Hamiltonian and Brownian systems with long-range interactions: V. Stochastic kinetic equations and theory of fluctuations

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Abstract

We develop a theory of fluctuations for Brownian systems with weak long-range interactions. For these systems, there exists a critical point separating a homogeneous phase from an inhomogeneous phase. Starting from the stochastic Smoluchowski equation governing the evolution of the fluctuating density field of the Brownian particles, we determine the expression of the correlation function of the density fluctuations around a spatially homogeneous equilibrium distribution. In the stable regime, we find that the temporal correlation function of the Fourier components of the density fluctuations decays exponentially rapidly with the same rate as the one characterizing the damping of a perturbation governed by the deterministic mean field Smoluchowski equation (without noise). On the other hand, the amplitude of the spatial correlation function in Fourier space diverges at the critical point $T = T_c$ (or at the instability threshold $k = k_m$) implying that the mean field approximation breaks down close to the critical point and that the phase transition from the homogeneous phase to the inhomogeneous phase occurs sooner. By contrast, the correlations of the velocity fluctuations remain finite at the critical point (or at the instability threshold). We give explicit examples for the Brownian Mean Field (BMF) model and for Brownian particles interacting via the gravitational potential and via the attractive Yukawa potential. We also introduce a stochastic model of chemotaxis for bacterial populations generalizing the deterministic mean field Keller-Segel model by taking into account fluctuations and memory effects.

1 Introduction

In a recent series of papers [1, 2, 3, 4, 5], we have considered some theoretical aspects of the dynamics and thermodynamics of systems with weak long-range interactions [6, 7]. In these systems, the interaction potential u(r) decays with a rate slower than $1/r^d$ at large distances, where d is the dimension of space (these potentials are sometimes called "non-integrable"). As a result, any particle feels a potential dominated by interactions with far away particles (i.e. the interaction is not restricted to nearest neighbours) and the energy is *non-additive*. This can lead to striking properties (absent in systems with short-range interactions) such as

inequivalence of statistical ensembles and negative specific heats in the microcanonical ensemble. On the other hand, the usual thermodynamic limit $N \to +\infty$ with N/V fixed is not relevant for these systems and must be reconsidered. If we write the potential of interaction as $u(|\mathbf{r}-\mathbf{r}'|) = k\tilde{u}(|\mathbf{r}-\mathbf{r}'|)$ where k is the coupling constant, then the appropriate thermodynamic limit for weak long-range interactions corresponds to $N \to +\infty$ in such a way that the coupling constant $k \sim 1/N \to 0$ and the volume $V \sim 1$. In that limit, we have an *extensive* scaling of the energy $E \sim N$ and entropy $S \sim N$ (while the temperature $T \sim 1$), but the system remains fundamentally *non-additive*. Other equivalent combinations of the parameters are possible to define the thermodynamic limit as discussed in [1, 5] and in various contributions of [7]. For systems with weak long-range interactions, it is often claimed that the mean field approximation is a very good approximation and that it becomes exact in the proper thermodynamic limit $N \to +\infty$. In fact, this is true only if we are far from a critical point. Close to a critical point, the fluctuations become large and cannot be ignored. In that case, the two-body correlation function does not factor out in a product of two one-body distribution functions and the mean field approximation breaks down. It is therefore highly desirable to derive stochastic kinetic equations that go beyond the mean field approximation and that take full account of fluctuations. This is the main object of the present paper.

In our previous studies, we have distinguished two types of systems: Hamiltonian and Brownian. Hamiltonian systems with long-range interactions are *isolated* and evolve at fixed energy. The dynamics of the particles is described by N coupled deterministic Newton equations. Since the energy is conserved, the relevant statistical ensemble is the microcanonical ensemble. The evolution of the N-body distribution function is governed by the Liouville equation and the statistical equilibrium state is described by the microcanonical distribution. Examples of such systems are provided by stellar systems [8, 9, 10, 11, 12], two-dimensional vortices [13, 14, 15, 16] and the Hamiltonian Mean Field (HMF) model [17, 18]; see also the important list of references in these papers. Brownian systems with long-range interactions, on the other hand, are dissipative and evolve at fixed temperature. The particles are subject to their mutual longrange interactions but they experience, in addition, a friction force and a stochastic force which mimic the interaction with a thermal bath (that is due to *short-range* interactions). Therefore, the dynamics of the particles is described by N coupled stochastic Langevin equations. The temperature is defined through the Einstein relation as the ratio between the diffusion coefficient and the friction coefficient, and it measures the strength of the stochastic force. Since the temperature is fixed, the relevant statistical ensemble is the canonical ensemble. The evolution of the N-body distribution function is governed by the Fokker-Planck equation and the statistical equilibrium state is described by the canonical distribution. Examples of such systems are provided by self-gravitating Brownian particles [19, 20], bacterial populations experiencing chemotaxis [21, 22, 23, 24] and the Brownian Mean Field (BMF) model [18, 25].

Systems with long-range interactions have a very peculiar dynamics and thermodynamics [6, 7]. When the interaction is attractive, there exists a critical point separating a spatially homogeneous (gaseous) phase from a spatially inhomogeneous (clustered) phase. In the microcanonical ensemble (MCE), the "clustered" phase appears below a critical energy E_c and in the canonical ensemble (CE), it appears below a critical temperature T_c . The homogeneous phase may still exist for $E < E_c$ or $T < T_c$ but is dynamically and thermodynamically unstable. One fundamental illustration of this type of phase transitions corresponds to the famous Jeans instability in astrophysics [26, 9]. For the Jeans problem ¹, the critical temperature $T_c = +\infty$

¹As is well-known, the standard Jeans analysis assumes that an infinite and homogeneous self-gravitating system is a steady state of the hydrodynamical equations (the so-called Euler-Poisson system), which is not correct. This inconsistency in the stability analysis is referred to as the *Jeans swindle* [9]. However, the Jeans treatment can be justified in cosmology [27] if we take into account the expansion of the universe because this

so that the system is always in the "clustered" phase: this corresponds to the universe that we know, filled of galaxies. A spatially homogeneous distribution of matter is always unstable to sufficiently large wavelengths, above the Jeans length $k_J^{-1} = (4\pi G n m^2 \beta)^{-1/2}$, and this leads to gravitational collapse and clustering. If we formally consider a screened Newtonian interaction (attractive Yukawa potential) with screening length k_0^{-1} , we find that the phase transition occurs at a finite critical temperature $T_c = 4\pi G n m^2 / k_0^2$ separating a homogeneous phase from a clustered phase [1]. For $T > T_c$ the homogeneous phase is stable and for $T < T_c$ it becomes unstable to wavenumbers $k < k_m \equiv k_0 (T_c/T - 1)^{1/2}$. Alternatively, we can consider a spatially inhomogeneous self-gravitating gas in a spherical box of radius R. In that case, there exists a critical energy $E_c = -0.335 GM^2/R$ in MCE (discovered by Antonov [28, 29] for stellar systems) and a critical temperature $T_c = GMm/(2.52k_BR)$ in CE (discovered by Emden [30] for isothermal stars and recently emphasized by the author for molecular clouds in contact with a thermal bath [31] and for self-gravitating Brownian particles [19]) below which the system passes from a slightly inhomogeneous gaseous phase to a highly inhomogeneous condensed phase². Interestingly, the analogue of these "gravitational phase transitions" also takes place in the context of the chemotaxis of bacterial populations in biology [32, 33]. In these systems, an infinite and uniform distribution of cells is a steady state of the equations of motion (the socalled Keller-Segel model and its generalizations) so there is no "Jeans swindle". Furthermore, the screened Yukawa potential enters naturally in the problem and the screening length has a clear physical interpretation as it takes into account the degradation of the secreted chemical. Another illustration of this type of phase transitions is given by the study of toy models like the Hamiltonian Mean Field (HMF) and the Brownian Mean Field (BMF) models [17, 18]. For these systems, there exists a critical energy $E_c = kM^2/(8\pi)$ and a critical temperature $T_c = kM/(4\pi)$ separating a spatially homogeneous (non-magnetized) phase from a spatially inhomogeneous (magnetized) phase. The magnetization plays the role of the order parameter and the phase transition is second order in that case [17, 18]. We expect the above-mentioned types of phase transitions, and their generalizations [34, 35, 36, 37, 38, 12, 1], to occur for a large class of mean field systems with different potentials of interaction.

The critical point separating the homogeneous phase from the inhomogeneous phase is often called a *spinodal point*. The instability threshold of the homogeneous phase can be obtained from different classical methods: (i) by studying the sign of the second order variations of the thermodynamical potential (entropy in MCE or free energy in CE) and determining when the spatially uniform distribution becomes an unstable saddle point (see, e.g., Sec. 4.4. of Paper I or Appendix C of [24]) (ii) by studying the bifurcation (from homogeneous to inhomogeneous) of the solutions of the meanfield integrodifferential equation (I-19) determining the statistical equilibrium state (see, e.g., Sec. 2.3 of Paper I and Appendix C of [24]) (iii) by studying the linear and nonlinear dynamical stability of the homogeneous state with respect to kinetic equations: Vlasov, Euler, Landau, Kramers, Smoluchowski... (see, e.g., Paper II and [33, 24]). In the homogeneous phase, the one-body distribution function is trivial (it is spatially uniform with a Maxwellian velocity distribution) and the state of the system is usually characterized by the two-body distribution function. For weak long-range interactions, the two-body correlation function $h(|\mathbf{r}_1 - \mathbf{r}_2|)$ can be obtained from the equilibrium BBGKY-hierarchy at the order O(1/N) by neglecting the three-body correlation function that is of order $O(1/N^2)$ [1]. This is similar to the Debye-Hückel approximation in plasma physics. It is found however that, for

creates a sort of "neutralizing background" in the comoving frame expanding with the system allowing for infinite and homogeneous steady states.

 $^{^{2}}$ In order to have a well-defined condensed phase, one has to introduce a small-scale regularization of the gravitational potential. More fundamentally, we can invoke quantum mechanics (Pauli exclusion principle) and consider the case of self-gravitating fermions (e.g. white dwarfs, neutron stars and fermion balls) [12].

large but fixed N, the Fourier transform of the correlation function diverges at the critical point (or at the instability threshold) so that the mean field approximation breaks down close to the critical point. This implies that the phase transition should take place *strictly before* the critical point (or strictly before the instability threshold) predicted by mean field theory. In [1, 18], we have reached this conclusion from the study of the equilibrium BBGKY hierarchy. In the present work, we would like to complement our previous studies by developing a theory of fluctuations starting directly from the stochastic kinetic equation governing the evolution of the fluctuating density field. In this paper, we restrict ourselves to the case of Brownian systems.

The paper is organized as follows. In Sec. 2, we consider a gas of Brownian particles in interaction in an overdamped limit where inertial effects are neglected. In Sec. 2.1, using a BBGKY-like hierarchy, we give the deterministic kinetic equation (5) satisfied by the smooth density profile and the Smoluchowski equation (8) resulting from a mean field approximation. In Sec. 2.2, we give the stochastic kinetic equation (17) satisfied by the exact density distribution (expressed in terms of δ -functions) and, averaging over the noise, we recover the equation obtained from the BBGKY-like hierarchy. In Sec. 2.3, we give the stochastic kinetic equation (27) satisfied by the coarse-grained density distribution obtained by averaging the exact density distribution over a small spatio-temporal window (thus keeping track of fluctuations). In Appendix B, we provide another derivation of this equation by using the general theory of fluctuations exposed by Landau & Lifshitz [39]. In Sec. 2.4, we derive generalized stochastic Cahn-Hilliard equations in the limit of short-range interactions. In Sec. 2.5, starting from the stochastic Smoluchowski equation (44)-(45), we develop a theory of fluctuations for Brownian particles with weak long-range interactions. Specifically, we study the correlations $\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle$ of the Fourier transform of the density fluctuations and show that it behaves like $A(k)e^{\sigma(k)\tau}$. In the stable regime, the decay rate $\sigma(k) < 0$ of the temporal correlations coincides with the decay rate of the perturbations $\delta \hat{\rho}_k(t) \sim e^{\sigma(k)t}$ governed by the deterministic mean field Smoluchowski equation (without noise). On the other hand, for $T < T_c$, the amplitude of the correlation function A(k) diverges as we approach the instability threshold $k \to k_m$, suggesting that the instability takes place sooner than predicted by mean field theory (for $T > T_c$, considering the "dangerous" mode $k = k_*$, the correlation function diverges as we approach the critical point $T \to T_c^+$). These results clearly demonstrate that fluctuations cannot be ignored close to a critical point. In Sec. 3, we generalize our results to the case of an inertial model of particles in interaction including a friction force and a stochastic force. The overdamped model is recovered in a strong friction limit. Again, the correlations of the density fluctuations diverge as we approach the instability threshold but, in sharp contrast, the correlations of the velocity fluctuations remain finite at this threshold. In Sec. 3.4, we derive a generalized Smoluchowski equation taking into account memory effects and make contact with the Cattaneo model and the telegraph equation. In Sec. 3.5, we derive a stochastic model of chemotaxis generalizing the deterministic mean field Keller-Segel model by keeping track of fluctuations. Finally, in Sec. 4, we extend our study in phase space taking full account of the inertia of the particles. We derive a stochastic Kramers equation and make the connection with the diffusive and hydrodynamic models of the previous sections.

2 The overdamped case

2.1 The smooth density distribution

We consider an overdamped system of N Brownian particles in interaction whose dynamics is governed by the coupled stochastic equations (see Paper II):

$$\frac{d\mathbf{r}_i}{dt} = -\mu m^2 \nabla_i U + \sqrt{2D_*} \mathbf{R}_i(t), \tag{1}$$

where $\mu = 1/(m\xi)$ is the mobility (ξ denotes the friction coefficient), $U(\mathbf{r}_1, ..., \mathbf{r}_N) = \sum_{i < j} u(|\mathbf{r}_i - \mathbf{r}_j|)$ is the potential of interaction, D_* is the diffusion coefficient and $\mathbf{R}_i(t)$ is a white noise such that $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$ and $\langle R_i^{\alpha}(t) R_j^{\beta}(t') \rangle = \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$ where i = 1, ..., N refer to the particles and $\alpha = 1, ..., d$ to the coordinates of space. We assume that the particles interact via a binary potential $u(|\mathbf{r}_i - \mathbf{r}_j|)$ depending only on the absolute distance between the particles. As discussed in the Introduction, this stochastic model can describe self-gravitating Brownian particles [19, 20], bacterial populations experiencing chemotaxis [22, 23, 24] (see also Sec. 3.5) or toy models like the BMF model [18]. The N-body distribution $P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ is solution of the Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_i} \cdot \left(D_* \frac{\partial P_N}{\partial \mathbf{r}_i} + \mu m^2 P_N \frac{\partial U}{\partial \mathbf{r}_i} \right).$$
(2)

The stationary solution of this equation is the Gibbs canonical distribution

$$P_N = \frac{1}{Z} e^{-\beta m^2 U},\tag{3}$$

provided that the inverse temperature $\beta = 1/(k_B T)$ is related to the mobility and the diffusion coefficient through the Einstein relation

$$\beta = \frac{\mu}{D_*}.\tag{4}$$

From the Fokker-Planck equation (2), we can construct a BBGKY-like hierarchy [2]. The exact first equation of the hierarchy is

$$\frac{\partial P_1}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \cdot \left[D_* \frac{\partial P_1}{\partial \mathbf{r}_1} + \mu m^2 (N-1) \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_2 d\mathbf{r}_2 \right].$$
(5)

The two-body distribution function can be written as

$$P_2(\mathbf{r}_1, \mathbf{r}_2, t) = P_1(\mathbf{r}_1, t) P_1(\mathbf{r}_2, t) + P_2'(\mathbf{r}_1, \mathbf{r}_2, t),$$
(6)

where $P'_2(\mathbf{r}_1, \mathbf{r}_2, t)$ is the correlation function. Let us consider a weak long-range potential of interaction $u(\mathbf{r}_{12}) = k\tilde{u}(\mathbf{r}_{12})$ in a proper thermodynamic limit $N \to +\infty$ in such a way that $k \sim 1/N$ and $V \sim 1$. Using scaling arguments, it can be shown that $P'_2 = O(1/N)$ except close to a critical point (see [1, 5] for details). Therefore, if we are far from a critical point and if Nis sufficiently large, we can make the mean field approximation $P_2(\mathbf{r}_1, \mathbf{r}_2, t) \simeq P_1(\mathbf{r}_1, t)P_1(\mathbf{r}_2, t)$ and we obtain

$$\frac{\partial P_1}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \cdot \left[D_* \frac{\partial P_1}{\partial \mathbf{r}_1} + \mu m^2 N P_1(\mathbf{r}, t) \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_2, t) d\mathbf{r}_2 \right].$$
(7)

Introducing the smooth density distribution $\rho(\mathbf{r}, t) = NmP_1(\mathbf{r}, t)$, this equation can be rewritten as

$$\frac{\partial \rho}{\partial t}(\mathbf{r},t) = D_* \Delta \rho(\mathbf{r},t) + \mu m \nabla \cdot \left(\rho(\mathbf{r},t) \nabla \int \rho(\mathbf{r}',t) u(\mathbf{r}-\mathbf{r}') d\mathbf{r}' \right).$$
(8)

It can be put in the form of a mean field Smoluchowski equation

$$\frac{\partial \rho}{\partial t} = D_* \Delta \rho + \mu m \nabla \cdot (\rho \nabla \Phi), \tag{9}$$

where $\Phi(\mathbf{r}, t)$ is a smooth potential produced by the particles themselves

$$\Phi(\mathbf{r},t) = \int \rho(\mathbf{r}',t)u(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}'.$$
(10)

If we introduce the mean field Boltzmann free energy functional

$$F = E - TS = \frac{1}{2} \int \rho \Phi \, d\mathbf{r} + k_B T \int \frac{\rho}{m} \ln \frac{\rho}{m} \, d\mathbf{r}, \tag{11}$$

we can rewrite the mean field Smoluchowski equation (9) in the form

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} \rho(\mathbf{r}, t) \nabla \frac{\delta F}{\delta \rho} \right].$$
(12)

This equation satisfies an H-theorem appropriate to the canonical ensemble

$$\dot{F} = \int \frac{\delta F}{\delta \rho} \frac{\partial \rho}{\partial t} d\mathbf{r} = \int \frac{\delta F}{\delta \rho} \nabla \cdot \left[\frac{1}{\xi} \rho \nabla \frac{\delta F}{\delta \rho} \right] d\mathbf{r} = -\int \frac{1}{\xi} \rho \left(\nabla \frac{\delta F}{\delta \rho} \right)^2 d\mathbf{r} \le 0.$$
(13)

The steady solution of the mean field Smoluchowski equation (9) or (12) corresponds to a uniform $\mu = \delta F / \delta \rho$ leading to the mean field Boltzmann distribution

$$\rho = A e^{-\beta m \Phi},\tag{14}$$

where $\Phi(\mathbf{r})$ is given by Eq. (10). Finally, we note that the mean field Smoluchowski equation (8) can be written in Fourier space as

$$\frac{\partial \hat{\rho}}{\partial t}(\mathbf{k},t) = -D_* k^2 \hat{\rho}(\mathbf{k},t) - (2\pi)^d \mu m \int \mathbf{k} \cdot \mathbf{k}' \hat{\rho}(\mathbf{k}-\mathbf{k}',t) \hat{u}(\mathbf{k}') \hat{\rho}(\mathbf{k}',t) d\mathbf{k}'.$$
(15)

For weak long-range potentials of interaction, the mean field approximation usually provides a good and useful description of the system as a first approach ³. We must, however, recall its domain of validity: (i) first, it assumes that the number of particles in the system is large (mathematically speaking it is valid in the limit $N \to +\infty$). Therefore, we can expect deviations from mean field theory due to finite N effects. These deviations will become manifest for

³It is often advocated that long-range potentials of interaction exhibit lack of temperedness and stability. Some potentials, like the cosine potential in the HMF model are well-behaved. By contrast, the gravitational potential is singular and, strictly speaking, there is no statistical equilibrium state (no global maximum of entropy in MCE and no global minimum of free energy in CE). However, there exist long-lived metastable states (local maxima of entropy or local minima of free energy) that can be adequately described by the mean field approximation [12]. The formation of a Dirac peak in CE, which can be viewed as the "equilibrium state" of the system, can also be described by the mean field Smoluchowski-Poisson system [40]. By contrast, the formation of binary stars in MCE requires going beyond the mean field approximation.

sufficiently large times (see discussion at the end of Sec. 2.3). (ii) Close to a critical point $T \to T_c$, the correlation function diverges (see [1] and Secs. 2.5 and 2.6). Typically, we expect a scaling of the form $P'_2 \sim N^{-1}(T - T_c)^{-1}$ so that the limits $N \to +\infty$ and $T \to T_c$ do not commute (see Sec. 2.7 in [18]). Therefore, even for large N, the mean field approximation is expected to break down close to a critical point because P'_2 is not necessarily small (the mean field approximation is valid for $N \gg (T - T_c)^{-1}$, which requires larger and larger particle numbers as $T \to T_c$). In the two cases (i) and (ii) mentioned above, we must take fluctuations into account and consider stochastic kinetic equations as discussed in the sequel.

2.2 The exact density distribution

The exact density distribution of the particles is expressed as a sum of Dirac distributions in the form

$$\rho_d(\mathbf{r}, t) = m \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i(t)).$$
(16)

It was shown by Dean [41] (see also [42, 43, 44]) that the exact density field satisfies a stochastic equation of the form

$$\frac{\partial \rho_d}{\partial t}(\mathbf{r},t) = D_* \Delta \rho_d(\mathbf{r},t) + \mu m \nabla \cdot \left(\rho_d(\mathbf{r},t) \nabla \int \rho_d(\mathbf{r}',t) u(\mathbf{r}-\mathbf{r}') d\mathbf{r}' \right)
+ \nabla \cdot \left(\sqrt{2D_* m \rho_d(\mathbf{r},t)} \mathbf{R}(\mathbf{r},t) \right),$$
(17)

where $\mathbf{R}(\mathbf{r},t)$ is a Gaussian random field such that $\langle \mathbf{R}(\mathbf{r},t) \rangle = \mathbf{0}$ and $\langle R^{\alpha}(\mathbf{r},t)R^{\beta}(\mathbf{r}',t') \rangle = \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$. Note that the noise is multiplicative [41]. Introducing the exact potential

$$\Phi_d(\mathbf{r},t) = \int \rho_d(\mathbf{r}',t) u(\mathbf{r}-\mathbf{r}') \, d\mathbf{r}',\tag{18}$$

the stochastic equation (17) can be rewritten as

$$\frac{\partial \rho_d}{\partial t} = D_* \Delta \rho_d + \mu m \nabla \cdot (\rho_d \nabla \Phi_d) + \nabla \cdot (\sqrt{2D_* m \rho_d} \mathbf{R}).$$
(19)

For $D_* = 0$ and $\mu \neq 0$, we get the exact deterministic equation

$$\frac{\partial \rho_d}{\partial t} = \mu m \nabla \cdot (\rho_d \nabla \Phi_d). \tag{20}$$

If we introduce the discrete Boltzmann free energy functional

$$F_d = E_d - TS_d = \frac{1}{2} \int \rho_d \Phi_d \, d\mathbf{r} + k_B T \int \frac{\rho_d}{m} \ln \frac{\rho_d}{m} \, d\mathbf{r}, \tag{21}$$

we can rewrite the stochastic equation (19) in the form

$$\frac{\partial \rho_d}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} \rho_d(\mathbf{r}, t) \nabla \frac{\delta F_d}{\delta \rho_d} \right] + \nabla \cdot \left(\sqrt{\frac{2k_B T \rho_d}{\xi}} \mathbf{R} \right).$$
(22)

This equation can be viewed as a Langevin equation for the field $\rho_d(\mathbf{r}, t)$. From this equation, it can be shown that the probability of the density distribution $W[\rho_d, t]$ is governed by the Fokker-Planck equation

$$\frac{\partial W[\rho_d, t]}{\partial t} = -\int \frac{\delta}{\delta \rho_d(\mathbf{r}, t)} \left\{ \nabla \cdot \rho_d(\mathbf{r}, t) \nabla \left[D_* \frac{\delta}{\delta \rho_d} + \mu \frac{\delta F_d}{\delta \rho_d} \right] W[\rho_d, t] \right\} d\mathbf{r}.$$
 (23)

At equilibrium, we get the Boltzmann distribution $W[\rho_d] \propto e^{-\beta (F_d[\rho_d] - \mu \int \rho_d d\mathbf{r})}$ [42, 41, 43, 44].

If we average Eq. (17) over the noise, we find that the evolution of the smooth density field $\rho(\mathbf{r}, t) = \langle \rho_d \rangle$ is governed by an equation of the form

$$\frac{\partial \rho}{\partial t}(\mathbf{r},t) = D_* \Delta \rho(\mathbf{r},t) + \mu m \nabla \cdot \int \langle \rho_d(\mathbf{r},t) \rho_d(\mathbf{r}',t) \rangle \nabla u(\mathbf{r}-\mathbf{r}') d\mathbf{r}'.$$
(24)

Using the identity (see Appendix A):

$$\langle \rho_d(\mathbf{r},t)\rho_d(\mathbf{r}',t)\rangle = Nm^2 P_1(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}') + N(N-1)m^2 P_2(\mathbf{r},\mathbf{r}',t),$$
(25)

and assuming that $\nabla u(\mathbf{0}) = \mathbf{0}$, we find that Eq. (24) coincides with the exact equation (5) of the BBGKY-like hierarchy giving the evolution of the one-body distribution function. Furthermore, if we make the mean field approximation $\langle \rho_d(\mathbf{r},t)\rho_d(\mathbf{r}',t)\rangle \simeq \rho(\mathbf{r},t)\rho(\mathbf{r}',t)$, we recover the mean field Smoluchowski equation (9).

2.3 The coarse-grained density distribution

Equation (5) (or equivalently Eq. (24)) for the ensemble averaged density field $\rho(\mathbf{r}, t)$ is a deterministic equation since we have averaged over the noise. In contrast, Eq. (17) for the exact density field $\rho_d(\mathbf{r}, t)$ is a stochastic equation taking into account fluctuations. However, it is not very useful in practice since the field $\rho_d(\mathbf{r}, t)$ is a sum of Dirac distributions, not a regular function. Therefore, it is easier to directly solve the stochastic equations (1) rather than the equivalent Eq. (17). Following [44], we can keep track of fluctuations while avoiding the problem of δ -functions by defining a "coarse-grained" density field $\overline{\rho}(\mathbf{r}, t)$ obtained by averaging the exact density field on a spatio-temporal window of finite resolution. The "coarse-grained" density field satisfies a stochastic equation of the form

$$\frac{\partial\overline{\rho}}{\partial t}(\mathbf{r},t) = D_*\Delta\overline{\rho}(\mathbf{r},t) + \mu m\nabla \cdot \left(\int\overline{\rho}^{(2)}(\mathbf{r},\mathbf{r}',t)\nabla u(\mathbf{r}-\mathbf{r}')d\mathbf{r}'\right) \\
+\nabla \cdot \left(\sqrt{2D_*m\overline{\rho}(\mathbf{r},t)}\mathbf{R}(\mathbf{r},t)\right).$$
(26)

where $\overline{\rho}^{(2)}(\mathbf{r}, \mathbf{r}', t)$ is a two-body correlation function. For a weak long-range potential of interaction and for a sufficiently small spatio-temporal window, we propose to make the approximation $\overline{\rho}^{(2)}(\mathbf{r}, \mathbf{r}', t) \simeq \overline{\rho}(\mathbf{r}, t)\overline{\rho}(\mathbf{r}', t)$. In that case, we obtain a stochastic equation of the form

$$\frac{\partial\overline{\rho}}{\partial t}(\mathbf{r},t) = D_*\Delta\overline{\rho}(\mathbf{r},t) + \mu m\nabla \cdot \left(\overline{\rho}(\mathbf{r},t)\int\overline{\rho}(\mathbf{r}',t)\nabla u(\mathbf{r}-\mathbf{r}')d\mathbf{r}'\right) \\
+\nabla \cdot \left(\sqrt{2D_*m\overline{\rho}(\mathbf{r},t)}\mathbf{R}(\mathbf{r},t)\right).$$
(27)

Introducing the smooth potential

$$\overline{\Phi}(\mathbf{r},t) = \int \overline{\rho}(\mathbf{r}',t)u(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}',\tag{28}$$

the stochastic equation (27) can be rewritten

$$\frac{\partial \overline{\rho}}{\partial t} = D_* \Delta \overline{\rho} + \mu m \nabla \cdot (\overline{\rho} \nabla \overline{\Phi}) + \nabla \cdot (\sqrt{2D_* m \overline{\rho}} \mathbf{R}).$$
(29)

This will be called the *stochastic Smoluchowski equation* for the smoothed-out density field $\overline{\rho}(\mathbf{r}, t)$. This equation is intermediate between Eqs. (8) and (17). It keeps track of fluctuations

while dealing with a continuous density field instead of a sum of δ -functions. This equation will be central in the rest of the paper. We will see that it can reproduce the equilibrium density correlation function (70) that was obtained in Paper I from the equilibrium BBGKYlike hierarchy [1]. Therefore, it represents an improvement with respect to the mean field Smoluchowski equation (8). We stress that this equation is physically distinct from Eq. (17). In Appendix B, we propose an alternative derivation of Eq. (29) by using the general theory of fluctuations exposed by Landau & Lifshitz [39].

If we introduce the coarse-grained Boltzmann free energy functional

$$F_{c.g.} = E_{c.g.} - TS_{c.g.} = \frac{1}{2} \int \overline{\rho} \overline{\Phi} \, d\mathbf{r} + k_B T \int \frac{\overline{\rho}}{m} \ln \frac{\overline{\rho}}{m} \, d\mathbf{r}, \tag{30}$$

we can write the stochastic equation (29) in the form

$$\frac{\partial \overline{\rho}}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} \overline{\rho}(\mathbf{r}, t) \nabla \frac{\delta F_{c.g.}}{\delta \overline{\rho}} \right] + \nabla \cdot \left(\sqrt{\frac{2k_B T \overline{\rho}}{\xi}} \mathbf{R} \right).$$
(31)

This equation can be viewed as a Langevin equation for the field $\overline{\rho}(\mathbf{r}, t)$. The evolution of the probability of the density distribution $W[\overline{\rho}, t]$ is governed by a Fokker-Planck equation of the form (23) where F_d and ρ_d are replaced by $F_{c.g.}$ and $\overline{\rho}$. At equilibrium, we have $W[\overline{\rho}] \propto$ $e^{-\beta(F_{c.g.}[\overline{\rho}]-\mu\int \overline{\rho} d\mathbf{r})}$. For $N \to +\infty$, the equilibrium distribution $W[\overline{\rho}]$ is strongly peaked around the global minimum of $F_{c.q.}[\overline{\rho}]$ at fixed mass. However, the system can remain trapped in a metastable state (local minimum of $F_{c.g.}[\overline{\rho}]$) for a very long time. Let us be more precise. If we ignore the noise term, Eq. (31) reduces to Eq. (12). In that case, the system tends to a steady state that is a minimum (global or local) of the free energy functional $F_{c.a.}[\overline{\rho}]$ at fixed mass (maxima or saddle points of free energy are linearly dynamically unstable with respect to mean field Fokker-Planck equations [24]). If the free energy admits several local minima, the selection of the steady state will depend on a notion of *basin of attraction*. Without noise, the system remains on a minimum of free energy forever. Now, in the presence of noise, the fluctuations can induce a *dynamical phase transition* from one minimum to the other. Thus, accounting correctly for fluctuations is very important when there exists metastable states. The probability of transition scales as $e^{-\Delta F/k_BT}$ where ΔF is the barrier of free energy between two minima. Therefore, in an infinite time, the system will explore all the minima and will spend most time in the global minimum for which ΔF is the largest. Now, for systems with long-range interactions, the barrier of free energy ΔF scales as N so that the probability of escape from a local minimum is very small and behaves as e^{-N} . Therefore, even if the global minimum is in principle the most probable state, metastable states are highly robust in practice since their lifetime scales like e^N . They are thus fully relevant for $N \gg 1$ [47]. These interesting features (basin of attraction, dynamical phase transitions, metastability,...) would be interesting to study in more detail in the case of systems with long-range interactions. Some results in this direction have been reported in [45, 46, 47] in the gravitational case.

2.4 Generalized Cahn-Hilliard equations

Let us assume that $u(|\mathbf{r} - \mathbf{r}'|)$ is a short-range potential of interaction and that the preceding equation (29) remains valid (to simplify the notations, we drop the bar on the coarse-grained fields). Setting $\mathbf{q} = \mathbf{r}' - \mathbf{r}$ and writing

$$\Phi(\mathbf{r},t) = \int u(q)\rho(\mathbf{r}+\mathbf{q},t)d\mathbf{q},$$
(32)

we can Taylor expand $\rho(\mathbf{r} + \mathbf{q}, t)$ to second order in \mathbf{q} so that

$$\rho(\mathbf{r} + \mathbf{q}, t) = \rho(\mathbf{r}, t) + \sum_{i} \frac{\partial \rho}{\partial x_{i}} q_{i} + \frac{1}{2} \sum_{i,j} \frac{\partial^{2} \rho}{\partial x_{i} \partial x_{j}} q_{i} q_{j}.$$
(33)

Substituting this expansion in Eq. (32), we obtain [48]:

$$\Phi(\mathbf{r},t) = -a\rho(\mathbf{r},t) - \frac{b}{2}\Delta\rho(\mathbf{r},t), \qquad (34)$$

with $a = -S_d \int_0^{+\infty} u(q)q^{d-1}dq$ and $b = -\frac{1}{d}S_d \int_0^{+\infty} u(q)q^{d+1}dq$. Note that $l = (b/a)^{1/2}$ has the dimension of a length corresponding to the range of the interaction. Substituting Eq. (34) in Eq. (30), we can put the free energy in the form

$$F_{c.g.}[\rho] = \int \left[\frac{1}{2}(\nabla\rho)^2 + V(\rho)\right] d\mathbf{r},$$
(35)

where V is the effective potential

$$V(\rho) = -\frac{a}{b}\rho^2 + \frac{2k_BT}{mb}\rho\ln\rho + V_0.$$
 (36)

In that case, Eq. (31) can be rewritten

$$\frac{\partial \rho}{\partial t} = -A\nabla \cdot \left[\rho \nabla \left(\Delta \rho - V'(\rho)\right)\right] + \nabla \cdot \left(\sqrt{\frac{2k_B T \rho}{\xi}} \mathbf{R}\right),\tag{37}$$

with $A = b/(2\xi)$. Substituting Eq. (36) in Eq. (37) we explicitly obtain

$$\xi \frac{\partial \rho}{\partial t} = \frac{k_B T}{m} \Delta \rho - \frac{a}{2} \Delta \rho^2 - \frac{b}{2} \nabla \cdot \left(\rho \nabla(\Delta \rho)\right) + \nabla \cdot \left(\sqrt{2k_B T \xi \rho} \mathbf{R}\right).$$
(38)

Without the noise term, the steady state of Eq. (31), (37) or (38) corresponds to a uniform $\mu = \delta F / \delta \rho$ leading to

$$\Delta \rho = V'(\rho) - \mu = -\frac{2a}{b}\rho + \frac{2k_BT}{mb}\ln\rho + \frac{2k_BT}{mb} - \mu.$$
(39)

In particular, at T = 0, Eq. (38) reduces to

$$\xi \frac{\partial \rho}{\partial t} = -\frac{a}{2} \Delta \rho^2 - \frac{b}{2} \nabla \cdot (\rho \nabla (\Delta \rho)), \tag{40}$$

and its steady state is solution of the Helmholtz equation

$$\Delta \rho + \frac{2a}{b}\rho = -\mu. \tag{41}$$

Morphologically, Eq. (31) with Eq. (35), or equivalently Eq. (37), is similar to the stochastic Cahn-Hilliard equation [49] for model B (conserved dynamics):

$$\xi \frac{\partial \rho}{\partial t} = \Delta \frac{\delta F}{\delta \rho} + \sqrt{2\xi k_B T} \nabla \cdot \mathbf{R}, \qquad F[\rho] = \int \left[\frac{1}{2} (\nabla \rho)^2 + V(\rho) \right] d\mathbf{r}, \tag{42}$$

or explicitly

$$\xi \frac{\partial \rho}{\partial t} = -\Delta (\Delta \rho - V'(\rho)) + \sqrt{2\xi k_B T} \nabla \cdot \mathbf{R}.$$
(43)

There are, however, crucial differences between Eqs. (37) and (43). First, the presence of the density $\rho(\mathbf{r}, t)$ in the deterministic current and in the noise term. Secondly, in the usual Cahn-Hilliard problem, the potential has a double-well shape of the typical form $V(\rho) = (1 - \rho^2)^2$ leading to a phase separation while, in the present case, the potential (36) is of a different nature.

2.5 Theory of fluctuations

Let us return to the stochastic Smoluchowski equation (31) satisfied by the coarse-grained density distribution that we write in the form (for convenience, we drop the bars on the coarse-grained fields):

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) \right] + \nabla \cdot \left(\sqrt{\frac{2k_B T \rho}{\xi}} \mathbf{R} \right),\tag{44}$$

$$\Phi(\mathbf{r},t) = \int \rho(\mathbf{r}',t)u(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}'.$$
(45)

We wish to study the fluctuations of the density around an infinite and homogeneous equilibrium distribution. To that purpose, we consider small perturbations $\delta\rho(\mathbf{r}, t)$ and $\delta\Phi(\mathbf{r}, t)$ around the steady state $\rho(\mathbf{r}) = \rho$, $\Phi(\mathbf{r}) = \Phi$ with $\Phi = \rho \int u(x) d\mathbf{x}$. The linearized equations for the perturbations are

$$\xi \frac{\partial \delta \rho}{\partial t} = \frac{k_B T}{m} \Delta \delta \rho + \rho \Delta \delta \Phi + \sqrt{2k_B T \xi \rho} \nabla \cdot \mathbf{R}, \tag{46}$$

$$\delta\Phi(\mathbf{r},t) = \int \delta\rho(\mathbf{r}',t)u(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}'.$$
(47)

We now decompose the perturbations in Fourier modes in the form

$$\delta\rho(\mathbf{r},t) = \int \delta\hat{\rho}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d\mathbf{k}d\omega,$$
(48)

and similar expressions for $\delta \Phi(\mathbf{r}, t)$ and $\mathbf{R}(\mathbf{r}, t)$. Taking the Fourier transform of Eqs. (46) and (47), we obtain the algebraic equations

$$-i\xi\omega\delta\hat{\rho}_{k\omega} = -\frac{k_BT}{m}k^2\delta\hat{\rho}_{k\omega} - \rho k^2\delta\hat{\Phi}_{k\omega} + \sqrt{2k_BT\xi\rho}\nabla\cdot\hat{\mathbf{R}}_{k\omega},\tag{49}$$

$$\delta \hat{\Phi}_{k\omega} = (2\pi)^d \hat{u}(k) \delta \hat{\rho}_{k\omega},\tag{50}$$

where we have used the fact that the integral in Eq. (47) is a convolution. Solving for $\delta \hat{\rho}_{k\omega}$, we obtain

$$\left[\frac{k_B T}{m}k^2 + (2\pi)^d \hat{u}(k)k^2\rho - i\xi\omega\right]\delta\hat{\rho}_{k\omega} = \sqrt{2k_B T\xi\rho} \ ik^{\mu}\hat{R}^{\mu}_{k\omega},\tag{51}$$

where the correlations of the Fourier transform of the noise are given by

$$\langle \hat{R}^{\mu}_{k\omega} \hat{R}^{\nu}_{k'\omega'} \rangle = \frac{1}{(2\pi)^{d+1}} \delta^{\mu\nu} \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega').$$
(52)

Without noise $(\mathbf{R} = \mathbf{0})$, Eq. (51) gives the dispersion relation (see Paper II) associated with the mean field Smoluchowski equation (9), i.e.:

$$Z(k,\omega) \equiv \frac{k_B T}{m} k^2 + (2\pi)^d \hat{u}(k) k^2 \rho - i\xi\omega = 0.$$
(53)

The function $Z(k,\omega)$ plays a role similar to the dielectric function in plasma physics. The perturbation evolves exponentially rapidly like $\delta \hat{\rho}_k(t) \propto e^{\sigma(k)t}$ with a rate given by $\sigma(k) = -\omega_0^2(k)/\xi$ where

$$\omega_0^2(k) \equiv \frac{k_B T}{m} k^2 + (2\pi)^d \hat{u}(k) k^2 \rho.$$
(54)

Thus, we find that the spatially homogeneous phase is stable with respect to the mean field Smoluchowski equation (9) if $\omega_0^2(k) > 0$, i.e.

$$1 + (2\pi)^d \beta \rho m \hat{u}(k) > 0, \tag{55}$$

for all modes k and unstable (to some modes) otherwise. If $\hat{u} > 0$, the homogeneous phase is always stable. Otherwise, a necessary condition of instability is that

$$k_B T < k_B T_c \equiv (2\pi)^d \rho m |\hat{u}(k)|_{max}.$$
(56)

If this condition is fulfilled, the range of unstable wavenumbers is determined by

$$(2\pi)^d |\hat{u}(k)| > \frac{k_B T}{\rho m}.$$
(57)

Some explicit examples of potentials of interaction, and the corresponding conditions of instability, are given in Paper I and in [33].

Let us now determine the correlations of the fluctuations around a stable equilibrium homogeneous distribution in the presence of noise. If we take the noise into account $(\mathbf{R} \neq \mathbf{0})$, the Fourier transform of the density fluctuations is given by

$$\delta\hat{\rho}_{k\omega} = \frac{\sqrt{2k_B T\xi\rho} \ ik^{\mu}\hat{R}^{\mu}_{k\omega}}{Z(k,\omega)}.$$
(58)

Therefore, the correlations of the fluctuations in Fourier space are

$$\langle \delta \hat{\rho}_{k\omega} \delta \hat{\rho}_{k'\omega'} \rangle = \frac{-2k_B T \xi \rho k^{\mu} k'^{\nu} \langle \hat{R}^{\mu}_{k\omega} \hat{R}^{\nu}_{k'\omega'} \rangle}{Z(k,\omega) Z(k',\omega')}.$$
(59)

Using Eq. (52), we get

$$\langle \delta \hat{\rho}_{k\omega} \delta \hat{\rho}_{k'\omega'} \rangle = \frac{1}{(2\pi)^{d+1}} \frac{2k_B T \xi \rho k^2}{|Z(k,\omega)|^2} \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega'), \tag{60}$$

or, more explicitly,

$$\left\langle \delta\hat{\rho}_{k\omega}\delta\hat{\rho}_{k'\omega'}\right\rangle = \frac{1}{(2\pi)^{d+1}} \frac{2k_B T\xi\rho k^2}{\left[\frac{k_B T}{m}k^2 + (2\pi)^d\hat{u}(k)k^2\rho\right]^2 + \xi^2\omega^2} \delta(\mathbf{k} + \mathbf{k}')\delta(\omega + \omega').$$
(61)

The temporal correlation function of the Fourier components of the density fluctuations is given by

$$\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \int \langle \delta \hat{\rho}_{k\omega} \delta \hat{\rho}_{k'\omega'} \rangle e^{i\omega t} e^{i\omega'(t+\tau)} d\omega d\omega'.$$
(62)

Using Eq. (61), the integral on ω' is trivial and yields

$$\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \frac{1}{(2\pi)^{d+1}} 2k_B T \xi \rho k^2 \delta(\mathbf{k} + \mathbf{k}') \int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\tau}}{|Z(k,\omega)|^2}.$$
 (63)

The integral on ω can be performed by using the Cauchy residue theorem. The poles of the integrand are the zeros of the functions $Z(k,\omega)$ and $Z(k,\omega)^*$, i.e. they are solutions of the dispersion relation (53) and its complex conjugate. If $\omega_0^2(k) > 0$, which is required for the stability of the homogeneous phase, the integrand has a single pole in the upper-half plane at $\omega = i\omega_0^2(k)/\xi$ and the residue is $e^{-\omega_0^2(k)\tau/\xi}/[2i\xi\omega_0^2(k)]$. Therefore, we obtain

$$\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \frac{1}{(2\pi)^d} \frac{k_B T \rho k^2}{\omega_0^2(k)} \delta(\mathbf{k} + \mathbf{k}') e^{-\omega_0^2(k)\tau/\xi},\tag{64}$$

or, more explicitly,

$$\langle \delta \hat{\rho}_{k}(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \frac{1}{(2\pi)^{d}} \frac{k_{B}T\rho}{\frac{k_{B}T}{m} + (2\pi)^{d} \hat{u}(k)\rho} \delta(\mathbf{k} + \mathbf{k}') e^{-\left[\frac{k_{B}T}{m}k^{2} + (2\pi)^{d} \hat{u}(k)k^{2}\rho\right]\tau/\xi}.$$
 (65)

This formula is one of the most important results of this paper. We shall come back to its physical interpretation in Sec. 2.6. The equal time correlation function (corresponding to $\tau = 0$) is given by

$$\langle \delta \hat{\rho}_k \delta \hat{\rho}_{k'} \rangle = \frac{1}{(2\pi)^d} \frac{\rho m}{1 + (2\pi)^d \beta \rho m \hat{u}(k)} \delta(\mathbf{k} + \mathbf{k}').$$
(66)

In the absence of any interaction between the particles (u = 0), we get

$$\langle \delta \hat{\rho}_k \delta \hat{\rho}_{k'} \rangle = \frac{1}{(2\pi)^d} \rho m \delta(\mathbf{k} + \mathbf{k}'), \tag{67}$$

which corresponds to the standard result [39]:

$$\langle (\delta\rho)^2 \rangle = \frac{m\rho}{\Delta V}.$$
(68)

On the other hand, using Eq. (66) and the identity (see Appendix A):

$$\left\langle \delta \hat{\rho}_k \delta \hat{\rho}_{k'} \right\rangle = \frac{1}{(2\pi)^d} \rho m \left[1 + (2\pi)^d n \hat{h}(\mathbf{k}) \right] \delta(\mathbf{k} + \mathbf{k}'), \tag{69}$$

we find that the Fourier transform of the correlation function is

$$\hat{h}_{eq}(\mathbf{k}) = \frac{-\beta m^2 \hat{u}(k)}{1 + (2\pi)^d \beta n m^2 \hat{u}(k)}.$$
(70)

This is precisely the result (I-54) obtained in Paper I by analysing the second equation of the equilibrium BBGKY-like hierarchy and neglecting the three-body correlation function. Therefore, the stochastic Smoluchowski equation (44)-(45) is able to reproduce the equilibrium twobody correlation function. On the other hand, from Eqs. (65) and (69), the Fourier transform of the equilibrium temporal correlation function is

$$\hat{h}(\mathbf{k}, t, t+\tau) = \hat{h}_{eq}(\mathbf{k})e^{-\left[\frac{k_BT}{m}k^2 + (2\pi)^d \hat{u}(k)k^2\rho\right]\tau/\xi}.$$
(71)

This can be compared to the out-of-equilibrium temporal evolution of the equal-time spatial correlation function $\hat{h}(\mathbf{k}, t)$ given by Eq. (II-165) of Paper II. Note that the condition of stability (55) is implied by Eq. (65), Eq. (71), Eq. (II-165) and Eq. (66). This completes the discussion given in Sec. 4.2 of Paper I.

2.6 Specific examples

The physical content of formula (64) is very instructive. First, we note that the temporal correlation function of the Fourier components of the density fluctuations decreases exponentially rapidly with a decay rate $\sigma(k) = -\omega_0^2(k)/\xi$ that coincides with the decay rate of a perturbation of the density governed by the deterministic mean field Smoluchowski equation (9), i.e. without noise ⁴. According to this temporal factor, or according to the mean field theory based on the Smoluchowski equation (9), the modes satisfying the inequality $\omega_0^2(k) \geq 0$ should be stable. For $T \leq T_c$, the threshold of instability corresponds to the wavenumber(s) $k = k_m$ where k_m is defined by $\omega_0^2(k_m) = 0$. Now, we note that the amplitude of the fluctuations in formula (64) behaves like $\omega_0^2(k)^{-1}$, so that it diverges as we approach the instability threshold $k \to k_m$. On the other hand, if we denote by k_m^* the value of the critical wavenumber at $T = T_c$ satisfying $1 + (2\pi)^d \beta_c \rho m \hat{u}(k_m^*) = 0$ and if we consider the mode $k = k_m^*$ in Eq. (65), we get for $T \geq T_c$:

$$\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \frac{1}{(2\pi)^d} \frac{T\rho m}{T-T_c} \delta(\mathbf{k} + \mathbf{k}') e^{-k_B (T-T_c)(k_m^*)^2 \tau/(\xi m)}.$$
(72)

This formula clearly shows that the correlation function diverges at the critical point $T = T_c$ for the "dangerous" mode $k = k_m^*$. These results imply that the mean field approximation breaks down close to the critical point (or close to the instability threshold) and that the instability triggering the phase transition can occur *sooner* than what is predicted by mean field theory (i.e. from the stability analysis of the mean field Smoluchowski equation). Some results in this direction have been reported in [45, 46, 47] in the gravitational case.

Let us consider specific examples for illustration (we use the notations of Paper I). For the BMF model (the Brownian version of the HMF model) [18], there exists a critical temperature $T_c = kM/(4\pi)$. Considering the linear dynamical stability of a spatially homogeneous distribution with respect to the mean field Smoluchowski equation (196), we find that $\omega_0^2(n) = Tn^2 + 2\pi \hat{u}_n \rho n^2$ where \hat{u}_n is given by Eq. (198). The modes $n \neq \pm 1$ decay exponentially rapidly as $e^{-Tn^2t/\xi}$ so they are always stable. By contrast, the modes $n = \pm 1$ evolve in time like $e^{-(T-T_c)t/\xi}$. For $T > T_c$, the perturbation is damped and for $T < T_c$ the perturbation has exponential growth. In that case, the homogeneous phase is unstable to the $n = \pm 1$ modes (see Appendix C). According to formula (64), the correlation function of the density fluctuations can be written for the stable modes $n \neq \pm 1$:

$$\langle \delta \hat{\rho}_n(t) \delta \hat{\rho}_m(t+\tau) \rangle = \frac{M}{4\pi^2 n^2} e^{-Tn^2 \tau/\xi} \delta_{n,-m},\tag{73}$$

and for the "dangerous modes" $n = \pm 1$:

$$\left\langle \delta \hat{\rho}_{\pm 1}(t) \delta \hat{\rho}_m(t+\tau) \right\rangle = \frac{M}{4\pi^2} \frac{T}{T - T_c} e^{-(T - T_c)\tau/\xi} \delta_{m,\mp 1}.$$
(74)

This simple toy model, where the potential of interaction is restricted to one Fourier mode, is very interesting for pedagogical purposes because it clearly illustrates the discussion given above. Considering the temporal factor in Eq. (74), we see that the correlations decay for $T > T_c$ with the rate given by mean field theory. However, as we approach the critical temperature from above $(T \to T_c^+)$, the amplitude of the fluctuations diverges like $(T - T_c)^{-1}$ implying that

⁴Similarly, in Sec. 2.9 of Paper II, we noted that, for Hamiltonian systems with long-range interactions, the Fourier modes of the temporal correlation function of the force decay exponentially rapidly with a decay rate that coincides with the decay rate of a perturbation of the distribution function governed by the Vlasov equation.

the mean field approximation breaks down ⁵ and that the phase transition should occur for T strictly above T_c . We had also reached this conclusion in [18] from the study of the equilibrium BBGKY-like hierarchy.

Let us now consider the attractive Yukawa potential [1]. A detailed stability analysis of the homogeneous phase with respect to the mean field Smoluchowski equation (and generalizations) has been performed in [33]. There exists a critical temperature $k_B T_c = S_d G \rho m/k_0^2$ depending on the screening length k_0^{-1} . Furthermore, in the linear regime, the perturbation evolves exponentially rapidly as $\delta \hat{\rho}_k(t) \propto e^{\sigma(k)t}$ with a rate

$$\sigma(k) = -\frac{\omega_0^2(k)}{\xi} = \frac{k_B T}{m\xi} \frac{k^2}{k^2 + k_0^2} [k_0^2(T_c/T - 1) - k^2].$$
(75)

For $T > T_c$, the homogeneous phase is always stable and for $T < T_c$ it is unstable to wavenumbers $k < k_m(T) \equiv k_0(T_c/T - 1)^{1/2}$. Taking into account the fluctuations, formula (64) shows that the correlation function of the density fluctuations is

$$\left\langle \delta \hat{\rho}_{k}(t) \delta \hat{\rho}_{k'}(t+\tau) \right\rangle = \frac{\rho m}{(2\pi)^{d}} \frac{k^{2} + k_{0}^{2}}{k^{2} + k_{0}^{2}(1-T_{c}/T)} \delta(\mathbf{k} + \mathbf{k}') e^{-\frac{k_{B}T}{m} \frac{k^{2}}{k^{2} + k_{0}^{2}} [k^{2} + k_{0}^{2}(1-T_{c}/T)]\frac{\tau}{\xi}}.$$
 (76)

Considering the mode $k = k_m^* = 0$, we see that the correlation function diverges like $(1 - T_c/T)^{-1}$ as we approach the critical temperature $T \to T_c^+$. On the other hand, for $T < T_c$, we see that the amplitude diverges like $(k^2 - k_m^2)^{-1}$ as we approach the critical wavenumber $k \to k_m^+(T)$. This is particularly true for the gravitational interaction $(k_0 = 0)$ for which Eq. (76) reduces to

$$\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \frac{\rho m}{(2\pi)^d} \frac{k^2}{k^2 - k_J^2} \delta(\mathbf{k} + \mathbf{k}') e^{-\frac{k_B T}{m} (k^2 - k_J^2) \frac{\tau}{\xi}},\tag{77}$$

where $k_J = (S_d Gm\beta\rho)^{1/2}$ is the Jeans wavenumber. According to the standard Jeans analysis [26], the homogeneous phase is stable against perturbations with wavenumbers $k > k_J$ and it becomes unstable for $k \leq k_J$. However, the divergence of the correlation function as $k \to k_J^+$ suggests that the gravitational instability will take place *sooner*, i.e. for smaller wavelengths than the Jeans length. This conclusion was previously reached by Monaghan [45] on the basis of a hydrodynamical model of self-gravitating system incorporating viscosity and fluctuations.

3 The inertial model

In this section, we generalise the previous results to the case of a stochastic model taking into account inertial effects. This corresponds to the damped Euler equations with a long-range potential of interaction and a stochastic forcing. This generalization allows us to study the correlations of the fluctuations of the velocity field and their behaviour close to the critical point. The stochastic Smoluchowski equation (44)-(45) is recovered in a strong friction limit $\xi \to +\infty$ by neglecting the inertial term (l.h.s.) in Eq. (79). The stochastic damped Euler equations can find applications in certain biological models of chemotaxis where inertial effects are relevant [50, 51, 23] (see also Sec. 3.5).

3.1 The density correlations

We consider the stochastic damped Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{78}$$

⁵This implies that the limits $N \to +\infty$ (mean field) and $T \to T_c$ do not commute.

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\frac{k_B T}{m} \nabla \rho - \rho \nabla \Phi - \xi \rho \mathbf{u} - \sqrt{2k_B T \xi \rho} \,\mathbf{R}(\mathbf{r}, t), \tag{79}$$

$$\Phi(\mathbf{r},t) = \int u(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}',t) \, d\mathbf{r}',\tag{80}$$

where the quantities have their usual meaning. Without the stochastic term ($\mathbf{R} = \mathbf{0}$), we recover the damped Euler equations introduced in [52] (see also [20, 23, 33, 24]). With the stochastic term, we obtain a more general model taking into account fluctuations. The form of the noise is justified in Appendix B using the general theory of fluctuations of Landau & Lifshitz [39]. For $\xi = 0$, we get the usual Euler equations and for $\xi \to +\infty$, neglecting the inertial term (l.h.s.) in Eq. (79) and substituting the resulting expression $\xi \rho \mathbf{u} \simeq -(k_B T/m)\nabla \rho - \rho \nabla \Phi - \sqrt{2k_B T \xi \rho} \mathbf{R}$ in the continuity equation (78), we recover the stochastic Smoluchowski equation (44)-(45). Note that Eq. (79) can be written in terms of the free energy (11) as

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\rho \nabla \frac{\delta F}{\delta \rho} - \xi \rho \mathbf{u} - \sqrt{2k_B T \xi \rho} \mathbf{R}(\mathbf{r}, t).$$
(81)

Considering small perturbations around a uniform distribution with $\rho(\mathbf{r}) = \rho$, $\Phi(\mathbf{r}) = \Phi$ and $\mathbf{u} = \mathbf{0}$, we find that the linearized equations for the perturbations are

$$\frac{\partial \delta \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} = 0, \tag{82}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\frac{k_B T}{m} \nabla \delta \rho - \rho \nabla \delta \Phi - \xi \rho \mathbf{u} - \sqrt{2k_B T \xi \rho} \mathbf{R}(\mathbf{r}, t), \tag{83}$$

$$\delta\Phi(\mathbf{r},t) = \int u(\mathbf{r}-\mathbf{r}')\delta\rho(\mathbf{r}',t)\,d\mathbf{r}'.$$
(84)

They can be combined to give

$$\frac{\partial^2 \delta \rho}{\partial t^2} + \xi \frac{\partial \delta \rho}{\partial t} = \frac{k_B T}{m} \Delta \delta \rho + \rho \Delta \delta \Phi + \sqrt{2k_B T \xi \rho} \nabla \cdot \mathbf{R}.$$
(85)

If we decompose the perturbations in Fourier modes of the form $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, we obtain the system of algebraic equations

$$-i\omega\delta\hat{\rho}_{k\omega} + i\rho\mathbf{k}\cdot\hat{\mathbf{u}}_{k\omega} = 0, \tag{86}$$

$$-i\omega\rho\hat{\mathbf{u}}_{k\omega} = -\frac{k_BT}{m}i\mathbf{k}\delta\hat{\rho}_{k\omega} - i\mathbf{k}\rho\delta\hat{\Phi}_{k\omega} - \xi\rho\hat{\mathbf{u}}_{k\omega} - \sqrt{2k_BT\xi\rho}\,\hat{\mathbf{R}}_{k\omega},\tag{87}$$

$$\delta \hat{\Phi}_{k\omega} = (2\pi)^d \hat{u}(k) \delta \hat{\rho}_{k\omega}.$$
(88)

Solving for the density perturbation, we get

$$\left[\frac{k_B T}{m}k^2 + (2\pi)^d \hat{u}(k)\rho k^2 - \omega(\omega + i\xi)\right]\delta\hat{\rho}_{k\omega} = \sqrt{2k_B T\xi\rho} \ ik^{\mu}\hat{R}^{\mu}_{k\omega}.$$
(89)

Without noise $(\mathbf{R} = \mathbf{0})$, Eq. (89) gives the dispersion relation associated with the mean field damped Euler equation [20], i.e.

$$Z(k,\omega) \equiv \frac{k_B T}{m} k^2 + (2\pi)^d \hat{u}(k) \rho k^2 - \omega(\omega + i\xi) = 0.$$
(90)

Using the definition (54), the dispersion relation can be rewritten

$$\omega^2 + i\xi\omega - \omega_0^2(k) = 0, \qquad (91)$$

with solutions

$$\omega(k) = \frac{-i\xi \pm \sqrt{-\xi^2 + 4\omega_0^2(k)}}{2} \equiv -i\frac{\xi}{2} \pm \Omega(k).$$
(92)

The perturbation evolves in time as $e^{-i\omega(k)t}$. If $\omega_0^2(k) < 0$, the perturbation grows exponentially rapidly in time with a rate $\left[-\xi + \sqrt{\xi^2 + 4|\omega_0^2|}\right]/2$. If $\omega_0^2(k) > \xi^2/4$, the perturbation decays exponentially rapidly in time with a rate $-\xi/2$ and oscillates with a pulsation $\sqrt{4\omega_0^2 - \xi^2/2}$. If $0 < \omega_0^2(k) < \xi^2/4$, the perturbation decays exponentially rapidly in time with a rate $\left[-\xi + \sqrt{\xi^2 - 4\omega_0^2}\right]/2$ without oscillating. Therefore, according to mean field theory, the system is stable iff $\omega_0^2(k) > 0$ for all k. This returns the condition (55) studied in Sec. 2.5. For $T < T_c$, the onset of instability corresponds to the wavenumber(s) k_m such that $\omega_0(k_m) = 0$.

We now consider stable modes $(\omega_0^2(k) > 0)$ and study the correlations of the density fluctuations in the presence of noise. Repeating the steps of Sec. 2.5 going from Eq. (58) to Eq. (63), the density correlation function can be written

$$\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \frac{1}{(2\pi)^{d+1}} 2k_B T \xi \rho k^2 I(k,\tau) \delta(\mathbf{k}+\mathbf{k}'), \tag{93}$$

where $I(k, \tau)$ is the integral

$$I = \int_{-\infty}^{+\infty} f(\omega)e^{i\omega\tau}d\omega, \qquad f(\omega) = \frac{1}{|Z(k,\omega)|^2}.$$
(94)

We can evaluate the integral with the Cauchy residue theorem. The poles of $f(\omega)$ are the zeros of the functions $Z(k,\omega)$ and $Z(k,\omega)^*$, i.e. they are solutions of the dispersion relation (91) and its complex conjugate. If $\omega_0^2(k) > 0$, the function $f(\omega)$ has two simple poles in the upper-half plane at $\omega = i\xi/2 + \Omega$ and $\omega = i\xi/2 - \Omega$. The residues of $f(\omega)e^{i\omega\tau}$ at $\omega = i\xi/2 + \Omega$ and at $\omega = i\xi/2 - \Omega$ are

$$\frac{e^{-\frac{\xi}{2}\tau}e^{i\Omega\tau}}{i\xi(i\xi+2\Omega)(2\Omega)}, \qquad \frac{e^{-\frac{\xi}{2}\tau}e^{-i\Omega\tau}}{(i\xi-2\Omega)i\xi(-2\Omega)}.$$
(95)

Using the Cauchy residue theorem, and recalling that

$$\Omega^2 = \omega_0^2 - \frac{1}{4}\xi^2, \tag{96}$$

we obtain after simplification

$$I = \frac{\pi}{\xi\omega_0^2} e^{-\frac{\xi}{2}\tau} \left[\cos(\Omega\tau) + \frac{\xi}{2\Omega}\sin(\Omega\tau) \right],\tag{97}$$

where Ω can be complex. In conclusion, the density correlation function for the inertial model is

$$\left\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \right\rangle = \frac{1}{(2\pi)^d} \frac{k_B T \rho k^2}{\omega_0^2(k)} e^{-\frac{\xi}{2}\tau} \left[\cos(\Omega \tau) + \frac{\xi}{2\Omega} \sin(\Omega \tau) \right] \delta(\mathbf{k} + \mathbf{k}'). \tag{98}$$

The interpretation of this formula is similar to the one given in Sec. 2.6. In particular, we see that the amplitude of the fluctuations diverges when $k \to k_m$, corresponding to $\omega_0^2(k) \to 0$. Thus, instability should set in slightly before we reach the range of unstable wavenumbers (57). In particular, the amplitude diverges at the critical point $T \to T_c^+$ for the "dangerous" wavenumber k_m^* . For $\xi \to 0$ (Euler), we can make the approximation $\Omega \simeq \omega_0$, and we obtain

$$\langle \delta \hat{\rho}_k(t) \delta \hat{\rho}_{k'}(t+\tau) \rangle = \frac{1}{(2\pi)^d} \frac{k_B T \rho k^2}{\omega_0^2(k)} e^{-\frac{\xi}{2}\tau} \cos(\omega_0 \tau) \delta(\mathbf{k} + \mathbf{k}').$$
(99)

For $\xi \to +\infty$ (Smoluchowski), we can make the approximations

$$\Omega \simeq \frac{\xi}{2}i\left(1 - \frac{2\omega_0^2}{\xi^2}\right), \quad e^{-\frac{\xi}{2}\tau}\cos(\Omega\tau) \simeq \frac{1}{2}e^{-\frac{\omega_0^2}{\xi}\tau}, \quad e^{-\frac{\xi}{2}\tau}\sin(\Omega\tau) \simeq -\frac{1}{2i}e^{-\frac{\omega_0^2}{\xi}\tau}, \tag{100}$$

and we recover the result of Eq. (64) obtained in the overdamped limit.

3.2 The velocity correlations

Let us now turn to the correlations of the velocity fluctuations. From Eq. (87), the Fourier components of the velocity fluctuations satisfy the relation

$$\rho(\xi - i\omega)\hat{\mathbf{u}}_{k\omega} = -\frac{\omega_0^2}{k^2}i\mathbf{k}\delta\hat{\rho}_{k\omega} - \sqrt{2k_B T\xi\rho}\,\hat{\mathbf{R}}_{k\omega}.$$
(101)

Therefore, the velocity correlations can be expressed as

$$\rho^{2}(\xi - i\omega)(\xi - i\omega') \langle \hat{u}^{\mu}_{k\omega} \hat{u}^{\nu}_{k'\omega'} \rangle = -\frac{\omega_{0}^{4}}{k^{2}k'^{2}} k^{\mu} k'^{\nu} \langle \delta \hat{\rho}_{k\omega} \delta \hat{\rho}_{k'\omega'} \rangle + 2k_{B}T\xi \rho \langle \hat{R}^{\mu}_{k\omega} \hat{R}^{\nu}_{k'\omega'} \rangle + \frac{\omega_{0}^{2}}{k^{2}} ik^{\mu} \sqrt{2k_{B}T\xi} \rho \langle \delta \hat{\rho}_{k\omega} \hat{R}^{\nu}_{k'\omega'} \rangle + \frac{\omega_{0}^{'2}}{k'^{2}} ik'^{\nu} \sqrt{2k_{B}T\xi} \rho \langle \delta \hat{\rho}_{k'\omega'} \hat{R}^{\mu}_{k\omega} \rangle.$$
(102)

Using the relation

$$\delta\hat{\rho}_{k\omega} = \frac{i\sqrt{2k_B T\xi\rho}k^{\alpha}\hat{R}^{\alpha}_{k\omega}}{Z(k,\omega)},\tag{103}$$

and Eq. (52), we get

$$\rho^{2}(\xi^{2}+\omega^{2})\langle\hat{u}_{k\omega}^{\mu}\hat{u}_{k'\omega'}^{\nu}\rangle = -2k_{B}T\xi\rho\frac{\omega_{0}^{4}}{k^{4}}\frac{k^{\mu}k^{\nu}}{|Z(k,\omega)|^{2}}k^{\alpha}k^{\beta}\langle\hat{R}_{k\omega}^{\alpha}\hat{R}_{k'\omega'}^{\beta}\rangle + 2k_{B}T\xi\rho\langle\hat{R}_{k\omega}^{\mu}\hat{R}_{k'\omega'}^{\nu}\rangle - 2k_{B}T\xi\rho\frac{\omega_{0}^{2}}{k^{2}}\frac{k^{\mu}k^{\alpha}}{Z(k,\omega)^{*}}\langle\hat{R}_{k'\omega}^{\alpha}\hat{R}_{k\omega}^{\mu}\rangle, \qquad (104)$$

or, more explicitly,

$$\rho^{2}(\xi^{2} + \omega^{2}) \left\langle \hat{u}_{k\omega}^{\mu} \hat{u}_{k'\omega'}^{\nu} \right\rangle = \frac{1}{(2\pi)^{d+1}} 2k_{B}T\xi\rho\delta(\mathbf{k} + \mathbf{k}')\delta(\omega + \omega') \\ \times \left\{ -\frac{\omega_{0}^{4}}{k^{2}} \frac{k^{\mu}k^{\nu}}{|Z(k,\omega)|^{2}} + \delta^{\mu\nu} - \frac{\omega_{0}^{2}}{k^{2}} \frac{k^{\mu}k^{\nu}}{Z(k,\omega)} - \frac{\omega_{0}^{2}}{k^{2}} \frac{k^{\mu}k^{\nu}}{Z(k,\omega)^{*}} \right\}.$$
(105)

Finally, using the identity

$$\frac{1}{Z(k,\omega)} + \frac{1}{Z(k,\omega)^*} = \frac{2\operatorname{Re}(Z)}{|Z(k,\omega)|^2} = \frac{2(\omega_0^2 - \omega^2)}{|Z(k,\omega)|^2},$$
(106)

the foregoing relation can be rewritten

$$\langle \hat{u}_{k\omega}^{\mu} \hat{u}_{k'\omega'}^{\nu} \rangle = \frac{1}{(2\pi)^{d+1}} \frac{2k_B T \xi}{\rho} \frac{1}{\xi^2 + \omega^2} \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') \\ \times \left\{ -\frac{3\omega_0^4}{k^2} \frac{k^{\mu} k^{\nu}}{|Z(k,\omega)|^2} + \frac{2\omega_0^2 \omega^2}{k^2} \frac{k^{\mu} k^{\nu}}{|Z(k,\omega)|^2} + \delta^{\mu\nu} \right\}.$$
(107)

Taking the inverse Fourier transform in ω -space of this relation, we find that the temporal correlations of the velocity fluctuations are given by

$$\left\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t+\tau)\right\rangle = \frac{1}{(2\pi)^{d+1}}\frac{2k_{B}T\xi}{\rho}\delta(\mathbf{k}+\mathbf{k}')\left\{-3\omega_{0}^{4}\frac{k^{\mu}k^{\nu}}{k^{2}}K + 2\omega_{0}^{2}\frac{k^{\mu}k^{\nu}}{k^{2}}K' + K''\delta^{\mu\nu}\right\},\ (108)$$

where we have introduced the integrals

$$K = \int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\tau}}{(\xi^2 + \omega^2)|Z(k,\omega)|^2}, \quad K' = \int_{-\infty}^{+\infty} d\omega \frac{\omega^2 e^{i\omega\tau}}{(\xi^2 + \omega^2)|Z(k,\omega)|^2}, \tag{109}$$

$$K'' = \int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega\tau}}{\xi^2 + \omega^2}.$$
(110)

These integrals are easily calculated with the Cauchy residue theorem. After simplification, we obtain

$$\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t+\tau)\rangle = \frac{1}{(2\pi)^{d}} \frac{k_{B}T}{\rho} \delta(\mathbf{k}+\mathbf{k}') \bigg\{ \omega_{0}^{2} \frac{k^{\mu}k^{\nu}}{k^{2}} \frac{1}{\omega_{0}^{2}+2\xi^{2}} e^{-\frac{\xi}{2}\tau} \bigg[\bigg(\frac{2\xi^{2}}{\omega_{0}^{2}}-1\bigg) \cos(\Omega\tau) \\ -\frac{\xi}{\Omega} \bigg(\frac{7}{2}+\frac{\xi^{2}}{\omega_{0}^{2}}\bigg) \sin(\Omega\tau) - \bigg(3+\frac{2\xi^{2}}{\omega_{0}^{2}}\bigg) e^{-\frac{\xi}{2}\tau} \bigg] + e^{-\xi\tau} \delta^{\mu\nu} \bigg\}.$$
(111)

We note that, at variance with the density correlation function (98), the velocity correlation function does *not* diverge when $k \to k_m$. Indeed, for $\omega_0^2 = 0$, we have

$$\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t+\tau)\rangle = \frac{1}{(2\pi)^{d}}\frac{k_{B}T}{\rho}e^{-\xi\tau}\delta(\mathbf{k}+\mathbf{k}')\delta^{\mu\nu}.$$
(112)

On the other hand, taking $\tau = 0$ in Eq. (111), we obtain the equal time velocity correlation function

$$\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t)\rangle = \frac{1}{(2\pi)^{d}}\frac{k_{B}T}{\rho}\delta(\mathbf{k}+\mathbf{k}')\left(\delta^{\mu\nu} - \frac{4\omega_{0}^{2}}{\omega_{0}^{2}+2\xi^{2}}\frac{k^{\mu}k^{\nu}}{k^{2}}\right).$$
(113)

Contracting the indices, we get

$$\langle \hat{\mathbf{u}}_{k}(t) \cdot \hat{\mathbf{u}}_{k'}(t) \rangle = \frac{1}{(2\pi)^{d}} \frac{k_{B}T}{\rho} \delta(\mathbf{k} + \mathbf{k}') \frac{2d\xi^{2} - (4 - d)\omega_{0}^{2}}{\omega_{0}^{2} + 2\xi^{2}}.$$
(114)

For $\xi \to 0$ (Euler), Eq. (111) can be simplified into

$$\left\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t+\tau)\right\rangle = \frac{1}{(2\pi)^{d}}\frac{k_{B}T}{\rho}\delta(\mathbf{k}+\mathbf{k}')\left\{-\frac{k^{\mu}k^{\nu}}{k^{2}}e^{-\frac{\xi}{2}\tau}\left[\cos(\omega_{0}\tau)+3e^{-\frac{\xi}{2}\tau}\right]+e^{-\xi\tau}\delta^{\mu\nu}\right\}.$$
 (115)

Alternatively, for $\xi \to +\infty$ (Smoluchowski), we get

$$\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t+\tau)\rangle = -\frac{1}{(2\pi)^{d}}\frac{3k_{B}T}{\rho}\delta(\mathbf{k}+\mathbf{k}')\frac{\omega_{0}^{2}}{\xi^{2}}\frac{k^{\mu}k^{\nu}}{k^{2}}e^{-\frac{\omega_{0}^{2}}{\xi}\tau}.$$
(116)

This result can be obtained directly from the study of the stochastic Smoluchowski equation in Sec. 2.5 by defining the velocity field

$$\xi \rho \mathbf{u} \equiv -\frac{k_B T}{m} \nabla \rho - \rho \nabla \Phi - \sqrt{2k_B T \xi \rho} \,\mathbf{R}.$$
(117)

Note that the velocity correlations in Eq. (116) tend to zero when $k \to k_m$ contrary to Eq. (112). This shows that the limits $\xi \to +\infty$ and $k \to k_m$ do not commute.

3.3 Specific examples

Let us discuss specific examples by restricting ourselves, for brevity, to the overdamped limit (116). For the stochastic BMF model, we get for the stable modes $n \neq \pm 1$:

$$\langle \hat{u}_n(t)\hat{u}_m(t+\tau)\rangle = -\frac{3T^2n^2}{M\xi^2}\delta_{m,-n}e^{-Tn^2\tau/\xi},$$
(118)

and for the "dangerous" modes $n = \pm 1$:

$$\langle \hat{u}_{\pm 1}(t)\hat{u}_m(t+\tau)\rangle = -\frac{3T}{M\xi^2}\delta_{m,\mp 1}(T-T_c)e^{-(T-T_c)\tau/\xi}.$$
 (119)

The velocity correlations tend to zero when $T \to T_c^+$. For the attractive Yukawa potential, we get

$$\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t+\tau)\rangle = -\frac{3(k_{B}T)^{2}}{(2\pi)^{d}\xi^{2}\rho m}\frac{k^{2}+k_{0}^{2}(1-T_{c}/T)}{k^{2}+k_{0}^{2}}k^{\mu}k^{\nu}e^{-\frac{k_{B}T}{m}\frac{k^{2}}{k^{2}+k_{0}^{2}}[k^{2}+k_{0}^{2}(1-T_{c}/T)]\frac{\tau}{\xi}}\delta(\mathbf{k}+\mathbf{k}').$$
(120)

For $T \leq T_c$, the amplitude tends to zero as we approach the critical wavenumber $k \to k_m^+(T)$. For the gravitational interaction $(k_0 = 0)$, Eq. (120) reduces to

$$\langle \hat{u}_{k}^{\mu}(t)\hat{u}_{k'}^{\nu}(t+\tau)\rangle = -\frac{3(k_{B}T)^{2}}{(2\pi)^{d}\xi^{2}\rho m}(k^{2}-k_{J}^{2})\frac{k^{\mu}k^{\nu}}{k^{2}}e^{-\frac{k_{B}T}{m}(k^{2}-k_{J}^{2})\frac{\tau}{\xi}}\delta(\mathbf{k}+\mathbf{k}').$$
(121)

The amplitude goes to zero as $k \to k_J^+$. The fact that the velocity correlation function does not diverge when $k \to k_J^+$ was previously observed by Monaghan [45] with his hydrodynamic model.

3.4 Stochastic model with memory

The stochastic damped Euler equations (78)-(79) can be rewritten

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad (122)$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla(\rho \mathbf{u} \otimes \mathbf{u}) = -\frac{k_B T}{m} \nabla \rho - \rho \nabla \Phi - \xi \rho \mathbf{u} - \sqrt{2k_B T \xi \rho} \mathbf{R}(\mathbf{r}, t).$$
(123)

If we neglect the inertial term (l.h.s.) in Eq. (123) and substitute the resulting expression for $\rho \mathbf{u}$ in Eq. (122), we obtain the stochastic Smoluchowski equation (44). This is valid in a strong friction limit $\xi \to +\infty$. We can obtain a more general model taking into account some memory effects. If we neglect only the nonlinear term $\nabla(\rho \mathbf{u} \otimes \mathbf{u})$ in Eq. (123), we get

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) = -\frac{k_B T}{m} \nabla \rho - \rho \nabla \Phi - \xi \rho \mathbf{u} - \sqrt{2k_B T \xi \rho} \mathbf{R}(\mathbf{r}, t).$$
(124)

Taking the time derivative of Eq. (122) and substituting Eq. (124) in the resulting expression, we obtain a simplified stochastic model keeping track of memory effects

$$\frac{\partial^2 \rho}{\partial t^2} + \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) + \nabla \cdot \left(\sqrt{2k_B T \xi \rho} \mathbf{R} \right).$$
(125)

In terms of the free energy (11), we have

$$\frac{\partial^2 \rho}{\partial t^2} + \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \frac{\delta F}{\delta \rho} \right) + \nabla \cdot \left(\sqrt{2k_B T \xi \rho} \mathbf{R} \right).$$
(126)

This equation, which is second order in time, is analogous to the *telegraph equation* which generalizes the diffusion equation by introducing memory effects. We note that in the linear regime $|\mathbf{u}| \ll 1$ considered in Sec. 3, the nonlinear term $\nabla(\rho \mathbf{u} \otimes \mathbf{u})$ in Eq. (123) is negligible so that Eq. (125) can be justified rigorously from the damped Euler equations in this regime. This implies that the theory of fluctuations that we have developed in Sec. 3 directly applies to the stochastic equation (125). In particular, the linearization of Eq. (125) around a homogeneous distribution returns Eq. (85). However, the stochastic Smoluchowski equation with memory (125) may also be relevant in the nonlinear regime as a heuristic equation. Indeed, although we have neglected the nonlinear term $\nabla(\rho \mathbf{u} \otimes \mathbf{u})$ in Eq. (123), we have kept the full nonlinearities in the right hand side. Therefore, Eq. (125) is a semi-linear model intermediate between the fully nonlinear hydrodynamical model (122)-(123) and the linearized hydrodynamical model (85).

Finally, we note that Eq. (126) can be viewed as a form of stochastic Cattaneo model. The deterministic Smoluchowski equation can be written as a continuity equation $\partial_t \rho = -\nabla \cdot \mathbf{J}$ where the current $\mathbf{J} = -(1/\xi)\rho\nabla\mu$ is proportional to the gradient of a chemical potential $\mu = \delta F/\delta\rho$ [24]. This is similar to Fick's law for the diffusion of particles or to Fourier's law for the diffusion of heat. In the context of heat conduction, Cattaneo [53] has proposed a modification of Fourier's law in order to describe heat conduction with finite speed. He assumed that the current is not instantaneously equal to the gradient $\nabla\mu$ but relaxes to it with a time constant $1/\tau$. In the present situation, this would lead to a model of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \qquad (127)$$

$$\tau \frac{\partial \mathbf{J}}{\partial t} + \mathbf{J} = -\frac{1}{\xi} \rho \nabla \frac{\delta F}{\delta \rho} - \sqrt{\frac{2k_B T \rho}{\xi}} \mathbf{R}, \qquad (128)$$

where we have included the stochastic term for completeness. These equations are equivalent to the semi-linear model formed by Eqs. (122) and (124) if we set $\mathbf{J} = \rho \mathbf{u}$ and $\tau = 1/\xi$. For $\tau = 0$, we recover the stochastic Smoluchowski equation (31). More generally, taking the time derivative of Eq. (127) and using Eq. (128), we obtain

$$\tau \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\frac{1}{\xi} \rho \nabla \frac{\delta F}{\delta \rho}\right) + \nabla \cdot \left(\sqrt{\frac{2k_B T \rho}{\xi}} \mathbf{R}\right),\tag{129}$$

which coincides with Eq. (126) provided that we take $\tau = 1/\xi$.

3.5 Application to chemotaxis

In this section, we briefly mention the application of the preceding results to the problem of chemotaxis in biology [54]. A more detailed discussion is given in a specific paper [55] with complements and amplification. The standard Keller-Segel (KS) model [21] of chemotaxis can be viewed as a form of mean field Smoluchowski equation [24]. It describes the diffusion of bacteria (or other chemotactic species) in the concentration gradient of a chemical produced by the particles themselves. As we have seen in this paper, the correlation function diverges close to a critical point. In that case, the mean field approximation breaks down and the fluctuations must be taken into account. Fluctuations also play an important role when the particle number N is small and when there exist metastable states (local minima of free energy). In that case, fluctuations can trigger dynamical phase transitions from one state to the other (see Sec. 2.3). For these different reasons, it is important to derive a chemotactic model going beyond the mean field approximation and taking into account fluctuations.

We start from a microscopic model of chemotaxis where the dynamics of the particles is governed by N coupled stochastic equations of the form

$$\frac{d\mathbf{r}_i}{dt} = \chi \nabla c_d(\mathbf{r}_i(t), t) + \sqrt{2D_*} \mathbf{R}_i(t), \qquad (130)$$

$$\frac{\partial c_d}{\partial t} = -kc_d + D_c \Delta c_d + h \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i(t)), \qquad (131)$$

where $\mathbf{r}_i(t)$, with i = 1, ..., N, denote the positions of the particles and $c_d(\mathbf{r}, t)$ is the exact field of secreted chemical. In these equations, χ and D_* represent the mobility and the diffusion coefficient of the organisms and k, h and D_c represent the degradation rate, the production rate and the diffusion coefficient of the secreted chemical. By extending Dean's approach (and the results of Sec. 2.3) to the case of chemotactic species, we obtain a stochastic Keller-Segel model of chemotaxis:

$$\frac{\partial \rho}{\partial t}(\mathbf{r},t) = D_* \Delta \rho(\mathbf{r},t) - \chi \nabla \cdot \left(\rho(\mathbf{r},t) \nabla c(\mathbf{r},t)\right) + \nabla \cdot \left(\sqrt{2D_*\rho(\mathbf{r},t)} \mathbf{R}(\mathbf{r},t)\right), \quad (132)$$

$$\frac{\partial c}{\partial t}(\mathbf{r},t) = -kc(\mathbf{r},t) + D_c \Delta c(\mathbf{r},t) + h\rho(\mathbf{r},t), \qquad (133)$$

generalizing the deterministic mean field Keller-Segel model. This model fully takes into account the effect of fluctuations ⁶. On the other hand, there exists situations in biology where inertial effects must be taken into account [50]. In that case, parabolic models like the Keller-Segel model must be replaced by hyperbolic models similar to hydrodynamic equations [50, 51, 23]. By extending the results of Sec. 3, we obtain a hydrodynamic model of chemotaxis taking into account inertial effects and fluctuations in the form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad (134)$$

⁶Note added: Until now, fluctuations have been ignored by people working on chemotaxis. Therefore, our paper is the first attempt to include fluctuations in the Keller-Segel model. However, after submission of this paper [arXiv:0803.0263], a paper by Tailleur & Cates [arXiv:0803.1069] came out on a related subject. These authors also consider the effect of fluctuations in the motion of bacteria. They derive transport coefficients from microscopic models but do not take into account the long-range interaction between bacteria due to chemotaxis. Alternatively, in our approach, the coefficients D_* and χ appearing in the Langevin equations are phenomenological coefficients but chemotaxis is fully taken into account. Therefore, these two independent studies are complementary to each other.

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla(\rho \mathbf{u} \otimes \mathbf{u}) = -\xi D_* \nabla \rho + \rho \nabla c - \xi \rho \mathbf{u} - \sqrt{2D_* \xi^2 \rho} \mathbf{R}(\mathbf{r}, t), \qquad (135)$$

coupled to the field equation (133). In the strong friction limit $\xi \to +\infty$ where the inertial term in Eq. (135) can be neglected, it returns the stochastic KS model (132) with $\chi = 1/\xi$. On the other hand, if we only neglect the term $\nabla(\rho \mathbf{u} \otimes \mathbf{u})$ in Eq. (135) like in Sec. 3.4, we obtain a stochastic equation of the form

$$\chi \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(D_* \nabla \rho - \chi \rho \nabla c \right) + \nabla \cdot \left(\sqrt{2D_* \rho} \mathbf{R} \right), \tag{136}$$

It can be viewed as a stochastic Cattaneo model of chemotaxis (or a stochastic telegraph equation).

4 The stochastic Kramers equation

In this section, we generalize the results of Secs. 2.1-2.3 in phase space. This is the rigorous way to take into account inertial effects and fluctuations in the problem. The motion of the Brownian particles is described by N coupled stochastic Langevin equations of the form (see Paper I):

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i,\tag{137}$$

$$\frac{d\mathbf{v}_i}{dt} = -\xi \mathbf{v}_i - m\nabla_i U(\mathbf{r}_1, ..., \mathbf{r}_N) + \sqrt{2D} \mathbf{R}_i(t).$$
(138)

The friction coefficient ξ and the diffusion coefficient D are related to each other by the Einstein relation $\xi = D\beta m$ where $\beta = 1/(k_B T)$ is the inverse temperature [1]. In the strong friction limit $\xi \to +\infty$, we can neglect the inertial term in Eq. (138) and we obtain the overdamped equations (1) of Sec. 2.1 with $\mu = 1/(\xi m)$ and $D_* = D/\xi^2$.

Extending Dean's approach [41] in phase space, we find that the exact distribution function $f_d(\mathbf{r}, \mathbf{v}, t) = m \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t))$ expressed in terms of δ -functions satisfies a stochastic equation of the form

$$\frac{\partial f_d}{\partial t} + \mathbf{v} \cdot \frac{\partial f_d}{\partial \mathbf{r}} - \nabla \Phi_d \cdot \frac{\partial f_d}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left(D \frac{\partial f_d}{\partial \mathbf{v}} + \xi f_d \mathbf{v} \right) + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\sqrt{2Dmf_d} \mathbf{Q}(\mathbf{r}, \mathbf{v}, t) \right), \quad (139)$$

where $\mathbf{Q}(\mathbf{r}, \mathbf{v}, t)$ is a Gaussian random field such that $\langle \mathbf{Q}(\mathbf{r}, \mathbf{v}, t) \rangle = \mathbf{0}$ and $\langle Q_{\alpha}(\mathbf{r}, \mathbf{v}, t) Q_{\beta}(\mathbf{r}', \mathbf{v}', t') \rangle = \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{v} - \mathbf{v}') \delta(t - t')$ and $\Phi_d(\mathbf{r}, t)$ is defined by Eq. (18). If we average over the noise, we obtain

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{r}' d\mathbf{v}' [\nabla u(\mathbf{r} - \mathbf{r}')] \langle f_d(\mathbf{r}, \mathbf{v}, t) f_d(\mathbf{r}', \mathbf{v}', t) \rangle = \frac{\partial}{\partial \mathbf{v}} \cdot \left(D \frac{\partial f}{\partial \mathbf{v}} + \xi f \mathbf{v} \right).$$
(140)

Using $f = NmP_1$ and the identity

$$\langle f_d(\mathbf{r}, \mathbf{v}, t) f_d(\mathbf{r}', \mathbf{v}', t) \rangle = Nm^2 P_1(\mathbf{r}, \mathbf{v}, t) \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{v} - \mathbf{v}') + N(N-1)m^2 P_2(\mathbf{r}, \mathbf{v}, \mathbf{r}', \mathbf{v}', t), (141)$$

we find that Eq. (140) is equivalent to Eq. (II-139) obtained from the BBGKY-like hierarchy. If we implement a mean field approximation $\langle f_d(\mathbf{r}, \mathbf{v}, t) f_d(\mathbf{r}', \mathbf{v}', t) \rangle \simeq f(\mathbf{r}, \mathbf{v}, t) f(\mathbf{r}', \mathbf{v}', t)$, we obtain the mean field Kramers equation [2]:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left(D \frac{\partial f}{\partial \mathbf{v}} + \xi f \mathbf{v} \right), \tag{142}$$

where $\Phi(\mathbf{r}, t)$ is defined by Eq. (10). Finally, we can heuristically propose a stochastic kinetic equation for the evolution of the coarse-grained distribution function $\overline{f}(\mathbf{r}, \mathbf{v}, t)$ obtained by averaging $f_d(\mathbf{r}, \mathbf{v}, t)$ over a small spatio-temporal window. This leads to the stochastic Kramers equation

$$\frac{\partial \overline{f}}{\partial t} + \mathbf{v} \cdot \frac{\partial \overline{f}}{\partial \mathbf{r}} - \nabla \overline{\Phi} \cdot \frac{\partial \overline{f}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left(D \frac{\partial \overline{f}}{\partial \mathbf{v}} + \xi \overline{f} \mathbf{v} \right) + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\sqrt{2Dm\overline{f}} \mathbf{Q}(\mathbf{r}, \mathbf{v}, t) \right), \tag{143}$$

where $\overline{\Phi}(\mathbf{r}, t)$ is defined by Eq. (28). This equation keeps track of fluctuations but applies to a continuous distribution function instead of a sum of Dirac distributions. An alternative derivation of this equation is proposed in Appendix B using the general theory of fluctuations of Landau & Lifshitz [39].

Let us now try to make the link with the parabolic and hydrodynamic models considered in Secs. 2 and 3. Taking the hydrodynamic moments on the stochastic Kramers equation (139) and proceeding as in [52, 20], we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad (144)$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P_{ij}}{\partial x_j} - \rho\frac{\partial\Phi}{\partial x_i} - \xi\rho u_i - \int\sqrt{2Dmf}Q_i d\mathbf{v},\tag{145}$$

where $\rho(\mathbf{r},t) = \int f d\mathbf{v}$ is the density, $\mathbf{u}(\mathbf{r},t) = (1/\rho) \int f \mathbf{v} d\mathbf{v}$ is the local velocity, $\mathbf{w} = \mathbf{v} - \mathbf{u}(\mathbf{r},t)$ is the relative velocity and $P_{ij} = \int f w_i w_j d\mathbf{v}$ is the pressure tensor. Defining $\mathbf{g}(\mathbf{r},t) = \int \sqrt{2Dmf} \mathbf{Q} d\mathbf{v}$, it is clear that \mathbf{g} is a Gaussian noise and that its correlation function is

$$\langle g_i(\mathbf{r},t)g_j(\mathbf{r}',t')\rangle = 2Dm \int \sqrt{f(\mathbf{r},\mathbf{v},t)f(\mathbf{r}',\mathbf{v}',t')} \langle Q_i(\mathbf{r},\mathbf{v},t)Q_j(\mathbf{r}',\mathbf{v}',t')\rangle d\mathbf{v}d\mathbf{v}'$$

= $2Dm\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\int f(\mathbf{r},\mathbf{v},t)d\mathbf{v} = 2Dm\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\rho(\mathbf{r},t).$ (146)

Therefore, the equation for the momentum (145) can be rewritten

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P_{ij}}{\partial x_j} - \rho\frac{\partial\Phi}{\partial x_i} - \xi\rho u_i - \sqrt{2Dm\rho}R_i(\mathbf{r}, t).$$
(147)

This equation is not closed since the pressure tensor depends on the next order moment of the velocity. If, following [52, 20], we make a local thermodynamic equilibrium (L.T.E.) approximation $f_{LTE}(\mathbf{r}, \mathbf{v}, t) \simeq (\beta m/2\pi)^{d/2} \rho(\mathbf{r}, t) e^{-\beta m w^2/2}$ to compute the pressure tensor, we find that $P_{ij} \simeq (k_B T/m) \rho \delta_{ij}$. In that case, Eqs. (144) and (147) yield the stochastic damped Euler equations (78)-(79). We recall, however, that there is no rigorous justification for this local thermodynamic equilibrium approximation. Therefore, it does not appear possible to rigorously derive the damped Euler equations (78)-(79) from the Kramers equation (143). Alternatively, if we consider the strong friction limit $\xi \to +\infty$ for fixed β , leading to $D = \xi/(\beta m) \to +\infty$, the first term in the r.h.s. of Eq. (139) implies that $f(\mathbf{r}, \mathbf{v}, t) \simeq (\beta m/2\pi)^{d/2} \rho(\mathbf{r}, t) e^{-\beta m v^2/2} + O(1/\xi)$, $\mathbf{u} = O(1/\xi)$ and $P_{ij} = (k_B T/m) \rho \delta_{ij} + O(1/\xi)$. To leading order in $1/\xi$, Eq. (147) becomes

$$\rho \mathbf{u} \simeq -\frac{1}{\xi} \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi + \sqrt{2Dm\rho} \mathbf{R}(\mathbf{r}, t) \right).$$
(148)

Inserting Eq. (148) in the continuity equation (144) and defining $\mu = 1/(\xi m)$ and $D_* = D/\xi^2 = k_B T/(\xi m)$, we obtain the stochastic Smoluchowski equation (29). This equation can thus be derived from Eq. (143) in the limit $\xi \to +\infty$.

5 Conclusion

In this paper, we have developed a theory of fluctuations for a system of Brownian particles with weak long-range interactions. Starting from the *stochastic* Smoluchowski equation (44)-(45), justified in Appendix B from the Landau & Lifshitz general theory, we have obtained a simple formula (65) for the temporal correlation function of the Fourier components of the density fluctuations at equilibrium (for an infinite and homogeneous distribution). This formula shows that the correlations decay in time with the same damping rate as the one obtained from the study of the normal modes of the *deterministic* Smoluchowski equation (9), without noise. Furthermore, the amplitude of the correlation function diverges at the critical point T_c (or at the instability threshold $k = k_m$) leading to a failure of the mean field approximation in that case. As a result, the limits $N \to +\infty$ and $T \to T_c$ do not commute and the instability occurs strictly before the critical point as discussed in [45, 46, 47] for gravitational systems. In future works, we shall extend this theory of fluctuations to more general models. Indeed, the method developed in this paper can be generalized to any type of kinetic equations including fluctuations. In particular, the structure of formula (63) where $Z(k, \omega)$ is a sort of "dielectric function" obtained from the linearized kinetic equation without noise, has a general scope.

A Correlation functions

Considering the correlation function of the exact density field (16), and introducing the one and two-body distributions, we find that

$$\langle \rho_d(\mathbf{r})\rho_d(\mathbf{r}')\rangle = \langle m^2 \sum_{i,j} \delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{r}' - \mathbf{r}_j)\rangle = \langle m^2 \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{r}' - \mathbf{r})\rangle + \langle m^2 \sum_{i\neq j} \delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{r}' - \mathbf{r}_j)\rangle = Nm^2 P_1(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') + N(N-1)m^2 P_2(\mathbf{r}, \mathbf{r}').$$
(149)

Denoting by $\rho(\mathbf{r}) = \langle \rho(\mathbf{r}) \rangle = NmP_1(\mathbf{r})$ the equilibrium averaged distribution and introducing the fluctuations $\delta \rho(\mathbf{r}) = \rho_d(\mathbf{r}) - \rho(\mathbf{r})$, we get

$$\langle \delta \rho(\mathbf{r}) \delta \rho(\mathbf{r}') \rangle = \langle \rho_d(\mathbf{r}) \rho_d(\mathbf{r}') \rangle - \rho(\mathbf{r}) \rho(\mathbf{r}').$$
(150)

Starting from the identity (149) and introducing the correlation function $h(\mathbf{r}, \mathbf{r}')$ through the defining relation $P_2(\mathbf{r}, \mathbf{r}') = P_1(\mathbf{r})P_1(\mathbf{r}')[1 + h(\mathbf{r}, \mathbf{r}') + 1/N]$, we obtain at the order O(1/N):

$$\langle \delta \rho(\mathbf{r}) \delta \rho(\mathbf{r}') \rangle = m \rho(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') + \rho(\mathbf{r}) \rho(\mathbf{r}') h(\mathbf{r}, \mathbf{r}').$$
(151)

For a spatially homogeneous equilibrium distribution where $\rho(\mathbf{r}) = \rho = mn$ and $h(\mathbf{r}, \mathbf{r}') = h(|\mathbf{r} - \mathbf{r}'|)$, the Fourier transform of the density fluctuations is

$$\left\langle \delta \hat{\rho}_{\mathbf{k}} \delta \hat{\rho}_{\mathbf{k}'} \right\rangle = \frac{1}{(2\pi)^d} \rho m \left[1 + (2\pi)^d n \hat{h}(\mathbf{k}) \right] \delta(\mathbf{k} + \mathbf{k}').$$
(152)

The equilibrium correlation function can be obtained from the equilibrium BBGKY-like hierarchy using a Debye-Hückel-type of approximation or from field theoretical methods using the Landau approximation. Starting from the Gibbs canonical distribution in configuration space (I-44), which is the steady state of the N-body Smoluchowski equation (2), we can obtain [1] the equilibrium BBGKY-like hierarchy (I-45). The first two equations of this hierarchy are

$$\frac{\partial P_1}{\partial \mathbf{r}_1} = -(N-1)\beta m^2 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_2 \, d\mathbf{r}_2,\tag{153}$$

$$\frac{\partial P_2}{\partial \mathbf{r}_1} = -\beta m^2 P_2 \frac{\partial u_{12}}{\partial \mathbf{r}_1} - (N-2)\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_3 \, d\mathbf{r}_3. \tag{154}$$

Introducing the decomposition (I-14) in Eq. (153), we first obtain

$$\frac{\partial P_1}{\partial \mathbf{r}_1} = -(N-1)\beta m^2 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) d\mathbf{r}_2 - (N-1)\beta m^2 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_2'(\mathbf{r}_1,\mathbf{r}_2) d\mathbf{r}_2.$$
(155)

Then, introducing the decomposition (I-14)-(I-15) in Eq. (154), and using Eq. (155) to simplify some terms, we get

$$\frac{\partial P_2'}{\partial \mathbf{r}_1} = -\beta m^2 P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} - \beta m^2 P_2'(\mathbf{r}_1, \mathbf{r}_2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} +\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3 - (N-2)\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'(\mathbf{r}_1, \mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3 +\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'(\mathbf{r}_1, \mathbf{r}_3) P_1(\mathbf{r}_2) d\mathbf{r}_3 - (N-2)\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'(\mathbf{r}_2, \mathbf{r}_3) P_1(\mathbf{r}_1) d\mathbf{r}_3 -(N-2)\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_3'(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) d\mathbf{r}_3.$$
(156)

At the order 1/N in the thermodynamic limit defined in Paper I where P_1 , β , m, $|\mathbf{r}|$ are O(1), P'_2 , u are O(1/N) and P'_3 are $O(1/N^2)$, the foregoing equations reduce to ⁷:

$$\frac{\partial P_1}{\partial \mathbf{r}_1} = -(N-1)\beta m^2 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) d\mathbf{r}_2 - N\beta m^2 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_2'(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_2, \qquad (157)$$

$$\frac{\partial P_2'}{\partial \mathbf{r}_1} = -\beta m^2 P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} + \beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3$$
$$-N\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'(\mathbf{r}_1, \mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3 - N\beta m^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'(\mathbf{r}_2, \mathbf{r}_3) P_1(\mathbf{r}_1) d\mathbf{r}_3.$$
(158)

Now, introducing $P'_2(1,2) = P_1(1)P_1(2)[h(1,2) + \frac{1}{N}]$ in Eq. (158), using Eq. (157), and neglecting terms of order $O(1/N^2)$ or smaller, we find that the correlation function satisfies

$$h(\mathbf{r}_1, \mathbf{r}_2) = -\beta m^2 u_{12} - N\beta m^2 \int u_{13} h(\mathbf{r}_2, \mathbf{r}_3) P_1(\mathbf{r}_3) d\mathbf{r}_3, \qquad (159)$$

where $P_1(\mathbf{r})$ is given by the zeroth order Eqs. (I-20) and (I-21). We emphasize that this relation, which was not given in Paper I, is valid for a possibly spatially *inhomogeneous* equilibrium state. For a homogeneous distribution, we recover Eq. (I-51) which can be solved in Fourier space yielding Eq. (70).

It is very instructive to recover these results in a different manner using methods of field theory [56]. The equilibrium probability of the density distribution governed by the stochastic Smoluchowski equation (29) is $W[\rho] = \frac{1}{Z} e^{-\beta(F[\rho] - \mu \int \rho d\mathbf{r})}$ with $Z = \int \mathcal{D}\rho \ e^{-\beta(F[\rho] - \mu \int \rho d\mathbf{r})}$ (to simplify the notations, we drop the bars on the coarse-grained fields). To compute the correlation function $G(\mathbf{r}, \mathbf{r}') = \langle \delta \rho(\mathbf{r}) \delta \rho(\mathbf{r}') \rangle$, it proves convenient to introduce an auxiliary field $\mu(\mathbf{r})$ and write

$$W[\rho] = \frac{1}{Z} e^{-\beta (F[\rho] - \int \mu(\mathbf{r})\rho d\mathbf{r})}.$$
(160)

⁷Note that some terms of order 1/N were missing in Paper I because we made the approximation $N-1 \simeq N$ everywhere which is incorrect.

The equilibrium corresponds to $\mu(\mathbf{r}) = \mu$. In the Landau (mean field) approximation, the free energy $F = -k_B T \ln Z$ is given by $F \simeq F[\rho] - \int \mu(\mathbf{r})\rho d\mathbf{r}$ where $\rho(\mathbf{r})$ is the most probable distribution of $W[\rho]$. The maximum of $W[\rho]$ satisfies the condition $\mu(\mathbf{r}) = \delta F/\delta\rho(\mathbf{r})$. Using the expression (30) of the free energy, we obtain

$$\mu(\mathbf{r}) = \int u(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' + \frac{k_B T}{m}\ln\rho(\mathbf{r}).$$
(161)

At equilibrium, taking $\mu(\mathbf{r}) = \mu$, we recover the mean field Boltzmann distribution (14). On the other hand, taking the functional derivative of Eq. (161) with respect to $\mu(\mathbf{r}')$ and using the fundamental identity $G(\mathbf{r}, \mathbf{r}') = k_B T \delta \rho(\mathbf{r}) / \delta \mu(\mathbf{r}')$ [56], we find that the equilibrium correlation function is solution of

$$\delta(\mathbf{r} - \mathbf{r}') = \beta \int u(\mathbf{r} - \mathbf{r}'') G(\mathbf{r}', \mathbf{r}'') d\mathbf{r}'' + \frac{1}{m\rho(\mathbf{r})} G(\mathbf{r}, \mathbf{r}').$$
(162)

Finally, substituting Eq. (151) in Eq. (162) and simplifying some terms, we recover Eq. (159). Noting that the partition function of the N-body problem can be written

$$Z = \int e^{-\beta m^2 U} d\mathbf{r}_1 \dots d\mathbf{r}_N = \int \mathcal{D}\rho(\mathbf{r}) e^{S[\rho]} e^{-\beta E[\rho]} = \int \mathcal{D}\rho(\mathbf{r}) e^{-\beta F[\rho]},$$
(163)

where the sum runs over the macrostates $\rho(\mathbf{r})$ with mass $\int \rho(\mathbf{r}) d\mathbf{r} = M$ and $e^{S[\rho]}$ denotes the number of microstates associated with the macrostates $\rho(\mathbf{r})$, we see the link between the two previously exposed methods.

B Application of the Landau-Lifshitz general theory of fluctuations

In this Appendix, we derive the stochastic Smoluchowski equation (44)-(45) by using the general theory of fluctuations exposed in Landau & Lifshitz (see [39], Chap. XVII). We write the equation for the density in the conservative form

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J},\tag{164}$$

where \mathbf{J} is the current:

$$\mathbf{J} = -\frac{1}{\xi} \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) - \mathbf{g}(\mathbf{r}, t).$$
(165)

The first term is the deterministic Smoluchowski current (see, e.g., [2]) and the second term is a stochastic term that takes into account fluctuations. The problem at hand consists in characterizing the stochastic term $\mathbf{g}(\mathbf{r}, t)$. In order to use the general theory of fluctuations [39], we divide the fluid volume in small elements ΔV and take the average of each quantity in each element. The continuum limit $\Delta V \rightarrow 0$ will be performed in the final expressions. Equations (164) and (165) correspond to the equations

$$\dot{x}_a = -\sum_b \gamma_{ab} X_b + y_a, \tag{166}$$

of the general theory [39] provided that we make the identifications $\dot{x}_a \to -J_\alpha$ and $y_a \to g_\alpha$. The X_a can be obtained from the expression of the rate of production of entropy. In fact, since we are working in the canonical ensemble, the proper thermodynamical potential is the free energy F = E - TS or, equivalently, the Massieu function J = S - E/T which is the Legendre transform of the entropy. It can be written explicitly

$$J = -k_B \int \frac{\rho}{m} \ln \frac{\rho}{m} d\mathbf{r} - \frac{1}{2T} \int \rho \Phi \, d\mathbf{r}.$$
 (167)

Taking the time derivative of this expression, using Eq. (164), and integrating by parts, we obtain the expression

$$\dot{J} = -\int \frac{1}{T\rho} \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) \cdot \mathbf{J} \, d\mathbf{r}.$$
(168)

Note that for $\mathbf{g} = \mathbf{0}$ (no noise) we recover the appropriate form of the H-theorem valid in the canonical ensemble [2, 20]:

$$\dot{J} = \int \frac{1}{T\rho\xi} \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi\right)^2 d\mathbf{r} \ge 0.$$
(169)

If we replace the integral in Eq. (168) by a summation on ΔV , we obtain

$$\dot{J} = -\sum \frac{1}{T\rho} \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) \cdot \mathbf{J} \, \Delta V.$$
(170)

According to the general theory [39], we must also have

$$\dot{J} = -k_B \sum_{a} X_a \dot{x}_a. \tag{171}$$

Comparing Eq. (170) with the general expression (171), we find that the X_a are given by

$$X_a \to -\frac{1}{k_B T \rho} \left(\frac{k_B T}{m} \frac{\partial \rho}{\partial x_\alpha} + \rho \frac{\partial \Phi}{\partial x_\alpha} \right) \Delta V.$$
 (172)

It is now easy to find the expression of the coefficients γ_{ab} that appear in Eq. (166). Comparing Eqs. (165), (166) and (172), we find that

$$\gamma_{ab} = 0 \quad (\text{if} \quad a \neq b); \qquad \gamma_{aa} = \frac{k_B T \rho}{\xi \Delta V}.$$
 (173)

Now, the general theory of fluctuations [39] gives

$$\langle y_a(t_1)y_b(t_2)\rangle = (\gamma_{ab} + \gamma_{ba})\delta(t_1 - t_2).$$
(174)

Therefore, the correlation function of the stochastic field $g(\mathbf{r}, t)$ satisfies

$$\langle g_{\alpha}(\mathbf{r},t)g_{\beta}(\mathbf{r}',t')\rangle = 0 \quad (\text{if } \mathbf{r} \neq \mathbf{r}'),$$
 (175)

$$\langle g_{\alpha}(\mathbf{r},t)g_{\beta}(\mathbf{r},t')\rangle = \frac{2k_{B}T\rho}{\xi\Delta V}\delta_{\alpha\beta}\delta(t-t').$$
(176)

Taking the limit $\Delta V \rightarrow 0$, we can condense the above formulae under the form

$$\langle g_{\alpha}(\mathbf{r},t)g_{\beta}(\mathbf{r}',t')\rangle = \frac{2k_{B}T\rho}{\xi}\delta_{\alpha\beta}\delta(t-t')\delta(\mathbf{r}-\mathbf{r}').$$
(177)

We thus recover the expression of the stochastic term appearing in Eq. (44) by a method different from Dean [41].

We can repeat the same arguments for the inertial model (78)-(80). We write Eq. (79) in the form

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\frac{k_B T}{m} \nabla \rho - \rho \nabla \Phi - \xi \rho \mathbf{u} - \mathbf{g}'(\mathbf{r}, t), \qquad (178)$$

where $\mathbf{g}'(\mathbf{r},t)$ is the noise term to be determined. Comparing Eq. (178) with Eq. (166), we have the correspondences $\dot{x}_a \to -\xi \rho u_\alpha - g'_\alpha$ and $y_a \to -g'_\alpha$. The Massieu function J = S - E/T for the inertial model is [20]:

$$J = -k_B \int \frac{\rho}{m} \ln \frac{\rho}{m} d\mathbf{r} - \frac{1}{2T} \int \rho \Phi \, d\mathbf{r} - \frac{1}{2T} \int \rho \mathbf{u}^2 \, d\mathbf{r}.$$
 (179)

Taking the time derivative of this expression and using Eqs. (78)-(80), we obtain after some elementary calculations (see, e.g., Appendix G of [24]) the expression

$$\dot{J} = \frac{1}{T} \int \mathbf{u} \cdot \left(\xi \rho \mathbf{u} + \mathbf{g}'\right) d\mathbf{r}.$$
(180)

For $\mathbf{g}' = \mathbf{0}$ (no noise), we recover the appropriate form of the *H*-theorem valid in the canonical ensemble for the mean field damped Euler equation [20]:

$$\dot{J} = \frac{1}{T} \int \xi \rho \mathbf{u}^2 \, d\mathbf{r} \ge 0. \tag{181}$$

The discrete expression of Eq. (180) is

$$\dot{J} = \frac{1}{T} \sum \mathbf{u} \cdot (\xi \rho \mathbf{u} + \mathbf{g}') \Delta V.$$
(182)

Comparing Eq. (182) with the general expression (171), we find that the X_a are given by

$$X_a \to \frac{\Delta V}{k_B T} u_\alpha. \tag{183}$$

Then, comparing Eqs. (178), (166) and (183), we find that

$$\gamma_{ab} = 0 \quad (\text{if} \quad a \neq b); \qquad \gamma_{aa} = \frac{k_B T \xi \rho}{\Delta V}.$$
 (184)

Finally, using Eq. (174), we obtain the correlation function

$$\langle g'_{\alpha}(\mathbf{r},t)g'_{\beta}(\mathbf{r}',t')\rangle = 2k_B T \xi \rho \delta_{\alpha\beta} \delta(t-t')\delta(\mathbf{r}-\mathbf{r}'), \qquad (185)$$

which coincides with the expression given in Eq. (79).

Let us finally briefly consider the kinetic model of Sec. 4. It can be written in the form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J},$$
(186)

with the current

$$\mathbf{J} = -D\left(\frac{\partial f}{\partial \mathbf{v}} + \beta m f \mathbf{v}\right) - \mathbf{g}(\mathbf{r}, \mathbf{v}, t).$$
(187)

The corresponding free energy (Massieu function) J = S - E/T is explicitly given by [2]:

$$J = -k_B \int \frac{f}{m} \ln \frac{f}{m} d\mathbf{r} d\mathbf{v} - \frac{1}{T} \int f \frac{v^2}{2} d\mathbf{r} d\mathbf{v} - \frac{1}{2T} \int \rho \Phi d\mathbf{r}.$$
 (188)

It production rate is

$$\dot{J} = -k_B \int \frac{1}{mf} \left(\frac{\partial f}{\partial \mathbf{v}} + \beta m f \mathbf{v} \right) \cdot \mathbf{J} \, d\mathbf{r} d\mathbf{v}.$$
(189)

For $\mathbf{g} = \mathbf{0}$ (no noise), we recover the appropriate form of the *H*-theorem valid in the canonical ensemble for the mean field Kramers equation [2, 20]:

$$\dot{J} = k_B \int \frac{D}{mf} \left(\frac{\partial f}{\partial \mathbf{v}} + \beta m f \mathbf{v} \right)^2 d\mathbf{r} d\mathbf{v} \ge 0.$$
(190)

On the other hand, repeating the general procedure developed previously, we find that

$$X_a \to -\frac{1}{mf} \left(\frac{\partial f}{\partial v_\alpha} + \beta m f v_\alpha \right) \Delta \mathcal{V},\tag{191}$$

$$\gamma_{ab} = 0 \quad (\text{if} \quad a \neq b); \qquad \gamma_{aa} = \frac{Dmf}{\Delta \mathcal{V}},$$
(192)

where $\Delta \mathcal{V}$ is the elementary volume in phase space. Passing to the limit $\Delta \mathcal{V} \to 0$, this leads to a correlation function of the form

$$\langle g_{\alpha}(\mathbf{r}, \mathbf{v}, t) g_{\beta}(\mathbf{r}', \mathbf{v}', t') \rangle = 2Dm f \delta_{\alpha\beta} \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{v} - \mathbf{v}'),$$
(193)

which coincides with the expression given in Eq. (139).

C Dispersion relation for the inertial BMF model

In this Appendix, we complement the discussion of Sec. 3.1 by studying the dispersion relation associated with the damped Euler equations (78)-(80) without noise ($\mathbf{R} = \mathbf{0}$) for the inertial BMF model [18]. These equations can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho u) = 0, \tag{194}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \theta}\right) = -T\frac{\partial\rho}{\partial\theta} - \frac{k\rho}{2\pi}\int_0^{2\pi}\sin(\theta - \theta')\rho(\theta', t)d\theta' - \xi\rho u.$$
(195)

For $\xi = 0$, they reduce to the Euler equations (see Eq. (67) of [18]) and for $\xi \to +\infty$, we can neglect the inertial term in Eq. (195) and we obtain the mean field Smoluchowski equation (see Eq. (234) of [18]):

$$\xi \frac{\partial \rho}{\partial t} = T \frac{\partial^2 \rho}{\partial \theta^2} + \frac{k}{2\pi} \frac{\partial}{\partial \theta} \left\{ \rho \int_0^{2\pi} \sin(\theta - \theta') \rho(\theta', t) d\theta' \right\}.$$
 (196)



Figure 1: Evolution of the dangerous modes $n = \pm 1$ for the BMF model with $\xi = 0$ (Euler). For $T > T_c$, the homogeneous phase is stable and the perturbation oscillates with pulsation ω . For $T < T_c$, the homogeneous phase is unstable and the perturbation increases exponentially rapidly with a growth rate $\gamma > 0$.

Considering the linear dynamical stability of a homogeneous distribution with respect to the damped Euler equations (194)-(195), and decomposing the perturbation in normal modes $\delta f \sim e^{i(n\theta-\omega t)}$, the dispersion relation (90) can be written

$$\omega(\omega + i\xi) = Tn^2 + 2\pi\hat{u}_n\rho n^2, \qquad (197)$$

where

$$\hat{u}_n = -\frac{k}{4\pi} (\delta_{n,1} + \delta_{n,-1}).$$
(198)

Let us first consider the case $\xi = 0$ (Euler). For $n \neq \pm 1$, the dispersion relation becomes $\omega^2 = Tn^2$ so that the perturbation oscillates with a pulsation $\omega = \sqrt{T}|n|$. For $n = \pm 1$, the dispersion relation becomes $\omega^2 = T - T_c$ where $T_c = kM/(4\pi)$ is the critical temperature of the BMF model [18]. For $T > T_c$, the perturbation oscillates with a pulsation $\omega = \sqrt{T - T_c}$ and for $T < T_c$, the perturbation grows exponentially with a gowth rate $\gamma = \sqrt{T_c - T}$. In that case, the homogeneous phase is unstable (see Fig. 1).

Let us now consider the overdamped case $\xi \to +\infty$ (Smoluchowski). For $n \neq \pm 1$, the dispersion relation becomes $i\xi\omega = Tn^2$ so that the perturbation decays exponentially with a rate $\gamma = -Tn^2/\xi$. For $n = \pm 1$, the dispersion relation becomes $i\xi\omega = T - T_c$. For $T > T_c$, the perturbation decays with a decay rate $\gamma = -(T - T_c)/\xi$ and for $T < T_c$, the perturbation grows exponentially with a rate $\gamma = (T_c - T)/\xi$. In that case, the homogeneous phase is unstable (see Fig. 2).

In the general case, setting $\sigma = -i\omega$ so that the perturbations behave as $\delta f \sim e^{\sigma t}$, the dispersion relation (197) can be rewritten

$$\sigma^2 + \xi \sigma + Tn^2 + 2\pi \hat{u}_n \rho n^2 = 0.$$
(199)

For $n \neq \pm 1$, it reduces to $\sigma^2 + \xi \sigma + T n^2 = 0$. The solutions of this equation are $\sigma_{\pm} =$



Figure 2: Evolution of the dangerous modes $n = \pm 1$ for the BMF model with $\xi \to +\infty$ (Smoluchowski). For $T > T_c$, the homogeneous phase is stable and the perturbation decays with a rate $\gamma < 0$. For $T < T_c$, the homogeneous phase is unstable and the perturbation grows with a rate $\gamma > 0$.



Figure 3: Evolution of the dangerous modes $n = \pm 1$ for the inertial BMF model described by the damped Euler equations. A homogeneous distribution is unstable for $T < T_c$ and stable for $T > T_c$. For $T < T_c$, the perturbation grows with a rate $\gamma > 0$. For $T_c < T < T_*$, the perturbation decays with a rate $\gamma < 0$ without oscillating. For $T > T_*$, the perturbation undergoes damped oscillations with a decay rate $\xi/2$ and a pulsation ω .

 $(-\xi \pm \sqrt{\Delta_n})/2$ where $\Delta_n = \xi^2 - 4Tn^2$. Let us introduce the wavenumber

$$n_* = \left(\frac{\xi^2}{4T}\right)^{1/2}.$$
(200)

For $\Delta_n < 0$, corresponding to $n^2 > n_*^2$, the perturbation presents damped oscillations with a pulsation and decay rate

$$\omega = \sqrt{T} (n^2 - n_*^2)^{1/2}, \qquad \gamma = -\xi/2.$$
(201)

For $\Delta_n > 0$, corresponding to $n^2 < n_*^2$ (since we have assumed $n \neq \pm 1$, this regime is accessible iff $|n_*| \ge 2$ i.e. $T \le \xi^2/16$), the perturbation has a pure exponential decay with a damping rate

$$\gamma = -\frac{\xi}{2} + \sqrt{T}(n_*^2 - n^2)^{1/2}.$$
(202)

The modes $n \neq \pm 1$ are always stable, whatever the temperature. For the "dangerous" modes $n = \pm 1$, the dispersion relation becomes $\sigma^2 + \xi \sigma + T - T_c = 0$. The solutions are $\sigma_{\pm} = (-\xi \pm \sqrt{\Delta_1})/2$ where $\Delta_1 = \xi^2 - 4(T - T_c)$. Let us introduce the temperature

$$T_* = T_c + \frac{\xi^2}{4}.$$
 (203)

For $\Delta_1 < 0$, corresponding to $T > T_*$, the perturbation undergoes damped oscillations with a pulsation and decay rate

$$\omega = \sqrt{T - T_*}, \qquad \gamma = -\xi/2. \tag{204}$$

For $0 < \Delta_1 < \xi^2$, corresponding to $T_c < T < T_*$, the perturbation has a pure exponential decay with a damping rate

$$\gamma = -\frac{\xi}{2} + \sqrt{T_* - T}.$$
 (205)

For $\Delta_1 > \xi^2$, corresponding to $T < T_c$, the perturbation grows exponentially rapidly with a growth rate

$$\gamma = -\frac{\xi}{2} + \sqrt{T_* - T}.$$
 (206)

Therefore, for $T < T_c$, the homogeneous phase is unstable to the modes $n = \pm 1$. The growth rate is maximum for T = 0 with value $\gamma_* = \gamma(0) = -\xi/2 + \sqrt{T_*}$. The dependence of the growth rate, damping rate and pulsation as a function of the temperature are plotted in Fig. 3. This figure should be compared with Fig. 1 of [33] obtained for the gravitational potential (in this analogy, the critical temperature T_c plays the same role as the Jeans wavenumber k_J^2).

It can be useful to introduce a dimensionless number

$$F = \frac{\xi^2/4}{T_c} = \frac{\xi^2}{2k\rho} = \frac{\pi\xi^2}{kM},$$
(207)

which measures the efficiency of the friction force (a similar number has been introduced in [20, 33]). It can be written as $F \sim (\xi t_D)^2$ where $t_D \sim 1/\sqrt{k\rho}$ is a typical dynamical time (see Sec. 2.2. of [5]). Thus, F is the ratio of the dynamical time on the friction time $\tau \sim 1/\xi$. In terms of this parameter, the temperature (203) marking the appearance of oscillations can be written $T_* = T_c(1+F)$.

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