Trans-Planckian Effects in Inflationary Cosmology and the Modified Uncertainty Principle

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Abstract

There are good indications that fundamental physics gives rise to a modified spacemomentum uncertainty relation that implies the existence of a minimum length scale. We implement this idea in the scalar field theory that describes density perturbations in flat Robertson-Walker space-time. This leads to a non-linear timedependent dispersion relation that encodes the effects of Planck scale physics in the inflationary epoch. Unruh type dispersion relations naturally emerge in this approach, while unbounded ones are excluded by the minimum length principle. We also find red-shift induced modifications of the field theory, due to the reduction of degrees of freedom at high energies, that tend to dampen the fluctuations at trans-Planckian momenta. In the specific example considered, this feature helps determine the initial state of the fluctuations, leading to a flat power spectrum.

1 Introduction

According to the inflationary scenario [1],[2],[3] in the past the universe has undergone a period of rapid expansion during which minute quantum fluctuations have been magnified to cosmic sizes. This gives rise to the inhomogeneities that are observed today, in the form of anisotropies in the cosmic microwave background radiation (CMBR) and large scale structures, superimposed on a homogeneous and isotropic background. To describe the primordial quantum fluctuations, it has so far sufficed to use a prototype free scalar field theory with a linear dispersion relation in the expanding background [4]. The prototype scalar field describes the tensor and scalar metric (or inflaton) fluctuations. The success of the inflationary scenario rests on its ability to explain not only the homogeneity of the background, but also the characteristics of the inhomogeneities superimposed upon it.

However, as pointed out in [5], in many versions of inflation, most notably chaotic inflation, the period of inflation lasts for so long that the co-moving scales of cosmological

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interest today correspond to quantum fluctuations at the beginning of inflation with physical wavenumbers larger than the Planck mass, or equivalently, with physical wavelengths smaller than the Planck length. It is clear that for such fluctuations the effects of Planck scale physics cannot be ignored and the use of a field theory with a linear dispersion relation is not fully justified. This has been referred to as the trans-Planckian problem of inflation. It opens up the interesting possibility of searching for the imprints of Planck scale physics in the large scale structure of the universe.

Of late, it has been customary to assume that the effects of Planck scale physics can be incorporated in the theory of cosmological perturbations by using a time dependent non-linear dispersion relation in the free field theory. In the absence of a first principle approach, in practice these dispersion relations are chosen in an *ad hoc* manner. In the context of cosmology, the study of the consequences of such modifications was taken up in [6],[7]. The modifications introduced were inspired by earlier work in the context of Hawking radiation [8],[9]. Subsequently, various aspects of this issue were further investigated in [10] - [22].

A direct derivation of the trans-Planckian effects from a fundamental theory, e.g., string theory is not yet possible. However, Planck scale physics does seem to have a robust characteristic feature, which is the existence of a minimum length scale. For a review, see [23]. In particular, in perturbative string theory, this feature has been known for some time [24], [25]. There, the existence of a minimum length uncertainty is essentially due to the fact that as one increases the energy of a string probe beyond the Planck mass, the string can get excited and develop a non-zero extension [26]. The existence of a minimum length uncertainty can be related to a modification of the standard space-momentum canonical commutation relations. For an implementation of this idea, see [27], [28]. Thus, one way of incorporating Planck scale physics in a field theory is to reformulate the theory such that it is consistent with the minimum length uncertainty relation. This is the strategy that we will follow in this paper. Earlier an attempt in this direction was made in [29], followed by further analysis in [30], [31]. This resulted in a rather complicated theory which cannot easily be compared with other models and in which the identification of physical variables is not straightforward. In this paper, we follow a different approach to implement the minimum length uncertainty principle in a field theory in the expanding universe. This results in a theory with a transparent physical content. It has a non-linear time-dependent dispersion relation which is no longer arbitrary, but is determined by the modified commutation relations. We show that this naturally leads to Unruh type dispersion relations. Besides this, the theory contains a trans-Planckian damping term due to the reduction of the degrees of freedom at high momenta. This tends to suppress the fluctuations at momenta sufficiently above the Planck scale, although the actual amount of damping depends on the form of the dispersion relation. It is interesting to note that there are independent arguments for trans-Planckian damping due to quantum gravity effects [32],[33].

The paper is organized as follows: In section 2 we describe a class of modified \mathbf{x} , \mathbf{p} commutation relations that lead to a minimum length uncertainty principle in flat space-time.

Then we construct a free scalar field theory based on this and interpret the modification in terms of higher derivative corrections. We also discuss the modified relation between wavelength and wavenumber. In section 3, we generalize the notion of modified commutators and the associated minimum length uncertainty to the flat Robertson-Walker space-time. In section 4, we apply this to a scalar field theory in flat Robertson-Walker space-time and obtain its modified equation of motion. We discuss the resulting non-linear and time-dependent dispersion relations which are now constrained by the minimum length principle. The time evolution of wavelengths is also discussed. In section 5, we analyze a new feature of the modified field theory which tends to dampen the fluctuations at trans-Planckian energies. For the dispersion relation used there, this feature allows us to determine the initial state, and leads to a scale invariant power spectrum for the fluctuations. The issue of back reaction is also discussed and the exact solution in the trans-Planckian region is presented. In subsection 5.3, we discuss the issue of the initial state dependence of the power spectrum and argue that the main criterion is whether the effective frequency at the initial time is real or imaginary. This is related to, but not the same as, the violation of adiabaticity during time evolution. Throughout this paper we assume de Sitter inflation so that the Hubble parameter is constant.

2 The minimum length uncertainty principle

Conventional quantum mechanics allows us to use trans-Planckian momenta to probe distances shorter than a Planck length, ignoring the issue of gravitational stability of matter. The problem can be avoided if one assumes that nature admits a description, at least in an effective sense, in terms of a theory with a minimum length scale. Such a minimum length scale follows from (or results in) a modified uncertainty relation. There are indications that fundamental physics [23], *e.g.*, string theory effects [24],[25],[26] *effectively* modify the usual space-momentum uncertainty relation to

$$\Delta x \ge \frac{1}{2} \left(\frac{1}{\Delta p} + l_s^2 \,\Delta p + \cdots \right). \tag{1}$$

This implies the existence of a lower bound on space uncertainty $(\Delta x)_{min} \sim l_s$. The scale l_s is a minimum length that we take as the Planck length, although it could also be some other related scale. Our goal in this paper is to explore the implications of such a minimum length uncertainty principle for the physics of the early inflationary period. Before dealing with the expanding universe in section 3, we describe the case of flat space-time in this section. The aim is to write down a field theory based on the modified uncertainty (1). This theory is expected to have the correct structure to describe Planck scale phenomena. As such, the mathematics of the problem does not suggest a unique way of incorporating (1) in field theory and one has to specify a prescription for doing so. One such prescription is used in some of the earlier work on the subject [27],[28]. Here, we will follow a different prescription which is suggested by, and is self-consistent with, the physics of the problem.

2.1 The dynamical nature of the modification

In ordinary quantum mechanics, the Heisenberg uncertainty principle $\Delta x \Delta p \geq \frac{1}{2}$ is a consequence of the commutation relation $[\mathbf{x}^i, \mathbf{p}^j] = i \,\delta^{ij}$. This is purely kinematic and is independent of the dynamics of the theory. The derivation of the uncertainty principle involves the construction of the space of states on which the operators \mathbf{x}^i and \mathbf{p}^j act.

It is also possible to obtain the minimum length uncertainty principle from a modified $[\mathbf{x}^i, \mathbf{p}^j]$ commutation relation. In fact, in the next section we will use such a modified commutator, following [27],[28], which leads to (1) in a low-energy expansion. However, a subtlety arises in this case. The space of states on which the operators \mathbf{x}^i and \mathbf{p}^j (now satisfying the modified commutators) act could have an involved structure, rendering the necessary manipulations difficult. Such representations were constructed in [27],[28] to which the reader can refer for details. Fortunately, as we argue below, unlike in ordinary quantum mechanics, now the actual details of the representation are not relevant.

The appearance of the scale l_s in (1) indicates that the modification to the standard Heisenberg uncertainty principle is dynamical in nature. In other words, if we have a fundamental theory capable of describing physics at the Planck scale, then the modification arises as a consequence of the dynamics and of the particular structure of the fundamental theory; without ever having to invoke modified \mathbf{x}^i , \mathbf{p}^j commutators ¹. Thus, although one may invent \mathbf{x}^i , \mathbf{p}^j commutators that result in (1), in practice, the modifications arise totally independent of the existence of such commutators and hence, of the structure of the function spaces on which they are realized.

Suppose we could derive an effective field theory description of some high-energy process from our fundamental theory, for example, by summing up, to all orders, the relevant terms of a perturbative expansion in powers of energy. Such an effective field theory would automatically incorporate the minimum length feature of (1) in its structure. Unfortunately, at present we do not know how to carry out such a computation in practice, especially in the case of interest which is a field theory in an expanding universe. Now the notion of a modified commutator comes handy. Instead of adding the high-energy corrections that encode (1) to the low-energy effective field theory, we regard them as getting absorbed in a modification of the $\mathbf{x}^i, \mathbf{p}^j$ commutators. This will leave the apparent structure of the field theory unchanged while modifying the relation between coordinates and momenta. Whatever the details, this should finally result in modified commutators consistent with the minimum length principle. We then transcribe this effect back from the commutators to the field theory, enabling us to surmise its high-energy modification. It is evident that in this approach the modified commutator has no fundamental significance. It is only used as a trick to guess the form of the high-energy corrections to a field theory consistent with the minimal length principle. This procedure is implemented explicitly in sections 2.3 and 4.1.

¹For example, this is how (1) is inferred in [24], [25] from the behaviour of the string scattering cross-sections at high energies. The string theory itself is based on standard commutators.

2.2 Modified commutation relations in flat space

The uncertainty relation (1) can also be obtained from modified $\mathbf{x}^i, \mathbf{p}^j$ commutators. Here we consider a general class of modified commutation relations, consistent with rotational invariance, that lead to a minimum length uncertainty² [27],[28],

$$[\mathbf{x}^{i}, \mathbf{p}^{j}] = i \,\delta^{ij} f(\mathbf{p}) + i \,g(\mathbf{p}) \,\mathbf{p}^{i} \mathbf{p}^{j} , \qquad [\mathbf{x}^{i}, \mathbf{x}^{j}] = 0 , \qquad [\mathbf{p}^{i}, \mathbf{p}^{j}] = 0 .$$
(2)

The information about the modification is entirely contained in the function f(p). The term with g(p) is required by the Jacobi identities and is fully determined in terms of f(p). The obvious restriction on f(p) is that for small enough p it should reduce to 1, leading back to the standard commutators. It is understood that p is measured in units of some high-energy scale, say, the Planck mass, l_s^{-1} .

To elucidate the implications of the modified commutator, we exploit a useful construction introduced in [28]: Introduce auxiliary variables ρ^i such that on functions $\phi(\rho)$,

$$\mathbf{x}^{i}\phi(\rho) = i\frac{\partial}{\partial\rho^{i}}\phi(\rho).$$
(3)

The ρ^i are given in terms of the momenta p^i as,

$$\rho^i = \frac{p^i}{f(p)} \,. \tag{4}$$

Now, using a formal power series expansion for f(p), it is easy to verify that (2) and (3) are equivalent. Further, let us assume that f(p) is such that as p varies from 0 to ∞ , ρ stays bounded between 0 and some maximum value ρ_{max} . Here, p and ρ denote the magnitudes of p^i and ρ^i , respectively.

One can now understand the essential consequence of (2) using a simple representation (one corresponding to a particle of momentum "x" in a rigid box of size " $2\rho_{max}$ "), without getting into the details of the construction of the more complicated function spaces. Equation (3) implies the commutator

$$[\mathbf{x}^i, \boldsymbol{\rho}^j] = i\,\delta^{ij}\,,\tag{5}$$

and hence, the associated uncertainty relation $\Delta x^i \ge 1/(2\Delta \rho^i)$. Since we assume ρ to be bounded by ρ_{max} , the maximum uncertainty in ρ^i is $2\rho_{max}$. Thus, there is an associated minimum length uncertainty,

$$(\Delta x)_{min} \sim \frac{1}{\rho_{max}}.$$
 (6)

This is required to be of order one, in units of the l_s appearing in (1). The conditions on f(p) are summarized below:

• As $p \to 0$, $f(p) \to 1$ and $\rho \to p$.

²Throughout this paper, bold-face letters $\mathbf{x}^i, \mathbf{p}^i, \mathbf{\rho}^i, \cdots$ denote quantum mechanical operators corresponding to $x^i, p^i, \mathbf{\rho}^i, \cdots$. Symbols p, ρ, \cdots denote the magnitudes of vectors $p^i, \mathbf{\rho}^i, etc$.

- $\rho = p/f(p)$ is bounded by 1 in units of l_s^{-1} , which we take to be the Planck mass.
- The most natural functions f satisfying the last condition are those for which ρ increases monotonically with p, approaching its maximum value ρ_{max} as p → ∞. An example is

$$\rho = \tanh^{1/\gamma} \left(p^{\gamma} \right). \tag{7}$$

We close this subsection with two comments. In formulating a theory based on the modified commutator, it may be tempting to speculate on identifying ρ as the physical momentum with a cut-off, as in [29]. However, a momentum cut-off is not natural from the point of view of fundamental physics. For example, in string theory loop diagrams are naturally regulated as a consequence of modular invariance, and not by a momentum cut-off. Moreover, if the modified commutator (2) is obtained by analyzing physical processes in some fundamental theory, say, through a calculation leading to equation (9) below, then p will manifestly correspond to the physical momentum.

Furthermore, as in ordinary quantum mechanics, one faces ordering ambiguities when dealing with products of x^i and p^j . Now, the problem is more severe since the ambiguity is itself *p*-dependent. It is therefore safest to avoid coordinate systems in which the Lagrangian involves such products.

2.3 Scalar fields in flat space with modified commutators

A field theory based on the modified commutation relation (2) will contain new effects that become relevant at large momenta. As explained in subsection 2.1, this is expected to reproduce an effective field theory that provides a description of high energy interactions in a fundamental theory. In this subsection we will consider the implications for the scalar field propagator in flat space-time. In particular we will discuss two effects that will be relevant to the trans-Planckian problem of inflationary cosmology, i) the modification of dispersion relation and the associated minimum bound on the wavelengths, and ii) the increase in phase space volume occupied by a quantum state at high momenta. The generalization to field theory in the Robertson-Walker universe will be considered in the coming sections.

Consider a massless scalar field $\phi(t, x)$ with the standard action S, and equation of motion (ignoring interaction terms),

$$S = -\frac{1}{2} \int d^4x \,\partial^\mu \phi \,\partial_\mu \phi \,, \qquad \left(\partial_t^2 - \partial^i \partial_i\right) \,\phi(t,x) = 0\,. \tag{8}$$

The justification for choosing this as the starting point was given in subsection 2.1: Imagine that the field theory is derived from some fundamental theory. Then, for low-energy processes one obtains (8) along with interaction terms not considered here. To describe very high energy processes, the field theory is expected to get modified such that its dynamics leads to the minimum length uncertainty (1). Since such modifications have not yet been derived from first principle, we assume, alternatively, that the high energy corrections coming from the fundamental theory can also be accommodated by modifying the ordinary $\mathbf{x}^i, \mathbf{p}^j$ commutators to the form (2), keeping the action unchanged. In this case, the action preserves its low-energy form (8), although now the relation between "coordinate" and "momentum" has changed and they are no longer conjugate variables, *i.e.*, $-i\partial/\partial x$ does not represent p.

In fact, the discussion in the previous subsection shows that the coordinates x^i are now conjugate to the variables ρ_i . Thus, we interpret the symbol " $\partial/\partial x^i$ " in (8) simply as a representation of $i\rho_i$ on the appropriate functions $\phi(x)$, irrespective of the details of the construction (for a discussion of the representations see [27, 28]). In this sense, the commutation relation (5) allows us to make a Fourier transformation to a variable $\tilde{\phi}(t,\rho)$ through $\phi(t,x) = N \int_{-\rho_{max}}^{\rho_{max}} d^3\rho \,\tilde{\phi}(t,\rho) e^{ix^i \rho_i}$, where N is a normalization constant. Then, the free massless equation of motion becomes

$$\left(\partial_t^2 + \rho^2(p)\right)\tilde{\phi}(t,\rho) \equiv \left(\partial_t^2 + \frac{p^2}{f^2(p)}\right)\tilde{\phi}(t,p) = 0, \qquad (9)$$

where, ρ has been expressed in terms of p using (4). We have used the same notation ϕ for the field as a function of both ρ and p; the difference should be clear from the context.

To further clarify the origin of the modification, let us introduce new variables $\hat{\mathbf{x}}^i$ such that their commutators with \mathbf{p}^i have the standard form,

$$\left[\hat{\mathbf{x}}^{i}, \mathbf{p}^{j}\right] = i\,\delta^{ij}\,.\tag{10}$$

In the \hat{x} -representation, $p_i = -i\partial/\partial \hat{x}^i$. Then, in terms of \hat{x} the modified equation of motion is equivalent to the higher-derivative equation,

$$\left[\partial_t^2 + \rho^2(-i\partial_{\hat{x}})\right]\hat{\phi}(t,\hat{x}) = 0.$$
(11)

The new variables \hat{x}^i are identified as the ordinary space coordinates.

One may take the approach that (9) is the basic equation that emerges in the effective field theory limit of some fundamental theory in a momentum-space computation, in the spirit of [34]. This could then be given a space-time representation either in terms of the two-derivative field equation (8) with modified commutators, or in terms of the infinite-derivative field equation (11) with the standard commutators. In this sense, the use of the modified commutators enables us to express the higher derivative corrections to the lowest order equation of motion in a closed form, consistent with the notion of a minimal length scale. The closed form of the expression is crucial since a truncated power series expansion is not a good approximation at large momenta. In the string theory context, this is analogous to summing up the α' corrections to all orders (albeit for the issue of Lorentz invariance on which we will comment below). To summarize, at low energies one may start with (8) and the standard commutation relations. Then high energy corrections, assuming that they are summable, will convert this to equation (11), keeping the commutators unchanged. Alternatively, one may incorporate the effect of high energy corrections into the commutation relation keeping the form of the action unchanged.

The interpretation of (9) as the basic equation will turn out to be particularly useful when we consider the Robertson-Walker space-time later. Then the analogue of this equation is still well defined even though it no longer has a space-time representation analogous to equation (8).

As indicated above, Lorentz invariance is broken by the modified commutator. As a result, the time evolution of the solutions is still determined by a second order equation, in spite of the higher space derivatives. For comments on the possibility of a dynamical breakdown of Lorentz invariance near the Planck scale, see [35]. One may also circumvent the issue of Lorentz invariance and regard the flat space field theory of this section only as a toy example employed to illustrate the ideas. Our aim is to finally use the modified commutators in Robertson-Walker space-times, where Lorentz invariance is anyway broken by the background. For a related setup, where a time-dependent background leads to a modified dispersion relation, see [36].

Independently of its relevance to cosmology, equation (9) defines an interesting field theory in its own right. The built-in minimum length uncertainty in the field theory may also render it more finite as can be inferred from the finite range of ρ -integration, although the physical momentum remains unbounded.

Equation (9) contains the non-linear dispersion relation $\omega = \rho(p)$ which is fully determined by the function f(p) and satisfies the restrictions discussed above. For example, for the choice (7) one gets the Unruh dispersion relation which has been used to mimic the effects of trans-Planckian physics in the contexts of Hawking radiation [8] and in inflationary cosmology [6]. Let us now consider the minimum wavelength bound implied by the modified dispersion relation. From (8) and (5) it is obvious that the plane wave in the theory with the modified commutator is given by $e^{i\omega t + i\rho_i x^i}$. This has a wavelength,

$$\lambda = \frac{2\pi}{\rho} = 2\pi \frac{f(p)}{p}.$$
(12)

Since ρ is bounded by $1/l_s$, the wavelength of a fluctuation is never smaller than the Planck length regardless of how high the value of its momentum may be. Also, the energy $\omega = \rho$ is always bounded by the Planck mass.

When the Fourier transform is implemented in the action S, then the change of variable from ρ to p also induces a Jacobian which can be computed as $J = \partial(\rho^3)/\partial(p^3)$,

$$S = \frac{1}{2} \int dt \, d^3 p \, J(p) \left(\partial_t \tilde{\phi}^* \, \partial_t \tilde{\phi} - \rho^2(p) \, \tilde{\phi}^* \, \tilde{\phi} \right). \tag{13}$$

In flat space-time, the Jacobian does not affect the equation of motion for $\tilde{\phi}(t, p)$. However, since $\int d^3p J(p)/\hbar^3$ corresponds to a sum over momentum states (where \hbar has been reinstated), it indicates that now a momentum mode p occupies phase space volume $\sim \hbar^3/J$ as opposed to $\sim \hbar^3$ of ordinary quantum mechanics. For a $\rho(p)$ consistent with the minimum length uncertainty, J decreases for large p and hence the phase space volume occupied by a momentum mode p increases. Equivalently, the total number of degrees of freedom at high energies decreases. This is consistent with the results of [37] which finds a similar behaviour for the degrees of freedom in string theory at high temperatures. This further supports our prescription for incorporating the modified commutation relations in field theory. As we will see later, in an expanding universe, the Jacobian affects the equation of motion in a non-trivial way and has some interesting consequences.

The modification of the dispersion relation and the associated bound on the wavelength, makes the modified commutator theory very appealing from the point of view of the trans-Planckian problem in inflationary cosmology. In this latter context, modified dispersion relations similar to (9) have been proposed [6],[7], [10]-[14],[19], albeit in an *ad hoc* manner (except for [16], and in a related context [38]), to encode the effects of trans-Planckian physics on the dynamics of scalar fields. Furthermore, the form of (12) may be taken to indicate that wavelengths corresponding to the present day scales and CMBR anisotropies, when blue-shifted backward in time to the beginning of the inflationary period, will never attain sub-Planckian lengths, regardless of how long the inflation may last. As we will see, these features carry over to the case of inflationary universe, although the time dependence of the physical momenta, due to the red-shift, introduces some subtleties as well as new effects.

3 The modified commutator in flat RW space-time

To apply the modified commutator to the early universe, it should first be generalized to the flat Robertson-Walker (RW) space-time. This generalization will be carried out in the present section. The underlying assumption, of course, is that a geometric description of the space-time still remains valid, at least in some approximation. This is reasonable since the effects of the modified commutator will be perceived by the high-energy fluctuations and not by the slowly varying background fields.

We recall the flat RW metric in the conformal frame,

$$ds^{2} = a^{2}(\eta) \left(-d\eta^{2} + dy^{i}dy^{j}\delta_{ij}\right) .$$

$$(14)$$

Here, η is the conformal time and y^i are comoving coordinates. Physical distances at time η are simply given by $a(\eta) y^i$. Let us denote the comoving momentum (wave number) conjugate to y^i by k_i . Then, the physical momentum undergoes a red-shift with time and, in the metric above, is given by $k_i/a(\eta)$. The minimum length uncertainty should be time independent and apply to the physical lengths. At the same time, the modified commutator should be consistent with the residual symmetries of the metric (14), in terms of comoving coordinates.

Only a subset of general coordinate transformations keep the form of the RW metric in comoving coordinates unchanged. Besides rigid spatial rotations and translations, which are in common with flat space, this subset also contains constant rescalings of the coordinates that amount to rescaling a, keeping the form of the metric unchanged. Then, the analogue of the modified commutator (2) in the flat RW space, consistent with the residual covariance of the metric $g_{\mu\nu}$ in the comoving coordinates, is

$$[\mathbf{y}^{i}, \mathbf{k}^{j}] = i \mathbf{g}^{ij} f(\mathbf{k}) + i g(\mathbf{k}) \, \mathbf{k}^{i} \, \mathbf{k}^{j} \,, \tag{15}$$

where, $k^i = g^{ij}k_j = a^{-2}\delta^{ij}k_j$. This should not lead to a fixed minimum length uncertainty in the comoving coordinates y^i , which would result in a rather large uncertainty in the proper distance ay^i at the present epoch. It is easy to see that it is actually the physical distance that has a fixed minimum uncertainty: In order to compare with the flat space case, let us introduce $n_i = k_i$, $n^i = \delta^{ij}n_j = a^2k^i$, so that $k^2 = n^2/a^2$. In terms of this, the commutation relation takes the form

$$[a\mathbf{y}^{i}, \frac{\mathbf{n}^{j}}{a}] = i\delta^{ij} f(\frac{\mathbf{n}}{a}) + ig(\frac{\mathbf{n}}{a}) \frac{\mathbf{n}^{i}}{a} \frac{\mathbf{n}^{j}}{a}.$$
 (16)

This has the same structure as (2) and hence implies a minimum position uncertainty in the physical or proper distance ay^i as desired, with n^i/a as the physical momentum. More precisely, in analogy with the flat space case, one can introduce the "physical" auxiliary variables ρ_i as well as "comoving" ones, $\hat{\rho}_i$, given by

$$\rho_i = \frac{n_i/a}{f(n/a)}, \qquad \hat{\rho}_i = a \,\rho_i \,. \tag{17}$$

The commutator (16) now becomes

$$[a\mathbf{y}^i,\,\boldsymbol{\rho}^j] = i\,\delta^{ij} \tag{18}$$

This leads to a minimum length in ay^i as $(a\Delta y)_{min} \sim \rho_{max}^{-1}$. Since ρ is a function of a single variable, this value is η -independent. In terms of the "comoving" version of the auxiliary variable $\hat{\rho}_i = a\rho_i$, the commutator is $[\mathbf{y}^i, \hat{\boldsymbol{\rho}}_j] = i\,\delta_j^i$. Note that unlike the flat space case of the previous section, now f and ρ_i have become time dependent through the red-shift of the physical momentum. However, they still satisfy the restrictions described below equation (6).

4 Trans-Planckian effects in the inflationary era

Of late, it has been customary to assume that the effects of fundamental physics at Planckian energies can be effectively incorporated in field theory by modifying the free field dispersion relation in some appropriate way [6]-[21]. In practice, in the absence of a first principle approach, one ends up choosing the modified dispersion relations in an *ad hoc* manner. This approach has been used to investigate the possible effects of the Planck scale physics on the CMBR spectrum during the early inflationary epoch. In this section we attempt to achieve a more fundamental understanding of this issue based on the modified uncertainty principle in RW universe. It will be shown that the time-dependent non-linear dispersion relation emerges in a natural way and its form is determined by the modified commutator. There is also a further modification associated with the reduction in the number of high energy degrees of freedom. This could dominate at very high momenta with the effect of dampening the fluctuations. This new feature will be investigated in detail in section 5.

4.1 Fields in flat RW space with modified commutators

We will now consider the implications of the modified commutation relations for scalar field theory in flat Robertson-Walker space-time. The resulting theory includes the effects of Planck scale physics in an inflationary universe. The momentum red-shift associated with the rapid expansion induces new effects beyond those considered in section 2. Let us start with the standard massless scalar field theory in Robertson-Walker space-time,

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \,\partial_\nu \phi = \frac{1}{2} \int d\eta \,d^3y \,a^2 \Big(\partial_\eta \phi \,\partial_\eta \phi - \sum_{i=1}^3 \partial_{y^i} \phi \,\partial_{y^i} \phi \Big) \,. \tag{19}$$

Here, y is regarded as a variable conjugate to $a\rho$ and not to the comoving momentum n. Of course, it is not obvious that such a simple looking space-time description could survive near the Planck scale (we will see indications that it does not). But in the absence of a better formalism, we choose this as our starting point, in analogy with the flat space case considered in section 2. The picture described there is still valid with some refinements to be discussed.

Since \mathbf{y}^i and $\hat{\boldsymbol{\rho}}_i$ satisfy the standard commutation relation, we simply interpret the symbol " $-i\partial_{y^i}$ " in (19) as the operator conjugate to $\hat{\boldsymbol{\rho}}_i = a\boldsymbol{\rho}_i$. In order to rewrite the action in the ρ -representation, consider the Fourier transform

$$\phi(\eta, y) = a^3 \int_{-l_s^{-1}}^{l_s^{-1}} d^3 \rho \, \tilde{\phi}(\eta, \rho) \, e^{iy^i a\rho_i} \,, \tag{20}$$

which in the limit $l_s \to 0$, $\rho \to n/a$ reduces to the standard expression in the absence of Planck scale effects. Then by the above interpretation,

$$-i \,{}^{"}\partial_{y^{i}}{}^{"} \,\phi(\eta, y) = a^{3} \,\int_{-l_{s}^{-1}}^{l_{s}^{-1}} d^{3}\rho \,\tilde{\phi}(\eta, \rho) a\rho_{i} \,e^{iy^{i}a\rho_{i}} \,, \tag{21}$$

Also, since ρ is a dummy integration variable,

$$\partial_{\eta}\phi = a^3 \int_{-l_s^{-1}}^{l_s^{-1}} d^3\rho \left(\partial_{\eta}\tilde{\phi} + \left[i y^i a \rho_i + 3 \right] \frac{\partial_{\eta} a}{a} \tilde{\phi} \right) e^{i y^i a \rho_i} \,. \tag{22}$$

The correct $l_s \to 0$, $\rho \to n/a$ limit of this is reproduced by the term outside the square brackets alone. In this sense, the two terms within the square brackets conspire to cancel in this limit. To see this, note that both terms in the square brackets are needed to assemble the right hand side as the η -derivative of the right hand side of (20), where the limit can be easily taken. For example, if $3\partial_{\eta}a/a\tilde{\phi}$ is dropped, it will no longer be possible to recover the correct limit from (22).

To make sense of carrying out the Fourier transform in (19), using the above expressions, we choose the simple representation in which $y^i a$ is discrete and interpret $\int d^3 y$ in (19) as denoting a sum (as argued earlier the final result is independent of the representation used). Then at any given time η ,

$$y^i a \to \pi m^i l_s$$
, $\int d^3 y \to \frac{1}{a^3} \left(\frac{l_s}{2}\right)^3 \prod_{i=1}^3 \sum_{m^i = -\infty}^{+\infty}$. (23)

We can now use (20)-(23) to write the action in the ρ -representation. Let us first ignore the terms within the square brackets in (22) (we will argue below that the action cannot contain contributions of this form). Then, after expressing ρ in terms of n from (17), the ρ -space action yields,

$$S = \frac{1}{2} \int d\eta \, d^3 n \, J(\frac{n}{a}) \, a^2 \left(\partial_\eta \tilde{\phi}^* \, \partial_\eta \tilde{\phi} - \frac{n^2}{f^2(n/a)} \, \tilde{\phi}^* \, \tilde{\phi} \right). \tag{24}$$

We use the same notation ϕ for the field as a function of both ρ and n since the difference is clear form the context. Here, $n^2 = \sum_{i=1}^{3} n_i n_i$ and J(n/a) is the Jacobian for the change of variables determined by (17),

$$J(v) = \frac{\rho^2}{v^2} \frac{\partial \rho}{\partial v}, \qquad \left(v = \frac{n}{a}\right). \tag{25}$$

The restrictions on ρ fix the asymptotic behaviour of J such that $J \to 1$ for $v \ll 1$, and $J \to 0$ for large physical momentum v. The equation of motion following from (24) is

$$\left(\partial_{\eta}^{2} + a^{2}\rho^{2}(\frac{n}{a}) - \frac{\partial_{\eta}^{2}(\sqrt{J}a)}{\sqrt{J}a}\right)\tilde{\chi}(\eta, n) = 0, \qquad (26)$$

where $\tilde{\chi}(\eta, n) = a \sqrt{J} \tilde{\phi}(\eta, n)$. This is the generalization of equation (9) from flat space-time to the RW space-time.

Before proceeding further, let us get back to the terms within the square brackets in (22) that have been ignored. The first term in the square brackets will appear in (24) and (26) as $\rho_i \partial_{\rho_i} \tilde{\phi}$. It is not difficult to see that such expressions should not appear at all: We have been regarding our modified theory as an effective field theory description of high energy processes in some fundamental theory. An explicit derivation of the effective theory from the fundamental theory (if it could be carried out) would then involve momentum space computations, as is generically the case in such derivations (see [34] for a related approach). For a spatially flat metric, such computations cannot lead to expressions containing momentum space derivatives, $\partial_p \tilde{\phi}(p)$, which are required to produce the terms we have ignored. In the coordinate representation, such expressions would lead to space-dependent potentials which are not consistent with the homogeneity of the background and hence are ruled out by it.

If for the above reason we ignore the first term in the square brackets in (22) but retain the second term, then our modified theory will not reduce to the standard theory of scalar fields in RW space-time in the limit $l_s \rightarrow 0$, when Planck scale effects disappear. This is evident from the discussion below (22). Since this is not desirable, we conclude that both terms within the square brackets in (22) should be ignored.

To summarize, we have argued that scalar field theory in flat RW space-time, based on the modified commutators (15), or (16), is directly defined by (24) and (26), without the extra terms. This also was our preferred interpretation of the analogous equation (9) in flat space. However, while there the associated coordinate space equations could be written in the usual form (in terms of a coordinate conjugate to ρ), in the present example of an expanding universe this is not the case. Thus, the space-time action (19) is a useful starting point, although it is incomplete.

There is a further conceptual difference between (9) and (26): The modified uncertainty relation in flat space-time, which leads to (9), can be regarded as the outcome of a string theory computation. As for (26), a similar statement could in principle be made, although in practice string theory in the expanding universe is still poorly understood. For a recent discussion of the difficulties see [39].

4.2 Time-dependent non-linear dispersion relations

The standard equation of motion for a scalar field in RW metric corresponds to the small momentum limit of (26), where $a\rho \sim n$ and $J \sim 1$. The effects of Planck scale physics, as encoded in the modified commutation relations, result in the modification of the dispersion relation and in the appearance of J in the equation of motion. The J dependence is a novel feature and its consequence will be investigated in the next section. Here we discuss the dispersion relation.

In the equation of motion (26), the free field dispersion relation $\omega_{phys}^{free} = n/a$, is modified to a non-linear, time dependent one,

$$\omega_{phys} = \rho \equiv \frac{n/a}{f(n/a)}.$$
(27)

This is a generalization of the modified dispersion relations considered in [6],[7], [10]-[21]. However, the modification is no longer *ad hoc* (as is also the case in [16]), but is determined by the nature of the uncertainty principle (15), through the function f. As described in subsection 2.2 (following equation (6)), this function satisfies certain restrictions. In terms of ω_{phys}^2 , the restrictions translate to the following:

- The linear dispersion relation $\omega_{phys} = n/a$ emerges at small physical momenta n/a.
- $\omega_{phys}^2 \ge 0$ and ω_{phys} is always real.
- The quantity ω_{phys} is bounded by ρ_{max} which is fixed by the minimum length scale as $1/l_s$.

Thus, while there is no cut-off on the physical momentum n_i/a , there is always a cutoff on the effective frequency ω_{phys} . These restrictions exclude some of the dispersion relations that have appeared in the literature as being inconsistent with the minimum length uncertainty. Some often used dispersion relations are depicted in Figure 1.

The scalar field theory with the modified dispersion relation can now be second quantized in terms of $\tilde{\chi}(\eta, n)$, in the usual fashion. The modification of the commutation relation in the first quantized theory does not affect the field commutators of the second quantized theory.

As stated in section 2, the most natural class of solutions to the restrictions on f consist of ρ increasing monotonically with n/a. This avoids the problem of associating multiple momenta with the same frequency. For example, the class of solutions in (7), lead to

$$\omega_{phys} = \tanh^{1/\gamma} \left(\frac{n}{a}\right)^{\gamma} . \tag{28}$$

These correspond to the well known Unruh dispersion relations [8]. For these, equation (26) in the special case of J = 1, was analyzed in [6]. The conclusion was that this class of trans-Planckian modifications does not modify the spectrum of cosmological perturbations calculated with the linear dispersion relation. The case of $J \neq 1$ will be discussed in some detail in the next section.

Another form of f appearing in the literature is [28]

$$f(n/a) = \frac{2(n^2/a^2)}{\sqrt{1 + 4(n^2/a^2)} - 1}.$$
(29)

One can numerically verify that the dispersion relation corresponding to this function has the same qualitative behaviour as the Unruh form (28), with an appropriate value of In Figure 1, this curve is la- γ . beled as "soft". Hence one can infor that for J = 1, it leaves the spectrum of cosmological perturbations unchanged (later we comment on $J \neq 1$). This conclusion is at variance with the result of [29] (and the follow-up analysis in [30], [31]) which uses the same form of f, but implements the modified commutator in a different manner.



Figure 1: Dispersion relations

Next, we consider the generalized Corley-Jacobson type of dispersion relations [9],[6],

$$\rho^2 = \frac{n^2}{a^2} \left(1 + \sum_{q=1}^m b_q \left(\frac{n^2}{a^2}\right)^q \right) \,. \tag{30}$$

Since ρ is not bounded, such dispersion relations cannot be associated with a modified uncertainty principle with a minimum length scale. However, one could regard these as low-momentum expansions of expressions that are bounded. Then ρ will remain bounded as long as the expansion is valid. Contrary to the previous two cases, the Corley-Jacobson dispersion relations need not be monotonic functions of n. For J = 1 the effect of these on the cosmological perturbation spectrum has been studied in [6],[7],[10] - [13],[17] where it was found that they could give rise to deviations from the predictions of linear dispersion relation when $b_m < 0$.

4.3 Modified red-shift and evolution of wavelengths

As pointed out, for example, in [5],[6], in models where inflation lasts for a sufficiently long period of time, the CMBR anisotropies at the present epoch seem to have their origin in fluctuations whose wavelengths, in the beginning of inflation, were smaller than a Planck length. This assumes the validity of the standard red-shift formula for the physical wavelength, $\lambda_{phys} = 2\pi a(\eta)/n$, down to the Planck length and beyond. One may try to understand, at least heuristically, the time evolution of wavelengths, based on the modified commutators. As described in section 2.3, in flat space, the modified commutators lead to a wavelength (12) that cannot decrease below the Planck length. One may extrapolate the same result to the flat RW case, in an adiabatic approximation, simply by replacing the flat space momentum p by the physical momentum in the expanding universe, n/a,

$$\lambda = \frac{2\pi}{\rho} = 2\pi \frac{a}{n} f(\frac{n}{a}). \tag{31}$$

We take ρ to be a monotonically increasing function. At late times (large a), the function f tends to identity and one recovers the usual red-shift formula. As we go backward in time (decreasing a), λ tends to a constant 2π (in units of l_s). Near a = 0, a small wavelength band corresponds to a large momentum interval. This behaviour is shown in Fig. 2 for an f associated with the Unruh dispersion relation (28) with $\gamma = 1$. Thus equation (31) insures that at the end of the inflationary era one still finds fluctuations of all possible wavelengths within the observable universe without requiring sub-Planckian wavelengths in the early epochs. This argument, al-



Figure 2: Time evolution of wavelength λ associated with momentum mode n

though reasonable, still remains heuristic since a plane wave with the above wavelength

is not really a solution of the equation of motion (26), due to the time dependence of a. However, since the theory has an inbuilt minimum length uncertainty, this picture does capture the qualitative behaviour of length scales associated with the fluctuations.

5 Trans-Planckian damping and the initial state

In this section we discuss the consequences of the appearance of J in the last term of equation (26). To put our results in context, we first briefly review the relevant features of the J = 1 case analyzed in [6]. It is then argued that the presence of J can help us fix the initial state of the scalar field.

5.1 Review of the J = 1 case

When the dispersion relation is modified by hand, one misses the Jacobian J in the equations. This corresponds to J = 1 for which equation (26) reduces to

$$\left(\partial_{\eta}^{2} + a^{2}\rho^{2}(v) - \frac{\partial_{\eta}^{2}a}{a}\right)\tilde{\chi}_{MB}(\eta, n) = 0.$$
(32)

This is the equation that has so far been used to study the cosmological implications of modified dispersion relations. It was analyzed in detail by Martin and Brandenberger [6] for the Unruh and Corley-Jacobson type dispersion relations. Here we review some features of this model which are of interest to us. We assume de Sitter expansion for which $\partial_{\eta}^2 a/a = 2H^2a^2$. Let us follow a mode of comoving momentum n backward in time, as its physical momentum, v = n/a, is blue-shifted. The momentum range can be divided into three regions according to the behaviour of the solution:

- At late times, when $v \ll H$ (region III), the dispersion relation is linear and $\rho^2 = v^2$ can be neglected as compared to $\partial_{\eta}^2 a/a$. The relevant solution is $\tilde{\chi}_{MB}^{(\text{III})} = C_n^{(\text{III})} a$.
- When $H \ll v \ll 1$ (region II), the dispersion relation is still linear, but now $\partial_{\eta}^2 a/a$ can be neglected and one gets an oscillatory solution $\tilde{\chi}_{MB}^{(\text{II})} = C_{1n}^{(\text{II})} e^{in\eta} + C_{2n}^{(\text{II})} e^{-in\eta}$. If the dispersion relation had not been modified, then region II would have extended beyond the Planck scale, v = 1, all the way up to the beginning of inflation. This is the standard scenario that consequently suffers from the trans-Planckian problem.
- For non-linear dispersion relations, when v > 1 one enters region I. Then, for the Unruh dispersion relation, $\rho^2 \simeq 1$. In the de Sitter phase, $a^2 = (H\eta)^{-2}$ and the equation has an exact solution, $\tilde{\chi}_{MB}^{(I)} = C_{1n}^{(I)} |\eta|^{\delta_1} + C_{2n}^{(I)} |\eta|^{\delta_2}$. The exponents are determined in terms of H, $\delta_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{9 4H^{-2}}$, and contain imaginary components. Therefore, the solution in region I has an oscillatory nature. In the case of Corley-Jacobson dispersion with $b_m < 0$, at early enough times, ρ^2 becomes negative and the solution is damped.

The power spectrum of scalar fluctuations is given by $\mathcal{P} = n^3 |C_n^{(\text{III})}|^2 / 2\pi^2$. The coefficient $C_n^{(\text{III})}$ is determined in terms of $C_{1n,2n}^{(\text{II})}$ and $C_{1n,2n}^{(1)}$ by matching the solution and its first derivative across the boundaries of regions III, II and I. The coefficients $C_{1n,2n}^{(1)}$ are in turn determined by the initial conditions at some (arbitrary) time η_i in region I, which extends to the beginning of the inflationary era. Depending on the dispersion relation, the power spectrum may or may not acquire a non-trivial dependence on the initial state in the form of a modified dependence on n. As such, there is no natural choice for the initial state and, for oscillatory solutions, the best one can do is to pick up the local vacuum state at time η_i . This is based on the implicit assumption that the modes are created in their ground state and that nothing drastic happens from the beginning of inflation until time η_i . Below we consider a model based on the the equation of motion with $J \neq 1$ where the initial state problem can be addressed.

5.2 $J \neq 1$ and trans-Planckian damping

In this subsection we will study the equation of motion with the Jacobian factor J included. This factor takes into account the reduction in the number of degrees of freedom at high energies. As an explicit example, we will concentrate on the Unruh dispersion relation with $\gamma = 3$, although the discussion can be generalized to other cases. The solution is first discussed at a qualitative level. At the end we will comment on the dispersion relation based on (29) which leads to a qualitatively different result. For convenience we reproduce the relevant equation, (26), below,

$$\left(\partial_{\eta}^{2} + \omega_{total}^{2}\right)\tilde{\chi}(\eta, n) = 0, \qquad \omega_{total}^{2} = a^{2}\rho^{2}(v) - \frac{\partial_{\eta}^{2}(a\sqrt{J})}{a\sqrt{J}}.$$
(33)

Here, v = n/a is the magnitude of the physical momentum and $\tilde{\chi}(\eta, n)$ is the canonically normalized field given by

$$\tilde{\chi}(\eta, n) = a \sqrt{J} \,\tilde{\phi}(\eta, n) \,. \tag{34}$$

The second term in ω_{total}^2 is

$$\frac{\partial_{\eta}^{2}(a\sqrt{J})}{a\sqrt{J}} = \frac{v^{2}}{2} \left[\frac{\partial_{v}^{2}J}{J} - \frac{1}{2} \left(\frac{\partial_{v}J}{J} \right)^{2} \right] \left(\frac{\partial_{\eta}a}{a} \right)^{2} + \left[1 - \frac{v}{2} \frac{\partial_{v}J}{J} \right] \left(\frac{\partial_{\eta}a}{a} \right) \,. \tag{35}$$

The behaviour of this term depends on the choice of the dispersion relation which determines J through (25). During de Sitter expansion with Hubble parameter H one has

$$\left(\frac{\partial_{\eta}a}{a}\right)^2 = H^2 a^2, \qquad \left(\frac{\partial_{\eta}^2 a}{a}\right) = 2H^2 a^2.$$
(36)

Therefore the J-dependent term is suppressed by the small number H^2 .

For concreteness we will now consider the Unruh dispersion relation (28) for $\gamma = 3$. The variations of $\rho^2(v)$ and $a^{-2}\partial_{\eta}^2(a\sqrt{J})/(a\sqrt{J})$ with v are shown in Figure 3. We have taken $H = 10^{-5}$, in Planck units. Below the Planck scale, v < 1, (regions III and II) the modifications due to the modified uncertainty relation are absent. So we will concentrate on the trans-Planckian regime. As v increases above the Planck scale in region I (1 < v < 32), ρ^2 soon approaches 1 and J (not shown in the figure) decreases rapidly. However, the J-dependent term grows large in spite of the suppression by H^2 . At the boundary between regions I and 0, the two terms are equal and $\omega_{total}^2 = 0$.

Beyond this, we enter region 0 where the J-dependent term dominates and $\omega_{total}^2 < 0$. Note that the appearance of imaginary frequencies in this model is not built by hand into the disper-Rather, it is a consesion relation. quence of the rapid expansion during which the presence of J and the redshift of physical momenta induce the trans-Planckian damping term. In other words, in Planck scale processes at the present epoch, the frequencies are real and are given by $\rho^2 > 0$ alone. In this sense, the appearance of imaginary frequencies is not an unphysical feature of the theory. In particular, one avoids the multiple valuedness of momenta for a given energy which is a feature of nonmonotonic dispersion relations consid-



Figure 3: ρ^2 and $a^{-2}\partial_{\eta}^2 a_J/a_J$ $(a_J \equiv a\sqrt{J})$ for the Unruh dispersion relation for $\gamma = 3$, as a function of momentum v

ered earlier in the literature 3 . Region 0 is a new region that has not appeared in earlier models.

Deep in region 0, we can ignore ρ^2 in (33). The equation then has an obvious solution,

$$\tilde{\chi}_{1}^{(0)}(\eta, n) \approx a \sqrt{J} C_{1n}^{(0)} \sim a \, e^{-v^3} \, C_{1n}^{(0)} \tag{37}$$

where $C_{1n}^{(0)}$ is an *n*-dependent constant. This is similar to the solution in region III. But now the modes are frozen because, due to the presence of \sqrt{J} , they perceive the universe as expanding much faster than it really does. The other solution is

$$\tilde{\chi}_{2}^{(0)}(\eta, n) = a \sqrt{J} \int^{\eta} \frac{d\eta}{a^2 J} \approx \frac{a}{\sqrt{J}} C_{2n}^{(0)} \sim a \, e^{v^3} C_{2n}^{(0)} \tag{38}$$

Later it will be shown that the dependence on n is given by $C_{1,2n}^{(0)} = n^{-3/2}C_{1,2}^{(0)}$. Comparing with (34) one finds that $\tilde{\phi}_1^{(0)} = C_{1n}^{(0)}$ and $\tilde{\phi}_2^{(0)} = C_{2n}^{(0)}/J$. The energy density in terms of the field $\tilde{\phi}$ is given by,

$$\underline{\varepsilon(v)} \, d^3v \sim \frac{1}{2} \, v \, J \left((\partial_\eta \tilde{\phi})^2 + a^2 \rho^2 \tilde{\phi}^2 \right) d^3v \tag{39}$$

³To compare with the earlier models, note that although we started with the Unruh dispersion relation, ω_{total} is reminiscent of the Corley-Jacobson case with $b_m < 0$.

As we go backward in time, the energy density for the first solution vanishes as $J(\equiv 1-\rho^6)$, while the one corresponding to the second solution blows up as 1/J. Therefore, it is reasonable to impose $C_{2n}^{(0)} = 0$ as a boundary condition. Thus, deep in the trans-Planckian region, the field $\tilde{\chi}_1^{(0)}$ starts from an extremely small value and increases with time as the momentum is red-shifted. At the boundary between regions 0 and I, ω_{total} turns real and the solution starts oscillating. Since $\tilde{\chi}^{(0)}$ is real and has only one branch, $\tilde{\chi}^{(1)}$ is also real and oscillates as a cosine function. The exact solution is given in subsection 5.4. Note that the damping effect of the *J*-dependent term has fixed the "initial state" in region I in terms of the solution in region 0. For the Unruh dispersion relation, this is no longer the adiabatic vacuum used in the literature. The issue of the initial state dependence will be discussed in more detail in the next subsection.

The behavior of ω_{total} with v could change appreciably from the one depicted in Figure 3 for other dispersion relations. For the Unruh dispersion relations (28), the length of region I decreases with increasing γ . Another interesting case is the dispersion relation corresponding to (29), depicted as "soft" in Figure 1. In this case, the J dependent term in ω_{total} always remains very small and as a result in region I, well above the Planck scale, $\omega_{total}^2 \approx \rho^2 \approx 1$, and there will be no region 0. Thus, the problem is very similar to the case of Unruh dispersion relation analyzed in [6]. We recall that in the approach of [29], a modified commutation relation based on (29) lead to a different and much more complicated equation of motion.

5.3 The initial state

At physical momenta below the Planck scale (region II), the effect of J disappears and the dispersion relation is linear, leading to the standard free wave solutions. The coefficients $C_n^{(\text{II})}$ are determined by matching with the solution in the trans-Planckian regime. In region III, below the Hubble scale the modes freeze and the solution becomes $\tilde{\chi}^{(\text{III})}$ $C_n^{(\text{III})} a$. The power spectrum of fluctuations is finally given in terms of $|C_n^{(\text{III})}|^2$. If this goes as n^{-3} , one ends up with the flat spectrum. When the Hubble parameter is time dependent, the flat spectrum receives corrections even in the absence of trans-Planckian modifications. Therefore, here we concentrate on the case of constant H to isolate the purely trans-Planckian effects. The possibility of deviations from the flat spectrum due to the modified dispersion relation, and the choice of the initial state (boundary conditions), has been extensively discussed in the literature [6], [7], [10] - [21]. The initial state is defined in the adiabatic, *i.e.*, lowest order WKB, approximation. A modification of the flat spectrum by the choice of the initial state is then associated with the breakdown of this lowest order WKB approximation. Here, we formulate the problem in terms of a (formally) exact WKB solution and argue that the initial state dependence is determined by whether the frequency at the initial time is real or imaginary. The result can then be understood in terms of the breakdown of the lowest order adiabatic approximation. However, the two criteria are not exactly equivalent in the sense that the violation of adiabaticity does not always lead to initial state dependence of the power spectrum. This is not in contradiction with [11] and [14] since the initial state dependence there can be attributed to a time dependent Hubble parameter. In fact, their results show that for a constant Hubble parameter, the initial state dependence disappears while adiabaticity is still violated, which is further evidence for the assertion above.

During de Sitter expansion, $v = n/a = -nH\eta$. This can be used to write the equation of motion (33) in terms of v,

$$\left(\partial_v^2 + \omega_v^2\right)\tilde{\chi}_v = 0, \qquad \omega_v^2(v) = \frac{\omega_{total}^2(\eta, v)}{n^2 H^2}.$$
(40)

From equations (35) and (36) one can see that ω_v^2 depends only on v and not on n and η separately⁴. The discussion below applies to general ω_v and is not restricted to the situation depicted in Figure 3. The exact solutions of the above equation can be formally written in a WKB form [40],

$$\tilde{\chi}_{v\pm} = \frac{1}{\sqrt{2W_v}} \exp\left(\pm i \int_{v_i}^v W_{\bar{v}} d\bar{v}\right) \,, \tag{41}$$

where W_v is given by

$$W_v^2 = \omega_v^2 + Q_v(W_v), \qquad Q_v(W_v) = \frac{1}{2} \left(\frac{3(\partial_v W_v)^2}{2W_v^2} - \frac{\partial_v^2 W_v}{W_v} \right).$$
(42)

This solution is formally exact, but in practice W_v is determined iteratively in terms of $\omega_v(v)$ to some order of approximation. Near the classical turning points, where ω_v changes sign, the integral is to be analytically continued to the complex v-plane to allow for well defined solutions. This procedure, which mixes the two branches labeled by \pm , is equivalent to using matching conditions in the more familiar form of WKB [41]. Clearly, the solutions $\tilde{\chi}_{v\pm}$ are functions of v alone. Let us now turn to the solutions $\tilde{\chi}_{\pm}(\eta, v)$ of equation (33). These are also given by expressions analogous to the ones above,

$$\tilde{\chi}_{\pm} = \frac{1}{\sqrt{2W_{total}}} \exp\left(\pm i \int_{\eta_i}^{\eta} W_{total} d\bar{\eta}\right)$$
(43)

where, $W_{total} = nHW_v$ and $Q_{total} = nHQ_v$. It is then evident that the exponential is the same in terms of v and η . Therefore, the dependence of $\tilde{\chi}$ on n, v and $\eta_i = -v_i/nH$ is of the form

$$\tilde{\chi}_{\pm}(n,v,v_i) = \frac{1}{\sqrt{nH}} \,\tilde{\chi}_{v\pm}(v,v_i). \tag{44}$$

Thus, for a given v, the extra dependence on n (beyond the $1/\sqrt{n}$) could only come from the initial state at η_i . In particular, matching the WKB solution in region I to the solution in region II at a fixed $v = v_f$ does not introduce extra dependence on n.

⁴This will not be the case for a time dependent Hubble parameter. For example, for $a = \eta^{\lambda} \ell_0$, the equation in terms of v becomes $(\lambda^2 v^2 \partial_v^2 + \lambda(\lambda + 1)v \partial_v + \eta^{2\lambda+2} \ell_0^2 \rho^2(v) - F(v)) \chi = 0$ which explicitly depends on η , besides v.

The initial state dependence of the power spectrum depends on whether $W_{total}(\eta_i)$ is real or imaginary⁵. When this quantity is real, one can pick up the positive frequency branch $\tilde{\chi}_-$ as a generalization of the Minkowski vacuum. This is the standard way of choosing the initial state in such cases. Then the initial state dependence will only appear as a phase in $\tilde{\chi}_-$ through the lower limit of the integral in (43). This phase will carry over to $C_n^{(\text{III})}$ and will drop out of the final amplitude. The situation will not change if adiabaticity is violated during the evolution from η_i to η_f , as long as $W_{total}(\eta_i)$ and $W_{total}(\eta_f)$ are real. This corresponds to ω_{total} having an even number of classical turning points in this range.

However, if the initial state is chosen such that the solution also contains a small negative frequency component, then the initial state dependence does not disappear and the final amplitude will develop an oscillatory dependence on v_i or, equivalently, on n for fixed η_i . For an explicit realization of this, see [22].

When $W_{total}(\eta_i)$ is imaginary, the magnitudes, and not phases, of $\tilde{\chi}_{\pm}$ will depend on the initial state through the lower limit of the integral. This dependence will always carry over to region III and will show up in the power spectrum as long as η_i is a finite time in the past. The Corley-Jacobson dispersion relation for $b_m < 0$ analyzed in [6] falls in this class. Note that since $W_{total}(\eta_f)$ is taken to be real, ω_{total} will have an odd number of classical turning points between η_i and η_f and hence, adiabaticity is violated at least once during the evolution.

In any case, one way of avoiding explicit dependence on η_i (and the associated dependence on n) is to specify the initial state at $\eta_i = -\infty$. For oscillatory solutions, $(\omega^2(\eta_i) > 0)$ this implies extrapolating the validity of the equation of motion to the very beginning of inflation, which is not a very reasonable assumption. This will simply shift the trans-Planckian problem to higher momenta. On the other hand, for strongly damped solutions ($\omega^2(\eta_i) << 0$) it is a natural choice. Since the solution is damped away rapidly with decreasing η , it is enough to require the validity of the equation of motion up to a certain time η_i , far enough in the past, but well after the beginning of inflation. Then as far as the solution is concerned, the initial state is effectively fixed at $\eta_i = -\infty$. The equation of motion discussed in the last subsection admits solutions belonging to this class. The strategy can also be applied to the Corley-Jacobson dispersion relation with $b_m < 0$. Another example is the model in [19], although the damping there is not very strong. Both are based on non-monotonic dispersion relations.

5.4 Solution in regions I and 0

Let us now get back to our equation (33). For the Unruh dispersion relation (28) with $\gamma = 3$, one gets $J = 1 - \rho^6$, $\partial_v J/J = -6v^2 \rho^3$ and $\partial_v^2 J/J = 18v^4(3\rho^6 - 1) - 12v\rho^3$. This

⁵One often chooses the initial state in a region where Q_{total} is small so that ω_{total} is a good approximation to W_{total} (adiabatic approximation). Then W_{total} is real when ω_{total} , or ω_v , is real.

leads to the equation,

$$\left(\partial_{\eta}^{2} + a^{2}\rho^{2} - \left[9v^{6}(2\rho^{6} - 1) + 2\right]a^{2}H^{2}\right)\tilde{\chi} = 0.$$
(45)

The last two terms are plotted in Figure 3. In regions I and 0, above the Planck scale, we can make the approximation $\rho \approx 1$ and also drop the 2 in the square brackets. Then, the equation in terms of v, reduces to

$$\left(\partial_v^2 + \frac{H^{-2}}{v^2} - 9v^4\right)\tilde{\chi}_v = 0.$$

$$\tag{46}$$

This has exact solutions, $\sqrt{v}I_{\nu}(v^3)$ and $\sqrt{v}K_{\nu}(v^3)$, in terms of the modified Bessel's functions. The order $\nu = \sqrt{1 - 4H^{-2}}/6$ is a large imaginary number. Using (44), one gets

$$\tilde{\chi}^{(0,\mathrm{I})} = C_1 \sqrt{\frac{v}{n}} K_{\nu}(v^3) + C_2 \sqrt{\frac{v}{n}} I_{\nu}(v^3) \,. \tag{47}$$

 C_1 and C_2 are constants that are to be fixed by the boundary conditions in region 0. To this end, let us consider the asymptotic behaviour of the modified Bessel functions as v goes to infinity,

$$K_{\nu}(v^3) \approx \sqrt{\frac{\pi}{2v^3}} e^{-v^3}, \qquad I_{\nu}(v^3) \approx \sqrt{\frac{1}{2\pi v^3}} e^{v^3}, \qquad (\text{as } v \to \infty).$$
 (48)

Comparing (47) with (37) and (38) in this limit, one can fix the normalizations $C_{1,2n}^{(0)} = n^{-3/2}C_{1,2}^{(0)}$, as promised. As argued there, the growing solution gives rise to an energy density that diverges as v increases. Therefore, we impose $C_2 = 0$ as a boundary condition. The surviving solution decays rapidly with increasing v. As argued in the previous subsection, this allows us to regard the initial state in the far past as effectively corresponding to $\eta_i = -\infty$. The absence of an explicit η_i dependence insures the flatness of the power spectrum of the associated fluctuations.

Thus in this scenario, in the beginning of inflation, the oscillations are very heavily damped by the Jacobian factor J and the field $\tilde{\chi}$ is essentially zero. The expansion of the universe reduces J and the field starts growing in region 0. At the boundary of regions 0 and I, ω_{total} turns positive and the solution starts oscillating in region I. It remains real and behaves as a cosine function with its amplitude decreasing with time. The sinusoidal nature of the solution can be directly inferred from the application of the WKB connection formulae at the turning point. The solution becomes less accurate as we approach region II and can be matched to $\tilde{\chi}^{(II)} = C_{1n}^{(II)} e^{in\eta} + C_{2n}^{(II)} e^{-in\eta}$ and its first derivative at $v = v_p = 1$. The reality of the solution in region I implies that $C_{1n}^{(II)*} = C_{2n}^{(II)}$, so that $\tilde{\chi}^{(II)} = |C_n^{(II)}| \cos(n\eta + \varphi)$. The amplitude and the phase φ are determined by the matching and their explicit expressions can be easily worked out in terms of $K_{\nu}(v^3)$ and its derivatives. Finally, $C_n^{(III)}$ is determined by matching across the boundary of regions II and III. Clearly, its dependence on n is given by $n^{-3/2}$, leading to a flat spectrum. Note that in region 0 the field started in a WKB ground state (albeit with an imaginary frequency). But in region I the solution behaves as a cosine function, and is no longer in a WKB ground state. Ignoring the issue of negative frequencies, this may be interpreted as particle creation caused by the breakdown of adiabaticity near the classical turning point, at the boundary between the two regions. One could argue, however, that the back reaction problem is avoided because the energy density (39) for these modes is heavily suppressed by J. One may also note that the solution in region III belongs to the same branch as that in region 0 and therefore is in ground state with respect to it. This may be taken to suggest that there is no net particle creation in region III, with respect to region 0.

The appearance of imaginary frequencies in region 0 is one of the features of models with an Unruh dispersion relation and a trans-Planckian damping factor. A consequence of this is the appearance of real fluctuations in regions I and II. In region II, for example, this leads to $|C_1^{(II)}|^2 - |C_2^{(II)}|^2 = 0$ as opposed to 1, which is the usual Wronskian condition. As pointed out in the literature, the quantum field theory of modes with imaginary frequencies is not well understood. For a discussion, however, see [42] and [43].

In this section we have focused on the behaviour of equation (33) with the dispersion relation (28) for $\gamma = 3$. For other dispersion relations the behaviour could be drastically different. As an example, the case of the dispersion relation based on (29) was discussed at the end of subsection 5.2.

6 Conclusions

It is believed that fundamental physics at the Planck scale leads to a minimum length uncertainty principle. Such a principle can in turn be associated with a modified spacemomentum commutation relation⁶. We investigate a class of such commutators and show that the minimum length principle can be consistently incorporated into a field theory, both in flat space-time, as well as in the expanding universe. Planck scale physics modifies the linear dispersion relation to a non-linear one. Since the modification of the dispersion relation is related to that of the commutators, it is constrained by the requirement of minimum length. Of late, modified dispersion relations have been used in the literature to mimic the effects of trans-Planckian physics. However one can now see that not all of them are consistent with the minimum length principle. Our resulting field theory can also be understood as one in which higher-derivative corrections due to Planck scale physics have been summed up in a closed form. Another feature of this theory is the reduction in the number of degrees of freedom at high energies. It is interesting to investigate the quantization of such theories, especially when they are extended to include gravity, since they seem to be free of ultraviolet divergences. Also, the modified commutator theory leads to a (dynamical) breakdown of Lorentz invariance due to the emergence of a

⁶An alternative approach is based on space-time uncertainty relations [44],[45],[46],[47] which is related to non-perturbative features of string theory.

minimum length scale, although the background we finally consider is itself not invariant. On Lorentz invariance at Planck scale, especially in the context of cosmology see, for example, [35].

In an expanding universe, the physical momenta undergo a red-shift which makes the non-linear dispersion relation time dependent. The red-shift also induces a trans-Planckian damping factor in the equation of motion. This is due to the reduction in the number of degrees of freedom at ultra high momenta. This factor tends to suppress the field fluctuations at trans-Planckian momenta, although the actual effectiveness of the damping depends on the detailed form of the dispersion relation. The damping is strong in the example of the Unruh dispersion relation that we have considered, but can be insignificant in some other cases. The energy density of the fluctuations is also strongly suppressed at large momenta by the trans-Planckian damping factor, thereby avoiding the problem of back reaction.

The modified theory, which incorporates Planck scale physics, is then used to study the evolution of cosmological perturbations. It is shown that for de Sitter inflation it still leads to a flat power spectrum for the fluctuations. In principle, the non-linear form of the dispersion relation could complicate the choice of the initial state whose identification at some initial time η_i introduces an extra scale dependence, affecting the spectral tilt. We have discussed this in some detail and argued that, for de Sitter inflation, the main criterion is the real or imaginary nature of the frequency at the initial time. However, when the oscillations are strongly damped in the far past, as is the case here, then one can *effectively* treat the initial time as $\eta_i = -\infty$, even though inflation is not really required to be eternal in the past. This avoids the extra scale dependence and keeps the spectrum flat.

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