A Note on Coherent States with Quantum Gravity Effects

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Abstract

Existence of a minimal measurable length and an upper bound for the momentum fluctuations are the casting reasons for generalization of uncertainty principle and then reformulation of Hilbert space representation of quantum mechanics. In this paper, we study the consequences of the Generalized Uncertainty Principle (GUP) in the presence of both minimal length and maximal momentum. We consider a simple harmonic oscillator in the framework of GUP by introducing it's energy eigenstates and energy spectrum. Investigation of coherent states for a generalized harmonic oscillator and it's generic behavior are the other topics in our study.

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1 Introduction

It is a well-known idea that gravity may modify the uncertainty principle. Bearing the gravity in mind, field theory leads to an effective cutoff (minimum measurable length) in the ultraviolet energies. In fact, since high energies have major gravitational effects, they will change the spacetime structure in the small scales [1]. Theoretically, the minimum measurable length is proposed in various approaches to the quantum gravity such as string theory [2-6], loop quantum gravity [7] and quantum geometry [8], which has the same order as the Planck length $(l_{pl} \sim 10^{-35} \text{m})$ [9-11]. In the Heisenberg uncertainty principle (HUP), one cannot measure the momentum and position of a particulary particle with zero uncertainty, together. However, it is possible to measure one of these observable quantities by withdrawing the corresponding information about the other. It means that one can vanish the uncertainty in position yielding $(\Delta x)_{min} = 0$. But the story is changed by considering the modified uncertainty principle due to the quantum gravity which is now called the Generalized Uncertainty Principle (GUP). In fact, due to the non-zero effective cutoff, it is not possible to set $(\Delta x)_{min} = 0$ whenever the effects of quantum gravity become important meaning that we should replace HUP by GUP [12-26]. Moreover, Hamiltonian is changed due to GUP leading to the changes of the energy spectrum of quantum systems. Since GUP implies $(\Delta x)_{min} \neq 0$, one may conclude that spacetime has a non-commutative structure in the Planck scales leading to quantize the spacetime in the Planck scales. Therefore, one can reinterpret this cutoff as the quanta of space, which is due to the quantum fluctuations of background spacetime, leading to a new representation for the Hilbert space [27]. This new representation attracted more investigators to itself [28-30]. Moreover, considering the Planck length as the minimum permissible length, which is independent of the observer, changes the Special Relativity foundation that is yielding the Doubly Special Relativity (DSR) theory [31-34]. In the DSR theory, the minimum effective cutoff makes an upper bound on momentum. Therefore, the Planck length inspires an upper bound for the admissible energy in this theory. We should note that the above results are modified to the more common situations in which the spacetime curvature is taken into account. Indeed, the spacetime curvature induces a non-zero minimum to the momentum uncertainty. The non-zero uncertainty in the position and momentum lead to retire the wave functions in the position and momentum spaces, which are introduced in the quantum mechanics based on HUP, respectively. In order to avoid these shortcomings, bearing the states with maximum localization in mind, one should use the quasi-position and quasi-momentum representations and reformulate the quantum mechanics, and therefore the Hilbert space [27,35]. Our aim in this paper is investigating the effects of considering the minimal length and maximal momentum on the harmonic oscillator and its properties. The paper is organized as follows. In the next section, we review GUP by considering the minimum length and maximum momentum considerations. In section 3 we present the generalized harmonic oscillator as well as its eigenstates and eigenvalues. Coherent states of a harmonic oscillator is presented in section 4. In addition, we study the normalization coefficient and probability distribution of the coherent states. The last section is devoted to a summary and concluding remarks.

2 GUP with minimal length and maximal momentum

In the framework of a generalized uncertainty principle that predicts maximal observable momentum in addition to minimal observable length, we can write [19-24,35]

$$\Delta x \ \Delta p \ge \frac{\hbar}{2} (1 - 2\alpha +4\alpha^2 < p^2 >), \tag{1}$$

which α is the GUP parameter in the presence of the two aforesaid cutoffs. The above uncertainty relation can be obtained from the following algebraic structure

$$[x,p] = i\hbar(1 - \alpha p + 2\alpha^2 p^2).$$
⁽²⁾

In this relation there is a first order term in particle's momentum which has its origin on the existence of a maximal momentum, whereas the second order term in particle's momentum originates from the existence of a minimal length.

We can define position and momentum operators for the GUP case as

$$X = x$$
 , $P = p(1 - \alpha p + 2\alpha^2 p^2)$, (3)

where x and p ensure the Jacobi identities, and X and P satisfy the generalized commutation relation

$$[X,P] = i\hbar(1 - \alpha p + 2\alpha^2 p^2).$$
(4)

In this case, we interpret p as the momentum operator at low energies by a standard representation in position space, $p_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j}$, and P as the momentum operator at high energies which has the generalized representation in position space as

$$P_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j} \left[1 - \alpha \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j}\right) + 2\alpha^2 \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j}\right)^2\right].$$

To show how maximal momentum arises in this setup, we first find the minimal observable length, i.e. $\Delta x_0 \equiv \Delta x_{min} (\langle p \rangle = 0)$. We can write the inequality (1) on the boundary of the allowed region and use $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ to obtain a second order equation for Δp , which has the following solutions for Δp

$$\Delta p = \frac{\Delta x}{4\alpha^2 \hbar} \pm \sqrt{\left(\frac{\Delta x}{4\alpha^2 \hbar}\right)^2 - \frac{\langle p \rangle}{2\alpha}} (2\alpha \langle p \rangle - 1) - \frac{1}{4\alpha^2}.$$
(5)

The reality of solutions gives the minimum value of Δx as

$$\Delta x_{min}() = 2\alpha \hbar \sqrt{1 - 2\alpha +4\alpha^2 ^2}$$

Using $\langle p \rangle = 0$, absolutely smallest uncertainty in position (absolute minimal observable length) can be deduced from the latter equation as

$$\Delta x_0 = 2\alpha\hbar. \tag{6}$$

Due to duality of position and momentum operators, we can assume $\Delta x_{min} \propto \Delta p_{max}$. With this assumption, using the condition $\langle p \rangle = 0$ in Eq. (5) and making use of Eq. (6), we have

$$(\Delta p)_{max} = \frac{1}{2\alpha},\tag{7}$$

where we will assume this result as the maximal measurable momentum in our setup.

2.1 Representation on momentum space

It is to be noted that, if we assume the minimal observable length as minimal, nonzero uncertainty in position, we have no longer a Hilbert space representation on position space wave functions of the ordinary quantum mechanics. This is because one can not find any physical state which is a position eigenstate, since such an eigenstate would have zero uncertainty in position. Therefore we must construct a new Hilbert space representation compatible with relation (4). This representation can be achieved in a continuous momentum space. Now, in this space, momentum and position operators have the form

$$P = p$$
 , $X = (1 - \alpha p + 2\alpha^2 p^2)x$, (8)

where $x = i\hbar \frac{\partial}{\partial p}$. Due to the presence of the additional factor $(1 - \alpha p + 2\alpha^2 p^2)$, the scalar product in momentum representation should be rewritten as

$$\langle \phi \mid \varphi \rangle = \int_{-P_{pl}}^{+P_{pl}} \frac{dp}{1 - \alpha p + 2\alpha^2 p^2} \phi^*(p)\varphi(p) \,. \tag{9}$$

We note that appearance of the limits $-P_{pl}$ to $+P_{pl}$ (Planck momentum) originate from the maximal measurable momentum, so that in the absence of this cutoff the integrals must be calculated from $-\infty$ to $+\infty$ [27]. In present framework, the identity operator would be modified as

$$1 = \int_{-P_{pl}}^{+P_{pl}} \frac{dp}{1 - \alpha p + 2\alpha^2 p^2} \mid p >
(10)$$

2.2 Functional analysis of the position operator

The position operator (X in Eq. (8)) acting on position eigenstate in momentum space, $\varphi_{\xi}(p) = \langle p | \xi \rangle$, gives the following eigenvalue equation

$$i\hbar(1 - \alpha p + 2\alpha^2 p^2)\frac{\partial\varphi_{\xi}(p)}{\partial p} = \xi\varphi_{\xi}(p).$$
(11)

By solving this differential equation, we obtain the position eigenvectors in the presence of aforesaid cutoffs as

$$\varphi_{\xi}(p) = \varphi_{\xi}(0) \exp\left[-i\frac{2\xi}{\alpha\hbar\sqrt{7}} \left\{\tan^{-1}(\frac{1}{\sqrt{7}}) + \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}})\right\}\right].$$
 (12)

Using the normalization

$$1 = <\varphi \mid \varphi > = \int_{-P_{pl}}^{+P_{pl}} \frac{1}{1 - \alpha p + 2\alpha^2 p^2} \phi_{\xi}^*(p) \phi_{\xi}(p) dp$$

we can obtain the coefficients $\varphi_{\xi}(0)$ as

$$\varphi_{\xi}(0) = \sqrt{\frac{\alpha\sqrt{7}}{2}} \left[\tan^{-1}\left(\frac{4\alpha P_{pl} - 1}{\sqrt{7}}\right) + \tan^{-1}\left(\frac{4\alpha P_{pl} + 1}{\sqrt{7}}\right) \right]^{-1/2}.$$

2.3 Maximal localization

In the presence of minimum observable length, $l_{pl} = \Delta x_0 = 2\alpha\hbar$, it is not possible to probe distances less than Planck length. So, the notion of spacetime manifold should be revised for the finite resolution of the spacetime points. In this manner, we are obliged to introduce the states with maximal localization that are confined up to Planck length and it is impossible to localize them further. Now, we consider the states $|\varphi_{\xi}^{ml}\rangle$ of maximal localization around a position $\xi \geq l_{pl}$ and write

$$\langle X \rangle = \langle \varphi_{\xi}^{ml} \mid X \mid \varphi_{\xi}^{ml} \rangle = \xi.$$
(13)

By using the positivity of norm for each state $| \varphi \rangle$ in the representation of the Heisenberg algebra, namely

$$\| \left(X - \langle X \rangle + \frac{\langle [X, P] \rangle}{2(\Delta P)^2} (P - \langle P \rangle) \right) |\varphi\rangle \| \ge 0,$$

we can deduce [27] that the state $| \varphi >$ will be on the boundary of the physically allowed region only it obeys

$$\left(X - \langle X \rangle + \frac{\langle [X, P] \rangle}{2(\Delta P)^2} (P - \langle P \rangle)\right) |\varphi\rangle = 0.$$
(14)

Using Eq. (8), relation (14) takes the form of a differential equation in momentum space as

$$\left(i\hbar(1-\alpha p+2\alpha^2 p^2)\frac{\partial}{\partial p}-\langle X\rangle+i\hbar\frac{1+2\alpha^2(\Delta p)^2+2\alpha^2^2-\alpha}{2(\Delta p)^2}(p-\langle p\rangle)\right)|\varphi\rangle=0.$$
(15)

By taking into account that the states of absolutely maximal localization can only be obtained for $\langle p \rangle = 0$, and using Eqs. (7) and (13), Eq. (15) can be solved to obtain

$$\varphi_{\xi}^{ml}(p) = \varphi_{\xi}^{ml}(0) \frac{e^{-\frac{3}{2\sqrt{7}}(\frac{4i\xi}{3\alpha\hbar} + 1)(\tan^{-1}(\frac{1}{\sqrt{7}}) + \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{\frac{3}{4}}}.$$

Using normalization condition

$$1 = \langle \varphi_{\xi}^{ml}(p) | \varphi_{\xi}^{ml}(p) \rangle = \int_{-P_{pl}}^{+P_{pl}} \varphi_{\xi}^{ml}(0) \varphi_{\xi}^{ml^*}(0) \frac{e^{-\frac{3}{\sqrt{7}}(\tan^{-1}(\frac{1}{\sqrt{7}}) + \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{\frac{5}{2}}} dp,$$

we find $\varphi_{\xi}^{ml}(0)$ as

$$\phi_{\xi}^{ml}(0) = \sqrt{6\alpha} \Big[\sqrt{8}e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\frac{\eta}{3})} \Big]^{-\frac{1}{2}} e^{\frac{\eta}{2} \tan^{-1}(\frac{\eta}{3})},$$

where $\eta \equiv \frac{4\alpha P_{pl}-1}{\sqrt{7}}$. Since we have $P_{pl} = \frac{1}{2\alpha}$, we find $\eta = \frac{3}{\sqrt{7}}$. Finally, the momentum space wavefunction for maximally localized states around ξ can be written in the form

$$\phi_{\xi}^{ml}(p) = \frac{\sqrt{6\alpha} \left[\sqrt{8}e^{\eta \tan^{-1}(\eta)} - e^{-\eta \tan^{-1}(\frac{\eta}{3})} \right]^{-\frac{1}{2}}}{(1 - \alpha p + 2\alpha^2 p^2)^{\frac{3}{4}}} e^{-\frac{\eta}{2} \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}})} e^{-\frac{2i\xi}{\alpha\hbar\sqrt{7}}(\tan^{-1}(\frac{\eta}{3}) + \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}}))}.$$
 (16)

In ordinary quantum mechanics, we expand the states $|\varphi\rangle$ in the position eigenbasis $\{|x\rangle\}$ as $\langle x | \varphi \rangle$. But, there are now no physical states which would form a position eigenbasis. However, there is a possibility to project arbitrary states $|\varphi\rangle$ on maximally localized states, $|\varphi_{\xi}^{ml}\rangle$, to obtain the probability amplitude of maximal localization for the particle around the position ξ . We will call the collection of these projections, $\langle \varphi_{\xi}^{ml} | \phi \rangle$, the states " quasi-position wave function"

$$\phi(\xi) := <\varphi_{\xi}^{ml} |\phi>.$$

Using a generalization of the Fourier transformation that maps momentum space wavefunction into quasi-position space wavefunction, we can transform a state's wavefunction in momentum representation to it's quasi-position wavefunction as follows

$$\varphi(\xi) = \varphi_{\xi}^{ml}(0) \int_{-P_{pl}}^{+P_{pl}} \frac{e^{(-\frac{\eta}{2} + \frac{2i\xi}{\alpha\hbar\sqrt{7}})(\tan^{-1}(\frac{\eta}{3}) + \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}}))}}{(1 - \alpha p + 2\alpha^2 p^2)^{\frac{7}{4}}} \varphi(p) dp.$$
(17)

As in ordinary quantum mechanics, one can write

$$e^{i\frac{2\xi}{\alpha\hbar\sqrt{7}}(\tan^{-1}(\frac{\eta}{3})+\tan^{-1}(\frac{4\alpha p-1}{\sqrt{7}}))} \equiv e^{iK\xi}$$

to take into account the $K \equiv \frac{2}{\alpha \hbar \sqrt{7}} \left(tan^{-1} \left(\frac{\eta}{3} \right) + tan^{-1} \left(\frac{4\alpha p - 1}{\sqrt{7}} \right) \right)$ as modified wavenumber. So, the modified wavelength for quasi-position wavefunction of physical states has the form

$$\lambda(p) = \frac{\pi \alpha \hbar \sqrt{7}}{\tan^{-1}(\frac{\eta}{3}) + \tan^{-1}(\frac{4\alpha p - 1}{\sqrt{7}})}.$$

Since $\alpha \neq 0$ and p is limited to the Planck momentum, there is no wavelength smaller than λ_0

$$\lambda_0 = \lambda(P_{pl}) = \frac{\pi \alpha \hbar \sqrt{7}}{\tan^{-1}(\frac{\eta}{3}) + \tan^{-1}(\frac{4\alpha P_{pl} - 1}{\sqrt{7}})}.$$
(18)

Using the relation between momentum and energy, $E = \frac{p^2}{2m}$, we can write the maximum energy of the momentum eigenstates as

$$E(\lambda_0) = \frac{P_{pl}^2}{2m},\tag{19}$$

which for $m \approx M_{pl}$, the energy of short wavelength modes will be the Planck energy, $E(\lambda_0) \approx E_{pl}$. Note that there is not any energy divergency in λ_0 . This result is in agreement with ordinary quantum mechanics and is an important outcome of a GUP formalism in the presence of both minimal length and maximal momentum. By inverse Fourier transform of Eq. (17), we have

$$\phi(p) = \left(\varphi_{\xi}^{ml}(0)^{-1}\right) \int_{-\infty}^{+\infty} \frac{(1 - \alpha p + 2\alpha^2 p^2)^{\frac{3}{4}}}{2\pi\hbar} e^{\left(\frac{\eta}{2} - \frac{2i\xi}{\alpha\hbar\sqrt{7}}\right)(\tan^{-1}\left(\frac{\eta}{3}\right) + \tan^{-1}\left(\frac{4\alpha p - 1}{\sqrt{7}}\right))} \phi(\xi) d\xi , \qquad (20)$$

which because of the integration over ξ (not p), the integration interval will be over $-\infty$ to $+\infty$. Using Eq. (20) and following the customary method in ordinary quantum mechanics, we can deduce the generalized form of momentum operator in the quasi-position space. Since

$$\frac{\partial}{\partial\xi}e^{i\frac{2\xi}{\alpha\hbar\sqrt{7}}(\tan^{-1}(\frac{\eta}{3})+\tan^{-1}(\frac{4\alpha p-1}{\sqrt{7}})} = i\frac{2}{\alpha\hbar\sqrt{7}}\left(\tan^{-1}(\frac{\eta}{3})+\tan^{-1}(\frac{4\alpha p-1}{\sqrt{7}}\right)e^{i\frac{2\xi}{\alpha\hbar\sqrt{7}}(\tan^{-1}(\frac{\eta}{3})+\tan^{-1}(\frac{4\alpha p-1}{\sqrt{7}})},$$

one can infer the relation $\frac{\alpha\sqrt{7}}{2}\frac{\hbar}{i}\frac{\partial}{\partial\xi} \equiv \tan^{-1}(\frac{\eta}{3}) + \tan^{-1}(\frac{4\alpha p-1}{\sqrt{7}})$. Then, using $\tan(\tan^{-1}(\frac{\eta}{3}) + \tan^{-1}(\frac{4\alpha p-1}{\sqrt{7}})) = \frac{\sqrt{7}\alpha p}{2-\alpha p}$ momentum operator can be obtained as

$$P \equiv \frac{2}{\alpha} \frac{\tan(\frac{\alpha\sqrt{7}}{2}\frac{\hbar}{i}\frac{\partial}{\partial\xi})}{\sqrt{7} + \tan(\frac{\alpha\sqrt{7}}{2}\frac{\hbar}{i}\frac{\partial}{\partial\xi})}.$$
 (21)

3 Generalized harmonic oscillator

In this section we generalize the formulation of a linear harmonic oscillator in the presence of both minimal length and maximal momentum, and obtain the eigenvalue and eigenfunctions of harmonic oscillator by solving the Schrödinger equation. According to position and momentum operators in momentum space representation (Eq. (8)) and by using them in harmonic oscillator Hamiltonian, $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$, time-independent Schrödinger equation, $H\psi = E\psi$, can be written as

$$\frac{\partial^2 \psi(p)}{\partial p^2} + \frac{4\alpha^2 p - \alpha}{1 - \alpha p + 2\alpha^2 p^2} \frac{\partial \psi(p)}{\partial p} + \frac{1}{(1 - \alpha p + 2\alpha^2 p^2)^2} (\epsilon - \beta^2 p^2) \psi(p) = 0, \tag{22}$$

with

$$\epsilon = \frac{2E}{m\omega^2\hbar^2}$$
 , $\beta^2 = \frac{1}{(m\hbar\omega)^2}$

With solving the above differential equation, the eigenfunctions can be obtained in terms of the Legendre functions

$$\psi(p) = C_1 \exp\left[-\frac{1}{4} \frac{\sqrt{2} \tanh^{-1} \frac{\sqrt{2} \alpha p}{\sqrt{\alpha p - 1}}}{\sqrt{\alpha p - 1}}\right] P\left(\frac{1}{2} \frac{\sqrt{\alpha^4 + \beta^2} - \alpha^2}{\alpha^2}, \frac{1}{4} \frac{\sqrt{2} \sqrt{\alpha^4 - 4\epsilon \alpha^2 + 2\beta^2 (\alpha p - 1)}}{\alpha^2 \sqrt{\alpha p - 1}}, \frac{2\alpha p}{\sqrt{2\alpha p - 2}}\right) + C_2 \exp\left[-\frac{1}{4} \frac{\sqrt{2} \tanh^{-1} \frac{\sqrt{2} \alpha p}{\sqrt{\alpha p - 1}}}{\sqrt{\alpha p - 1}}\right] Q\left(\frac{1}{2} \frac{\sqrt{\alpha^4 + \beta^2} - \alpha^2}{\alpha^2}, \frac{1}{4} \frac{\sqrt{2} \sqrt{\alpha^4 - 4\epsilon \alpha^2 + 2\beta^2 (\alpha p - 1)}}{\alpha^2 \sqrt{\alpha p - 1}}, \frac{2\alpha p}{\sqrt{2\alpha p - 2}}\right)$$
(23)

where $P(\nu, u, x)$ and $Q(\nu, u, x)$ are the associated Legendre functions of the first and second kind, respectively.

To find the energy spectrum, let us use the annihilation and creation operators [36] and [37]. By substitution of

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^{\dagger} + a)$$
 , $p = i\sqrt{\frac{m\hbar\omega}{2}}(a^{\dagger} - a)$, (24)

in Eq. (3) we can write the Hamiltonian of generalized harmonic oscillator as

$$H = H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2,$$
 (25)

and then obtain the spectrum of oscillator, $E_n = \langle H \rangle_n$, as

$$\epsilon_n = 2n + 1 + 5\alpha^2 \gamma^2 [8\alpha^2 \gamma^2 (n + \frac{1}{2})(n^2 + n + \frac{3}{2}) + 3(n^2 + n + \frac{1}{2})],$$
(26)

where

$$\varepsilon_n = \frac{E_n}{\frac{\hbar\omega}{2}}$$
 , $\gamma = \sqrt{\frac{m\hbar\omega}{2}}$

4 Generalized Coherent States

Within the context of classical mechanics, a physical system is described by states which are points of its phase space. In quantum mechanics, the system is described by states which are vectors in a Hilbert space. There exist superpositions of quantum states which have many features (properties or dynamical behaviors) analogous to those of their classical counterparts: they are the so-called "coherent states". Coherent states were introduced by Schrödinger in 1926 [38] while he was studying the one-dimensional harmonic oscillator system. These states were rediscovered by Glauber [39] and Klauder [40] at the beginning of 1960s. The phrase "Coherent States" was proposed by Glauber in the context of quantum optics. Glauber found them while he was studying the electromagnetic correlation function. Indeed, these states are superpositions of Fock states of quantized electromagnetic field that, up to a complex factor, are not modified by the action of photon annihilation operators. He also realized that these states have the interesting property of minimizing the uncertainty Heisenberg relation. Thus, one could say that these states are the quantum states with the closest behavior to a classical system. Coherent states are localized wave packets in position and momentum spaces and in time are not broadening, in fact remain coherent. Although there are many ways to construct coherent states, in this paper we're looking for Klauder's approach and use the version of the generalized Heisenberg algebra [41]. So, for constructing the standard coherent states of the harmonic oscillator, we build a state which is an eigenstate of the annihilation operator of the generalized Heisenberg algebra. If $|\lambda\rangle$ be a coherent state, it can be described as an eigenstate of the annihilation operator

$$a|\lambda\rangle = \lambda|\lambda\rangle. \tag{27}$$

With expanding $|\lambda\rangle$ in terms of constant states $|n\rangle$, we can write coherent states in the following form

$$|\lambda\rangle = N(\lambda) \sum_{n=0}^{\infty} \frac{\lambda^n}{N_{n-1}!} |n\rangle, \qquad (28)$$

where $N(\lambda)$ is the normalization coefficient, by definition $N_n! \equiv N_0 N_1 \dots N_n$ and by consistency $N_{-1}! \equiv 1$. It is important to note that Klauder's coherent states should satisfy the following minimal set of conditions

I) Normalizability

$$<\lambda |\lambda>=1,$$

II) Continuity in the label

$$|\lambda - \lambda'| \to 0$$
 ; $||\lambda > -|\lambda' > || \to 0$,

III) Completeness

$$\int d^2 \lambda \, \omega(\lambda) |\lambda \rangle < \lambda| = 1.$$

Since the aforesaid approach implies $N_{n-1}^2 = \alpha_n - \alpha_0$ and $\alpha_n = \varepsilon_n$, using Eq. (26) we obtain

$$N_{n-1}^{2} = n \left\{ 2 + 5\alpha^{2}\gamma^{2} \left[8\alpha^{2}\gamma^{2} (n^{2} + \frac{3}{2}n + 2) + 3(n+1) \right] \right\}.$$
 (29)

To satisfy the normalizability condition and using Eq. (28), normalization coefficient can be written as

$$N^{2}(|\lambda|) = \frac{1}{\sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(N_{n-1}!)^{2}}}.$$
(30)

Now, we can study the behavior of normalization coefficient by depicting of $N(|\lambda|)$ for several values of $\beta \equiv \alpha \gamma$, see Fig. 1. As the figure shows, for $\alpha \to 0$ (harmonic oscillator without GUP) $N(|\lambda|)$ goes to $e^{-\frac{|\lambda|^2}{2}}$.

In the absence of GUP and in the ordinary quantum mechanics ($\alpha \rightarrow 0$), the probability distribution of photons in a coherent state is given by Poisson distribution

$$P(n, |\lambda|) = | < n |\lambda > |^2 = \frac{e^{-|\lambda|}}{n!} |\lambda|^n.$$
(31)

Now, in the presence of minimal length and maximal momentum, the probability is no longer Poissonian, namely

$$P(n, |\lambda|, \alpha) = |\langle n|\lambda \rangle|^2 = N^2(|\lambda|) \frac{|\lambda|^{2n}}{(N_{n-1}!)^2}.$$
(32)

In Fig. 2, the schematic behavior of the probability distribution for the GUP-corrected harmonic oscillator is shown for several values of β . As the figure shows, for $\alpha \to 0$ (harmonic oscillator without GUP) P tends to Poisson distribution.



Figure 1: Normalization Coefficient for the GUP-Corrected Harmonic Oscillator.



Figure 2: The Probability Distribution for the GUP-Corrected Harmonic Oscillator.

5 Summary and Conclusion

According to the existence of non-zero uncertainty in the position and momentum, various physical concepts need to review, which one of them is the coherence. So, in this paper, we reviewed the formulation of the generalized uncertainty principle and also Hilbert space representation of quantum mechanics in the presence of both minimal observable length and maximal observable momentum. Then, we have obtained the energy eigenfunctions and spectrum of energy for a generalized harmonic oscillator in the context of GUP which implies the both mentioned cutoffs. We showed that because of the GUP effects, there is a complex mass dependence in energy spectrum of oscillator. Afterwards, we have studied the coherent states of the generalized harmonic oscillator. Though, there is no difference in the definition of coherent states in the GUP framework, there are some considerable implications due to the gravitational effects. Therefore, we investigated the general behavior of normalization coefficient and probability distribution in terms of GUP parameter.

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