and it is easily verified that the operator identity $\hat{q}_{122} + \hat{q}_{121} = 0$ holds. It follows that

$$M_v = \left(\frac{3}{4}\right)^3 \frac{i}{4} [\epsilon(1, 2, 3) - \epsilon(1, 2, 4) + \epsilon(1, 3, 4) - \epsilon(2, 3, 4)] \hat{q}_{123} =: \sigma \hat{q}_{123}$$

which simplifies the discussion tremendously since we just need to compute the matrix elements of the single operator \hat{q}_{123} . We have to compute the $k(k+1)/2$ matrix elements

$$\langle j_{12} | \hat{q}_{123} | j'_{12} \rangle = [j_{12}(j_{12} + 1) - j'_{12}(j'_{12} + 1)] \sum_{j_{23}} j_{23}(j_{23} + 1) \langle j_{12} | j_{23} \rangle \langle j'_{12} | j_{23} \rangle$$

for $j_{12}, j'_{12} = j_1 + j_2 - l$, $l = 0, \dots, k$ and the sum extends over $j_{23} = j_2 + j_3 - l$, $l = 0, \dots, k$ which requires the computation of $(k+1)^2 6j$ symbols for $j_{123} = j = j_1 + j_2 + j_3 - k = j_4$. The computation is rather tedious. We will display only a few intermediate calculational results before giving the eigenvalue λ_k .

Let $j_{12} := j_1 + j_2$, $j_{23} := j_2 + j_3$, $J = 2(j_1 + j_2 + j_3)$.

Case $k = 1$.

The $6j$ symbols are

$$\begin{aligned} \langle j_{12} | j_{23} \rangle &= - \langle j_{12} - 1 | j_{23} - 1 \rangle = \sqrt{\frac{j_1 j_3}{j_{12} j_{23}}} \\ \langle j_{12} - 1 | j_{23} \rangle &= \langle j_{12} | j_{23} - 1 \rangle = \sqrt{\frac{j_2 J}{2 j_{12} j_{23}}} \end{aligned} \quad (5.18)$$

and so we have the only non-vanishing matrix element

$$\langle j_{12} | \hat{q}_{123} | j_{12} - 1 \rangle = 4 \sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}. \quad (5.19)$$

The corresponding eigenvalue of the volume operator is

$$\lambda_1 = 2\ell_p^3 \sqrt{|\sigma|} \sqrt[4]{j_1 j_2 j_3 (j_1 + j_2 + j_3)}. \quad (5.20)$$

Case $k = 2$:

The $6j$ symbols are

$$\begin{aligned} \langle j_{12} | j_{23} \rangle &= \sqrt{\frac{j_1(2j_1 - 1)j_3(2j_3 - 1)}{j_{12}(2j_{12} - 1)j_{23}(2j_{23} - 1)}} \\ \langle j_{12} - 1 | j_{23} \rangle &= \sqrt{2 \frac{(2j_1 - 1)j_2 j_3 (J - 1)}{j_{12}(2j_{12} - 2)j_{23}(2j_{23} - 1)}} \\ \langle j_{12} - 2 | j_{23} \rangle &= \sqrt{\frac{j_2(2j_2 - 1)(J - 1)(J - 2)}{(2j_{12} - 1)(2j_{12} - 2)j_{23}(2j_{23} - 1)}} \\ \langle j_{12} | j_{23} - 1 \rangle &= \sqrt{2 \frac{j_1 j_2 (2j_3 - 1)(J - 1)}{j_{12}(2j_{12} - 1)j_{23}(2j_{23} - 2)}} \end{aligned}$$

$$\begin{aligned}
\langle j_{12}|j_{23}-2\rangle &= \sqrt{\frac{j_2(2j_2-1)(J-1)(J-2)}{j_{12}(2j_{12}-1)(2j_{23}-1)(2j_{23}-2)}} \\
\langle j_{12}-1|j_{23}-1\rangle &= \frac{(2j_2-1)(J-1)-(2j_1-1)(2j_3-1)}{2\sqrt{j_{12}(2j_{12}-2)j_{23}(2j_{23}-2)}} \\
\langle j_{12}-2|j_{23}-1\rangle &= -\sqrt{2\frac{(2j_1-1)(2j_2-1)j_3(J-2)}{(2j_{12}-1)(2j_{12}-2)j_{23}(2j_{23}-2)}} \\
\langle j_{12}-1|j_{23}-2\rangle &= -\sqrt{2\frac{j_1(2j_2-1)(2j_3-1)(J-2)}{j_{12}(2j_{12}-2)(2j_{23}-1)(2j_{23}-2)}} \\
\langle j_{12}-2|j_{23}-2\rangle &= 2\sqrt{\frac{j_1(2j_1-1)j_3(2j_3-1)}{(2j_{12}-1)(2j_{12}-2)(2j_{23}-1)(2j_{23}-2)}} \quad (5.21)
\end{aligned}$$

and we find then the matrix elements

$$\begin{aligned}
\langle j_{12}|\hat{q}_{123}|j_{12}-1\rangle &= \sqrt{\frac{2j_1 2j_2(2j_3-1)(J-1)}{2(2j_{12}-1)(2j_{12}-2)}} \frac{1}{2(2j_{23}-1)} \times \\
&\quad \times [(2j_1-1)[6j_3+2j_{23}-1] + (2j_2-1)[3J+2j_{23}-7]] \\
\langle j_{12}|\hat{q}_{123}|j_{12}-2\rangle &= 0 \\
\langle j_{12}-1|\hat{q}_{123}|j_{12}-2\rangle &= \sqrt{\frac{(2j_1-1)(2j_2-1)2j_3(J-2)}{2(2j_{12})(2j_{12}-1)}} \frac{1}{2(2j_{23}-1)} \times \\
&\quad \times [(J-1)[6j_2+2j_{23}-1] + (2j_3-1)[6j_1-2j_{23}+1]] \quad (5.22)
\end{aligned}$$

and since the eigenvalues of any 3×3 skew matrix with entries a, b, c are given by $0, \pm i\sqrt{a^2+b^2+c^2}$ we find the eigenvalues $0, \lambda_2$ where

$$\lambda_2 = \ell_p^3 \sqrt{|\sigma|} \sqrt[4]{\langle j_{12}|\hat{q}_{123}|j_{12}-1\rangle^2 + \langle j_{12}-1|\hat{q}_{123}|j_{12}-2\rangle^2}. \quad (5.23)$$

Case $k=3$:

We will refrain from displaying the $6j$ -symbols explicitly in terms of the j_I , rather we consider them as given through the formulae of the previous section. Now it is easy to check that the eigenvalues of a 4×4 skew matrix with entries a, b, c, d, e, f are given by i times

$$t = \pm \frac{1}{\sqrt{2}} \sqrt{a^2+b^2+c^2+d^2+e^2+f^2 \pm \sqrt{[(a+f)^2+(c+d)^2+(e-f)^2] \times [(a-f)^2+(c-d)^2+(e+f)^2]}}$$

and notice that the argument of the inner square root can be written as $(a^2+b^2+c^2+d^2+e^2+f^2-4(af+cd+eb)^2)$ which shows that t is purely real. Substituting the matrix elements $\langle j_{12}-l|\hat{q}_{123}|j_{12}-l'\rangle$, $0 \leq l < l' \leq 3$ for a, b, c, d, e, f we obtain

$$\lambda_3 = \ell_p^3 \sqrt{|\sigma|} \sqrt{|t|}. \quad (5.24)$$

The reader should now have a feeling of how to compute the spectrum analytically. The formulae derived can serve for analytical estimates as well as the starting point for the implementation of an appropriate computer code for an algebraic manipulation programme.

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