Decoherence effects in Bose-Einstein condensate interferometry
I. General Theory

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Abstract

The present paper outlines a basic theoretical treatment of decoherence and dephasing effects in interferometry based on single component Bose-Einstein condensates in double potential wells, where two condensate modes may be involved. Results for both two mode condensates and the simpler single mode condensate case are presented. The approach involves a hybrid phase space distribution functional method where the condensate modes are described via a truncated Wigner representation, whilst the basically unoccupied non-condensate modes are described via a positive P representation..The Hamiltonian for the system is described in terms of quantum field operators for the condensate and non-condensate modes. The functional Fokker-Planck equation for the double phase space distribution functional is derived. Equivalent Ito stochastic equations for the condensate and non-condensate fields that replace the field operators are obtained, and stochastic averages of products of these fields give the quantum correlation functions that can be used to interpret interferometry experiments. The stochastic field equations are the sum of a deterministic term obtained from the drift vector in the functional Fokker-Planck equation, and a noise field whose stochastic properties are determined from the diffusion matrix in the functional Fokker-Planck equation. The stochastic properties of the noise field terms are similar to those for Gaussian-Markov processes in that the stochastic averages of odd numbers of noise fields are zero and those for even numbers of noise field terms are the sums of products of stochastic averages associated with pairs of noise fields. However each pair is represented by an element of the diffusion matrix rather than products of the noise fields themselves, as in the case of Gaussian-Markov processes. The treatment starts from a generalised mean field theory for two condensate modes, where generalised coupled Gross-Pitaevskii equations are obtained for the modes and matrix mechanics equations are derived for the amplitudes describing possible fragmentations of the condensate between the two modes. These self-consistent sets of equations are derived via the Dirac-Frenkel variational principle. Numerical studies for interferometry experiments would involve using the solutions from the generalised mean field theory in calculations for the stochastic fields from the Ito stochastic field equations.

1. Introduction

The creation of Bose-Einstein condensates (BEC) in cold atomic gases has enabled the realisation of a controllable quantum system on a macroscopic scale. With all bosons occupying the same single particle state (or mode) the BEC exhibits coherence somewhat analogous to the coherence for an idealised single mode laser and interference effects were soon observed [1], [2]. Interferometry using BECs was a natural outcome, and much research centres around developing BEC interferometric systems, motivated not only by wishing to study coherence, interference and entanglement in macroscopic systems but also because of their potential applications for precision measurement, including the development of BEC interferometry for measurements at the Heisenberg limit [3], [4], [5], [6], [7]. Experiments demonstrating precision beyond the standard quantum limit have recently been reported [8], [9]. Reviews covering general aspects of BEC interferometry include [10], [11], [12].

Interferometry with BECs is a quantum effect. In its simplest form quantum interferometry essentially involves transitions between an initial prepared state and a final measured state for the interferometer system, where the overall transition probability amplitude for transitions is split into two partial amplitudes associated with different intermediate states, which are then recombined. The two amplitudes must remain coherent but depend differently on the feature being measured. A variety of such features can produce interferometric effects, ranging from a transition frequency between states of interest to an asymmetry in a trapping potential due to gravity effects. The partial amplitudes for the differing intermediate states may result from various types of time evolution, including free evolution stages and interaction stages, where the system is subjected to external classical fields. As the feature changes, constructive and destructive interference between the partial amplitudes results, leading to the changes in measurement probability for the final state.

In the case of interferometry with single atoms, the review by Cronin *et al* [12] outlines how *Ramsey interferometry* can be described in these terms. Here the interferometric system is a two level atom with internal states $|a\rangle$, $|b\rangle$, the first being the initial state and the second is the final state. The feature that produces the interferometric effect is the transition frequency ω_{ba} and the interferometer is used to obtain a precise measurement of ω_{ba} - to use for example in an atomic clock. The atoms are in a beam with a fixed velocity and pass through two short interaction regions when a resonant classical field of pulse area $\pi/2$ couples the two internal states, turning each into different orthogonal linear superpositions of $|a\rangle$, $|b\rangle$ - say $|a\rangle \rightarrow (|a\rangle + |b\rangle)/\sqrt{2}$ and $|b\rangle \rightarrow (|a\rangle - |b\rangle)/\sqrt{2}$. Between the interaction regions the atoms undergo free evolution for time T, with $|a\rangle \rightarrow \exp(i\omega_a T)|a\rangle$ and $|b\rangle \rightarrow \exp(i\omega_b T)|b\rangle$. The states $|a\rangle$, $|b\rangle$ also act as two possible intermediate states for the process $a \rightarrow b$, and there are two distinct pathways $a \rightarrow b \rightarrow b \rightarrow b$ and $a \rightarrow a \rightarrow a \rightarrow b$ whose partial amplitudes

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interfere. In the first pathway the resonant classical field transition $a \to b$ occurs in the first step, in the second it is in the last step, and between the first and last steps free evolution occurs in different states - b for the first pathway and a for the second. The partial amplitudes are $(-1/\sqrt{2})\exp(i\omega_b T)(+1/\sqrt{2})$ for the first pathway and $(+1/\sqrt{2})\exp(i\omega_a T)(+1/\sqrt{2})$ for the second, giving a total amplitude proportional to $\sin(\omega_{ba}T/2)$ resulting from interference between the two partial amplitudes. This produces oscillations in the measurement probability, enabling ω_{ba} to be determined. Single atom *Mach-Zender interferometry* [13], [14] involving a *double well* is another case where a similar description applies. The initial state is the lowest symmetric state $|S(0)\rangle$ for an atom in a single well trap, the final state $|AS(T)\rangle$ is the lowest antisymmetric state in the same single well. The process $|S(0)\rangle \to |AS(T)\rangle$ involves splitting the single well to a slightly asymmetric double well and then recombining back to the single well during a time T. The intermediate state can be chosen as two localised states [14] for the actual double well, one $|L(T/2)\rangle$ being localised in the left well the other $|R(T/2)\rangle$ in the right well. The two pathways whose transition amplitudes interfere are $|S(0)\rangle \to |L(T/2)\rangle \to |AS(T)\rangle$ and $|S(0)\rangle \to |R(T/2)\rangle \to |AS(T)\rangle$, the overall process being driven by non-adiabatic evolution during the splitting and recombination stages. Asymmetry in the trapping potential produces the interferometric effect. In the case of single atom *Bragg interferometry* [15], [10] an atom in a zero momentum state is subjected to three Bragg pulses with pulse areas $\pi/2$, π, $\pi/2$, where each pulse involves counterpropagating photons of two slightly differing wave numbers k_λ, k_μ A two-photon off-resonant Raman process removes a photon from one of the laser beams in the Bragg pulse and adds a photon to the other beam. The momentum difference changes the atom's momentum from zero to $2\hbar k = k_\lambda + k_\mu$. For a given k the wave numbers k_λ, k_μ can be adjusted to satisfy energy as well as momentum conservation. Bragg interferometry can be described in terms of two momentum states $|p = 0\rangle$ and $|p = 2\hbar k\rangle$ for the atom. The $\pi/2$ pulses change each state.into linear combinations of these two states - say $|0\rangle \to (|0\rangle - \exp(-i\phi)\,|2\hbar k\rangle)/\sqrt{2}$ and $|2\hbar k\rangle \to (\exp(+i\phi)\,|0\rangle + |2\hbar k\rangle)/\sqrt{2}$. The π pulse changes each momentum state.into the other state - say $|0\rangle \to -\exp(-i\phi)\,|2\hbar k\rangle$ and $|2\hbar k\rangle \to \exp(+i\phi)\,|0\rangle$. Here ϕ is a phase factor for the Bragg pulse involved. For an overall process say $|0\rangle \to |0\rangle$ there are two pathways each with its own transition amplitude $|0\rangle \to |0\rangle \to |2\hbar k\rangle \to |0\rangle$ and $|0\rangle \to |2\hbar k\rangle \to |0\rangle \to |0\rangle$, the successive steps involving the pulses $\pi/2$, π, $\pi/2$ respectively. If we choose $\phi = 0$ in the first two steps and $\phi \neq 0$ in the final $\pi/2$ step, the transition probability is given by $(1 + \cos\phi)/2$, giving interferometric effects as ϕ is changed.

Ramsey, Mach-Zender and Bragg interferometry [15], [16], [10], [12] can also be carried out using BECs rather than single atoms, and a generalised version of the above approach could be used to describe these. Quantum interference in double well BEC interferometry is discussed qualitatively in [17] in terms of interfering transition amplitudes. However, since BECs involve a large number N of atoms rather than just one, there are a number of additional complexities that need to be taken into account, notably associated with the feature that macro-

scopic numbers of atoms may occupy each single particle state. Firstly, large numbers of partial transition amplitudes may now be involved in the overall process, and evaluating all the partial transition amplitudes and then recombining them becomes a formidable task. The analysis for the single atom case establishes the general point that for interferometry to occur there must be *at least two* different single particle states (or *modes*) that an atom can occupy - otherwise two or more pathways for the overall process to occur would not be available. This suggests immediately that interferometry using BECs must at least be based on a *two-mode theory*. For *single component* BECs, the two single particle states would be represented by two orthogonal, normalised spatial mode functions $\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$. Time dependences are left implicit. For double well interferometry, these could be either *localised* in each of the two potential wells or *delocalised* symmetrically or antisymmetrically over the two wells. For Bragg interferometry the two modes could be two different momentum eigenfunctions. For *two component* BECs, with internal (hyperfine) states $|F\rangle$, $|G\rangle$ the two single particle states would be represented by $\phi_F(\mathbf{r})|F\rangle$, $\phi_G(\mathbf{r})|G\rangle$, where the associated *normalised* spatial mode functions are $\phi_F(\mathbf{r})$, $\phi_G(\mathbf{r})$. The significance of two-mode theories for BECs is well recognised [18], [19] and points to the existence of *Josephson effect* [20] physics in cold quantum gases. The idea of the BEC being equivalent to a *giant spin system*, with direct linkages to angular momentum theory [21], spin squeezing [22] etc. stems from two-mode theory, as will be outlined below. For the quantum description of Ramsey, double well and Bragg interferometry with BECs however, even if each atom is restricted to one of two single particle states there are now $N + 1$ distinct ways of dividing the atoms between the two single particle states, corresponding to *Fock states* with occupancies given by $\frac{N}{2} - k$, $\frac{N}{2} + k$ with $(k = -N/2, -N/2+1, .., N/2)$ in the two modes. The Fock states can describe BECs that are *fragmented*, with two modes having macroscopic occupancy [18], [23]. Consequently, in any overall process there are a great many pathways involved, so the overall transition amplitude can contain many contributions. Having more interfering pathways raises both the possibility of sharper interferometric effects [24] but also the possibility that effects can be degraded depending on how the phases and magnitudes of the.partial amplitudes are related. These sorts of effects are also familiar from multiple slit optical interference. Secondly, the need to consider steps in the process where the intermediate states already have many atoms occupying each of the two single particle states raises the possibility of *bosonic enhancement* of contributions to the partial transition amplitude from the step involved. These sorts of effects are familiar from the theory of lasers and in super-radiance. The effects could occur because two-mode BECs are like a giant spin system rather than a collection of independent atoms, and implies that the simple analysis described above for single atoms is no longer valid. However, a closer analysis (see [25]) suggests that bosonic enhancement and super-radiance effects are not in fact present. Thirdly, the evolution is not as simple as in the single atom case, since collisions between the atoms need to be taken into account. Even with only two single particle states allowed, *dephasing* between the contributing amplitudes can occur - which tend to degrade interferometric features but which

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may also produce collapse and revival effects [26], [6]. Also, even if the BEC is close to zero temperature, collisions could remove atoms from the macroscopically occupied pair of single particle states and deposit them into previously unoccupied higher energy *thermal states*. The unoccupied thermal states act as a kind of environment (or reservoir) so the system defined in terms of the two macroscopically occupied single particle states suffers *decoherence*. Collective phonon-like states of the BEC called *Bogoliubov excitations* [27], [18], [23] can be created. These processes again generally result in degradation of interferometric effects. However, some aspects of the interferometric process will still be similar to the single atom interferometry case. These include the presence of interaction regions in which the BEC is subjected to external pulsed classical fields with pulse areas $\pi/2$, π that couple internal states, the effect of Bragg pulses that change the momentum of each atom, the presence of asymmetries in trapping potentials that confine the BEC, as well as periods of free evolution of the BEC - though now of course collisions need to be taken into account. The difference is though a more elaborate theory is needed to treat quantum interferometry in BECs allowing for all these effects.

Theories of BEC interferometry that take into account the many body nature of the system are of various levels of sophistication [19] depending on the range of effects taken into account. The Hamiltonian is often expressed in terms of *field operators*. For single component BECs the field annihilation operator $\widehat{\Psi}(\mathbf{r})$ destroys a bosonic atom at position $\mathbf{r}$, whilst for two component BECs the field annihilation operator $\widehat{\Psi}_a(\mathbf{r})$ ($a = F, G$) destroys a bosonic atom with internal state $|a\rangle$ at position $\mathbf{r}$. Interferometry experiments are generally interpreted in terms of *quantum correlation functions*, which are expectation values of products of field annihilation operators with the associated field creation operators, and are related to bosonic many atom position measurements [28]. Actual measurements of quantum correlation functions may be made via scattering a weak probe beams (atoms, light) off the system, [29]. If boson-boson interactions were absent and the BEC isolated from the environment, idealised forms of the quantum correlation functions would result, with clearly visible interferometric effects. Even where external environmental effects are absent, the internal boson-boson interactions can still result in dephasing (associated with interactions within the condensate modes) and decoherence effects (associated with interactions causing transitions from the condensate modes) that degrade the interference pattern. Many treatments of BEC interferometry are based on the simplest assumption, namely that during the interferometric process the condensate is *unfragmented*, with all bosons occupying the *same* single particle state $|\chi\rangle$. This situation is a special case of a two mode theory, with the occupied single particle state written as a linear superposition of the two modes. For single component BEC interferometry linear combinations of the form $\langle \mathbf{r} \,|\, \chi \rangle = \chi(\mathbf{r}) = \alpha_1 \phi_1(\mathbf{r}) + \alpha_2 \phi_2(\mathbf{r})$ are involved, for two component BEC interferometry superpositions $\langle \mathbf{r} \,|\, \chi \rangle = \chi_F(\mathbf{r}) \,|F\rangle + \chi_G(\mathbf{r}) \,|G\rangle$ of the two internal states occur. Equations for the spatial wave functions associated with these single particle states can be obtained using variational principles [30], [31]. For

the single component case the well-known *Gross-Pitaevskii equation* [32], [33] applies for the so-called *condensate wave function* $\chi(\mathbf{r})$, for the two component case *coupled Gross-Pitaevskii equations* [34], [35] apply for the condensate wave functions $\chi_F(\mathbf{r})$, $\chi_G(\mathbf{r})$ associated with the two internal states. The Gross-Pitaevskii equations are *non-linear*, with collision effects occuring via *mean field* terms. Treatments of BEC interferometry based on assuming the condensate is unfragmented include [36], [37] for the single component case and [38], [39] for the two component case.

However, there are two distinct single particle states each boson could occupy, and for N bosons the $N+1$ dimensional state space for two mode theories allows for more general quantum states that are *fragmented*, with macroscopic occupancy of two single particle states. The basis states can be chosen as Fock states $\left| \frac{N}{2}, k \right\rangle$ ($k = -N/2, .., N/2$) in which $\frac{N}{2} - k$ bosons occupy one of the two single particle states ($\phi_1(\mathbf{r})$ for the single component case, $\phi_F(\mathbf{r}) |F\rangle$ for the two component case) and $\frac{N}{2} + k$ bosons occupy the other two single particle states ($\phi_2(\mathbf{r})$ for the single component case, $\phi_G(\mathbf{r}) |G\rangle$ for the two component case). Each Fock state is a fragmented state, with definite numbers $\frac{N}{2} \mp k$ of bosons respectively in the two modes. In two mode theory the general quantum state of the N boson system is written as a superposition of the Fock states with general amplitudes b_k. The unfragmented states are just special cases called *binomial states* since the amplitudes b_k are determined from binomial coefficients. The Dirac-Frenkel variational principle [30], [31] can be used to obtain *matrix mechanics* equations for the $N+1$ general amplitudes b_k and *generalized Gross-Pitaevskii equations*.for the two mode functions ($\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$ for the single component case, $\phi_F(\mathbf{r})$, $\phi_G(\mathbf{r})$ for the two component case). The $N+1$ amplitude equations describe the system evolution amongst the possible Fock states, and involve Fock state Hamiltonian and rotation matrix elements which depend on the two mode functions. The two coupled Gross-Pitaevskii equations are again non-linear in the mode functions due to collision terms - which occur via *generalised* mean fields - and involve the trap potential, with an additional intercomponent coupling term in the two component case. They contain as coefficients one and two body correlation functions that depend quadratically on the amplitudes, and which reflect the relative importance of the different Fock states during the interference process. The combined amplitude and mode equations are *self-consistent*, and are more general than the equations for the unfragmented BEC case. It should be noted however that other authors [40], [41], [42] define the condensate mode functions via a diferent approch, namely in terms of the eigenfunctions of the first order quantum correlation function that have macroscopic eigenvalues. This approach is discussed below in Section 3. Two mode theories similar to the present treatment have previously been developed for single component BECs with two orthogonal spatial modes (such as in double-well interferometry) [43], [44], [17], [45], [46], [47], [48] and fragmentation effects shown in [46], [47], [48]. Two mode theories for the two component case have been presented in [49], [50] and elsewhere **(author?)** [25]. Two mode theories incorporate dephasing effects associated with transfers of bosons between the two modes, but decoherence effects and Bogoliubov excitations are outside

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the scope of the theory. Both the general two-mode theory and the single mode theory are referred to as *mean field theories*, since collisional effects occur via mean fields.

To allow for decoherence and Bogoliubov excitations the theory must include large numbers of non-condensate modes, which are modes with very small occupancy. Bogoliubov theory is perturbation theory in the interaction between condensate and non-condensate modes, and treatments of Bogoliubov excitations for BEC interferometry have been made [41] by adapting general BEC Bogoliubov theory [51], [40], [52], [42]to treat two-component BECs. Another approach that could be applied to BEC interferometry is a master equation method [53], [54], in which a condensate density operator is defined and a master equation is derived allowing for interactions with non-condensate modes, which constitute a reservoir. The quantum state is now non-pure so a density operator is needed to describe the system. The difficulty with this method is that it is hard to evaluate the non-condensate contributions to quantum correlation functions. A further approach could be based on the Heisenberg equation method that have been applied in numerous many-body theory problems. Heisenberg equations for field operators and products of field operators are derived, and taking the expectation values with the initial density operator results in a heirarchy of coupled equations for quantum correlation functions. An ansatz (such as assuming that a suitable high order correlation function factorises) produces a truncated set of coupled equations from which correlation functions of the required order can be calculated. The problem with this method is that it is hard to confirm the validity of the ansatz. In view of there being very large numbers of modes, phase space theories have also been developed with the density operator represented by a quasi-distribution functional in a phase space [55]. Quantum correlation functions are then expressed as functional integrals in the phase space, involving products of the distribution functional with the several field functions that replace the field operators. The Liouville-von Neumann equation for the density operator is replaced by a functional Fokker-Planck equation (FFPE) for the distribution functional. Finally, the FFPE are finally replaced by coupled Ito stochastic equations (c-number Langevin equations) for the field functions, where the Ito equations contain deterministic and random noise terms - identifiable from the FFPE. Stochastic averages of the field functions then give the quantum correlation functions. Phase space distribution functional treatments were originally developed to treat problems in quantum optical physics [56], [57], [58], [59], [60], but have since been adapted for BECs. There are several different phase space theories that have or could be used to treat BEC interferometry, depending on the nature of the distribution functional chosen to represent the density operator. The positive P representation has been used by [61] to treat spin squeezing in two component BECs. However, because most atoms will be in one or two highly occupied modes and these bosons can be treated approximately in terms of mean field theories, a more natural representation to use is the truncated Wigner representation. Such theories have been developed [55] and applied to BEC interferometry [62], [63]. In the truncated Wigner FFPE there are no second order functional derivatives, so there are no random noise

terms in the Ito equations. Quantum noise is embodied in the initial state, and Bogoliubov equations are used to describe this state. Based on the truncated Wigner representation, stochastic modifications of the Gross-Pitaevskii equation to allow for the effects due to non-condensate modes have been derived for the case where the condensate modes have macroscopic occupancy, and these methods could be applied to BEC interferometry. These approaches include the Projected Gross-Pitaevskii equation method [64], [65] and the Stochastic Gross-Pitaevskii equation theory [66], [67]. A review of these methods is given in [68]. In developing a quantum kinetic theory of BECs, Gardiner and Zoller [53], [54] divided the field operator for the bosonic system as a sum of condensate and non-condensate mode contributions. An alternative treatment also based on distinguishing condensate and non-condensate modes is the hybrid representation, with the highly occupied condensate modes described via a truncated Wigner representation (since the bosons in condensate modes behave like a classical mean field), whilst the basically unoccupied non-condensate modes are described via a positive P representation (these bosons should exhibit quantum effects). Such an approach has been developed by [69], [70], [71] and in the present paper. Finally, a more elaborate phase space treatment of BECs called the Gaussian quantum operator representation has been formulated [72] and could be applied to BEC interferometry. Pairs of bosonic annihilation, creation operators as well as single operators are represented by c-numbers in the phase space distribution function. The approach is based on representing the density operator via Gaussian rather than just coherent state projectors, as applies for the simpler phase space theories.

As well as being suitable for studying macroscopic decoherence and dephasing effects, interferometry with Bose-Einsten condensates is closely linked to another fundamental feature of the quantum physics in macroscopic systems - *entanglement*. Entanglement is linked to several important issues such as the EPR paradox, Bell inequalities and Schrodinger cats. A number of papers have discussed entanglement for two mode macroscopic systems, including [73], [74], [75], [76], [77], [78], [7] and [79]. Reviews include [80], [81]. Measures of entanglement are more straightforward for bi-partite systems such as bosonic systems based on two modes, where the two modes constitute the two subsystems. The *entropy of mode entanglement* is a useful measure, being the difference in entropy between that for the original state and that associated with the reduced density operator describing a sub-system, and thus related to the change of *quantum information*. The connection to interferometry can be seen with a simple example [74]. If $\widehat{a}$, $\widehat{b}$ are the annihilation operators for the modes a, b then the pure quantum state for the N boson system given in terms of the corresponding creation operators and the vacuum state $|0\rangle$ as

$$|\Phi\rangle_E = \frac{1}{\sqrt{N!}} \left(\frac{\widehat{a}^\dagger + \widehat{b}^\dagger}{\sqrt{2}} \right)^N |0\rangle = \left(\frac{1}{\sqrt{2}} \right)^N \sum_{n=0}^{N} \sqrt{C_n^N} \, |n\rangle_a \, |N-n\rangle_b \qquad (1)$$

is an *entangled state*, being a quantum superposition of separable states $|n\rangle_a \, |N-n\rangle_b$ in which there are n bosons in mode a and $N-n$ in mode b. This state is a

binomial state, since its form is determined by binomial coefficients C_n^N. The reduced density operator for the mode a subsystem is easily found to be

$$\widehat{\rho}_a^E = \left(\frac{1}{2}\right)^N \sum_{n=0}^{N} C_n^N \, |n\rangle_a \, \langle n|_a \tag{2}$$

which is clearly a mixed state and the entropy of entanglement is non-zero. Another pure state for the N boson system is

$$|\Phi\rangle_{NE} = \frac{1}{\sqrt{N!}} \left(\widehat{a}^\dagger\right)^N |0\rangle = |N\rangle_a \, |0\rangle_b \tag{3}$$

which is a *non-entangled state*, being a separable product of the states $|N\rangle_a$ and $|0\rangle_b$. The reduced density operator for the mode a subsystem is easily found to be

$$\widehat{\rho}_a^{NE} = |N\rangle_a \, \langle N|_a \tag{4}$$

which is clearly a pure state and the entropy of entanglement is zero. If we now consider an interferometry experiment applied to each of these two states, we will see that the entangled and non-entangled states leads to differing interferometric effects. The experiment involves applying a 50:50 beam–splitter process to each state and then measuring the number of bosons in modes a, b. The beam splitter process is associated with an evolution operator which transforms the mode annihilation operators as $\widehat{a} \rightarrow (\widehat{a} + \widehat{b})/\sqrt{2}$, $\widehat{b} \rightarrow (\widehat{a} - \widehat{b})/\sqrt{2}$. For single component BEC in a double well with modes localised in each well, such a process is associated with quantum tunneling through the potential barrier during a period short enough that collisions can be ignored. For two component BEC in a single well, the process is associated with applying a two-photon classical field during a similar short period. For the two initial states of interest the states change as

$$|\Phi\rangle_E \quad \rightarrow \quad \frac{1}{\sqrt{N!}} \left(\widehat{a}^\dagger\right)^N |0\rangle = |N\rangle_a \, |0\rangle_b$$

$$|\Phi\rangle_{NE} \quad \rightarrow \quad \frac{1}{\sqrt{N!}} \left(\frac{\widehat{a}^\dagger + \widehat{b}^\dagger}{\sqrt{2}}\right)^N |0\rangle = \left(\frac{1}{\sqrt{2}}\right)^N \sum_{n=0}^{N} \sqrt{C_n^N} \, |n\rangle_a \, |N-n\rangle_b \tag{5}$$

Measurements of the mean boson numbers in each mode give $\langle \widehat{a}^\dagger \widehat{a} \rangle = N$, $\langle \widehat{b}^\dagger \widehat{b} \rangle = 0$ for the initially entangled state and $\langle \widehat{a}^\dagger \widehat{a} \rangle = N/2$, $\langle \widehat{b}^\dagger \widehat{b} \rangle = N/2$ for the initially non-entangled state. Hence there is a difference in the interferometric results for the two cases. More generally, for an arbitrary *mixed non-entangled* state for N bosons the density operator is of the form

$$\widehat{\rho}^{NE} = \sum_{n=0}^{N} p_n \, |n\rangle_a \, \langle n|_a \otimes |N-n\rangle_b \, \langle N-n|_b \tag{6}$$

and the reduced density operator for mode a is

$$\widehat{\rho}_a^{NE} = \sum_{n=0}^{N} p_n \left| n \right\rangle_a \left\langle n \right|_a \qquad (7)$$

As there is no entropy change between the original state and the state for mode a, the entropy of entanglement is zero. In [74] it is shown that applying the beam-splitter process to this state gives a new state where again $\left\langle \widehat{a}^{\dagger}\widehat{a} \right\rangle = N/2$, $\left\langle \widehat{b}^{\dagger}\widehat{b} \right\rangle = N/2$. Hence *all* non-entangled states for for N bosons give no difference between the output measurements of boson numbers in the two modes. This contrasts the situation for entangled states, as our example has shown. Thus interferometry with BEC would be a possible measurement system for demonstrating entanglement effects.

In the present paper it will be assumed that the interferometry regime is such that at most two condensate modes have a macroscopic occupancy. The mean field theory treatment for this case is a time-dependent version of the approach in an earlier two-mode theory paper [17]. This approach leads to a set of self consistent equations for the two mode functions and for the probability amplitudes for finding the system in states with specific occupancies of the two modes. The mode equations are generalised time-dependent Gross-Pitaevskii equations involving non-linear mean field terms, and these equations include coefficients that depend on the amplitudes. The amplitude equations are matrix mechanics equations involving Hamiltonian and rotation matrix elements, that depend on the mode functions and their spatial and temporal derivatives. These self-consistent sets of equations are derived via the Dirac-Frenkel variational principle. This generalised mean field theory does allow for certain dephasing effects and for transitions between the two condensate modes. Thermal and decoherence effects are not included. For the purposes of the present paper it will be assumed that the solutions to the generalised mean field two mode theory have been obtained, and are available albeit in numerical form for applications of the present theory. Numerical solutions of equivalent equations have been published by Streltsov et al [46], [45], [47].

The present paper outlines a basic theoretical treatment of decoherence and dephasing effects in interferometry based on single component BECs in double potential wells, where we assume that only two condensate modes could have macroscopic occupancy. Results for both two mode condensates and the simpler single mode condensate case are presented. The approach involves a hybrid phase space distribution functional method where the condensate modes are described via a truncated Wigner representation, whilst the basically unoccupied non-condensate modes are described via a positive P representation [69], [70]..The Hamiltonian for the system is described in terms of quantum field operators for the condensate and non-condensate modes. The functional Fokker-Planck equation for the double phase space distribution functional is derived. Equivalent Ito stochastic equations for the condensate and non-condensate fields that replace the field operators are obtained and stochastic averages of products

of these fields give the quantum correlation functions that can be used to interpret interferometry experiments. The treatment starts from the generalised mean field theory for two condensate modes. Numerical studies for interferometry experiments would involve using the solutions from the generalised mean field theory in calculations for the stochastic fields from the Ito stochastic field equations.

Previous papers [56], [57], [58], [60], [55] using distribution functional and stochastic field approaches only contain brief explanations of the method, so the present paper is aimed at a more complete exposition. In Section 2 the Hamiltonian for the single component bosonic system is described in terms of field operators. The field operators are the sum of condensate and non-condensate mode contributions. The Hamiltonian is decomposed into contributions scaling with decreasing powers of $\sqrt{N}$, and within the weak interaction regime some terms are discarded, leaving a Hamiltonian which allows for Bogoliubov excitations. Certain linear coupling terms involving both condensate and non-condensate field operators are written in a new form based aroud the condensate mode functions as obtained from time-dependent Gross-Pitaevskii equations. In Section 3 phase space distribution functionals of a hybrid type are introduced (Wigner for condensate fields, P+ for non-condensate fields) starting with the characteristic functional, and quantum correlation functions (symmetric ordering for condensate fields, normal ordering for non-condensate fields) are expressed in terms of phase space functional integrals, with field functions replacing the field operators and the distribution functional replacing the density operator. The justification for these phase space functional integral results is carefully outlined. Correspondence rules and functional Fokker-Planck equations are obtained in Section 4, the key steps in the derivation of the correspondence rules and functional Fokker-Planck equations being explicitly covered. The derivation of the equivalent Ito stochastic field equations is fully set out in Section 5. Results for both two mode and single mode condensates are presented. The single mode condensate results are compared with equations recently presented in [71]. The paper is summarised in Section 6.

Online supplementary material and a website version of this paper [82] contains details for the derivations of results in this paper which are too lengthy to present in the journal version. Quantities involved in the two-mode theory equations are listed in Appendix A. In Appendix B the key ideas of functional calculus involving c-number functions are outlined. Results for quantum correlation functions are derived in Appendix C. The derivation of the correspondence rules and their application to deriving the functional Fokker-Planck equation is given in Appendix D and Appendix E respectively. The Ito stochastic equations details are in Appendix F.

2. Hamiltonian and field operators

In this section we describe the bosonic system in terms of field operators. The field operators are written as the sum of two contributions, one associated with the condensate modes, the other with the non-condensate modes. The Hamiltonian is introduced for the single component bosonic system within the zero range approximation for the boson-boson interactions. The situation is restricted in this paper to the weak interaction regime, and the Hamiltonian decomposed into contributions that scale with decreasing powers of $\sqrt{N}$, where N is the number of bosons. After discarding the two smallest contibutions that scale as $(\sqrt{N})^{-1}$ and $(\sqrt{N})^{-2}$, we are left with the Bogoliubov Hamiltonian [51], [40], [52], [42]. The condensate in this work dealing with applications in double well interferometry is assumed to involve at most two modes, and the Dirac-Frenkel principle [30], [31] is used to obtain two coupled generalised Gross-Pitaevskii equations for the two time-dependent mode functions. For the single condensate mode situation the same approach gives the standard Gross-Pitaevskii equation for the mode function. Our previous two mode theory [17] yielded adiabatic mode functions, rather than the time-dependent modes used here. Results from the Gross-Pitaevskki equations are then used to simplify one of the terms in the Bogoliubov Hamiltonian, thereby enabling functional Fokker-Planck equations to be derived.

2.1. Field Operators for Condensate, Non-Condensate Modes

For the application to double-well BEC interferometry most of the bosons occupy one or maybe two modes, and that all the other modes are essentially unoccupied. The two modes with macroscopic occupancy will be referred to as the condensate modes, the remaining modes are non-condensate modes. These physically based distinctions between the two types of modes will be embodied in the theoretical treatment, and it will be convenient to use two different phase space methods for the condensate and non-condensate bosons. In the present paper it is assumed that the interferometry regime is such that at most two condensate modes have a macroscopic occupation.

The *field operators* can be expanded in mode functions

$$\widehat{\Psi}(\mathbf{r}) = \sum_k \widehat{a}_k \phi_k(\mathbf{r}) \tag{8}$$

$$\widehat{\Psi}^\dagger(\mathbf{r}) = \sum_k \phi_k^*(\mathbf{r})\widehat{a}_k^\dagger \tag{9}$$

where the mode functions are orthonormal

$$\int d\mathbf{r}\phi_i^*(\mathbf{r})\phi_j(\mathbf{r}) = \delta_{ij} \tag{10}$$

Throughout this paper both the mode functions and their accompanying annihilation, creation operators are time dependent in general, but for simplicity of

notation the time dependence is usually left implicit. Note however that the field operators $\widehat{\Psi}(\mathbf{r})$ and $\widehat{\Psi}^{\dagger}(\mathbf{r})$ are always time independent.

In the mode expansion we will assume that there is a cut-off at some large mode number K (momentum cut-off). This is to be consistent with using the zero range approximation in the Hamiltonian. Accordingly the completeness expression for the mode functions does not give the ordinary delta function but a restricted delta function $\delta_K(\mathbf{r}, \mathbf{r}')$ which is no longer singular when $\mathbf{r} = \mathbf{r}'$.

$$\sum_k \phi_k(\mathbf{r})\phi_k^*(\mathbf{r}') = \delta_K(\mathbf{r}, \mathbf{r}') \tag{11}$$

Accordingly although the annihilation, creation operators satisfy the standard bosonic commutation rules, the field operators satisfy modified rules for which the non-zero results are

$$
\begin{aligned}
[\widehat{a}_k, \widehat{a}_l^{\dagger}] &= \delta_{kl} \\
[\widehat{\Psi}(\mathbf{r}), \widehat{\Psi}^{\dagger}(\mathbf{r}')] &= \delta_K(\mathbf{r}, \mathbf{r}')
\end{aligned}
\tag{12}
$$

In obtaining these rules those for the annihilation, creation operators are treated as fundamental and those for the field operators then derived. If the cut-off is made very large then the restricted delta function approaches the ordinary delta function.

To exploit the distinction between condensate modes with a macroscopic occupancy and non-condensate mode the field operator is written as the sum of a *condensate* term and a *non-condensate* term. In the *two-mode approximation* it is assumed that there are two condensate modes that may have macroscopic occupancy, in the standard *single mode approximation* only one.

For the *two mode* case

$$
\begin{aligned}
\widehat{\Psi}(\mathbf{r}) &= \widehat{\Psi}_C(\mathbf{r}) + \widehat{\Psi}_{NC}(\mathbf{r}) \tag{13} \\
\widehat{\Psi}_C(\mathbf{r}) &= \widehat{a}_1 \phi_1(\mathbf{r}) + \widehat{a}_2 \phi_2(\mathbf{r}) \tag{14} \\
\widehat{\Psi}_C^{\dagger}(\mathbf{r}) &= \phi_1^*(\mathbf{r})\widehat{a}_1^{\dagger} + \phi_2^*(\mathbf{r})\widehat{a}_2^{\dagger} \tag{15} \\
\widehat{\Psi}_{NC}(\mathbf{r}) &= \sum_{k \neq 1,2}^{K} \widehat{a}_k \phi_k(\mathbf{r}) \tag{16} \\
\widehat{\Psi}_{NC}^{\dagger}(\mathbf{r}) &= \sum_{k \neq 1,2}^{K} \phi_k^*(\mathbf{r})\widehat{a}_k^{\dagger} \tag{17}
\end{aligned}
$$

where the condensate is described via the two modes $\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$ and the non-condensate via the remaining modes $\phi_k(\mathbf{r})$, which are cut off for momenta greater than $K \sim \hbar/a$ where a is the distance scale of the.short range boson-boson interaction. In view of the orthogonality of the condensate and non-condensate modes, the contributions to the field operator commute. For the condensate

and non-condensate field operator components we have the following non-zero results

$$
\begin{aligned}
[\widehat{\Psi}_C(\mathbf{r}), \widehat{\Psi}_{NC}^\dagger(\mathbf{r})] &= 0 \\
[\widehat{\Psi}_C(\mathbf{r}), \widehat{\Psi}_C^\dagger(\mathbf{r}')] &= \phi_1(\mathbf{r})\phi_1^*(\mathbf{r}') + \phi_2(\mathbf{r})\phi_2^*(\mathbf{r}') \\
&= \delta_C(\mathbf{r}, \mathbf{r}') \qquad\qquad (18) \\
[\widehat{\Psi}_{NC}(\mathbf{r}), \widehat{\Psi}_{NC}^\dagger(\mathbf{r}')] &= \sum_{k\neq 1,2} \phi_k(\mathbf{r})\phi_k^*(\mathbf{r}') \\
&= \delta_{NC}(\mathbf{r}, \mathbf{r}') \qquad\qquad (19)
\end{aligned}
$$

The quantities $\delta_C(\mathbf{r}, \mathbf{r}')$ and $\delta_{NC}(\mathbf{r}, \mathbf{r}')$ act as restricted Dirac delta functions rather than ordinary delta functions, in that for functions $\psi_C(\mathbf{r})$ and $\psi_{NC}(\mathbf{r})$ only involving condensate or non-consensate modes respectively (and $\psi_C^+(\mathbf{r})$ and $\psi_{NC}^+(\mathbf{r})$ only involving their complex conjugates), we have

$$
\begin{aligned}
\psi_C(\mathbf{r}) &= \alpha_1\phi_1(\mathbf{r}) + \alpha_2\phi_2(\mathbf{r}) & \psi_C^+(\mathbf{r}) &= \phi_1^*(\mathbf{r})\alpha_1^+ + \phi_2^*(\mathbf{r})\alpha_2^+ & (20) \\
\psi_{NC}(\mathbf{r}) &= \sum_{k\neq 1,2} \alpha_k\phi_k(\mathbf{r}) & \psi_{NC}^+(\mathbf{r}) &= \sum_{k\neq 1,2} \phi_k^*(\mathbf{r})\alpha_k^+ & (21) \\
\psi_C(\mathbf{r}) &= \int d\mathbf{r}'\, \delta_C(\mathbf{r}, \mathbf{r}')\psi_C(\mathbf{r}') & \psi_C^+(\mathbf{r}) &= \int d\mathbf{r}'\, \psi_C^+(\mathbf{r}')\delta_C(\mathbf{r}', \mathbf{r}) & \\
\psi_{NC}(\mathbf{r}) &= \int d\mathbf{r}'\, \delta_{NC}(\mathbf{r}, \mathbf{r}')\psi_{NC}(\mathbf{r}') & \psi(\mathbf{r}) &= \int d\mathbf{r}'\, \psi_C^+(\mathbf{r}')\delta_C(\mathbf{r}', \mathbf{r}) & (22)
\end{aligned}
$$

Clearly

$$
\delta_K(\mathbf{r}, \mathbf{r}') = \delta_C(\mathbf{r}, \mathbf{r}') + \delta_{NC}(\mathbf{r}, \mathbf{r}') \qquad\qquad (23)
$$

These features involving restricted delta functions will be useful in deriving the functional Fokker-Planck equation.

In the *single mode* case the condensate, non-condensate field operators and restricted delta functions are now given by

$$
\begin{aligned}
\widehat{\Psi}_C(\mathbf{r}) &= \widehat{a}_1\phi_1(\mathbf{r}) & \widehat{\Psi}_C^\dagger(\mathbf{r}) &= \phi_1^*(\mathbf{r})\widehat{a}_1^\dagger & (24) \\
\widehat{\Psi}_{NC}(\mathbf{r}) &= \sum_{k\neq 1}^{K} \widehat{a}_k\phi_k(\mathbf{r}) & \widehat{\Psi}_{NC}^\dagger(\mathbf{r}) &= \sum_{k\neq 1}^{K} \phi_k^*(\mathbf{r})\widehat{a}_k^\dagger & (25) \\
\delta_C(\mathbf{r}, \mathbf{r}') &= \phi_1(\mathbf{r})\phi_1^*(\mathbf{r}') & \delta_{NC}(\mathbf{r}, \mathbf{r}') &= \sum_{k\neq 1} \phi_k(\mathbf{r})\phi_k^*(\mathbf{r}') & (26)
\end{aligned}
$$

with the time dependences of the mode functions and annihilation, creation operators left understood as usual. With obvious modifications, (18) - (23) also apply in the single mode case.

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2.2. Bogoliubov Hamiltonian

The full *Hamiltonian* in terms of field operators is given by

$$\widehat{H} = \int d\mathbf{r}(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}(\mathbf{r})^\dagger \cdot \nabla\widehat{\Psi}(\mathbf{r}) + \widehat{\Psi}(\mathbf{r})^\dagger V\widehat{\Psi}(\mathbf{r}) + \frac{g}{2}\widehat{\Psi}(\mathbf{r})^\dagger\widehat{\Psi}(\mathbf{r})^\dagger\widehat{\Psi}(\mathbf{r})\widehat{\Psi}(\mathbf{r}))$$

$$\tag{27}$$

$$= \widehat{K} + \widehat{V} + \widehat{U} \tag{28}$$

the sum of a kinetic energy, trap potential energy and boson-boson interaction energy terms. As usual the zero range approximation is made with $g = 4\pi\hbar^2 a_S/m$, where a_S is the s-wave scattering length.

The condensate mode occupation is of order the total boson number N. For bosons in a trap of frequency ω, with harmonic oscillator length scale $a_0 = \sqrt{(\hbar/2m\omega)}$, the density is of order $\rho = N/(a_0)^3$. In the *weak interaction regime* [40] we have $\rho(a_S)^3 \ll 1$, or

$$N(\frac{a_S}{a_0})^3 \ll 1 \tag{29}$$

For Rb^{87} in a trap with $\omega = 2\pi.58$ s^{-1} we have $a_0 = 1$ μm and $a_S = 5$ nm, so that the weak interaction regime applies for reasonably large boson numbers $N \ll 10^7$. Also, as has been shown [40], it is possible to consider a situation for the weak interaction regime where Na_S/a_0 and $k_BT/\hbar\omega$ are kept constant but with N becoming very large whilst a_S remains finite so that

$$gN = g_N \tag{30}$$

where g_N is constant. This can be achieved by decreasing the trap frequency.

In the weak interaction regime and with $g = g_N/N$ it is convenient to write the Hamiltonian as the sum of five terms in decreasing powers of $\sqrt{N}$, based on using Eq.(13) and assuming the condensate operators scale like $\sqrt{N}$. We can then express the Hamiltonian in the form

$$\widehat{H} = \widehat{H}_1 + \widehat{H}_2 + \widehat{H}_3 + \widehat{H}_4 + \widehat{H}_5 \tag{31}$$

where

$$\widehat{H}_1 = \int d\mathbf{r}(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_C^\dagger(\mathbf{r}) \cdot \nabla\widehat{\Psi}_C(\mathbf{r}) + \widehat{\Psi}_C^\dagger(\mathbf{r})V\widehat{\Psi}_C(\mathbf{r})$$

$$+\frac{g_N}{2N}\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) \tag{32}$$

$$\widehat{H}_2 = \int d\mathbf{r}(\widehat{\Psi}_{NC}(\mathbf{r})^\dagger \left\{ -\frac{\hbar^2}{2m}\nabla^2\widehat{\Psi}_C(\mathbf{r}) + V\widehat{\Psi}_C(\mathbf{r}) + \frac{g_N}{N}\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) \right\}$$

$$+\left\{ -\frac{\hbar^2}{2m}\nabla^2\widehat{\Psi}_C^\dagger(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})^\dagger V + \frac{g_N}{N}\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) \right\} \widehat{\Psi}_{NC}(\mathbf{r}))(33)$$

$$\widehat{H}_3 = \int d\mathbf{r}(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\cdot\nabla\widehat{\Psi}_{NC}(\mathbf{r}) + \widehat{\Psi}^\dagger_{NC}(\mathbf{r})V\widehat{\Psi}_{NC}(\mathbf{r})$$

$$+\frac{g_N}{2N}\left\{\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) + \widehat{\Psi}^\dagger_C(\mathbf{r})\widehat{\Psi}^\dagger_C(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\right\}$$

$$+\frac{g_N}{2N}\left\{4\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}^\dagger_C(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})\right\}) \tag{34}$$

$$\widehat{H}_4 = \int d\mathbf{r}\,\frac{g_N}{N}\left\{\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) + \widehat{\Psi}^\dagger_C(\mathbf{r})\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\right\} \tag{35}$$

$$\widehat{H}_5 = \int d\mathbf{r}\,\frac{g_N}{2N}\left\{\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})\right\} \tag{36}$$

The term $\widehat{H}_1$ is of order $N = (\sqrt{N})^2$ and is the Hamiltonian for the *condensate*. The term $\widehat{H}_2$ is of order $\sqrt{N} = (\sqrt{N})^1$ and describes part of the interaction between the *condensate* and the *non-condensate* that is linear in the non-condensate field. To obtain this term spatial integration by parts was used to have ∇ only operate on $\widehat{\Psi}_C(\mathbf{r})$ or $\widehat{\Psi}_C(\mathbf{r})^\dagger$. This term needs further development to avoid Fokker-Planck equations containing functional derivatives with respect to spatial derivatives of field functions, and this is accomplished in the next section. The term $\widehat{H}_3$ is of order $1 = (\sqrt{N})^0$ and describes part of the interaction between the condensate and the non-condensate that is quadratic in the non-condensate field.plus the kinetic and trap potential terms for the non-condensate. If the condensate fields are replaced by c-numbers, this term describes Bogoliubov excitations [40], [51]. The term $\widehat{H}_4$ is of order $1/\sqrt{N} = (\sqrt{N})^{-1}$ and describes part of the interaction between the condensate and the non-condensate that is cubic in the non-condensate field. The term $\widehat{H}_5$ is of order $1/N = (\sqrt{N})^{-2}$ and describes part of the interaction within the non-condensate, which is quartic in the non-condensate field.

We now make an approximation and neglect the terms $\widehat{H}_4$ and $\widehat{H}_5$. This leads to the so-called *Bogoliubov Hamiltonian*, albeit still in a number conserving form. This Hamiltonian would be adequate to describe Bogoliubov excitations, so we will use in to treat BEC interferometry in the weak interaction regime. The Bogoliubov Hamiltonian is

$$\widehat{H}_B = \widehat{H}_1 + \widehat{H}_2 + \widehat{H}_3 \tag{37}$$

The neglected terms would be needed in a theory for BEC interferometry in the strong interaction regime.

2.3. Two-Mode Theory and Generalised Gross-Pitaevski Equations

The development of a suitable form for the $\widehat{H}_2$ term in the case where *two condensate modes* are involved is based on a general two mode theory for one component BECs similar to that in [17], though here we apply the Dirac-Frenkel principle to the dynamic action and obtain Gross-Pitaevski equations for time-dependent mode functions, rather than the time independent Gross-Pitaevskii equations for adiabatic mode functions obtained in [17] by applying a variational

principle to the adiabatic action and involving Lagrange multipliers. In two-mode theories we write the *quantum state* $|\Phi(t)\rangle$ of the N boson system as a superposition of the $N + 1$ *basis states* $\left|\frac{N}{2}, k\right\rangle$, where there are $\frac{N}{2} - k$ and $\frac{N}{2} + k$ bosons (respectively) occupying the two modes with (time dependent) *mode functions* $\phi_1(\mathbf{r}, t)$ and $\phi_2(\mathbf{r}, t)$. The *amplitude* for this basis state is $b_k(t)$.

$$|\Phi(t)\rangle = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} b_k(t) \left|\frac{N}{2}, k\right\rangle. \tag{38}$$

and the basis states are *Fock states* given by

$$\left|\frac{N}{2}, k\right\rangle = \frac{\left(\widehat{a}_1(t)^\dagger\right)^{\left(\frac{N}{2}-k\right)}}{[(\frac{N}{2}-k)!]^{\frac{1}{2}}} \frac{\left(\widehat{a}_2(t)^\dagger\right)^{\left(\frac{N}{2}+k\right)}}{[(\frac{N}{2}+k)!]^{\frac{1}{2}}} |0\rangle \qquad (k = -N/2, -N/2+1, .., +N/2) \tag{39}$$

These basis states are *fragmented* or *number squeezed* states, allowing for both modes to have *macroscopic occupancy* when $|k| \ll N/2$.

The notation $\frac{N}{2}, k$ for the basis states reflects the feature that the two mode Bose condensate behaves like a giant spin system. *Spin angular momentum* operators can be defined by

$$\begin{aligned} \widehat{S}_x &= (\widehat{a}_2^\dagger \widehat{a}_1 + \widehat{a}_1^\dagger \widehat{a}_2)/2 \\ \widehat{S}_y &= (\widehat{a}_2^\dagger \widehat{a}_1 - \widehat{a}_1^\dagger \widehat{a}_2)/2i \\ \widehat{S}_z &= (\widehat{a}_2^\dagger \widehat{a}_2 - \widehat{a}_1^\dagger \widehat{a}_1)/2 \end{aligned} \tag{40}$$

which satisfy the standard angular momentum commutation rules. The *square* of the angular momentum $(\widehat{\underrightarrow{S}})^2$ is related to the total two mode boson number operator $\widehat{N} = (\widehat{a}_2^\dagger \widehat{a}_2 + \widehat{a}_1^\dagger \widehat{a}_1)$

$$(\widehat{\underrightarrow{S}})^2 = \sum_a (\widehat{S}_a)^2 = \frac{\widehat{N}}{2}\left(\frac{\widehat{N}}{2} + 1\right) \tag{41}$$

and the Fock state $\left|\frac{N}{2}, k\right\rangle$ is a simultaneous eigenstate of $(\widehat{\underrightarrow{S}})^2, \widehat{S}_z$

$$\begin{aligned} (\widehat{\underrightarrow{S}})^2 \left|\frac{N}{2}, k\right\rangle &= \frac{N}{2}\left(\frac{N}{2} + 1\right)\left|\frac{N}{2}, k\right\rangle \\ \widehat{S}_z \left|\frac{N}{2}, k\right\rangle &= k\left|\frac{N}{2}, k\right\rangle \end{aligned} \tag{42}$$

Hence the *total angular momentum* quantum number $j = \frac{N}{2}$ is *macroscopic*, and $k = -\frac{N}{2}, -\frac{N}{2}+1, .., \frac{N}{2}-1, +\frac{N}{2}$ specifies the *magnetic* quantum number as well as $2k$ determining the difference in *mode occupancy*.

In [17] equations for the amplitudes and adiabatic mode functions have been determined by applying Principles of Least Action , involving minimising the *dynamic* and *adiabatic actions* respectively for the state vector given by (38),

subject to the normalisation constraints for the amplitude and orthonormality constraints for the mode functions

$$\sum_{k=-\frac{N}{2}}^{\frac{N}{2}} |b_k(t)|^2 \quad = \quad 1$$

$$\int d\mathbf{r}\, \phi_i^*(\mathbf{r}, t)\, \phi_j(\mathbf{r}, t) \quad = \quad \delta_{ij} \qquad i, j = 1, 2 \tag{43}$$

In the present treatment, the Dirac-Frenkel principle [30], [31] is applied to the dynamic action to obtain equations for the amplitudes $b_k(t)$.and time-dependent mode functions $\phi_i(\mathbf{r}, t)$ ($i = 1, 2$). In applying the Dirac-Frenkel principle no Lagrange multipliers associated with the equations of constraint (43) are introduced, however mode orthonormality is used in the treatment and the final amplitude and mode equations can be shown to be consistent with both these constraints. Such variational principles are well-known in quantum physics, the Dirac-Frenkel principle applied to the dynamic action for an arbitrary unnormalised state vector gives the time-dependent Schrodinger equation. The Hartree-Fock equations for electrons in atoms and molecules and the Gross-Pitaevskii equations for a single mode condensate are two examples of their application based on state vectors with restricted forms. In the latter case the state vector assumed is a special case of (38) such as with just the single term $\left| \frac{N}{2}, -\frac{N}{2} \right\rangle$ or a special superposition (*binomial state*) corresponding to all bosons being in the same single particle state [17], itself a linear combination of the two original modes.

In the present case of *two condensate modes*, the mode functions satisfy the *coupled generalised Gross-Pitaevskii* equations

$$i\hbar \sum_j X_{ij}\, \frac{\partial}{\partial t}\phi_j \quad = \quad \sum_j X_{ij}(-\frac{\hbar^2}{2m}\nabla^2 + V)\,\phi_j$$

$$+ \sum_j (g \sum_{mn} Y_{im\,jn}\, \phi_m^*\, \phi_n)\, \phi_j \qquad (i = 1, 2). \tag{44}$$

These mode functions allow for boson-boson interactions and are time-dependent. They follow the changes in the time dependent potential $V(\mathbf{r}, t)$. The quantities X_{ij} and $Y_{im\,jn}$ are *one-body* and *two-body correlation functions*

$$X_{ij} \quad = \quad \langle \Phi | \widehat{a}_i^\dagger \widehat{a}_j | \Phi \rangle \tag{45}$$

$$Y_{im\,jn} \quad = \quad \langle \Phi | \widehat{a}_i^\dagger \widehat{a}_m^\dagger \widehat{a}_j \widehat{a}_n | \Phi \rangle \tag{46}$$

Detailed expressions given in the Appendix A in Eqs. (A.12) and (A.13), showing that X_{ij} and $Y_{im\,jn}$ are quadratic forms of the amplitudes b_k. These are of order N and N^2 respectively. The one body correlation functions can be expressed in terms of matrix elements between the Fock states of the spin operators, and the two body correlation functions as matrix elements products of two spin operators. The coupled Gross-Pitaevskii equations are non-linear in

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the mode functions. The non-linear terms $(g \sum_{mn} Y_{im\,jn}\, \phi_m^*\, \phi_n)$ that are present due to the boson-boson interactions scale like the boson particle density and behave as generalised mean fields. Hence the approach that produces generalised Gross-Pitaevskii equations is a form of *mean field theory*, though not of course as simple as in the case of a single mode theory. The kinetic energy and trap potential terms and the mean field terms may also be written as

$$\sum_j X_{ij}(-\frac{\hbar^2}{2m}\nabla^2 + V)\,\phi_j(\mathbf{r},t)$$

$$= \sum_j \langle\Phi|\widehat{a}_i^\dagger \left\{-\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2 + V(\mathbf{r},t)\right\}\widehat{a}_j\,|\Phi\rangle \;\phi_j(\mathbf{r},t) \qquad (47)$$

$$\sum_j (g\sum_{mn} Y_{im\,jn}\, \phi_m^*\, \phi_n)\,\phi_j(\mathbf{r},t)$$

$$= \sum_j \langle\Phi|\widehat{a}_i^\dagger \left\{g\widehat{\Psi}_C(\mathbf{r})^\dagger \widehat{\Psi}_C(\mathbf{r})\right\}\widehat{a}_j\,|\Phi\rangle \;\phi_j(\mathbf{r},t) \qquad (48)$$

showing the formal relationship of the terms to the state vector $|\Phi\rangle$.

For the present two mode condensate case the amplitudes satisfy *matrix mechanics* equations, as in [17]

$$i\hbar\frac{\partial b_k}{\partial t} = \sum_l (H_{kl} - \hbar U_{kl})b_l \qquad (k = -N/2, .., N/2). \qquad (49)$$

These $N+1$ equations (49) describe the system dynamics as it evolves amongst the possible fragmented states. The equations are similar to the standard amplitude equations obtained from matrix mechanics. In these equations the matrix elements H_{kl}, U_{kl} depend on the mode functions $\phi_i(r,t)$. Detailed expressions for H_{kl}, U_{kl} are given in Appendix A in Eqs.(A.10) and (A.7). The matrix elements H_{kl} are in fact the matrix elements of the Hamiltonian $\widehat{H}$ in equation (32) between the fragmented states $\left|\frac{N}{2},k\right\rangle$, $\left|\frac{N}{2},l\right\rangle$. The matrix elements U_{kl} are elements of the so-called *rotation matrix*, and allow for the time dependence of the mode functions.

$$H_{kl} = \left\langle\frac{N}{2},k\left|\widehat{H}\right|\frac{N}{2},l\right\rangle \qquad (50)$$

$$U_{kl} = \frac{1}{2i}\left(\left\{\partial_t\left\langle\frac{N}{2},k\right|\right\}\left|\frac{N}{2},l\right\rangle - \left\langle\frac{N}{2},k\right|\left\{\partial_t\left|\frac{N}{2},l\right\rangle\right\}\right) \qquad (51)$$

The specific forms of the X_{ij}, $Y_{im\,jn}$, H_{kl}, U_{kl} are not important in what follows, all that is required is that they have been determined. Equations for the mode functions and amplitudes similar to (44) and (49) have been obtained by Alon et al [47] for single component BECs.

From the amplitude and mode equations it can be shown that

$$\frac{\partial}{\partial t} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} |b_k(t)|^2 = 0 \qquad (52)$$

$$i\hbar \sum_{ij} X_{ij} \frac{\partial}{\partial t} \int d\mathbf{r}\, \phi_i^*(\mathbf{r},t)\, \phi_j(\mathbf{r},t) = 0 \qquad (53)$$

The first result shows that the amplitudes would remain normalised to unity and the second result is consistent with the modes remaining orthogonal and normalised, assuming they were so chosen at $t = 0$. The second result involves the trace of the product of a positive definite invertible matrix X with a matrix which is the time derivative of the mode orthogonality matrix.

Adiabatic solutions to the time dependent coupled Gross-Pitaevskii equations can be obtained for slowly varying trap potentials via the transformation to new *adiabatic modes* $\xi_k(\mathbf{r},t)$ $(k = 1, 2)$ in the form

$$\phi_i(\mathbf{r},t) = \sum_k \alpha_{ki}\, \exp(-i\mu_k t)\, \xi_k(\mathbf{r},t) \qquad (54)$$

where it is assumed that the coefficients α_{ki} and the new modes $\xi_k(\mathbf{r},t)$ are so slowly varying with time that their time derivatives can be ignored. All the time dependence is assumed to be carried in the oscillating exponential factors. The new modes are required to be orthonormal, and the frequency factors μ_k are required to be real, so that the transformation does not diverge for large $|t|$. The orthonormality condition shows that the α_{ki} form a unitary matrix.

$$\sum_k \alpha_{ki}^* \alpha_{kj} = \delta_{ij} \qquad \sum_i \alpha_{ki}^* \alpha_{li} = \delta_{kl} \qquad (55)$$

The condensate field operator can also be expressed in terms of the adiabatic mode functions and their associated annihilation, creation operators as

$$\widehat{\Psi}_C(\mathbf{r}) = \widehat{b}_1 \xi_1(\mathbf{r}) + \widehat{b}_2 \xi_2(\mathbf{r}) \qquad \widehat{\Psi}_C^\dagger(\mathbf{r}) = \xi_1^*(\mathbf{r})\widehat{b}_1^\dagger + \xi_2^*(\mathbf{r})\widehat{b}_2^\dagger \qquad (56)$$

where

$$\widehat{b}_k = \sum_i \alpha_{ki}\, \exp(-i\mu_k t)\, \widehat{a}_i \qquad \widehat{b}_k^\dagger = \sum_i \alpha_{ki}^*\, \exp(+i\mu_k t)\, \widehat{a}_i^\dagger \qquad (57)$$

and the standard commutation rules apply $[\widehat{b}_k, \widehat{b}_l^\dagger] = \delta_{kl}$.

Substituting for $\phi_j(\mathbf{r},t)$ in the coupled Gross-Pitaevskii equations (44), multiplying by $\alpha_{li}^* \exp(+i\mu_l t)$ and summing over i gives a pair of *time independent coupled Gross-Pitaevskii equations* for the adiabatic mode functions

$$\sum_k P_{lk}\, \hbar\mu_k\, \xi_k = \sum_k P_{lk}(-\frac{\hbar^2}{2m}\nabla^2 + V)\, \xi_k$$

$$+ \sum_k (g \sum_{mn} Q_{lr\,ks}\, \xi_r^* \xi_s)\, \xi_k \qquad (l = 1, 2). \qquad (58)$$

where now the one and two body correlation functions are

$$P_{lk} = \langle \Phi | \, \widehat{b}_l^\dagger \widehat{b}_k \, | \Phi \rangle \tag{59}$$

$$Q_{lr\,ks} = \langle \Phi | \, \widehat{b}_l^\dagger \widehat{b}_r^\dagger \widehat{b}_k \widehat{b}_s \, | \Phi \rangle \tag{60}$$

Equations (47) and (48) were also used in the derivation. These equations are only meaningful if the trap potential and the adiabatic mode functions are in fact slowly varying with time. The frequencies μ_1, μ_2 play the role of generalised chemical potentials. Noting that the N dependence in the mode equations is carried in the one and two body correlation functions - these being of $O(N)$ and $O(N^2)$ respectively, it is then possible to show that the chemical potential is given by

$$\mu = \frac{\partial}{\partial N} \langle \Phi | \, \widehat{H} \, | \Phi \rangle = \sum_k \hbar \mu_k \frac{\langle \Phi | \, \widehat{b}_k^\dagger \widehat{b}_k \, | \Phi \rangle}{N}. \tag{61}$$

As the quantity $\langle \Phi | \, \widehat{b}_k^\dagger \widehat{b}_k \, | \Phi \rangle / N$ is the fractional number of bosons occupying the adiabatic mode ξ_k it follows that $\hbar \mu_k$ is the *chemical potential* associated with that mode.

2.4. Single-Mode Theory and Standard Gross-Pitaevski Equation

For the case where there is just a *single condensate mode* the state vector becomes

$$| \, \Phi(t) \rangle = \frac{\left(\widehat{a}_1(t)^\dagger \right)^N}{[(N)!]^{\frac{1}{2}}} \, | \, 0 \rangle \tag{62}$$

The general Gross-Pitaevskii equations (44) then reduce to the single *Gross-Pitaevskii equation.*

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V + g(N-1) \, |\phi_1|^2 \right) \phi_1 = i\hbar \frac{\partial}{\partial t} \, \phi_1 \tag{63}$$

Note that in the regime of interest with N becoming very large, the factor $gY_{im\,jn}/N = g_N Y_{im\,jn}/N^2$ becomes approximately equal to g_N tiimes a factor of order unity. In deriving the single Gross-Pitaevskii equation from (44), the matrices with elements X_{ij} and $Y_{im\,jn}$ reduce to $1x1$ matrices with non-zero elements

$$X_{11} = N \qquad Y_{1111} = N(N-1) \tag{64}$$

since in this case $b_k = \delta_{k,-N/2}$. As there is now only one mode, there is now a single Fock state so amplitude equations, spin operators no longer apply.

For the single mode case an adiabatic solution can be obtained via the transformation

$$\phi_1(\mathbf{r}, t) = \exp(-i\mu_1 t) \, \xi_1(\mathbf{r}, t) \tag{65}$$

applied to (63), where it is assumed that the new mode $\xi_1(\mathbf{r}, t)$ is so slowly varying with time that its time derivative can be ignored. The time independent Gross-Pitaevskii equation for the adiabatic mode function becomes

$$\hbar \mu_1 \, \xi_1 = (-\frac{\hbar^2}{2m} \nabla^2 + V) \, \xi_1 + g(N-1) \, |\xi_1|^2 \, \xi_1 \tag{66}$$

and μ_1 is the chemical potential $\mu_1 = \frac{\partial}{\partial N} \langle \Phi| \widehat{H} |\Phi\rangle$.

2.5. Modified Form for $\widehat{H}_2$ Term

The previous form (33) of the $\widehat{H}_2$ term contains spatial derivatives and these would produce Fokker-Planck equations with functional derivatives with respect to spatial derivatives of field functions, which cannot be treated in the standard approach. However, the $\widehat{H}_2$ term can be put in a form in which spatial derivatives are absent.

2.5.1. Two-Mode Case

It is straightforward to show that the eigenvalues of the 2×2 matrix of the X_{ij} are both real, positive and their sum equals N. Apart from special cases where one of the eigenvalues is zero we see that the matrix of X_{ij} is invertible and hence we can write

$$\left(-\frac{\hbar^2}{2m} \sum_{\mu=x,y,z} \partial_\mu^2 + V\right) \phi_l = i\hbar\frac{\partial}{\partial t}\phi_l - \sum_{ij} X_{li}^{-1} Z_{ij}\phi_j \qquad (67)$$

where the *generalised mean field* that occurs in the mode equations is defined by

$$Z_{ij} = g \sum_{mn} Y_{im\,jn}\, \phi_m^* \,\phi_n \qquad (68)$$

and is quadratic in the mode functions. Thus we find that the condensate field annihilation operator satisfies the equation

$$\begin{aligned}
\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \widehat{\Psi}_C(\mathbf{r}) &= i\hbar\sum_{l}\widehat{a}_l\frac{\partial}{\partial t}\phi_l - \sum_{ijl} X_{li}^{-1}Z_{ij}\phi_j\widehat{a}_l \\
&= -i\hbar\sum_{l}\phi_l\frac{\partial}{\partial t}\widehat{a}_l \\
&\quad -g\sum_{ijmnl} X_{li}^{-1}Y_{im\,jn}\,\phi_m^*\phi_n\phi_j\,\widehat{a}_l \qquad (69)
\end{aligned}$$

using $\frac{\partial}{\partial t}\widehat{\Psi}_C(\mathbf{r}) = 0$.

However from (14)

$$\widehat{a}_l = \int d\mathbf{s}\, \phi_l^*(\mathbf{s})\widehat{\Psi}_C(\mathbf{s}) \qquad (l = 1, 2) \qquad (70)$$

so that the condensate field operator satisfies the integro-differential equation

$$\begin{aligned}
\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \widehat{\Psi}_C(\mathbf{r}) &= -i\hbar\sum_{l}\phi_l(\mathbf{r})\int d\mathbf{s}\,\frac{\partial}{\partial t}\phi_l^*(\mathbf{s})\,\widehat{\Psi}_C(\mathbf{s}) \\
&\quad -g\sum_{ijmnl} X_{li}^{-1}Y_{im\,jn}\,\phi_m^*(\mathbf{r})\phi_n(\mathbf{r})\phi_j(\mathbf{r}) \\
&\quad \times \int d\mathbf{s}\,\phi_l^*(\mathbf{s})\,\widehat{\Psi}_C(\mathbf{s}) \qquad (71)
\end{aligned}$$

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Hence operating from the left with $\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}$ and integrating over $\mathbf{r}$ we find that

$$
\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\left(-\frac{\hbar^2}{2m}\nabla^2+V\right)\widehat{\Psi}_C(\mathbf{r})
$$

$$
=\;-i\hbar\sum_l\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\phi_l(\mathbf{r})\int d\mathbf{s}\,\left(\frac{\partial}{\partial t}\phi_l^*(\mathbf{s})\right)\widehat{\Psi}_C(\mathbf{s})
$$

$$
-g\sum_{ijmnl}X_{li}^{-1}Y_{im\,jn}\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\phi_m^*(\mathbf{r})\phi_n(\mathbf{r})\phi_j(\mathbf{r})\int d\mathbf{s}\,\phi_l^*(\mathbf{s})\widehat{\Psi}_C(\mathbf{s})
$$

$$
=\;-g\sum_{ijmnl}X_{li}^{-1}Y_{im\,jn}\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\phi_m^*(\mathbf{r})\phi_n(\mathbf{r})\phi_j(\mathbf{r})
$$

$$
\times\int d\mathbf{s}\,\phi_l^*(\mathbf{s})\widehat{\Psi}_C(\mathbf{s}) \tag{72}
$$

since the first term on the right hand side is zero because the condensate mode functions $\phi_j(\mathbf{r})$ are orthogonal to the non-condensate mode functions $\phi_k^*(\mathbf{r})$ that are present in the expansion of the non-condensate field operator $\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}$. Thus we can write

$$
\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\left\{\left(-\frac{\hbar^2}{2m}\nabla^2+V\right)\widehat{\Psi}_C(\mathbf{r})\right\}=-\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\widehat{\Psi}_C(\mathbf{s}) \tag{73}
$$

where the kernel $F(\mathbf{r},\mathbf{s})$ is an ordinary spatial function of two positions and is defined by

$$
F(\mathbf{r},\mathbf{s})=\sum_{ijmnl}X_{li}^{-1}Y_{im\,jn}\phi_m^*(\mathbf{r})\phi_n(\mathbf{r})\phi_j(\mathbf{r})\,\phi_l^*(\mathbf{s}) \tag{74}
$$

Note that this kernel is *not* symmetric in $\mathbf{r},\mathbf{s}$.

Taking the adjoint of the last equation gives

$$
\int d\mathbf{r}\,\left\{\left(-\frac{\hbar^2}{2m}\nabla^2+V\right)\widehat{\Psi}_C^{\dagger}(\mathbf{r})\right\}\widehat{\Psi}_{NC}(\mathbf{r})=-\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F^*(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^{\dagger}\widehat{\Psi}_{NC}(\mathbf{s}) \tag{75}
$$

so the term $\widehat{H}_2$ can now be written as

$$
\widehat{H}_2\;=\;-\int\int d\mathbf{r}\,d\mathbf{s}\,\frac{g_N}{N}F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\widehat{\Psi}_C(\mathbf{s})
$$

$$
+\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\left\{+\frac{g_N}{N}\widehat{\Psi}_C(\mathbf{r})^{\dagger}\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})\right\}
$$

$$
+\int d\mathbf{r}\,\left\{+\frac{g_N}{N}\widehat{\Psi}_C(\mathbf{r})^{\dagger}\widehat{\Psi}_C(\mathbf{r})^{\dagger}\widehat{\Psi}_C(\mathbf{r})\right\}\widehat{\Psi}_{NC}(\mathbf{r}))
$$

$$
-\int\int d\mathbf{r}\,d\mathbf{s}\,\frac{g_N}{N}F^*(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^{\dagger}\widehat{\Psi}_{NC}(\mathbf{s}) \tag{76}
$$

This eliminates the awkward terms involving integrals of $\widehat{\Psi}_{NC}$ with the spatial derivative of $\widehat{\Psi}_C$ (and the adjoint expressions). These would lead to Fokker-Planck equations with functional derivatives with respect to spatial derivatives

of field functions, which cannot be treated in the standard approach. However, the term $\widehat{H}_2$ now involves *double spatial integrals* of field operators, and these require special treatment.

We can write $\widehat{H}_2$ as the sum of two terms, one involving field operators to the second order, the other involving field operators to the fourth order. Thus

$$\widehat{H}_2 = \widehat{H}_{2U4} + \widehat{H}_{2U2} \tag{77}$$

$$\widehat{H}_{2U4} = \frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\widehat{\Psi}^\dagger_C(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}))$$

$$\qquad +\frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}^\dagger_C(\mathbf{r})\widehat{\Psi}^\dagger_C(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})) \tag{78}$$

$$\widehat{H}_{2U2} = -\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\,\widehat{\Psi}_C(\mathbf{s})$$

$$\qquad -\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F^*(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{s}) \tag{79}$$

Note that both terms are proportional to the factor g_N/N. Thus we see that $\widehat{H}_2$ is now associated only with terms analogous to those for *boson-boson* interactions, both $\widehat{H}_{2U4}$ and $\widehat{H}_{2U2}$ being proportional to g_N/N.

2.5.2. Single Mode Case

If only a *single condensate mode* was involved the development of $\widehat{H}_2$ is simpler. From (63) and (24) similar procedure to the two mode case gives

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\widehat{\Psi}_C(\mathbf{r}) = -i\hbar\phi_1(\mathbf{r})\int d\mathbf{s}\,\left(\frac{\partial}{\partial t}\phi_1^*(\mathbf{s})\right)\widehat{\Psi}_C(\mathbf{s}) - g(N-1)\,|\phi_1(\mathbf{r})|^2\,\widehat{\Psi}_C(\mathbf{r}) \tag{80}$$

so that using the orthogonality of the condensate mode to all the non-condensate modes

$$\int d\mathbf{r}\,\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\widehat{\Psi}_C(\mathbf{r})$$

$$= -i\hbar\int d\mathbf{r}\,\widehat{\Psi}^\dagger_{NC}(\mathbf{r})\,\phi_1(\mathbf{r})\int d\mathbf{s}\,\left(\frac{\partial}{\partial t}\phi_1^*(\mathbf{s})\right)\widehat{\Psi}_C(\mathbf{s})$$

$$\qquad -\int d\mathbf{r}\,\widehat{\Psi}^\dagger_{NC}(\mathbf{r})g(N-1)\,|\phi_1(\mathbf{r})|^2\,\widehat{\Psi}_C(\mathbf{r})$$

$$= -\int d\mathbf{r}\,\widehat{\Psi}^\dagger_{NC}(\mathbf{r})g(N-1)\,|\phi_1(\mathbf{r})|^2\,\widehat{\Psi}_C(\mathbf{r}) \tag{81}$$

This result may also be recognised as a special case of (75). Using the special forms in (64) for the $X^{-1}_{li}Y_{im\,jn}$ we have

$$F(\mathbf{r},\mathbf{s}) = \frac{1}{N}N(N-1)\phi_1^*(\mathbf{r})\phi_1(\mathbf{r})\phi_1(\mathbf{r})\,\phi_1^*(\mathbf{s})$$

$$= (N-1)\phi_1^*(\mathbf{r})\phi_1(\mathbf{r})\delta_C(\mathbf{r},\mathbf{s}) \tag{82}$$

$$-\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\,\widehat{\Psi}_C(\mathbf{s}) = -\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^\dagger g(N-1)\,|\phi_1(\mathbf{r})|^2\,\widehat{\Psi}_C(\mathbf{r})$$

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as before, where a result from (22) has been used to evaluate the $\mathbf{s}$ integral..

We can use (81) and the related adjoint equation involving $\widehat{\Psi}_C^\dagger(\mathbf{r})g(N-1)\left|\phi_1(\mathbf{r})\right|^2\widehat{\Psi}_{NC}(\mathbf{r})$ to simplify $\widehat{H}_2$ into a form

$$
\begin{aligned}
\widehat{H}_2 \;=\;& \int d\mathbf{r}\,(\widehat{\Psi}_{NC}^\dagger(\mathbf{r})\frac{g_N}{N}\{\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r}) - \left\langle\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\right\rangle\}\widehat{\Psi}_C(\mathbf{r})) \\
& + \int d\mathbf{r}\,(\widehat{\Psi}_C^\dagger(\mathbf{r})\frac{g_N}{N}\{\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r}) - \left\langle\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\right\rangle\}\widehat{\Psi}_{NC}(\mathbf{r})) \quad (83)
\end{aligned}
$$

where we use the notation $\left\langle\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})\right\rangle = (N-1)\left|\phi_1(\mathbf{r})\right|^2$. We see that for the single mode condensate case $\widehat{H}_2$ is also the sum of a term $\widehat{H}_{2U4}$ which is fourth order in the field operators and a term $\widehat{H}_{2U2}$ which is second order.

$$
\begin{aligned}
\widehat{H}_2 \;=\;& \widehat{H}_{2U4} + \widehat{H}_{2U2} \qquad\qquad\qquad\qquad\qquad\qquad (84) \\
\widehat{H}_{2U4} \;=\;& \frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}_{NC}^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) \\
& + \frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})) \qquad\quad (85) \\
\widehat{H}_{2U2} \;=\;& -\frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}_{NC}^\dagger(\mathbf{r})\{\left\langle\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\right\rangle\}\widehat{\Psi}_C(\mathbf{r})) \\
& - \frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}_C^\dagger(\mathbf{r})\{\left\langle\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\right\rangle\}\widehat{\Psi}_{NC}(\mathbf{r})) \qquad (86)
\end{aligned}
$$

Thus we see that $\widehat{H}_2$ is now associated only with *boson-boson* interaction terms. However, unlike the two mode condensate case there is no double spatial integral involved. The general form of the development for $\widehat{H}_2$ is a simpler version of the result (77) for the two mode case.

3. Phase space distribution functional

In this section the phase space distribution functional is introduced starting with the characteristic functional. The distribution functional is of a mixed type, with the condensate fields involving a generalised Wigner form, whilst the non-condensate fields involving a positive P form. This is to reflect the feature that many bosons occupy the condensate modes, so a Wigner distribution is better suited since it descibes fields whose behaviour is close to a classical mean field situation. On the other hand, there will be few bosons occupying the non-condensate modes, hence a positive P distribution is better, since the non-condensate fields may display quantum behaviour. In this section we emphasise how the phase space distribution functionals determine the quantum correlation functions which are used to describe the probabilities for bosonic position measurements. The theory in this section is set out for the two-mode situation, but can be easily modified for the single mode condensate by just restricting the sums over condensate modes to a single term.

3.1. Characteristic Functional

From the density operator $\widehat{\rho}$ and by introducing four distinct functions $\xi_C^+(\mathbf{r}), \xi_C(\mathbf{r}), \xi_{NC}^+(\mathbf{r})$ and $\xi_{NC}(\mathbf{r})$ associated with the field operators $\widehat{\Psi}_C(\mathbf{r}), \widehat{\Psi}_C^\dagger(\mathbf{r}), \widehat{\Psi}_{NC}(\mathbf{r})$ and $\widehat{\Psi}_{NC}^\dagger(\mathbf{r})$ respectively, we define the *characteristic functional* $\chi[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r}), \xi_{NC}(\mathbf{r}), \xi_{NC}^+(\mathbf{r})]$ as

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] = Tr(\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \tag{87}$$

with

$$\widehat{\Omega} = \widehat{\Omega}_C\,\widehat{\Omega}_{NC} \tag{88}$$

$$\widehat{\Omega}_C = \exp \int d\mathbf{r}\, i\{\xi_C(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \tag{89}$$

$$\widehat{\Omega}_{NC} = \exp \int d\mathbf{r}\, i\{\xi_{NC}(\mathbf{r})\widehat{\Psi}_{NC}^\dagger(\mathbf{r})\} \exp \int d\mathbf{r}\, i\{\widehat{\Psi}_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\} \tag{90}$$

Thus this mixed characteristic functional is of the *Wigner* type for the condensate modes and of the *Positive P* (P^+) type for the non-condensate modes. The basic idea of a *functional* is explained in Appendix B ([82]) but essentially a functional $F[\psi(x)]$ of a field function $\psi(x)$ just defines a process that results in a c-number which depends on *all* the values of the field function, that is over the entire range of positions x.

If the mode expansions are used with

$$\xi_C(\mathbf{r}) = \xi_1\phi_1(\mathbf{r}) + \xi_2\phi_2(\mathbf{r}) \tag{91}$$

$$\xi_C^+(\mathbf{r}) = \phi_1^*(\mathbf{r})\xi_1^+ + \phi_2^*(\mathbf{r})\xi_2^+ \tag{92}$$

$$\xi_{NC}(\mathbf{r}) = \sum_{k\neq 1,2}^{K} \xi_k\,\phi_k(\mathbf{r}) \tag{93}$$

$$\xi_{NC}^+(\mathbf{r}) = \sum_{k\neq 1,2}^{K} \phi_k^*(\mathbf{r})\,\xi_k^+ \tag{94}$$

then we have

$$\widehat{\Omega}_C = \exp i\{\xi_1\widehat{a}_1^\dagger + \widehat{a}_1\xi_1^+ + \xi_2\widehat{a}_2^\dagger + \widehat{a}_2\xi_2^+\} \tag{95}$$

$$\widehat{\Omega}_{NC} = \exp i\sum_{k\neq 1,2}\xi_k\widehat{a}_k^\dagger \, \exp i\sum_{k\neq 1,2}\widehat{a}_k\xi_k^+ \tag{96}$$

This shows that the characteristic *functional* $\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+]$ is also a characteristic *function* $\chi_b(\xi_1,\xi_1^+,\xi_2,\xi_2^+,..,\xi_k,\xi_k^+,..)$ of the *c-number* expansion coefficients, a result that is important in deriving expressions based on functionals.

3.2. Quasi-Distribution Functional

For double phase space distributions as in the present case the *quasi-distribution functional* $P[\psi_C(\mathbf{r}),\psi_C^+(\mathbf{r}),\psi_{NC}(\mathbf{r}),\psi_{NC}^+(\mathbf{r}),\psi_C^*(\mathbf{r}),\psi_C^{+*}(\mathbf{r}),\psi_{NC}^*(\mathbf{r}),\psi_{NC}^{+*}(\mathbf{r})]$ involves four field functions $\psi_C(\mathbf{r}),\psi_C^+(\mathbf{r}),\psi_{NC}(\mathbf{r}),\psi_{NC}^+(\mathbf{r})$ corresponding to the field operators $\widehat{\Psi}_C(\mathbf{r}),\widehat{\Psi}_C^\dagger(\mathbf{r}),\widehat{\Psi}_{NC}(\mathbf{r})$ and $\widehat{\Psi}_{NC}^\dagger(\mathbf{r})$ respectively, plus their complex conjugate fields $\psi_C^*(\mathbf{r}),\psi_C^{+*}(\mathbf{r}),\psi_{NC}^*(\mathbf{r}),\psi_{NC}^{+*}(\mathbf{r})$. It is chosen to give the characteristic functional $\chi[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r}),\xi_{NC}(\mathbf{r}),\xi_{NC}^+(\mathbf{r})]$ via a *functional integration* process over the four complex field functions, the integration also incorporating an exponential factor, which may be written as
$\exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r})+\psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}\;\exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}$.
Thus

$$\chi[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r}),\xi_{NC}(\mathbf{r}),\xi_{NC}^+(\mathbf{r})]$$
$$= \iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+$$
$$\times P[\psi_C(\mathbf{r}),\psi_C^+(\mathbf{r}),\psi_{NC}(\mathbf{r}),\psi_{NC}^+(\mathbf{r}),\psi_C^*(\mathbf{r}),\psi_C^{+*}(\mathbf{r}),\psi_{NC}^*(\mathbf{r}),\psi_{NC}^{+*}(\mathbf{r})]$$
$$\times \exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}$$
$$\times \exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}. \tag{97}$$

The justification of this important result is set out below. Note that the quasi-distribution functional is not necessarily unique, it is only required that the above functional integral gives the characteristic functional, (which is unique).

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In the present case we will use a weight function of the form based on Eq.(B.60), but adapted to there being eight real fields, rather than four as in Appendix B.8.

$$w(\psi_1, \psi_1^+, .., \psi_i, \psi_i^+, .., \psi_n, \psi_n^+, \psi_1^*, \psi_1^{+*}, .., \psi_i^*, \psi_i^{+*}, .., \psi_n^*, \psi_n^{+*}) = \prod_i (\Delta \mathbf{r}_i)^4$$

(98)

The power law $(\Delta \mathbf{r}_i)^4$ arises because each field contributes $(\Delta \mathbf{r}_i)^{1/2}$.

Functional integration is fully explained in Appendix B ([82]), but a brief summary is as follows. If there are n modes then the range for each function $\psi(x)$ is divided up into n small intervals $\Delta x_i = x_{i+1} - x_i$ (the ith interval, where $\epsilon > |\Delta x_i|$), then we may specify the *value* ψ_i of the function $\psi(x)$ in the ith interval via the *average*

$$\psi_i = \frac{1}{\Delta x_i} \int\limits_{\Delta x_i} dx \, \psi(x)$$

(99)

and then any functional $F[\psi(x)]$ may be regarded as a *function $F(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$* of all the ψ_i, and ordinary integration over the ψ_i is used to define the functional integral. If each function $\psi(x) = \psi_x(x) + i\psi_y(x)$.is written in terms of its real and imaginary parts, then the functional integration becomes an ordinary integration over the values ψ_{ix}, ψ_{iy} of these components in each interval i of the function $F(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$ multiplied by a suitably chosen weight function $w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$. Thus

$$\int D^2\psi \, F[\psi(x)] = \lim_{n\to\infty} \lim_{\epsilon\to 0} \int ... \int d^2\psi_1 d^2\psi_2 .. d^2\psi_i .. d^2\psi_n \, w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$$
$$\times F(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$$

(100)

where the number of modes is increased to infinity along with the space interval decreasing to zero. The symbol $D^2\psi$ stands for $d^2\psi_1 d^2\psi_2 .. d^2\psi_i .. d^2\psi_n \, w(\psi_1, .., \psi_i, .., \psi_n)$, where the quantity $d^2\psi_i$ means $d\psi_{ix} d\psi_{iy}$. The present case involves a generalisation to treat four complex fields $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$.

To justify the characteristic functional result (97) mode expansions for the field functions are used with *c-number* expansion coefficients α_k, α_k^+

$$\psi_C(\mathbf{r}) = \alpha_1 \phi_1(\mathbf{r}) + \alpha_2 \phi_2(\mathbf{r})$$

(101)

$$\psi_C^+(\mathbf{r}) = \phi_1^*(\mathbf{r}) \alpha_1^+ + \phi_2^*(\mathbf{r}) \alpha_2^+$$

(102)

$$\psi_{NC}(\mathbf{r}) = \sum_{k\neq 1,2}^{K} \alpha_k \phi_k(\mathbf{r})$$

(103)

$$\psi_{NC}^+(\mathbf{r}) = \sum_{k\neq 1,2}^{K} \phi_k^*(\mathbf{r}) \alpha_k^+$$

(104)

The P$^+$ quasi-distribution *functional*
$P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ would then be

equivalent to a distribution *function* $P_b(\alpha_1, \alpha_1^+, .., \alpha_k, \alpha_k^+, .., \alpha_1^*, \alpha_1^{+*}, .., \alpha_k^*, \alpha_k^{+*}, ..)$ of the c-number expansion coefficients and their complex conjugates. For double phase space representations of bosonic systems the connection between the characteristic function $\chi_b(\xi_1, \xi_1^+, \xi_2, \xi_2^+ .., \xi_k, \xi_k^+, ..)$ and the distribution function via a phase space integral has been established by Drummond and Gardiner [83], [84]. The characteristic function is given by

$$
\begin{aligned}
&\chi_b(\xi_1, \xi_1^+, \xi_2, \xi_2^+ .., \xi_k, \xi_k^+, ..) \\
=\ & \int \ldots \int d^2\alpha_1 d^2\alpha_1^+ d^2\alpha_2 d^2\alpha_2^+ .. d^2\alpha_k d^2\alpha_k^+ .. d^2\alpha_n d^2\alpha_n^+ \\
&\times P_b(\alpha_1, \alpha_1^+, .., \alpha_k, \alpha_k^+, .., \alpha_1^*, \alpha_1^{+*}, .., \alpha_k^*, \alpha_k^{+*}, ..) \\
&\times \exp i \sum_{k=1}^{n} \{\xi_k\, \alpha_k^+\}\ \exp i \sum_{k=1}^{n} \{\alpha_k\, \xi_k^+\}
\end{aligned}
\tag{105}
$$

where $\alpha_k = \alpha_{kx} + i\alpha_{ky}$, $\alpha_k^+ = \alpha_{kx}^+ + i\alpha_{ky}^+$ and $d^2\alpha_k = d\alpha_{kx}\, d\alpha_{ky}$, $d^2\alpha_k^+ = d\alpha_{kx}^+ d\alpha_{ky}^+$. If the phase space integration is replaced by functional integration we can show that Eq.(105) leads to the result (97), which thus demonstrates that the distribution functional exists. The change from phase space integration to functional integration is outlined in Appendix B.8 (see Appendix B, [82]). In deriving the functional integration result for the characteristic function the expressions

$$
\begin{aligned}
\exp i \sum_{k=1}^{n} \{\xi_k\, \alpha_k^+\} &=\ \exp i \sum_{j=1,2} \{\xi_j\, \alpha_j^+\}\ \exp i \sum_{k \neq 1,2} \{\xi_k\, \alpha_k^+\} \\
&=\ \exp i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r})\}\ \exp i \int d\mathbf{r}\, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \\
\exp i \sum_{k=1}^{n} \{\alpha_k\, \xi_k^+\} &=\ \exp i \sum_{j=1,2} \{\alpha_j\, \xi_j^+\}\ \exp i \sum_{k \neq 1,2} \{\alpha_k\, \xi_k^+\} \\
&=\ \exp i \int d\mathbf{r}\, \{\psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}\ \exp i \int d\mathbf{r}\, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

are used.

Note that as each field can be expressed in terms of its real and imaginary components, the distribution functional involving the four fields and their conjugates may also be considered as a functional of the eight real, imaginary components.

$$
\begin{aligned}
&P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})] \\
\equiv\ &F[\psi_{CX}(\mathbf{r}), \psi_{CX}^+(\mathbf{r}), \psi_{NCX}(\mathbf{r}), \psi_{NCX}^+(\mathbf{r}), \psi_{CY}(\mathbf{r}), \psi_{CY}^+(\mathbf{r}), \psi_{NCY}(\mathbf{r}), \psi_{NCY}^+(\mathbf{r})
\end{aligned}
$$

This form of the distribution functional is analogous the corresponding form for the distribution function, which has been used as the basis for deriving Ito

stochastic equations for the real, imaginary parts of the phase variables [83], [84].

$$P_b(\alpha_1, \alpha_1^+, .., \alpha_k, \alpha_k^+, .., \alpha_1^*, \alpha_1^{+*}, .., \alpha_k^*, \alpha_k^{+*}, ..)$$
$$\equiv \quad F_b(\alpha_{1X}, \alpha_{1X}^+, .., \alpha_{kX}, \alpha_{kX}^+, .., \alpha_{1Y}, \alpha_{1Y}^+, .., \alpha_{kY}, \alpha_{kY}^+, ..) \qquad (106)$$

3.3. Interferometric Measurements

Coherence effects in BECs are described via *Quantum Correlation Functions*

$$G^N(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_N; \mathbf{s}_N, .., \mathbf{s}_2, \mathbf{s}_1) = \left\langle \widehat{\Psi}(\mathbf{r}_1)^\dagger .. \widehat{\Psi}(\mathbf{r}_N)^\dagger \widehat{\Psi}(\mathbf{s}_N) .. \widehat{\Psi}(\mathbf{s}_1) \right\rangle$$
$$= \quad Tr(\widehat{\rho}(t) \widehat{\Psi}(\mathbf{r}_1)^\dagger .. \widehat{\Psi}(\mathbf{r}_N)^\dagger \widehat{\Psi}(\mathbf{s}_N) .. \widehat{\Psi}(\mathbf{s}_1)) \qquad (107)$$

Various BEC *spatial interference effects* can be described via quantum correlation functions, which thereby specify the spatial coherence effects.

If we interchange coordinates of a pair of *bosons*, say $(\mathbf{r}_i, \mathbf{s}_i) \leftrightarrow (\mathbf{r}_j, \mathbf{s}_j)$ we see that because the commutation properties of the bosonic field operators, the quantum correlation function is unchanged. The *symmetrization principle* for bosonic systems is consistent with measured quantities remaining unchanged due to interchange of identical particles.

The quantum correlation function with $\mathbf{r}_i = \mathbf{s}_i \, (i = 1, ..., N)$ measures the simultaneous probability of detecting one boson at $\mathbf{r}_1$, a second at $\mathbf{r}_2$, .., the Nth at $\mathbf{r}_N$, ([28]. Actual measurements of quantum correlation functions may be made via scattering a weak probe beam (atoms, light) off the system, ([29]. If the field operators are written as the sum of condensate and non-condensate terms, then the quantum correlation functions will contain purely condensate terms, purely non-condensate terms and mixed terms involving both condensate and non-condensate operators.

The quantum averages of symmetrically ordered products of the condensate field operators $\{\widehat{\Psi}_C^\dagger(\mathbf{r}_1)\widehat{\Psi}_C^\dagger(\mathbf{r}_2)....\widehat{\Psi}_C^\dagger(\mathbf{r}_p)\widehat{\Psi}_C(\mathbf{s}_q)..\widehat{\Psi}_C(\mathbf{s}_1)\}$ and normally ordered products of the non-condensate field operators $\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_1)\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_2)....\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_r)\widehat{\Psi}_{NC}(\mathbf{v}_s)..\widehat{\Psi}_{NC}(\mathbf{v}_1)$ may then be expressed as functional integrals of the quasi-distribution functional $P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ with products of the field functions. Thus, with $\left\langle \widehat{\Xi} \right\rangle \equiv Tr(\widehat{\rho}\widehat{\Xi})$

$$\left\langle \{\widehat{\Psi}_C^\dagger(\mathbf{r}_1)....\widehat{\Psi}_C^\dagger(\mathbf{r}_p)\widehat{\Psi}_C(\mathbf{s}_q)..\widehat{\Psi}_C(\mathbf{s}_1)\} \widehat{\Psi}_{NC}^\dagger(\mathbf{u}_1)....\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_r)\widehat{\Psi}_{NC}(\mathbf{v}_s)..\widehat{\Psi}_{NC}(\mathbf{v}_1) \right\rangle$$
$$= \quad Tr\left(\widehat{\rho}\{\widehat{\Psi}_C^\dagger(\mathbf{r}_1)....\widehat{\Psi}_C^\dagger(\mathbf{r}_p)\widehat{\Psi}_C(\mathbf{s}_q)..\widehat{\Psi}_C(\mathbf{s}_1)\} \widehat{\Psi}_{NC}^\dagger(\mathbf{u}_1)....\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_r)\widehat{\Psi}_{NC}(\mathbf{v}_s)..\widehat{\Psi}_{NC}(\mathbf{v}_1) \right)$$
$$= \quad \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+$$
$$\times P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$$
$$\times \psi_C^+(\mathbf{r}_1) \, \psi_C^+(\mathbf{r}_2) .. \psi_C^+(\mathbf{r}_p) \, \psi_C(\mathbf{s}_q) .. \psi_C(\mathbf{s}_2) . \psi_C(\mathbf{s}_1)$$
$$\times \psi_{NC}^+(\mathbf{u}_1) \, \psi_{NC}^+(\mathbf{u}_2) .. \psi_{NC}^+(\mathbf{u}_r) \, \psi_{NC}(\mathbf{v}_s) .. \psi_{NC}(\mathbf{v}_2) \, \psi_{NC}(\mathbf{v}_1) \qquad (108)$$

and where

$$\{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\}$$

$$= \frac{1}{(p+q)!}\sum_R \Re(\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)). \qquad (109)$$

In Eq.(109) the sum over R is over all $(p+q)!$ orderings $\Re$ of the factors $\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)$. T hus, the condensate field operator $\widehat{\Psi}_C^\dagger(\mathbf{r}_i)$ is replaced by $\psi_C^+(\mathbf{r}_i)$ and $\widehat{\Psi}_C(\mathbf{s}_j)$ is replaced by $\psi(\mathbf{s}_j)$, with analogous replacements for the non-condensate field operators. The proof is given in Appendix C ([82]) and involves functional differentiation, which is explained in Appendix B ([82]).

These results together with the equal time commutation rules give the quantum correlation functions. For example, the *first order* quantum correlation function (which is used to exhibit macroscopic spatial coherence in a BEC) is given by

$$G^1(\mathbf{r}_1;\mathbf{s}_1)$$

$$= \left\langle \widehat{\Psi}(\mathbf{r}_1)^\dagger \widehat{\Psi}(\mathbf{s}_1) \right\rangle$$

$$= \left\langle \widehat{\Psi}_C(\mathbf{r}_1)^\dagger \widehat{\Psi}_C(\mathbf{s}_1) \right\rangle + \left\langle \widehat{\Psi}_C(\mathbf{r}_1)^\dagger \widehat{\Psi}_{NC}(\mathbf{s}_1) \right\rangle$$

$$+ \left\langle \widehat{\Psi}_{NC}(\mathbf{r}_1)^\dagger \widehat{\Psi}_C(\mathbf{s}_1) \right\rangle + \left\langle \widehat{\Psi}_{NC}(\mathbf{r}_1)^\dagger \widehat{\Psi}_{NC}(\mathbf{s}_1) \right\rangle$$

$$= \left\langle \left(\{\widehat{\Psi}_C(\mathbf{r}_1)^\dagger \widehat{\Psi}_C(\mathbf{s}_1)\} - \frac{1}{2}\delta(\mathbf{r}_1-\mathbf{s}_1) \right) \right\rangle + \left\langle \{\widehat{\Psi}_C(\mathbf{r}_1)^\dagger\} \widehat{\Psi}_{NC}(\mathbf{s}_1) \right\rangle$$

$$+ \left\langle \widehat{\Psi}_{NC}(\mathbf{r}_1)^\dagger \{\widehat{\Psi}_C(\mathbf{s}_1)\} \right\rangle + \left\langle \widehat{\Psi}_{NC}(\mathbf{r}_1)^\dagger \widehat{\Psi}_{NC}(\mathbf{s}_1) \right\rangle$$

$$= -\frac{1}{2}\delta(\mathbf{r}_1-\mathbf{s}_1)$$

$$+ \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+$$

$$\times P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$$

$$\times (\psi_C^+(\mathbf{r}_1) + \psi_{NC}^+(\mathbf{r}_1))(\psi_C(\mathbf{s}_1) + \psi_{NC}(\mathbf{s}_1)) \qquad (110)$$

and includes pure condensate terms, pure non-condensate terms and mixed terms. Note the delta function term which arises because of the difference between normal and symmetric ordering that applies for the condensate terms.

It is worth noting that some authors [40], [41], [42] determine the mode functions as the eigenfunctions of the first order quantum correlation function $\left\langle \widehat{\Psi}(\mathbf{r}_1)^\dagger \widehat{\Psi}(\mathbf{s}_1) \right\rangle$. Thus the mode functions $\varphi_i(\mathbf{r})$ satisfy the eigenvalue equations

$$\int d\mathbf{s}_1 \left\langle \widehat{\Psi}(\mathbf{r}_1)^\dagger \widehat{\Psi}(\mathbf{s}_1) \right\rangle \varphi_i(\mathbf{s}_1) = \lambda_i \varphi_i(\mathbf{r}_1) \qquad (111)$$

The mode functions can be shown to be orthonormal and the eigenvalues real and positive. The eigenvalues give the occupancy of the modes. For two mode

condensates in a general fragmented state, two such eigenvalues will have macroscopic values $\sim N$ and the other modes will have small eigenvalues. This approach to determining the mode functions has certain formal advantages, such as leading to the $\widehat{H}_2$ term in the Hamiltonian being zero. However, the method would require knowing the first order correlation function, and it is not clear how this could be done prior to knowing the mode functions. In the present approach the formalism is designed as a way to determine all the quantum correlation functions.

4. Functional Fokker-Planck equation

In this section we show how the Liouville-von Neumann equation for the quantum density operator describing the state of the bosonic system is equivalent to a functional Fokker-Planck equation for the phase space distribution functional. This is accomplished via the use of correspondence rules, wherein the product of the quantum density operator with the various condensate and non-condensate field operators (for both product orders) is equivalent to the operation of functional derivatives or field functions on the distribution function. The actual results for the functional Fokker-Planck equation in the case of the present two mode BEC condensate system are set out at the end of the section. For completeness the corresponding simpler results for a single mode condensate are also obtained.

4.1. Dynamics

The state of the bosonic system is described by the *density operator* $\widehat{\rho}$ which satisfies the *Liouville-von Neumann equation*

$$i\hbar\frac{\partial}{\partial t}\widehat{\rho} = [\widehat{H},\widehat{\rho}] \tag{112}$$

where the Bogoliubov Hamiltonian 37 will be used.

The approach used will be to turn the Liouville-von Neumann equation for the density operator $\widehat{\rho}$ into a *functional Fokker-Planck equation* for a quasi distribution functional $P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ and then replace this by *stochastic equations* for stochastic field functions $\widetilde{\psi}_C(\mathbf{r},t), \widetilde{\psi}_C^+(\mathbf{r},t), \widetilde{\psi}_{NC}(\mathbf{r},t), \widetilde{\psi}_{NC}^+(\mathbf{r},t)$. The latter are c-number Langevin equations of the Ito type, and in general will contain *random noise* terms as well as *deterministic* terms coupling the field functions.

4.2. Correspondence Rules

We now wish to replace the Liouville-von Neumann equation for the density operator by a *Functional Fokker-Planck Equation* for the quasi distribution functional. To do this we make use of so-called *correspondence rules*, in which the effect of a field operator on the density operator corresponds to the effects of *functional differentiation* and/or function *multiplication* on the distribution functional.

Functional differentiation is fully explained in Appendix B ([82]), but a summary is as follows. For a functional $F[\psi(x)]$ of a field $\psi(x)$ the functional derivative $\frac{\delta F[\psi(x)]}{\delta\psi(x)}$ is defined by

$$F[\psi(x) + \delta\psi(x)] \doteq F[\psi(x)] + \int dx\,\delta\psi(x)\left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_x \tag{113}$$

where $\delta\psi(x)$ is small. In this equation the left side is a functional of $\psi(x)+\delta\psi(x)$ and the first term on the right side is a functional of $\psi(x)$. The second term on

the right side is a functional of $\delta\psi(x)$ and thus the functional derivative must be a function of x, hence the subscript x. In most situations this subscript will be left understood. If we write $\delta\psi(x) = \epsilon\delta(x-y)$ for small ϵ then an equivalent result for the functional derivative at $x = y$ is

$$\left(\frac{\delta F[\psi(x)]}{\delta\psi(x)} \right)_{x=y} = \lim_{\epsilon\to 0} \left(\frac{F[\psi(x) + \epsilon\delta(x-y)] - F[\psi(x)]}{\epsilon} \right). \tag{114}$$

Note that for functionals involving both $\psi(x)$, $\psi^*(x)$ we treat these complex fields as independent, and functional derivatives with respect to both $\psi(x)$, $\psi^*(x)$ exist. Thus

$$F[\psi(x) + \delta\psi(x), \psi^*(x) + \delta\psi^*(x)]$$
$$\doteq\ F[\psi(x), \psi^*(x)] + \int dx\,\delta\psi(x)\,\left(\frac{\delta F[\psi(x), \psi^*(x)]}{\delta\psi(x)} \right)_x$$
$$+ \int dx\,\delta\psi^*(x)\,\left(\frac{\delta F[\psi(x), \psi^*(x)]}{\delta\psi^*(x)} \right)_x \tag{115}$$

For the equivalent functional $G[\psi_X(x), \psi_Y(x)] \equiv F[\psi(x), \psi^*(x)]$ involving the real, imaginary components $\psi_X(x), \psi_Y(x)$ the functional derivatives are defined by

$$G[\psi_X(x) + \delta\psi_X(x), \psi_Y(x) + \delta\psi_Y(x)]$$
$$\doteq\ G[\psi_X(x), \psi_Y(x)] + \int dx\,\delta\psi_X(x)\,\left(\frac{\delta G[\psi_X(x), \psi_Y(x)]}{\delta\psi_X(x)} \right)_x$$
$$+ \int dx\,\delta\psi_Y(x)\,\left(\frac{\delta G[\psi_X(x), \psi_Y(x)]}{\delta\psi_Y(x)} \right)_x \tag{116}$$

The present case involves a generalisation to treat four complex fields $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$.

4.2.1. Notation

As the notation is now getting rather cumbersome we will designate

$$\underset{\rightarrow}{\psi}(\mathbf{r}) \equiv \{\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})\} \tag{117}$$

$$\underset{\rightarrow}{\psi^*}(\mathbf{r}) \equiv \{\psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})\} \tag{118}$$

$$P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \equiv P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$$
$$\tag{119}$$

$$\underset{\rightarrow}{\xi}(\mathbf{r}) \equiv \{\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r}), \xi_{NC}(\mathbf{r}), \xi_{NC}^+(\mathbf{r})\} \tag{120}$$

$$\chi[\underset{\rightarrow}{\xi}(\mathbf{r})] \equiv \chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \tag{121}$$

for the fields and the distribution, characteristic functionals. For the expansion coefficients and the distribution function we introduce the notation

$$\underset{\rightarrow}{\alpha} \equiv \{\alpha_k, \alpha_k^+\} \tag{122}$$

$$\underset{\rightarrow}{\alpha}^* \equiv \{\alpha_k^*, \alpha_k^{+*}\} \tag{123}$$

$$P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*) \equiv P_b(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*}) \tag{124}$$

$$P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})] \equiv P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*) \tag{125}$$

where the original functional
$P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ of the fields
$\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$ and their complex conjugates $\psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})$
is equivalent to the function $P_b(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})$ of the expansion amplitudes
α_k, α_k^+ and their complex conjugates $\alpha_k^*, \alpha_k^{+*}$.

4.2.2. Correspondence Rules for Condensate and Non-Condensate Fields
   For the condensate operators we have

$$\widehat{\Psi}_C(\mathbf{s})\widehat{\rho} \leftrightarrow \left(\psi_C(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_C(\mathbf{s}) \leftrightarrow \left(\psi_C(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})]$$

$$\widehat{\Psi}_C^\dagger(\mathbf{s})\widehat{\rho} \leftrightarrow \left(\psi_C^+(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_C^\dagger(\mathbf{s}) \leftrightarrow \left(\psi_C^+(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})] \tag{126}$$

and for the non-condensate operators

$$\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\rho} \leftrightarrow (\psi_{NC}(\mathbf{s})) \, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_{NC}(\mathbf{s}) \leftrightarrow \left(\psi_{NC}(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})]$$

$$\widehat{\Psi}_{NC}^\dagger(\mathbf{s})\widehat{\rho} \leftrightarrow \left(\psi_{NC}^+(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_{NC}^\dagger(\mathbf{s}) \leftrightarrow (\psi_{NC}^+(\mathbf{s})) \, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})] \tag{127}$$

whilst for the density operator

$$\frac{\partial\widehat{\rho}}{\partial t} \rightarrow \frac{\partial P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^*(\mathbf{r})]}{\partial t} \tag{128}$$

4.2.3. Deriving the Correspondence Rules
   The proof of these correspondence rules is dealt with in Appendix D ([82]). Key steps in the derivation include first establishing the following changes to

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the characteristic functional. For the condensate modes we have

$$\widehat{\Psi}_C(\mathbf{s})\widehat{\rho} \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_C^+(\mathbf{s})} + \frac{1}{2}\xi_C(\mathbf{s})\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_C(\mathbf{s}) \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_C^+(\mathbf{s})} - \frac{1}{2}\xi_C(\mathbf{s})\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})]$$

$$\widehat{\Psi}_C^\dagger(\mathbf{s})\widehat{\rho} \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_C(\mathbf{s})} - \frac{1}{2}\xi_C^+(\mathbf{s})\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_C^\dagger(\mathbf{s}) \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_C(\mathbf{s})} + \frac{1}{2}\xi_C^+(\mathbf{s})\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})] \qquad (129)$$

and for the non-condensate modes

$$\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\rho} \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}^+(\mathbf{s})}\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_{NC}(\mathbf{s}) \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}^+(\mathbf{s})} - \xi_{NC}(\mathbf{s})\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})]$$

$$\widehat{\Psi}_{NC}^\dagger(\mathbf{s})\widehat{\rho} \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}(\mathbf{s})} - \xi_{NC}^+(\mathbf{s})\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})]$$

$$\widehat{\rho}\widehat{\Psi}_{NC}^\dagger(\mathbf{s}) \quad \leftrightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}(\mathbf{s})}\right)\chi[\underset{\rightarrow}{\xi}(\mathbf{r})] \qquad (130)$$

As can be seen from eqs. (101, 102, 103, 104) the distribution function $P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ is a functional of restricted functions (see Appendix B, [82]). It can also be considered as a function $P_b(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})$ of all the expansion coefficients α_k, α_k^+ in eqs. (101, 102, 103, 104) and their complex conjugates $\alpha_k^*, \alpha_k^{+*}$. Hence in applying the correspondence rules the following operator identities for the various functional derivatives can be used

$$\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k=1,2}\phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k} \qquad \left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k\neq 1,2}^{K}\phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k}$$

$$\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k=1,2}\phi_k(\mathbf{s})\frac{\partial}{\partial\alpha_k^+} \qquad \left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k\neq 1,2}^{K}\phi_k(\mathbf{s})\frac{\partial}{\partial\alpha_k^+}$$

$$(131)$$

where it is understood that the left side operates on the distribution functional $P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ of the restricted functions $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})$ and the right side operates on the equivalent function $P_b(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})$. The related identities for the functional differentiation with respect to the complex conjugate fields also exist, but are not needed because the correspondence rules only involve functional derivation with respect to $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$,

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and any functions arising from the multiplications are only functions of these fields and not their complex conjugates.

In deriving the correspondence rules that result in functional differentiation a key step involves a functional integration by parts of the form

$$\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \left(\frac{\delta G[\underrightarrow{\psi}(\mathbf{r})]}{\delta \psi(\mathbf{r})} \right) P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]$$

$$= -\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, G[\underrightarrow{\psi}(\mathbf{r})] \left(\frac{\delta P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]}{\delta \psi(\mathbf{r})} \right)$$

(132)

where

$$G[\underrightarrow{\psi}(\mathbf{r})] = \exp i \int d\mathbf{r} \left\{ \xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r}) \right\}$$

$$\times \exp i \int d\mathbf{r} \left\{ \xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) \right\} \exp i \int d\mathbf{r} \left\{ \xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r}) \right\} \quad (133)$$

is a functional of the four fields $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$ and $\psi(\mathbf{r})$ refers to any one of these. This step relies on the distribution function $P_b(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})$ going to zero on the boundaries of phase space, an assumption common to all correspondence rule derivations. Note that the functional differentiation of $P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ is well-defined, since $P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]$ is a functional of both the fields and their complex conjugates.

4.2.4. Real and Imaginary Field Components

Note that because $G[\underrightarrow{\psi}(\mathbf{r})]$ does not depend on the conjugate fields, its functional derivative with respect to any $\psi^*(\mathbf{r})$ is zero. Thus we also have

$$0 = \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \left(\frac{\delta G[\underrightarrow{\psi}(\mathbf{r})]}{\delta \psi^*(\mathbf{r})} \right) P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]$$

$$= -\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, G[\underrightarrow{\psi}(\mathbf{r})] \left(\frac{\delta P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]}{\delta \psi^*(\mathbf{r})} \right) \quad (134)$$

Adding an arbitrary multiple λ of the last equation to each side of (132) gives

$$\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \left(\frac{\delta G[\underrightarrow{\psi}(\mathbf{r})]}{\delta \psi(\mathbf{r})} \right) P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]$$

$$= -\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, G[\underrightarrow{\psi}(\mathbf{r})]$$

$$\times \left(\frac{\delta P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]}{\delta \psi(\mathbf{r})} + \lambda \frac{\delta P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]}{\delta \psi^*(\mathbf{r})} \right) \quad (135)$$

Noting that we can write the field in terms of its real, imaginary components and replace the distribution functional $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$ with an equivalent functional $F[\underset{\rightarrow}{\psi_X}(\mathbf{r}), \underset{\rightarrow}{\psi_Y}(\mathbf{r})]$ of the components

$$
\begin{aligned}
\psi(\mathbf{r}) &= \psi_X(\mathbf{r}) + i\psi_Y(\mathbf{r}) & \psi^*(\mathbf{r}) = \psi_X(\mathbf{r}) - i\psi_Y(\mathbf{r}) \\
\psi_X(\mathbf{r}) &= (\psi(\mathbf{r}) + \psi^*(\mathbf{r}))/2 & \psi_Y(\mathbf{r}) = (\psi(\mathbf{r}) - \psi^*(\mathbf{r}))/2i \\
\underset{\rightarrow}{\psi_X}(\mathbf{r}) &\equiv \{\psi_{CX}(\mathbf{r}), \psi_{CX}^+(\mathbf{r}), \psi_{NCX}(\mathbf{r}), \psi_{NCX}^+(\mathbf{r})\} \\
\underset{\rightarrow}{\psi_Y}(\mathbf{r}) &\equiv \{\psi_{CY}(\mathbf{r}), \psi_{CY}^+(\mathbf{r}), \psi_{NCY}(\mathbf{r}), \psi_{NCY}^+(\mathbf{r})\} & (136)
\end{aligned}
$$

$$
P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \equiv F[\underset{\rightarrow}{\psi_X}(\mathbf{r}), \underset{\rightarrow}{\psi_Y}(\mathbf{r})] \tag{137}
$$

then a straightford application of functional differentiation rules shows that by choosing $\lambda = 1$ or $\lambda = -1$ we have

$$
\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \left(\frac{\delta G[\underset{\rightarrow}{\psi}(\mathbf{r})]}{\delta\psi(\mathbf{r})} \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]
$$

$$
= -\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, G[\underset{\rightarrow}{\psi}(\mathbf{r})] \left(\frac{\delta F[\underset{\rightarrow}{\psi_X}(\mathbf{r}), \underset{\rightarrow}{\psi_Y}(\mathbf{r})]}{\delta\psi_X(\mathbf{r})} \right)
$$

$$\tag{138}$$

$$
= -\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, G[\underset{\rightarrow}{\psi}(\mathbf{r})] \left(\frac{\delta F[\underset{\rightarrow}{\psi_X}(\mathbf{r}), \underset{\rightarrow}{\psi_Y}(\mathbf{r})]}{\delta(i\psi_Y(\mathbf{r}))} \right)
$$

$$\tag{139}$$

This shows that functional differentiation of the distribution functional with respect to $\psi(\mathbf{r})$ is equivalent to functional differentiation of the related distribution functional $F[\underset{\rightarrow}{\psi_X}(\mathbf{r}), \underset{\rightarrow}{\psi_Y}(\mathbf{r})]$ with respect to either $\psi_X(\mathbf{r})$ or $i\psi_Y(\mathbf{r})$. This feature is useful if we wish to replace the fields by their real, imaginary components.

4.2.5. Applying the Correspondence Rules

In dealing with terms in the Liouville-von Neumann equation the density operator is often operated on by more than one field operator. To determine the overall effect on the quasi distribution functional it is necessary to carry out the above replacements in succession. A couple of examples illustrate the proceedure.

$$
\widehat{\Psi}_{NC}^{\dagger}(\mathbf{s}_1) \widehat{\rho} \widehat{\Psi}_C(\mathbf{s}_2)
$$

$$
\rightarrow \left(\psi_{NC}^+(\mathbf{s}_1) - \frac{\delta}{\delta\psi_{NC}(\mathbf{s}_1)} \right) \left(\psi_C(\mathbf{s}_2) - \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s}_2)} \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]
$$

$$
\widehat{\Psi}_C^{\dagger}(\mathbf{s}_1) \widehat{\rho} \widehat{\Psi}_C(\mathbf{s}_2)
$$

$$
\rightarrow \left(\psi_C^+(\mathbf{s}_1) - \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s}_1)} \right) \left(\psi_C(\mathbf{s}_2) - \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s}_2)} \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})].
$$

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Using the rules for functional differentiation we see that the differentiations can be carried out in either order.

In applying these rules to the BEC problem, the following functional derivative results can be obtained (see Appendix B, [82]) The general functions $\psi(\mathbf{r})$ and $\psi^+(\mathbf{r})$ each were used to cover the results for condensate and non-condensate modes. For the case where $\psi(\mathbf{r}) \equiv \psi_C(\mathbf{r})$ the restricted set K refers to the modes $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$, and for the non-condensate case where $\psi(\mathbf{r}) \equiv \psi_{NC}(\mathbf{r})$ the restricted set refers to the remaining modes $\phi_k(\mathbf{r})$. For the case where $\psi^+(\mathbf{r}) \equiv \psi_C^+(\mathbf{r})$ the restricted set K refers to the conjugate modes $\phi_1^*(\mathbf{r})$ and $\phi_2^*(\mathbf{r})$, and for the non-condensate case where $\psi^+(\mathbf{r}) \equiv \psi_{NC}^+(\mathbf{r})$ the restricted set K^* refers to the remaining conjugate modes $\phi_k^*(\mathbf{r})$. Because the coefficients are unrelated we are dealing with functionals such as the distribution functional $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$ in which the functions $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$ are mutually independent.

$$
\begin{aligned}
\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_C(\mathbf{r}) &= \delta_C(\mathbf{r}, \mathbf{s}) \\
\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_C^+(\mathbf{r}) &= \delta_{C+}(\mathbf{r}, \mathbf{s}) = \delta_C(\mathbf{s}, \mathbf{r}) \\
\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_C^+(\mathbf{r}) &= 0 \qquad \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_C(\mathbf{r}) = 0
\end{aligned}
\tag{140}
$$

with four other results obtained by replacing C by NC. Note the reverse order of $\mathbf{r}, \mathbf{s}$ in the second result. Similarly the functional derivatives of condensate fields with respect to non-condensate fields are zero, and vice-versa. Thus

$$
\begin{aligned}
\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_{NC}(\mathbf{r}) &= 0 \qquad \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_{NC}^+(\mathbf{r}) = 0 \\
\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_{NC}^+(\mathbf{r}) &= 0 \qquad \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_{NC}(\mathbf{r}) = 0
\end{aligned}
\tag{141}
$$

with four other results obtained by interchanging C and NC.

The product rule for functional derivatives

$$
\begin{aligned}
&\frac{\delta}{\delta\psi(\mathbf{s})}(F[\psi(\mathbf{r}), \psi^+(\mathbf{r})]G[\psi(\mathbf{r}), \psi^+(\mathbf{r})]) \\
=& (\frac{\delta}{\delta\psi(\mathbf{s})}F[\psi(\mathbf{r}), \psi^+(\mathbf{r})])G[\psi(\mathbf{r}), \psi^+(\mathbf{r})] + F[\psi(\mathbf{r}), \psi^+(\mathbf{r})](\frac{\delta}{\delta\psi(\mathbf{s})}G[\psi(\mathbf{r}), \psi^+(\mathbf{r})]) \\
&\frac{\delta}{\delta\psi^+(\mathbf{s})}(F[\psi(\mathbf{r}), \psi^+(\mathbf{r})]G[\psi(\mathbf{r}), \psi^+(\mathbf{r})]) \\
=& (\frac{\delta}{\delta\psi^+(\mathbf{s})}F[\psi(\mathbf{r}), \psi^+(\mathbf{r})])G[\psi(\mathbf{r}), \psi^+(\mathbf{r})] + F[\psi(\mathbf{r}), \psi^+(\mathbf{r})](\frac{\delta}{\delta\psi^+(\mathbf{s})}G[\psi(\mathbf{r}), \psi^+(\mathbf{r})])
\end{aligned}
\tag{142}
$$

is also needed. Here $\psi(\mathbf{r})$ refers to either $\psi_C(\mathbf{r})$ or $\psi_{NC}(\mathbf{r})$ and $\psi^+(\mathbf{r})$ refers to either $\psi_C^+(\mathbf{r})$ or $\psi_{NC}^+(\mathbf{r})$.

In addition the standard approach to space integration gives the result

$$\int d\mathbf{s}\, \{\partial_\mu C(\mathbf{s})\,\} = 0 \tag{143}$$

for functions $C(\mathbf{s})$ that become zero on the boundary. This then leads to the useful result involving product functions $C(\mathbf{s}) = A(\mathbf{s})B(\mathbf{s})$ enabling the spatial derivative to be applied to either $A(\mathbf{s})$ or $B(\mathbf{s})$

$$\int d\mathbf{s}\, \{\partial_\mu A(\mathbf{s})\,\} B(\mathbf{s}) = -\int d\mathbf{s}\, A(\mathbf{s})\, \{\partial_\mu B(\mathbf{s})\,\} \tag{144}$$

We can assume that the $\psi(\mathbf{s})$ and $\psi^+(\mathbf{s})$ become zero on the boundary, since they both involve condensate mode functions or their conjugates that are localised due to the trap potential. Also the functional derivatives produce linear combinations of either the condensate mode functions or their conjugates (see (131)) so the various $C(\mathbf{s})$ that will be involved should become zero on the boundary.

The results in this section also apply to the single mode case with obvious modifications, the sums over condensate modes now restricted to $k = 1$.

4.3. Condensate Functional Fokker-Planck Equation

The *functional Fokker-Planck equation* may be written in the form

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)\\
&= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H1} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H2}\\
&+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H3}
\end{aligned} \tag{145}$$

This is the sum from the terms in the Bogoliubov Hamiltonian of order N, $\sqrt{N}$ and $1/\sqrt{N}$ respectively. The derivation of the results for the Fokker-Planck equation is carried out in Appendix E ([82]).

4.3.1. The $\widehat{H}_1$ Terms

The contributions to the *functional Fokker-Planck equation* from the $\widehat{H}_1$ term, which is equal to the *condensate Hamiltonian*, may be written in the form

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H1}\\
&= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H1K} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H1V}\\
&+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H1U}
\end{aligned} \tag{146}$$

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of the sum of terms from the kinetic energy, the trap potential and the boson-boson interaction. Derivations of the form for each term are given in Appendix E ([82]). Here and elsewhere ∂_μ is short for $\frac{\partial}{\partial s_\mu}$.

H1K Terms - Single and Two-Mode Condensates. The contribution to the functional Fokker-Planck equation from the *condensate kinetic energy* is given by

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H1K}
$$
$$
= \frac{-i}{\hbar} \left\{ -\int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi_C^+(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ +\int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_C(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi_C(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \right\} \quad (147)
$$

H1V Terms - Single and Two-Mode Condensates. The contribution to the functional Fokker-Planck equation from the *condensate trap potential* is given by

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H1V}
$$
$$
= \frac{-i}{\hbar} \left\{ -\int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_C(\mathbf{s})} \{V(\mathbf{s}) \psi_C(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ +\int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \{V(\mathbf{s}) \psi_C^+(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \quad (148)
$$

H1U Terms - Single and Two-Mode Condensates . The contribution to the functional Fokker-Planck equation from the *condensate boson-boson interaction* is given by

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H1U}
$$
$$
= \frac{-i}{\hbar} \left\{ -\frac{g_N}{N} \int d\mathbf{s} \frac{\delta}{\delta \psi_C(\mathbf{s})} \left\{ (\psi_C^+(\mathbf{s}) \psi_C(\mathbf{s}) - \delta_C(\mathbf{s}, \mathbf{s})) \psi_C(\mathbf{s}) \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ +\frac{g_N}{N} \int d\mathbf{s} \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \left\{ (\psi_C^+(\mathbf{s}) \psi_C(\mathbf{s}) - \delta_C(\mathbf{s}, \mathbf{s})) \psi_C^+(\mathbf{s}) \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ \frac{g_N}{N} \int d\mathbf{s} \frac{\delta}{\delta \psi_C(\mathbf{s})} \frac{\delta}{\delta \psi_C(\mathbf{s})} \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \{ \frac{1}{4} \psi_C(\mathbf{s}) \} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ -\frac{g_N}{N} \int d\mathbf{s} \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \frac{\delta}{\delta \psi_C(\mathbf{s})} \{ \frac{1}{4} \psi_C^+(\mathbf{s}) \} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \quad (149)
$$

which involves first order and third order functional derivatives. The quantity $\delta_C(\mathbf{s}, \mathbf{s})$ is a diagonal element of the restricted delta function for condensate modes.

For the one mode case we note that

$$\int d\mathbf{s}\,\delta_C(\mathbf{s},\mathbf{s}) \quad = \quad 1 \tag{150}$$

$$\delta_C(\mathbf{s},\mathbf{s}) \quad = \quad |\phi_1(\mathbf{s})|^2 \tag{151}$$

corresponding to there being a single occupied condensate mode.

For the two mode case we have instead

$$\int d\mathbf{s}\,\delta_C(\mathbf{s},\mathbf{s}) \quad = \quad 2 \tag{152}$$

$$\delta_C(\mathbf{s},\mathbf{s}) \quad = \quad |\phi_1(\mathbf{s})|^2 + |\phi_2(\mathbf{s})|^2 \tag{153}$$

corresponding to there being two occupied condensate modes.

The total condensate number given by

$$\iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+ \int d\mathbf{s}(\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}))P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})] \tag{154}$$

which is of order N. The result of order N for the last expression indicates that the important contributions to the functional integral are where the condensate fields are of order $\sqrt{N}$. Similar considerations for the much smaller total non-condensate number indicate that the most important contributions are where the non-condensate fields are much smaller than $\sqrt{N}$.

Similar expressions for the functional Fokker-Planck equation in the case of a pure Wigner representation (but not involving a doubled phase space) are given in the paper by Steel et al [55] (see Eq. (23)). Comparisons can be made after substituting $\psi_C^+(\mathbf{s})$ with $\psi_C^*(\mathbf{s})$. In their result however, the restricted delta function $\delta_C(\mathbf{s},\mathbf{s})$ term in the condensate interaction contribution is replaced by 1. For the single condensate mode case unity is of course the integral of the restricted delta function, but it is not equal to it.

4.3.2. The $\widehat{H}_2$ Term

The contributions to the *functional Fokker-Planck equation* from the $\widehat{H}_2$ term, which is equal to terms in the interaction between the *condensate* and *non-condensate Hamiltonian* that are *linear* in the non-condensate fields, may be written in the form

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{H2}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{H2U4} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{H2U2} \tag{155}$$

These two contributions may be written as the sum of terms which are linear, quadratic, cubic and quartic in the number of functional derivatives. Derivations of the form for each term are given in Appendix E ([82]).

H2U4 Terms - Single and Two-Mode Condensates. The contribution to the functional Fokker-Planck equation from the $\widehat{H}_{2U4}$ term is given by

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H2U4}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{H2U4} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{H2U4}$$

$$+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{H2U4} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^4_{H2U4} \qquad (156)$$

where

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{H2U4}$$

$$= \frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\{[2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{\mathbf{C}}^+(\mathbf{s})}\right)\{[\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{\mathbf{C}}(\mathbf{s})}\right)\{[2\psi_{\mathbf{C}}(\mathbf{s})\psi_C^+(\mathbf{s})-\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{[\psi_C(\mathbf{s})\psi_C(\mathbf{s})]\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{[\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{[\psi_C(\mathbf{s})\psi_C^+(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(157)$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{H2U4}$$

$$= \frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_C(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(158)$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{H2U4}$$

$$= \frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(159)$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{4}_{H2U4}$$

$$= \frac{-i}{\hbar}\left\{\frac{g_{N}}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_{N}}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^{+}(\mathbf{s})}\right)\{\tfrac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

$$(160)$$

The contribution to the functional Fokker-Planck equation from the $\widehat{H}_{2U2}$ term is

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)_{H2U2}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{1}_{H22} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{2}_{H22} \qquad (161)$$

H2U2 Terms - Two-Mode Condensate. For the two mode condensate case

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{1}_{H2U2}$$

$$= \frac{-i}{\hbar}\left\{-\frac{g_{N}}{N}\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{u})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi_{NC}^{+}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_{N}}{N}\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\{F(\mathbf{u},\mathbf{s})^{*}\,\psi_{NC}(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_{N}}{N}\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{NC}^{+}(\mathbf{u})}\right)\{F(\mathbf{u},\mathbf{s})^{*}\,\psi_{C}^{+}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_{N}}{N}\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi_{C}(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

and

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{2}_{H2U2}$$

$$= \frac{-i}{\hbar}\left\{+\frac{g_{N}}{N}\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{u})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}F(\mathbf{s},\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_{N}}{N}\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^{+}(\mathbf{u})}\right)\{\tfrac{1}{2}\mathbf{F}(\mathbf{u},\mathbf{s})^{*}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}$$

These terms now involve double spatial integrals, and in the case of the quadratic term there are second order functional derivatives with respect to field functions at different spatial positions. This is different to the standard functional Fokker-Planck equation and requires special considerations for conversion to Ito stochastic equations for the field functions. The linear term is not so difficult to

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treat, though it still leads to an integro-differential equation. By changing the spatial variables we see that the linear term is

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)^1_{H2U2}
$$
$$
= \frac{-i}{\hbar} \left\{ -\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_C^+(\mathbf{s})} \right) \left\{ \int d\mathbf{u}\, F(\mathbf{u}, \mathbf{s})\, \psi_{NC}^+(\mathbf{u}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ +\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_C(\mathbf{s})} \right) \left\{ \int d\mathbf{u}\, F(\mathbf{u}, \mathbf{s})^*\, \psi_{NC}(\mathbf{u}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ -\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}^+(\mathbf{s})} \right) \left\{ \int d\mathbf{u}\, F(\mathbf{s}, \mathbf{u})^*\, \psi_C^+(\mathbf{u}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ +\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}(\mathbf{s})} \right) \left\{ \int d\mathbf{u}\, F(\mathbf{s}, \mathbf{u})\, \psi_C(\mathbf{u}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$

$$(162)$$

so the quantity inside the inner brackets is just another functional. The quadratic term is left unchanged except for interchanging positions to make the expression more symmetrical

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)^2_{H2U2}
$$
$$
= \frac{-i}{\hbar} \left\{ +\frac{g_N}{N} \int\int d\mathbf{s}\, d\mathbf{u} \left\{ \left(\frac{\delta}{\delta \psi_C^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi_{NC}(\mathbf{u})} \right) \left\{ \frac{1}{2} F(\mathbf{u}, \mathbf{s}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ -\frac{g_N}{N} \int\int d\mathbf{s}\, d\mathbf{u} \left\{ \left(\frac{\delta}{\delta \psi_C(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi_{NC}^+(\mathbf{u})} \right) \left\{ \frac{1}{2} \mathbf{F}(\mathbf{u}, \mathbf{s})^* \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$

$$(163)$$

H2U2 Term - Single Mode Condensate. For the case of a single mode condensate the result is simpler

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)^1_{H2U2}
$$
$$
= \frac{-i}{\hbar} \left\{ -\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_C^+(\mathbf{s})} \right) \left\{ \left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle \psi_{NC}^+(\mathbf{s}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ +\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_C(\mathbf{s})} \right) \left\{ \left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle \psi_{NC}(\mathbf{s}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ -\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}^+(\mathbf{s})} \right) \left\{ \left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle \psi_C^+(\mathbf{s}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ +\frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}(\mathbf{s})} \right) \left\{ \left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle \psi_C(\mathbf{s}) \right\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$

$$(164)$$

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$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{H2U2}$$

$$= \frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\frac{1}{2}\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(165)$$

Derivations of the form for each term are given in Appendix E ([82]). We can show using the particular form of $F(\mathbf{r},\mathbf{s})$ for a single mode condensate, that the results for the single mode condensate can be obtained from those for the two mode condensate (see Appendix E, [82]).

4.3.3. The $\widehat{H}_3$ Term

The contributions to the *functional Fokker-Planck equation* from the $\widehat{H}_3$ term, which is equal to the sum of the kinetic energy and trap potential terms in the *non-condensate* Hamiltonian plus the terms in the interaction between the *condensate* and *non-condensate* that are *quadratic* in the non-condensate fields, may be written in the form

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H3}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H3K} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H3V}$$

$$+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H3U} \qquad (166)$$

Derivations of the form for each term are given in Appendix E ([82]).

H3K Terms - Single and Two-Mode Condensates. The contribution to the functional Fokker-Planck equation from the *non-condensate kinetic energy* term is

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{H3K}$$

$$= \frac{-i}{\hbar}\left\{-\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\left(\sum_\mu\frac{\hbar^2}{2m}\partial_\mu^2\psi_{NC}^+(\mathbf{s})\right)P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}\right\}$$

$$+\frac{-i}{\hbar}\left\{+\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\left(\sum_\mu\frac{\hbar^2}{2m}\partial_\mu^2\psi_{NC}(\mathbf{s})\right)P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})])\right\}\right\}$$

$$(167)$$

H3V Terms - Single and Two-Mode Condensates. The contribution to the functional Fokker-Planck equation from the *non-condensate trap potential* term is

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H3V}
$$
$$
= \frac{-i}{\hbar} \left\{ - \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_{NC}(\mathbf{s})} \{V(\mathbf{s})\psi_{NC}(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ + \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_{NC}^+(\mathbf{s})} V(\mathbf{s})\psi_{NC}^+(\mathbf{s}) \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \quad (168)
$$

H3U Terms - Single and Two-Mode Condensates. The contribution to the functional Fokker-Planck equation from the $\widehat{H}_{3U}$ term is

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H3U}
$$
$$
= \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H3U}^1 + \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H3U}^2
$$
$$
+ \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H3U}^3 + \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H3U}^4 \quad (169)
$$

where

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{H3U}^1
$$
$$
= \frac{-i}{\hbar} \left\{ + \frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_C^+(\mathbf{s})} \right) \{ [\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s}) + 2\psi_C^+(\mathbf{s})\psi_{NC}(\mathbf{s})]\psi_{NC}^+(\mathbf{s}) \} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ - \frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_C(\mathbf{s})} \right) \{ [\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s}) + 2\psi_C(\mathbf{s})\psi_{NC}^+(\mathbf{s})]\psi_{NC}(\mathbf{s}) \} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ + \frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}^+(\mathbf{s})} \right) \{ [\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})]\psi_{NC}(\mathbf{s}) \} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ - \frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}(\mathbf{s})} \right) \{ [\psi_C(\mathbf{s})\psi_C(\mathbf{s})]\psi_{NC}^+(\mathbf{s}) \} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ + \frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}^+(\mathbf{s})} \right) \{ [2\psi_C(\mathbf{s})\psi_C^+(\mathbf{s}) - \delta_C(\mathbf{s}, \mathbf{s})]\psi_{NC}^+(\mathbf{s}) \} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
+ \frac{-i}{\hbar} \left\{ - \frac{g_N}{N} \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi_{NC}(\mathbf{s})} \right) \{ [2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - \delta_C(\mathbf{s}, \mathbf{s})]\psi_{NC}(\mathbf{s}) \} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}
$$
$$
(170)
$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{H3U}$$

$$= \frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s})+\psi_C^+(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s})+\psi_C(\mathbf{s})\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_C^+(\mathbf{s})\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\tag{171}$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{H3U}$$

$$= \frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\tag{172}$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^4_{H3U}$$

$$= \frac{-i}{\hbar}\left\{\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\frac{g_N}{N}\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\tag{173}$$

Derivations of the form for each term are given in Appendix E ([82]).

5. Ito stochastic equations

In this section we show how the functional Fokker-Planck equations for the phase space distribution functional are equivalent to Ito stochastic equations for stochastic fields. This first involves truncating the Fokker-Planck equations to only include terms with at most second order functional derivatives. The stochastic fields are defined via the expansion of the phase space field functions in terms of mode functions and then treating the expansion coefficients as stochastic variables. The derivation of the Ito equations for the stochastic fields is based on well-known Ito equations for stochastic expansion coefficients. The Ito stochastic field equations are the sum of a deterministic term associated with the first order functional derivatives in the FFPE (the drift terms) and a quantum noise term associated with the second order functional derivatives in the FFPE (the diffusion terms). The two mode condendsate case results in non-local drift and diffusion terms, so a special treatment is required to derive the Ito equations. Results for the Ito equations for the stochastic condensate and non-condensate fields are obtained for the two mode condensate case. Also, the corresponding simpler Ito equations for the single mode condensate case are presented. In this section we emphasise how the phase space distribution functionals which determine the quantum correlation functions can then be replaced by stochastic averages involving products of the stochastic condensate and non-condensate fields.

5.1. General Results

The derivation of Ito stochastic equations the the condensate and non-condensate fields is based on approximating the functional Fokker-Planck equation by neglecting all terms involving *third* and *fourth order* functional derivatives. The justification for this is as follows. The condensate fields are of order $\sqrt{N}$ in the regions of phase space important to the determination of the correlation functions via the functional integrals (108), whereas the non-condensate fields are much smaller. Hence terms like the third order functional derivatives in (149) scale like $1/N^2$ whereas the second order functional derivatives in (158) scale like $1/\sqrt{N}$. This enables all such third and fourth order terms from the functional Fokker-Planck equation based on the Bogoliubov Hamiltonian to be discarded. The resulting functional Fokker-Planck equation is then in a standard form involving just first and second order functional derivatives, from which Ito stochastic equations can be obtained.

The remaining *first* and *second order* functional derivative terms that are left are referred to as the drift and diffusion terms respectively, and the Ito stochastic equations for the stochastic fields can expressed in terms of the drift and diffusion terms. The stochastic fields will be indicated with a tilde, $\widetilde{\psi}_C(\mathbf{s},t)$, $..,\widetilde{\psi}^+_{NC}(\mathbf{s},t)$. The Ito stochastic field equations are the sum of two terms. The first is obtained from the drift term in the functional Fokker-Planck equation and is the so-called deterministic term, the second is obtained from the diffusion term and is the stochastic noise term. The stochastic fields are expanded in terms of

a convenient set of real,orthonormal mode functions, with the expansion coefficients regarded as stochastic quantities. The original stochastic noise terms in the Ito stochastic field equations depends on two types of stochastic quantities. One type are stochastic space dependent fields that involve the mode functions and quantities depending on the stochastic expansion coefficients that are obtained from the diffusion terms. The other type are time dependent stochastic Gaussian-Markov noise terms that would be the noise terms in Ito equations for the expansion coefficients. The derivation of the Ito equations for the stochastic fields is based on well-known Ito equations for stochastic expansion coefficients. Details of the derivation of the Ito stochastic equations are given in Appendix F ([82]). Here we will summarise the key features and results.

5.1.1. Symmetric Form of Functional Fokker-Planck Equation

The derivation begins with the functional Fokker-Planck equation set out in Section 4, but now with all terms having functional derivatives of third and fourth order ignored. For convenience we now introduce a simpler notation for listing the fields, namely we list $\psi_C \equiv \psi_{C-}, \psi_C^+ \equiv \psi_{C+}, \psi_{NC} \equiv \psi_{NC-}, \psi_{NC}^+ \equiv \psi_{NC+}$ as $\psi_1, \psi_2, \psi_3, \psi_4$ respectively. Now with $\underset{\rightarrow}{\psi}(\mathbf{r}) \equiv \{\psi_1(\mathbf{r}), \psi_2(\mathbf{r}), \psi_3(\mathbf{r}), \psi_4(\mathbf{r})\} \equiv \{\psi_K(\mathbf{r})\}$ and $\underset{\rightarrow}{\psi^*}(\mathbf{r}) \equiv \{\psi_1^*(\mathbf{r}), \psi_2^*(\mathbf{r}), \psi_3^*(\mathbf{r}), \psi_4^*(\mathbf{r})\} \equiv \{\psi_K^*(\mathbf{r})\}$ the functional Fokker-Planck equations from Section 4 are as follows.

For the *two mode* condensate case we have.

$$
\begin{aligned}
\frac{\partial P}{\partial t} &= \sum_A \int dx \frac{\delta}{\delta\psi_A(x)} A_A(\underset{\rightarrow}{\psi}(x), x) P \\
&+ \sum_{A \leq B} \int \int dx\, dy \frac{\delta}{\delta\psi_A(x)} \frac{\delta}{\delta\psi_B(y)} H_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y) P \quad (174)
\end{aligned}
$$

and for the *single mode* condensate case

$$
\begin{aligned}
\frac{\partial P}{\partial t} &= \sum_A \int dx \frac{\delta}{\delta\psi_A(x)} A_A(\underset{\rightarrow}{\psi}(x), x) P \\
&+ \sum_{A \leq B} \int dx \frac{\delta}{\delta\psi_A(x)} \frac{\delta}{\delta\psi_B(x)} H_{AB}(\underset{\rightarrow}{\psi}(x), x) P \quad (175)
\end{aligned}
$$

Here we use x, y to denote the spatial variables and in accord with the expressions in Section 4 the restriction to $A \leq B$ in the double sum is to avoid repetition of double functional derivatives. Since there are four fields involved $\int A, B = 1, 2, 3, 4$. In both cases the distribution functional is $P[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi^*}]$ and $A_A(\underset{\rightarrow}{\psi}(x), x)$ is the A element of a *drift column vector*. For the single mode condensate case $H_{AB}(\underset{\rightarrow}{\psi}(x), x)$ is the A, B element of a *local diffusion matrix*, and for the two mode condensate case $H_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y)$ is the $A; B$ element of a *non-local diffusion matrix*. In the latter case a double spatial integral is involved. Also, A_A and H_{AB} may depend on spatial derivatives $\partial_x \psi_K(x)$.etc. but in order to avoid too many symbols we have not shown this. For simplicity

the Fokker-Planck equation has been written with just one-dimensional spatial variables x, y, but the generalisation to three dimensional variables $\mathbf{r}, \mathbf{s}$ is straight-forward.

To proceed further the functional Fokker-Planck equations need to be recast with a symmetrical diffusion term. The details are covered in Appendix F ([82]). If we define a new diffusion matrix such that

$$
\begin{aligned}
D_{AB}(\underrightarrow{\psi}(x), x, \underrightarrow{\psi}(y), y) &= H_{AB}(\underrightarrow{\psi}(x), x, \underrightarrow{\psi}(y), y) && A < B \\
D_{AB}(\underrightarrow{\psi}(x), x, \underrightarrow{\psi}(y), y) &= H_{BA}(\underrightarrow{\psi}(y), y, \underrightarrow{\psi}(x), x) && A > B \\
D_{AA}(\underrightarrow{\psi}(x), x, \underrightarrow{\psi}(y), y) &= H_{AA}(\underrightarrow{\psi}(x), x, \underrightarrow{\psi}(y), y) + H_{AA}(\underrightarrow{\psi}(y), y, \underrightarrow{\psi}(x), x) && A = B
\end{aligned}
\tag{176}
$$

we see that the functional Fokker-Planck equation for the two mode case becomes

$$
\begin{aligned}
\frac{\partial P}{\partial t} =\ & \sum_A \int dx\, \frac{\delta}{\delta \psi_A(x)} A_A(\underrightarrow{\psi}(x), x)\, P \\
& + \frac{1}{2} \sum_{A,B} \int\int dx\, dy\, \frac{\delta}{\delta \psi_A(x)} \frac{\delta}{\delta \psi_B(y)} D_{AB}(\underrightarrow{\psi}(x), x, \underrightarrow{\psi}(y), y)\, P
\end{aligned}
\tag{177}
$$

The expressions have been defined so that D_{AB} is symmetric. For the two mode condensate case

$$
D_{AB}(\underrightarrow{\psi}(x), x, \underrightarrow{\psi}(y), y) = D_{BA}(\underrightarrow{\psi}(y), y, \underrightarrow{\psi}(x), x)
\tag{178}
$$

For the *single mode* condensate case we may also write the functional Fokker-Planck equation in the symmetric form

$$
\begin{aligned}
\frac{\partial P}{\partial t} =\ & \sum_A \int dx\, \frac{\delta}{\delta \psi_A(x)} A_A(\underrightarrow{\psi}(x), x)\, P \\
& + \frac{1}{2} \sum_{A,\,B} \int dx\, \frac{\delta}{\delta \psi_A(x)} \frac{\delta}{\delta \psi_B(x)} D_{AB}(\underrightarrow{\psi}(x), x)\, P
\end{aligned}
\tag{179}
$$

The proof is similar but now

$$
\begin{aligned}
D_{AB}(\underrightarrow{\psi}(x), x) &= H_{AB}(\underrightarrow{\psi}(x), x) && A < B \\
D_{AB}(\underrightarrow{\psi}(x), x) &= H_{BA}(\underrightarrow{\psi}(x), x) && A > B \\
D_{AA}(\underrightarrow{\psi}(x), x) &= 2H_{AA}(\underrightarrow{\psi}(x), x) && A = B
\end{aligned}
\tag{180}
$$

and again D_{AB} is symmetric.

$$
D_{AB}(\underrightarrow{\psi}(x), x) = D_{BA}(\underrightarrow{\psi}(x), x)
\tag{181}
$$

Results (180) and (176) enable us to identify the diffusion coefficients in the general forms (179) and (177) from those in the original functional Fokker-Planck equation forms (175) and (174).

5.1.2. Fokker-Planck Equation for Distribution Function

The field functions $\psi_A(x)$ may be expanded

$$\psi_A(x) = \sum_i \alpha_i^A \, \xi_i^A(x) \tag{182}$$

where the $\xi_i^A(x)$ are a convenient set of orthonormal mode functions for the A field satisfying

$$\int dx \xi_i^A(x)^* \xi_j^A(x) \;\; = \;\; \delta_{ij} \tag{183}$$

$$\sum_i \xi_i^A(x) \xi_i^A(y)^* \;\; = \;\; \delta(x-y) \tag{184}$$

For the various $\psi_A(x)$ these orthonormal mode functions may be interrelated. Thus if for $\psi_1(x) \equiv \psi_C(x)$ the mode functions are $\xi_i(x)$ $(i=1,2)$, then those for $\psi_2(x) \equiv \psi_C^+(x)$ are $\xi_i(x)^*$ $(i=1,2)$. Mode functions for different fields also may be orthogonal, thus for $\psi_3(x) \equiv \psi_{NC}(x)$ if the mode functions are $\xi_i(x)$ $(i \neq 1,2)$, and those for $\psi_4(x) \equiv \psi_{NC}^+(x)$ are $\xi_i(x)^*$ $(i \neq 1,2)$, then the $\xi_i^1(x)$ and $\xi_i^3(x)$ are mutually orthogonal, as are $\xi_i^2(x)$ and $\xi_i^4(x)$. However, these features are not required, the main requirement is that the mode functions for each specific field are orthonormal. The mode functions may be time dependent, but this will not be made explicit.

The derivation of the Ito stochastic field equation is based on first converting the functional Fokker-Planck equation to an ordinary *Fokker-Planck equation* via expanding the field functions and replacing the functional derivatives with ordinary derivatives

$$A_A(\underset{\rightarrow}{\psi}(x), x) \;\; \rightarrow \;\; \mathcal{A}^A(\underset{\rightarrow}{\alpha})$$

$$D_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y) \text{ or } D_{AB}(\underset{\rightarrow}{\psi}(x), x) \;\; \rightarrow \;\; \mathcal{D}^{AB}(\underset{\rightarrow}{\alpha})$$

$$P[\underset{\rightarrow}{\psi}, \underset{\rightarrow}{\psi}^*] \;\; \rightarrow \;\; P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*)$$

$$\frac{\delta}{\delta \psi_A(x)} \;\; \rightarrow \;\; \sum_i \xi_i^A(x)^* \frac{\partial}{\partial \alpha_i^A} \tag{185}$$

where $\mathcal{A}^A$ is the *drift vector*, $\mathcal{D}^{AB}$ is the symmetric *diffusion matrix* and $P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*)$ is the phase space *distribution function*. The drift and diffusion elements depend on the expansion coefficients $\underset{\rightarrow}{\alpha} \equiv \{\alpha_k, \alpha_k^+\}$ and the distribution function depends on $\underset{\rightarrow}{\alpha}^* \equiv \{\alpha_k^*, \alpha_k^{+*}\}$ also. The explicit expressions are

$$\mathcal{A}_i^A(\underset{\rightarrow}{\alpha}) \;\; = \;\; \int dx \, \xi_i^A(x)^* A_A(\underset{\rightarrow}{\psi}(x), x) \tag{186}$$

$$\mathcal{D}_{ij}^{AB}(\underset{\rightarrow}{\alpha}) \;\; = \;\; \int \int dx \, dy \, \xi_i^A(x)^* D_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y) \, \xi_j^B(y)^* \qquad Two\,Mode$$

$$\mathcal{D}_{ij}^{AB}(\underset{\rightarrow}{\alpha}) \;\; = \;\; \int \int dx \, \xi_i^A(x)^* D_{AB}(\underset{\rightarrow}{\psi}(x), x) \, \xi_j^B(x)^* \qquad One\,Mode \tag{187}$$

These relationships can be inverted using the completeness relationships to give

$$A_A(\underset{\rightarrow}{\psi}(x), x) = \sum_i \xi_i^A(x) \mathcal{A}_i^A(\underset{\rightarrow}{\alpha}) \tag{188}$$

$$D_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y) = \sum_{ij} \xi_i^A(x) \mathcal{D}_{ij}^{AB}(\underset{\rightarrow}{\alpha}) \xi_j^B(y) \qquad Two\ Mode$$

$$D_{AB}(\underset{\rightarrow}{\psi}(x), x)\delta(x-y) = \sum_{ij} \xi_i^A(x) \mathcal{D}_{ij}^{AB}(\underset{\rightarrow}{\alpha}) \xi_j^B(y) \qquad One\ Mode \tag{189}$$

The diffusion matrix is symmetric

$$\mathcal{D}_{ji}^{BA}(\underset{\rightarrow}{\alpha}) = \mathcal{D}_{ij}^{AB}(\underset{\rightarrow}{\alpha}) \tag{190}$$

this result being easily obtained from (181) or (178). As a result we can always write the diffusion matrix $\mathcal{D}$ in the form

$$\mathcal{D} = \mathcal{B}\mathcal{B}^T \tag{191}$$

where $\mathcal{B}$ has the same dimension as $\mathcal{D}$. This result is known as the Takagi factorisation [85]. A proof may be found in the textbook by Horn et al. [86]. A non-square matrix $\mathcal{B}$ can also be found, this is shown in Appendix F ([82]).

The ordinary *Fokker-Planck equation* that is obtained is given by

$$
\begin{aligned}
\frac{\partial P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*)}{\partial t} &= \sum_{Ai} \frac{\partial}{\partial \alpha_i^A} \mathcal{A}_i^A(\underset{\rightarrow}{\alpha}) P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*) \\
&\quad + \frac{1}{2} \sum_{Ai\,Bj} \frac{\partial}{\partial \alpha_i^A} \frac{\partial}{\partial \alpha_j^B} \mathcal{D}_{i;j}^{A;B}(\underset{\rightarrow}{\alpha}) P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*)
\end{aligned} \tag{192}
$$

This Fokker-Planck equation is equivalent to Ito stochastic equations, as is described in standard textbooks (see [83], [84]). The procedure involves replacing the time independent phase space variables α_i^A by time dependent stochastic variables $\widetilde{\alpha}_i^A(t)$. The Ito stochastic equations for the $\widetilde{\alpha}_i^A(t)$ are such that phase space averages of functions of the α_i^A give the same result as stochastic averages of the same functions of the $\widetilde{\alpha}_i^A(t)$. The derivation of the Ito stochastic equations requires that the complex diffusion matrix $\mathcal{D}$ is symmetric, a result we have now obtained.

5.1.3. Ito Equations for Stochastic Expansion Coefficients

The *Ito equations* for the stochastic expansion coefficients $\widetilde{\alpha}_i^C$ can be written in several forms

$$
\begin{aligned}
\delta\widetilde{\alpha}_i^A(t) &= \widetilde{\alpha}_i^A(t+\delta t) - \widetilde{\alpha}_i^A(t) \\
&= -\mathcal{A}_i^A(\underset{\rightarrow}{\widetilde{\alpha}}(t))\delta t + \sum_{Dk} \mathcal{B}_{i;k}^{A;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t)) \int_t^{t+\delta t} dt_1 \Gamma_k^D(t_1) \tag{193}
\end{aligned}
$$

$$\frac{d}{dt}\widetilde{\alpha}_i^A(t) = -\mathcal{A}_i^A(\underset{\rightarrow}{\widetilde{\alpha}}(t)) + \sum_{Dk} \mathcal{B}_{i;k}^{A;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t)) \frac{d}{dt} w_k^D(t) \tag{194}$$

$$= -\mathcal{A}_i^A(\underset{\rightarrow}{\widetilde{\alpha}}(t)) + \sum_{Dk} \mathcal{B}_{i;k}^{A;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t)) \Gamma_k^D(t_+) \tag{195}$$

where $\underset{\rightarrow}{\widetilde{\alpha}}(t) \equiv \{\widetilde{\alpha}_i^A(t)\} \equiv \{\widetilde{\alpha}_k(t), \widetilde{\alpha}_k^+(t)\}$ and the matrix $\mathcal{B}$ is related to the diffusion matrix $\mathcal{D}$ as in (191).

$$\mathcal{D}_{ij}^{A;B}(\underset{\rightarrow}{\widetilde{\alpha}}(t)) = \sum_{Dk} \mathcal{B}_{i;k}^{A;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t)) \mathcal{B}_{j;k}^{B;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t)) \qquad (196)$$

The matrix elements $\mathcal{B}_{i;k}^{A;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t))$ are functions of the $\widetilde{\alpha}_i^A(t)$. The quantity t_+ is to indicate that if the Ito stochastic equation is integrated from t to $t + \delta t$, the Gaussian-Markoff noise term is integrated over this interval whilst the $\mathcal{A}_i^A(\widetilde{\alpha}_j^C(t))$ and $\mathcal{B}_{i;k}^{A;D}(\widetilde{\alpha}_j^C(t))$ are left at time $t..$

The quantities $w_k^D(t)$ and $\Gamma_k^D(t)$ are Wiener and Gaussian-Markoff stochastic variables. The *Gaussian-Markoff* quantities Γ_k^D satisfy the stochastic averaging results

$$
\begin{aligned}
\overline{\Gamma_k^D(t_1)} &= 0 \\
\overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\}} &= \delta_{DE}\delta_{kl}\delta(t_1 - t_2) \\
\overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\Gamma_m^F(t_3)\}} &= 0 \\
\overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\Gamma_m^F(t_3)\Gamma_n^G(t_4)\}} &= \overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\}}\, \overline{\{\Gamma_m^F(t_3)\Gamma_n^G(t_4)\}} \\
&\quad + \overline{\{\Gamma_k^E(t_1)\Gamma_m^F(t_3)\}}\, \overline{\{\Gamma_l^E(t_2)\Gamma_n^G(t_4)\}} \\
&\quad + \overline{\{\Gamma_k^D(t_1)\Gamma_n^G(t_4)\}}\, \overline{\{\Gamma_l^E(t_2)\Gamma_m^F(t_3)\}} \\
& \quad ...
\end{aligned}
\qquad (197)
$$

with stochastic averages being denote with a bar. The stochastic average of an odd number of noise terms is always zero, whilst that for an even number is the sum of all products of stochastic averages of two noise terms. The Gaussian-Markoff noise terms Γ_k^D are related to the *Wiener* stochastic variables w_k^D via

$$w_k^D(t) = \int_0^t dt_1\, \Gamma_k^D(t_1) \qquad (198)$$

$$\delta w_k^D(t) = w_k^D(t + \delta t) - w_k^D(t) = \int_t^{t+\delta t} dt_1\, \Gamma_k^D(t_1) \qquad (199)$$

$$\frac{d}{dt} w_k^D(t) = \lim_{\delta t \to 0} \left(\frac{\delta w_k^D(t)}{\delta t} \right) = \Gamma_k^D(t_+) \qquad (200)$$

One of the rules in stochastic averaging is

$$\overline{\sum_a F_a(\underset{\rightarrow}{\widetilde{\alpha}}(t))} = \sum_a \overline{F_a(\underset{\rightarrow}{\widetilde{\alpha}}(t))} \qquad (201)$$

so the stochastic average of the sum is the sum of the stochastic averages. Also,in Ito stochastic calculus the noise terms $\Gamma_k^D(t_1)$ within the interval $t, t + \delta t$ are uncorrelated with any function of the $\widetilde{\alpha}_i^C(t)$ at the earlier time t, so that the stochastic average of the product of such a function with a product of the noise

terms factorises

$$\overline{F(\widetilde{\underset{\rightarrow}{\alpha}}(t_1))\{\Gamma_k^D(t_2)\Gamma_l^E(t_3)\Gamma_m^F(t_4)...\Gamma_a^X(t_l)\}}$$
$$= \overline{F(\widetilde{\underset{\rightarrow}{\alpha}}(t_1))}\,\overline{\{\Gamma_k^D(t_2)\Gamma_l^E(t_3)\Gamma_m^F(t_4)...\Gamma_a^X(t_l)\}} \qquad t_1 < t_2, t_3, .., t_l \quad (202)$$

These key features of Ito stochastic calculus are important in deriving the properties of the noise fields in the stochastic field equations.

5.1.4. Derivation of Ito Stochastic Field Equations

The *stochastic fields* $\widetilde{\psi}_A(x,t)$ are defined via the same expansion as for the time independent field functions $\psi_A(x)$ by replacing the time independent phase space variables α_i^A by time dependent stochastic variables $\widetilde{\alpha}_i^A(t)$

$$\widetilde{\psi}_A(x,t) = \sum_i \widetilde{\alpha}_i^A(t)\xi_i^A(x) \qquad (203)$$

The expansion coefficents in (203) are restricted to those required in expanding the particular field function $\psi_A(x)$. Also, stochastic variations in $\widetilde{\psi}_A(x,t)$ are chosen as to only being due to stochastic fluctuations in the $\widetilde{\alpha}_i^A(t)$. Although the mode functions may be time dependent, their time variations are not stochastic in origin, so the stochastic field equations for the $\widetilde{\psi}_A(x,t)$ do not allow for time variations in the mode functions.

The *Ito stochastic equation* for the stochastic fields $\widetilde{\psi}_A(x,t)$ can then be derived from the Ito stochastic equations for the expansion coefficients. Using (188) the drift term in the stochastic equation gives

$$-\sum_i \mathcal{A}_i^A(\widetilde{\underset{\rightarrow}{\alpha}}(t))\,\xi_i^A(x)\,\delta t = -A_A(\widetilde{\underset{\rightarrow}{\psi}}(x,t))\delta t \qquad (204)$$

which involves the drift vector A_A evaluated at the stochastic fields $\widetilde{\underset{\rightarrow}{\psi}}(x,t)$. The diffusion term in the stochastic equation gives

$$\sum_i \sum_{Dk} \mathcal{B}_{i;k}^{A;D}(\widetilde{\underset{\rightarrow}{\alpha}}(t))\,\xi_i^A(x)\int_t^{t+\delta t}dt_1\Gamma_k^D(t_1) = \sum_{Dk}\eta_k^{A;D}(\widetilde{\underset{\rightarrow}{\psi}}(x,t))\int_t^{t+\delta t}dt_1\Gamma_k^D(t_1)$$
$$(205)$$

where

$$\eta_k^{A;D}(\widetilde{\underset{\rightarrow}{\psi}}(x,t)) = \sum_i \mathcal{B}_{i;k}^{A;D}(\widetilde{\underset{\rightarrow}{\alpha}}(t))\,\xi_i^A(x) \qquad (206)$$

is related via $\mathcal{B}_{i;k}^{A;D}(\widetilde{\underset{\rightarrow}{\alpha}}(t))$ to the diffusion matrix D_{AB} evaluated at the stochastic fields $\widetilde{\underset{\rightarrow}{\psi}}(x,t)$ or $\widetilde{\underset{\rightarrow}{\psi}}(x,t)$, $\widetilde{\underset{\rightarrow}{\psi}}(y,t)$.

The stochastic field equations can then be written in several ways

$$
\begin{aligned}
\delta\widetilde{\psi}_A(x,t) &= \widetilde{\psi}_A(x,t+\delta t) - \widetilde{\psi}_A(x,t) \\
&= -A_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t))\delta t + \sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))\int_t^{t+\delta t} dt_1\Gamma_k^D(t_1) \\
&= -A_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t))\delta t + \delta\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+)) \quad\quad (207) \\
\frac{\partial}{\partial t}\widetilde{\psi}_A(x,t) &= -A_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t)) + \sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))\frac{d}{dt}w_k^D(t) \\
&= -A_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t)) + \sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))\,\Gamma_k^D(t_+) \\
&= -A_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t)) + \frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+)) \quad\quad (208)
\end{aligned}
$$

Here we denote $\underset{\rightarrow}{\widetilde{\psi}}(x,t) \equiv \{\widetilde{\psi}_A(x,t)\} \equiv \{\widetilde{\psi}_1(x,t), \widetilde{\psi}_2(x,t), \widetilde{\psi}_3(x,t), \widetilde{\psi}_4(x,t)\}$ and $\underset{\rightarrow}{\Gamma}(t_+) \equiv \{\Gamma_k^1(t_+), \Gamma_k^2(t_+), \Gamma_k^3(t_+), \Gamma_k^4(t_+)\}$. The first form gives the change in the stochastic field over a small time integral $t.t+\delta t$, the second is in the form of a partial differential equation. The first term in the Ito equation for the stochastic fields (208) $-A_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t))$ is the *deterministic term* and is obtained from the drift vector in the functional Fokker-Planck equation and the second term $\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+))$ is the *quantum noise field* whose statistical properties are obtained from the diffusion matrix, and which depends both on the stochastic fields $\underset{\rightarrow}{\widetilde{\psi}}(x,t)$ and on the Gaussian-Markoff stochastic variables $\underset{\rightarrow}{\Gamma}(t_+)$.

The *noise field* term is

$$
\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+)) = \sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))\frac{d}{dt}w_k^D(t) = \sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))\,\Gamma_k^D(t_+)
$$

$$(209)$$

where the stochastic field $\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))$ is related to the diffusion matrix expressed in terms of the stochastic fields $\underset{\rightarrow}{\widetilde{\psi}}(x,t)$ or $\underset{\rightarrow}{\widetilde{\psi}}(x,t), \underset{\rightarrow}{\widetilde{\psi}}(y,t)$.

5.1.5. Properties of Noise Fields

To determine the properties of the noise field we first establish the connection between the $\eta_k^{A;D}$ and the D_{AB}.

$$\sum_{Dk} \eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t))\eta_k^{B;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t))$$

$$= \left[\eta(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t))\eta(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t))^T\right]_{AB}$$

$$= \sum_{Dkij} \xi_i^A(x_1)\mathcal{B}_{ik}^{A;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t))\,\xi_j^B(x_2)\mathcal{B}_{jk}^{B;D}(\underset{\rightarrow}{\widetilde{\alpha}}(t))$$

$$= \sum_{ij} \xi_i^A(x_1)\mathcal{D}_{ij}^{A;B}(\underset{\rightarrow}{\widetilde{\alpha}}(t))\,\xi_j^B(x_2)$$

$$= D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_2,t),x_2) \qquad Two\ Mode$$

$$(210)$$

$$= D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t),x_{1,2})\,\delta(x_1-x_2) \qquad One\ Mode$$

$$(211)$$

using (196) and (189). Thus for the single mode condensate $\left[\eta(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t))\eta(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t),t)^T\right]_{AB}$ is delta function correlated in space and equal to the local diffusion matrix element, whereas in the two-mode condensate case this quantity is equal to the non-local diffusion matrix element.

The *stochastic averages* of the noise field terms can now be obtained. These results follow from (211), (210) and the properties (197), (201), (202) and are derived in Appendix F ([82]). For the stochastic average of each noise term

$$\overline{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+))\right)} = 0 \qquad (212)$$

showing that the stochastic average of of each noise field is zero. For the stochastic average of the product of two noise terms we have

$$\overline{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)}$$

$$= \overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,2}),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{1,2}),x_2)}$$

$$\times\delta(t_1-t_2) \qquad Two\ Mode \qquad (213)$$

$$= \overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t_{1,2}),x_{1,2})}$$

$$\times\delta(x_1-x_2)\delta(t_1-t_2) \qquad One\ Mode \qquad (214)$$

The stochastic average of the product of two noise terms is always *delta function correlated in time*. In the single mode condensate case this average is also delta function correlated in space, and the spatial correlation is given by the stochastic average of the *local diffusion term* $D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t),x_{1,2})$ in the

original functional Fokker-Planck equation (179). However for the two mode condensate it is not delta function correlated in space. Instead the spatial correlation is given by the stochastic average of the *non-local diffusion term* $D_{AB}(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t),x_1,\widetilde{\underset{\rightarrow}{\psi}}(x_2,t),x_2)$.in the original functional Fokker-Planck equation (177).

However, although the noise fields have some of the features in (197), they are not themselves Gaussian-Markov processes. The stochastic averages of products of odd numbers of noise fields are indeed zero, but although averages of products of even numbers of noise fields can be written as sums of products of stochastic averages of pairs of stochastic quantities with the same delta function time correlations as in (197), the pairs involved are the diffusion matrix elements $D_{AB}(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t),x_1,\widetilde{\underset{\rightarrow}{\psi}}(x_2,t),x_2)$ rather than products of noise fields such as $\left(\frac{\partial}{\partial t}\widetilde{G}_A(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\widetilde{\underset{\rightarrow}{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)$. Nevertheless, the stochastic averages of the noise field terms are either zero or are determined from stochastic averages only involving the diffusion matrix elements $D_{AB}(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t),x_1,\widetilde{\underset{\rightarrow}{\psi}}(x_2,t),x_2)$. There is thus never any need to actually determine the matrices $\eta(\widetilde{\underset{\rightarrow}{\psi}}(x,t))$ such that $\eta(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t))\eta(\widetilde{\underset{\rightarrow}{\psi}}(x_2,t))^T = D(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t),x_1,\widetilde{\underset{\rightarrow}{\psi}}(x_2,t),x_2)$ or $D(\widetilde{\underset{\rightarrow}{\psi}}(x_{1,2},t),x_{1,2})\delta(x_1-x_2)$, so all the required expressions for treating the stochastic properties of the noise fields are provided in the functional Fokker-Planck equation. Detailed expressions for stochastic averages of more than two noise fields are derived in Appendix F ([82]) as Eqns. (F.17), (F.18), (F.23) and (F.24).

For the two mode condensate case the results are

$$\begin{aligned}
&\overline{\left\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\widetilde{\underset{\rightarrow}{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\right.}\\
&\qquad\qquad\overline{\left.\times\left(\frac{\partial}{\partial t}\widetilde{G}_C(\widetilde{\underset{\rightarrow}{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\right\}}\\
&= \quad 0
\end{aligned} \tag{215}$$

for three noise fields and

$$\begin{aligned}
&\overline{\left\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\widetilde{\underset{\rightarrow}{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\right.}\\
&\overline{\left.\times\left(\frac{\partial}{\partial t}\widetilde{G}_C(\widetilde{\underset{\rightarrow}{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_D(\widetilde{\underset{\rightarrow}{\psi}}(x_4,t_4),\underset{\rightarrow}{\Gamma}(t_{4+}))\right)\right\}}\\
&= \overline{\left[D_{AB}(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t_{1,2}),x_1,\widetilde{\underset{\rightarrow}{\psi}}(x_2,t_{1,2}),x_2)\right]}\;\overline{\left[D_{CD}(\widetilde{\underset{\rightarrow}{\psi}}(x_3,t_{3,4}),x_3,\widetilde{\underset{\rightarrow}{\psi}}(x_4,t_{3,4}),x_4)\right]}\\
&\quad\times\delta(t_1-t_2)\delta(t_3-t_4)\\
&+\overline{\left[D_{AC}(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t_{1,3}),x_1,\widetilde{\underset{\rightarrow}{\psi}}(x_3,t_{1,3}),x_3)\right]}\;\overline{\left[D_{BD}(\widetilde{\underset{\rightarrow}{\psi}}(x_2,t_{2,4}),x_2,\widetilde{\underset{\rightarrow}{\psi}}(x_4,t_{2,4}),x_4)\right]}\\
&\quad\times\delta(t_1-t_3)\delta(t_2-t_4)\\
&+\overline{\left[D_{AD}(\widetilde{\underset{\rightarrow}{\psi}}(x_1,t_{1,4}),x_1,\widetilde{\underset{\rightarrow}{\psi}}(x_4,t_{1,4}),x_2)\right]}\;\overline{\left[D_{BC}(\widetilde{\underset{\rightarrow}{\psi}}(x_2,t_{2,3}),x_2,\widetilde{\underset{\rightarrow}{\psi}}(x_3,t_{2,3}),x_3)\right]}\\
&\quad\times\delta(t_1-t_4)\delta(t_2-t_3)
\end{aligned} \tag{216}$$

for four noise fields. The result for the stochastic average of four noise field terms is not quite the same as

$$
\overline{\left\{ \left(\frac{\partial}{\partial t} \widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1, t_1), \underset{\rightarrow}{\Gamma}(t_{1+})) \right) \left(\frac{\partial}{\partial t} \widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2, t_2), \underset{\rightarrow}{\Gamma}(t_{2+})) \right) \right\}}
$$
$$
\times \overline{\left\{ \left(\frac{\partial}{\partial t} \widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3, t_3), \underset{\rightarrow}{\Gamma}(t_{3+})) \right) \left(\frac{\partial}{\partial t} \widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4, t_4), \underset{\rightarrow}{\Gamma}(t_{4+})) \right) \right\}}
$$
$$
+ \overline{\left\{ \left(\frac{\partial}{\partial t} \widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1, t_1), \underset{\rightarrow}{\Gamma}(t_{1+})) \right) \left(\frac{\partial}{\partial t} \widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3, t_3), \underset{\rightarrow}{\Gamma}(t_{3+})) \right) \right\}}
$$
$$
\times \overline{\left\{ \left(\frac{\partial}{\partial t} \widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2, t_2), \underset{\rightarrow}{\Gamma}(t_{2+})) \right) \left(\frac{\partial}{\partial t} \widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4, t_4), \underset{\rightarrow}{\Gamma}(t_{4+})) \right) \right\}}
$$
$$
+ \overline{\left\{ \left(\frac{\partial}{\partial t} \widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1, t_1), \underset{\rightarrow}{\Gamma}(t_{1+})) \right) \left(\frac{\partial}{\partial t} \widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4, t_4), \underset{\rightarrow}{\Gamma}(t_{4+})) \right) \right\}}
$$
$$
\times \overline{\left\{ \left(\frac{\partial}{\partial t} \widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2, t_2), \underset{\rightarrow}{\Gamma}(t_{2+})) \right) \left(\frac{\partial}{\partial t} \widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3, t_3), \underset{\rightarrow}{\Gamma}(t_{3+})) \right) \right\}} \quad (217)
$$

because in general the stochastic average of a product of two diffusion matrix elements is not the same as the product of the stochastic averages of each element. Results analogous to (215) and (216) apply also for the single mode condensate case. For four noise fields factors such as $D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1, t_{1,2}), x_1, \underset{\rightarrow}{\widetilde{\psi}}(x_2, t_{1,2}), x_2)$ are just replaced by $D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2}, t_{1,2}), x_1) \delta(x_1 - x_2)$ etc., (see Appendix F ([82]), Eqs.(F.23) and (F.24).

5.1.6. Classical Field Equations

Classical field equations can be obtained from the Ito equations by ignoring the quantum noise term. The classical field equations are

$$
\frac{\partial \psi_A^{class}(x, t)}{\partial t} = -A_A(\underset{\rightarrow}{\psi}{}^{class}(x, t), x) \quad (218)
$$

for both the single and two mode condensate cases. Such equations are not of course really classical as they involve Planck's constant. As will be seen in specific cases (see Eq. (244)) their leading terms are often similar to Gross-Pitaevskii equations, so they could be referred to as generalised mean field equations.

5.1.7. Noise Fields for Single Mode Condensate

Having now established the general results for the stochastic averages of products of one, two, .. noise fields we can show for single mode condensates that the noise field terms can be written in a different form in which the noise fields are just functions of the stochastic fields $\underset{\rightarrow}{\widetilde{\psi}}(x, t)$ and new fundamental Gaussian-Markoff stochastic fields $\underset{\rightarrow}{\Theta}(x, t_+) \equiv \{\Theta_k(x, t_+)\}$, pairs of which are delta function correlated in both space and time [55]. These now replace the

$\underset{\rightarrow}{\Gamma}(t_+)$. Similarly to the $\underset{\rightarrow}{\Gamma}(t_+)$ the $\underset{\rightarrow}{\Theta}(x,\,t_+)$ are defined by their stochastic averages

$$
\begin{aligned}
\overline{\Theta_k(x_1,t_1)} &= 0 \\
\overline{\{\Theta_k(x_1,t_1)\Theta_l(x_2,t_2)\}} &= \delta_{kl}\delta(t_1-t_2) \\
\overline{\{\Theta_k(x_1,t_1)\Theta_l(x_2,t_2)\Theta_m(x_3,t_3)\}} &= 0 \\
\overline{\{\Theta_k(x_1,t_1)\Theta_l(x_2,t_2)\Theta_m(x_3,t_3)\Theta_n(x_4,t_4)\}} &= \overline{\{\Theta_k(x_1,t_1)\Theta_l(x_2,t_2)\}}\,\overline{\{\Theta_m(x_3,t_3)\Theta_n(x_4,t_4)\}} \\
&\quad +\overline{\{\Theta_k(x_1,t_1)\Theta_m(x_3,t_3)\}}\,\overline{\{\Theta_l(x_2,t_2)\Theta_n(x_4,t_4)\}} \\
&\quad +\overline{\{\Theta_k(x_1,t_1)\Theta_n(x_4,t_{24})\}}\,\overline{\{\Theta_l(x_2,t_2)\Theta_m(x_3,t_3)\}} \\
&\quad \ldots
\end{aligned}
\tag{219}
$$

with stochastic averages being denoted with a bar. The stochastic average of an odd number of noise field terms is always zero, whilst that for an even number is the sum of all products of stochastic averages of two noise field terms. Also,in Ito stochastic calculus the noise terms $\Theta_k(x,t)$ within the interval $t,\,t+\delta t$ are uncorrelated with any function of the $\underset{\rightarrow}{\widetilde{\psi}}(x,t)$ at the earlier time t, so that the stochastic average of the product of such a function with a product of the noise field terms factorises

$$
\begin{aligned}
&\overline{F(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\{\Theta_k(x_2,t_2)\Theta_l(x_3,t_3)\Theta_m(x_4,t_4)...\Theta_a(x_l,t_l)\}} \\
&= \overline{F(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))}\,\overline{\{\Theta_k(x_2,t_2)\Theta_l(x_3,t_3)\Theta_m(x_4,t_4)...\Theta_a(x_l,t_l)\}} \qquad t_1 < t_2, t_3, ..., t_l
\end{aligned}
\tag{220}
$$

As previously, the stochastic average of a sum is the sum of stochastic averages. These key features of Ito stochastic calculus are important in deriving the properties of the noise fields in the stochastic field equations. In the case of the single mode condensate the diffusion matrix is symmetric (181). Hence we can write the diffusion matrix D in the form $D(\underset{\rightarrow}{\psi}(x,t),x) = B(\underset{\rightarrow}{\psi}(x,t),x)B(\underset{\rightarrow}{\psi}(x,t),x)^T$ so that

$$
D_{AB}(\underset{\rightarrow}{\psi}(x,t),x)) = \sum_k B_k^A(\underset{\rightarrow}{\psi}(x,t),x))B_k^B(\underset{\rightarrow}{\psi}(x,t),x))
\tag{221}
$$

Note that in this case only a single space variable is involved. Now consider the new stochastic noise field terms defined by

$$
\frac{\partial}{\partial t}\widetilde{H}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Theta}(x,\,t_+)) = \sum_k B_k^A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),x)\,\Theta_k(x,t_+)
\tag{222}
$$

This is a function of the stochastic fields $\underset{\rightarrow}{\psi}(x,t)$ and the Gaussian-Markoff stochastic fields $\underset{\rightarrow}{\Theta}(x,t)$. It is straightforward to determine results for the new stochastic noise field terms. For the stochastic average of each noise term

$$\overline{\left(\frac{\partial}{\partial t}\widetilde{H}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Theta}(x,t_+))\right)} = 0 \qquad (223)$$

showing that the stochastic average of of each new noise field is zero as before. For the stochastic average of the product of two new noise field terms we have

$$\overline{\left(\frac{\partial}{\partial t}\widetilde{H}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Theta}(x_1,t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{H}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Theta}(x_2,t_{2+}))\right)}$$

$$= \overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t_{1,2}),x_{1,2})} \times \delta(x_1-x_2)\delta(t_1-t_2) \qquad (224)$$

giving the same result as before. For products of three, four, .. new noise field terms the results are again as before, so we can now write the original noise field term as

$$\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+)) = \frac{\partial}{\partial t}\widetilde{H}_A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Theta}(x,\,t_+))$$

$$= \sum_k B_k^A(\underset{\rightarrow}{\widetilde{\psi}}(x,t),x)\,\Theta_k(x,t_+) \qquad (225)$$

This form of the noise field is useful when the diffusion matrix $D(\underset{\rightarrow}{\psi}(x,t),x)$is easily factorised, as in Section 5.3.

5.2. Ito Equations for Two-Mode Condensate

The theory involved in writing down Ito stochastic equation for the condensate and non-condensate fields is non-standard. From above, the terms can be written down from the general form (208) by identifying the relevant terms in the functional Fokker-Planck equations set out in Section 4. All stochastic fields depend on t, but this is left implicit.

For the *condensate stochastic field* the Ito equation is

$$\frac{\partial}{\partial t}\widetilde{\psi}_C(\mathbf{s},t)$$

$$= -\frac{i}{\hbar}[-\frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}_C(\mathbf{s}) + V(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) + \frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\widetilde{\psi}_C(\mathbf{s})$$

$$+\frac{g_N}{N}\{2\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\widetilde{\psi}_{NC}(\mathbf{s}) - \frac{g_N}{N}\int d\mathbf{u}\,F(\mathbf{u},\mathbf{s})^*\widetilde{\psi}_{NC}(\mathbf{u})$$

$$+\frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\}\widetilde{\psi}_{NC}^+(\mathbf{s})$$

$$+\frac{g_N}{N}\{2\widetilde{\psi}_{NC}^+(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s})\}\widetilde{\psi}_C(\mathbf{s}) + \frac{g_N}{N}\{\widetilde{\psi}_{NC}(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s})\}\widetilde{\psi}_C^+(\mathbf{s})]$$

$$+\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\underset{\rightarrow}{\Gamma}(t_+)) \qquad (226)$$

where $\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\underset{\rightarrow}{\Gamma}(t_+))$ is the *noise field*.

For the *non-condensate stochastic field* the Ito equation is

$$
\frac{\partial}{\partial t}\widetilde{\psi}_{NC}(\mathbf{s},t)
$$

$$
= -\frac{i}{\hbar}[+\frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\widetilde{\psi}_C(\mathbf{s}) - \frac{g_N}{N}\int d\mathbf{u}\, F(\mathbf{s},\mathbf{u})\widetilde{\psi}_C(\mathbf{u})
$$

$$
-\frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}_{NC}(\mathbf{s}) + V(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s}) + \frac{g_N}{N}\{2\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\widetilde{\psi}_{NC}(\mathbf{s})
$$

$$
+\frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\}\widetilde{\psi}_{NC}^+(\mathbf{s})]
$$

$$
+\frac{\partial}{\partial t}\widetilde{G}_{NC}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\underset{\rightarrow}{\Gamma}(t_+)) \tag{227}
$$

where $\frac{\partial}{\partial t}\widetilde{G}_{NC}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\underset{\rightarrow}{\Gamma}(t_+))$ is the *noise field*. Similar equations apply for $\widetilde{\psi}_C^+(\mathbf{s})$ and $\widetilde{\psi}_{NC}^+(\mathbf{s})$. The stochastic condensate and non-condensate fields are *coupled* together and each is affected by *stochastic noise fields*. For the condensate field, the first line in the equation reads like a *time-dependent Gross-Pitaevskii equation* if $\widetilde{\psi}_C(\mathbf{s},t)$ is regarded as the *order function*. The three terms are the kinetic energy, the trap potential energy and the non-linear mean field energy contributions. Note that for the condensate equation the condensate density $\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})$ is depleted by two bosons due to the $|\phi_1(\mathbf{s})|^2$ and $|\phi_2(\mathbf{s})|^2$ terms. Both Ito stochastic equations are integro-differential equations due to the terms involving $\int d\mathbf{u}\, F(\mathbf{s},\mathbf{u})$ or $\int d\mathbf{u}\, F(\mathbf{u},\mathbf{s})^*$ - thus on the right side there are terms depending on stochastic fields at different spatial points. The first line in the condensate equation comes from the $\widehat{H}_1$ term, the second and third from the $\widehat{H}_2$ term and the fourth from the $\widehat{H}_3$ term. The first line in the non-condensate equation is a term coupling in the condensate field and comes from the $\widehat{H}_2$ term, the second and third from the $\widehat{H}_3$ term. The latter two lines differ somewhat from the form of a time-dependent Gross-Pitaevskii equation, which is not surprising since these refer to the relatively unoccupied non-condensate modes.

The *stochastic averages* of the noise fields are given in (213), where the

non-zero *diffusion matrix elements* are

$$D_{C+;NC-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2)$$

$$= +\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s}_{1,2})\widetilde{\psi}_C(\mathbf{s}_{1,2}) - \frac{1}{2}(|\phi_1(\mathbf{s}_{1,2})|^2 + |\phi_2(\mathbf{s}_{1,2})|^2)\}\delta(\mathbf{s}_1 - \mathbf{s}_2)$$

$$+\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_{NC}^+(\mathbf{s}_{1,2})\widetilde{\psi}_C(\mathbf{s}_{1,2}) + \widetilde{\psi}_C^+(\mathbf{s}_{1,2})\widetilde{\psi}_{NC}(\mathbf{s}_{1,2})\}\delta(\mathbf{s}_1 - \mathbf{s}_2)$$

$$-\frac{i}{\hbar}\frac{g_N}{N}\{\frac{1}{2}F(\mathbf{s}_2,\mathbf{s}_1)\}$$

$$= D_{NC-;C+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1) \tag{228}$$

$$D_{C-;NC+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2)$$

$$= -\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s}_{1,2})\widetilde{\psi}_C^+(\mathbf{s}_{1,2}) - \frac{1}{2}(|\phi_1(\mathbf{s}_{1,2})|^2 + |\phi_2(\mathbf{s}_{1,2})|^2)\}\delta(\mathbf{s}_1 - \mathbf{s}_2)$$

$$-\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_{NC}(\mathbf{s}_{1,2})\widetilde{\psi}_C^+(\mathbf{s}_{1,2}) + \widetilde{\psi}_C(\mathbf{s}_{1,2})\widetilde{\psi}_{NC}^+(\mathbf{s}_{1,2})\}\delta(\mathbf{s}_1 - \mathbf{s}_2)$$

$$+\frac{i}{\hbar}\frac{g_N}{N}\{\frac{1}{2}F(\mathbf{s}_2,\mathbf{s}_1)^*\}$$

$$= D_{NC+;C-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1) \tag{229}$$

$$D_{C-;NC-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2)$$

$$= -\frac{i}{\hbar}\frac{g_N}{N}\{\frac{1}{2}\widetilde{\psi}_C(\mathbf{s}_{1,2})\widetilde{\psi}_C(\mathbf{s}_{1,2}) + \widetilde{\psi}_{NC}(\mathbf{s}_{1,2})\widetilde{\psi}_C(\mathbf{s}_{1,2})\}\delta(\mathbf{s}_1 - \mathbf{s}_2)$$

$$= D_{NC-;C-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1) \tag{230}$$

$$D_{C+;NC+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2)$$

$$= +\frac{i}{\hbar}\frac{g_N}{N}\{\frac{1}{2}\widetilde{\psi}_C^+(\mathbf{s}_{1,2})\widetilde{\psi}_C^+(\mathbf{s}_{1,2}) + \widetilde{\psi}_{NC}^+(\mathbf{s}_{1,2})\widetilde{\psi}_C^+(\mathbf{s}_{1,2})\}\delta(\mathbf{s}_1 - \mathbf{s}_2)$$

$$= D_{NC+;C+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1) \tag{231}$$

$$D_{NC-;NC-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2)$$

$$= -\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s}_{1,2})\widetilde{\psi}_C(\mathbf{s}_{1,2})\}\delta(\mathbf{s}_1 - \mathbf{s}_2) \tag{232}$$

$$D_{NC+;NC+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2)$$

$$= +\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s}_{1,2})\widetilde{\psi}_C^+(\mathbf{s}_{1,2})\}\delta(\mathbf{s}_1 - \mathbf{s}_2) \tag{233}$$

with the notation $D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_1,t),\mathbf{s}_1,\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s}_2,t),\mathbf{s}_2)$ for
$D_{AB}(\widetilde{\psi}_1(\mathbf{s}_1,t),\widetilde{\psi}_2(\mathbf{s}_1,t),\widetilde{\psi}_3(\mathbf{s}_1,t),\widetilde{\psi}_4(\mathbf{s}_1,t),\mathbf{s}_1,\widetilde{\psi}_1(\mathbf{s}_2,t),\widetilde{\psi}_2(\mathbf{s}_2,t),\widetilde{\psi}_3(\mathbf{s}_2,t),\widetilde{\psi}_4(\mathbf{s}_2,t),\mathbf{s}_2)$
and replacements for AB as follows: $1 \equiv C-, 2 \equiv C+, 3 \equiv NC-, 4 \equiv NC+$.
Also we write $\mathbf{s}_{1,2} = \mathbf{s}_1 = \mathbf{s}_2$ for the delta function terms. The presence of the
terms $F(\mathbf{s}_2,\mathbf{s}_1), F(\mathbf{s}_2,\mathbf{s}_1)^*$ reflects the non-local nature of the diffusion matrix
and also give an explicit $\mathbf{s}_1, \mathbf{s}_2$ dependence. We see that the average of the

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product of any pair of noise fields is delta function correlated in time but not in space, and is then given by the diffusion matrix element that appears in the functional Fokker-Planck equation. The stochastic averages of products of odd numbers of noise fields is zero and the stochastic averages of products of even numbers of noise fields can be written as sums of products of stochastic averages of pairs of diffusion matrix elements in accordance with (215) and (216).

The *classical field equations* for the condensate field are

$$
\frac{\partial}{\partial t}\psi_C^{class}(\mathbf{s},t)
$$
$$
= -\frac{i}{\hbar}[-\frac{\hbar^2}{2m}\nabla^2\psi_C(\mathbf{s}) + V(\mathbf{s})\psi_C(\mathbf{s}) + \frac{g_N}{N}\{\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\psi_C(\mathbf{s})
$$
$$
+\frac{g_N}{N}\{2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\psi_{NC}(\mathbf{s}) - \frac{g_N}{N}\int d\mathbf{u}\,F(\mathbf{u},\mathbf{s})^*\psi_{NC}(\mathbf{u})
$$
$$
+\frac{g_N}{N}\{\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\psi_{NC}^+(\mathbf{s})
$$
$$
+\frac{g_N}{N}\{2\psi_{NC}^+(\mathbf{s})\psi_{NC}(\mathbf{s})\}\psi_C(\mathbf{s}) + \frac{g_N}{N}\{\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\psi_C^+(\mathbf{s})]
$$

$$(234)$$

and for the non-condensate stochastic field

$$
\frac{\partial}{\partial t}\psi_{NC}^{class}(\mathbf{s},t)
$$
$$
= -\frac{i}{\hbar}[+\frac{g_N}{N}\{\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\psi_C(\mathbf{s}) - \frac{g_N}{N}\int d\mathbf{u}\,F(\mathbf{s},\mathbf{u})\psi_C(\mathbf{u})
$$
$$
-\frac{\hbar^2}{2m}\nabla^2\psi_{NC}(\mathbf{s}) + V(\mathbf{s})\psi_{NC}(\mathbf{s}) + \frac{g_N}{N}\{2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2 - |\phi_2(\mathbf{s})|^2\}\psi_{NC}(\mathbf{s})
$$
$$
+\frac{g_N}{N}\{\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\psi_{NC}^+(\mathbf{s})]
$$

$$(235)$$

with corresponding equations for $\psi_C^{+\,class}$ and $\psi_{NC}^{+\,class}$. These also are integro-differential equations.

5.3. Ito Equations for Single Mode Condensate

We will next consider the simpler case where the BEC only involves a single mode. Here the Ito stochastic equations are relatively standard. From above, the terms can be written down from the general form (208) by identifying the relevant terms in the functional Fokker-Planck equations set out in Section 4. All stochastic fields depend on t, but this is left implicit.

For the *condensate stochastic field* the Ito stochastic equation is

$$\frac{\partial}{\partial t}\widetilde{\psi}_C(\mathbf{s},t)$$

$$= -\frac{i}{\hbar}[-\frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}_C(\mathbf{s}) + V(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) + \frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2\}\widetilde{\psi}_C(\mathbf{s})$$

$$+ \frac{g_N}{N}\{2\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - N|\phi_1(\mathbf{s})|^2\}\widetilde{\psi}_{NC}(\mathbf{s}) + \frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\}\widetilde{\psi}_{NC}^+(\mathbf{s})$$

$$+ \frac{g_N}{N}\{2\widetilde{\psi}_{NC}^+(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s})\}\widetilde{\psi}_C(\mathbf{s}) + \frac{g_N}{N}\{\widetilde{\psi}_{NC}(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s})\}\widetilde{\psi}_C^+(\mathbf{s})]$$

$$+ \frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t), \underset{\rightarrow}{\Gamma}(t_+)) \tag{236}$$

where $\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t), \underset{\rightarrow}{\Gamma}(t_+))$ is the *noise field*.

For the *non-condensate stochastic field* the Ito stochastic equation is

$$\frac{\partial}{\partial t}\widetilde{\psi}_{NC}(\mathbf{s},t)$$

$$= -\frac{i}{\hbar}[+\frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - N|\phi_1(\mathbf{s})|^2\}\widetilde{\psi}_C(\mathbf{s})$$

$$- \frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}_{NC}(\mathbf{s}) + V(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s}) + \frac{g_N}{N}\{2\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2\}\widetilde{\psi}_{NC}(\mathbf{s})$$

$$+ \frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\}\widetilde{\psi}_{NC}^+(\mathbf{s})]$$

$$+ \frac{\partial}{\partial t}\widetilde{G}_{NC}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t), \underset{\rightarrow}{\Gamma}(t_+)) \tag{237}$$

where $\frac{\partial}{\partial t}\widetilde{G}_{NC}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t), \underset{\rightarrow}{\Gamma}(t_+))$ is the *noise field*. Similar equations apply for $\widetilde{\psi}_C^+(\mathbf{s})$ and $\widetilde{\psi}_{NC}^+(\mathbf{s})$. The stochastic condensate and non-condensate fields are *coupled* together and each is affected by *stochastic noise fields*. For the condensate field, the first line in the equation reads like a *time-dependent Gross-Pitaevskii equation* if $\widetilde{\psi}_C(\mathbf{s},t)$ is regarded as the *order function*. The three terms are the kinetic energy, the trap potential energy and the non-linear mean field energy contributions. Note that for the condensate equation the condensate density $\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})$ is depleted by one boson due to the $|\phi_1(\mathbf{s})|^2$ term. The first line in the condensate equation comes from the $\widehat{H}_1$ term, the second from the $\widehat{H}_2$ term and the third from the $\widehat{H}_3$ term. The first line in the non-condensate equation is a term coupling in the condensate field and comes from the $\widehat{H}_2$ term, the second and third from the $\widehat{H}_3$ term. The latter two lines differ somewhat from the form of a time-dependent Gross-Pitaevskii equation, which is not surprising since these refer to the relatively unoccupied non-condensate modes.

The *stochastic averages* of the noise fields are given in (214), where the

non-zero *diffusion matrix elements* are

$$D_{C+;NC-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) = \frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - \frac{1}{2}N\,|\phi_1(\mathbf{s})|^2\}$$

$$+\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_{NC}^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) + \widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s})\}$$

$$= D_{NC-;C+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) \qquad (238)$$

$$D_{C-;NC+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) = -\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C^+(\mathbf{s}) - \frac{1}{2}N\,|\phi_1(\mathbf{s})|^2\}$$

$$-\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_{NC}(\mathbf{s})\widetilde{\psi}_C^+(\mathbf{s}) + \widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_{NC}^+(\mathbf{s})\}$$

$$= D_{NC+;C-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) \qquad (239)$$

$$D_{C-;NC-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) = -\frac{i}{\hbar}\frac{g_N}{N}\{\frac{1}{2}\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) + \widetilde{\psi}_{NC}(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\}$$

$$= D_{NC-;C-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) \qquad (240)$$

$$D_{C+;NC+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) = +\frac{i}{\hbar}\frac{g_N}{N}\{\frac{1}{2}\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C^+(\mathbf{s}) + \widetilde{\psi}_{NC}^+(\mathbf{s})\widetilde{\psi}_C^+(\mathbf{s})\}$$

$$= D_{NC+;C+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) \qquad (241)$$

$$D_{NC-;NC-}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) = -\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\} \qquad (242)$$

$$D_{NC+;NC+}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s}) = +\frac{i}{\hbar}\frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C^+(\mathbf{s})\} \qquad (243)$$

with the notation $D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(\mathbf{s},t),\mathbf{s})$ for
$D_{AB}(\widetilde{\psi}_1(\mathbf{s},t),\widetilde{\psi}_2(\mathbf{s},t),\widetilde{\psi}_3(\mathbf{s},t),\widetilde{\psi}_4(\mathbf{s},t),\mathbf{s})$ and replacements for AB as follows: $1 \equiv C-, 2 \equiv C+, 3 \equiv NC-, 4 \equiv NC+$. Note that the $|\phi_1(\mathbf{s})|^2$ terms give an explicit $\mathbf{s}$ dependence as well as that in the stochastic fields. We see that the average of the product of any pair of noise fields is delta function correlated in both space and time, and is then given by the diffusion matrix element that appears in the functional Fokker-Planck equation. The stochastic averages of products of odd numbers of noise fields is zero and the stochastic averages of products of even numbers of noise fields can be written as sums of products of stochastic averages of pairs of diffusion matrix elements analogous to (215) and (216) (see 5.1.5).

The *classical field equations* for the condensate field are

$$\frac{\partial}{\partial t}\psi_C^{class}(\mathbf{s},t)$$

$$= -\frac{i}{\hbar}[-\frac{\hbar^2}{2m}\nabla^2\psi_C(\mathbf{s}) + V(\mathbf{s})\psi_C(\mathbf{s}) + \frac{g_N}{N}\{\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2\}\psi_C(\mathbf{s})$$

$$+\frac{g_N}{N}\{2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - N\,|\phi_1(\mathbf{s})|^2\}\psi_{NC}(\mathbf{s}) + \frac{g_N}{N}\{\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\psi_{NC}^+(\mathbf{s})$$

$$+\frac{g_N}{N}\{2\psi_{NC}^+(\mathbf{s})\psi_{NC}(\mathbf{s})\}\psi_C(\mathbf{s}) + \frac{g_N}{N}\{\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\psi_C^+(\mathbf{s})] \qquad (244)$$

and for the non-condensate stochastic field

$$\frac{\partial}{\partial t}\psi_{NC}^{class}(\mathbf{s},t)$$

$$= -\frac{i}{\hbar}[+\frac{g_N}{N}\{\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - N\,|\phi_1(\mathbf{s})|^2\}\psi_C(\mathbf{s})$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi_{NC}(\mathbf{s}) + V(\mathbf{s})\psi_{NC}(\mathbf{s}) + \frac{g_N}{N}\{2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2\}\psi_{NC}(\mathbf{s})$$

$$+\frac{g_N}{N}\{\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\psi_{NC}^+(\mathbf{s})] \tag{245}$$

with corresponding equations for $\psi_C^{+\,class}$ and $\psi_{NC}^{+\,class}$. If the coupling terms to the non-condensate modes are ignored then the equation for $\psi_C^{class}(\mathbf{s},t)$ has a solution $\psi_C^{class}(\mathbf{s},t) = \sqrt{N}\,\phi_1(\mathbf{s}), \psi_C^{+\,class}(\mathbf{s},t) = \sqrt{N}\,\phi_1^*(\mathbf{s})$ for large N , where $\phi_1(\mathbf{s})$ satisfies the standard single mode Gross-Pitaevskii equation (63). Assuming the effects of coupling with the non-condensate field are small, this result shows that $\psi_C^{class}(\mathbf{s},t)$ is similar to the usual mean field solution.

5.4 Approximate Solutions - Single Mode Condensate

In general the coupled stochastic field equations are difficult to solve, even numerically. Approximate solutions can however be obtained which enable some features of the physics to be explored. As an illustration of how such approximate solutions can be obtained we consider the single mode condensate case for large N. By applying certain approximations to (236) - (245) the equations obtained by Krachmalnicoff et al. [71] can be obtained. Their approach is also based on a hybrid Wigner P+ distribution functional.

Firstly, we ignore all but the first line in of the Ito equation for the stochastic condensate field (236). Thus the noise field term is ignored as are the coupling terms involving non-condensate stochastic fields. The latter are higher order in $(\sqrt{N})^{-1}$, so this a reasonable first approximation. Consistency in neglecting the noise field term then requires that the only non-zero diffusion matrix elements in (236) that are retained are those just involving the non-condensate stochastic fields, $D_{NC-;NC-}$ and $D_{NC+;NC+}$. Consistency with the classical condensate field equation (244) also requires neglecting the coupling terms involving the non-condensate fields. The condensate stochastic field then satisfies

$$i\hbar\frac{\partial}{\partial t}\widetilde{\psi}_C(\mathbf{s},t)$$

$$= -\frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}_C(\mathbf{s}) + V(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) + \frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s}) - |\phi_1(\mathbf{s})|^2\}\widetilde{\psi}_C(\mathbf{s})$$

$$\tag{246}$$

We see from the Gross-Pitaevskii equation (63) that a solution is given by $\widetilde{\psi}_C(\mathbf{s},t) = \psi_C^{class}(\mathbf{s},t) = \sqrt{N}\,\phi_1(\mathbf{s}), \widetilde{\psi}_C^+(\mathbf{s},t) = \psi_C^{+\,class}(\mathbf{s},t) = \sqrt{N}\,\phi_1^*(\mathbf{s})$. Hence the condensate field now becomes non-stochastic.

Secondly, the first line in the Ito equation (237) for the stochastic non-condensate field then becomes zero leaving just the second and third line together with the noise field term. As the diffusion matrix is now diagonal then using (225) we can write the noise field as

$$\frac{\partial}{\partial t}\widetilde{G}_{NC}(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+)) = \sqrt{\frac{-i}{\hbar}\frac{g_N}{N}}(\widetilde{\psi}_C)^2\,\Theta_{NC}^-(x,t) \qquad (247)$$

$$\frac{\partial}{\partial t}\widetilde{G}_{NC}^+(\underset{\rightarrow}{\widetilde{\psi}}(x,t),\underset{\rightarrow}{\Gamma}(t_+)) = \sqrt{\frac{+i}{\hbar}\frac{g_N}{N}}(\widetilde{\psi}_C^+)^2\,\Theta_{NC}^+(x,t) \qquad (248)$$

where with $a,b = +,-$ we introduce two Gaussian-Markoff stochastic fields $\Theta_{NC}^\pm(x,t)$. The stochastic average for two stochastic fields is

$$\overline{\Theta_{NC}^a(x_1,t_1)\,\Theta_{NC}^b(x_2,t_2)} = \delta(x_1-x_2)\,\delta(t_1-t_2)\,\delta_{ab} \quad (a,b=+,-) \qquad (249)$$

and the results for products of other numbers of fields satisfy the standard Gaussian-Markoff rules. It is then straightforward to show that the two noise fields $\frac{\partial}{\partial t}\widetilde{G}_{NC}$ and $\frac{\partial}{\partial t}\widetilde{G}_{NC}^+$ satisfy the correct results in (214) etc. for stochastic averages.

For large N the $-|\phi_1(\mathbf{s})|^2$ term can be neglected, so the Ito equation (237) for the stochastic non-condensate field is then

$$i\hbar\frac{\partial}{\partial t}\widetilde{\psi}_{NC}(\mathbf{s},t) = -\frac{\hbar^2}{2m}\nabla^2\widetilde{\psi}_{NC}(\mathbf{s}) + V(\mathbf{s})\widetilde{\psi}_{NC}(\mathbf{s}) + 2\frac{g_N}{N}\{\widetilde{\psi}_C^+(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\}\widetilde{\psi}_{NC}(\mathbf{s})$$

$$+ \frac{g_N}{N}\{\widetilde{\psi}_C(\mathbf{s})\widetilde{\psi}_C(\mathbf{s})\}\widetilde{\psi}_{NC}^+(\mathbf{s}) + \sqrt{+i\hbar\frac{g_N}{N}}(\widetilde{\psi}_C)^2\,\Theta_{NC}^-(x,t) \quad (250)$$

This equation is equivalent to Eq.(5) in the paper by Krachmalnicoff et al. [71]. Note however that the derivation involves making approximations to the actual stochastic field equations for single mode condensates, in particular the neglect of noise terms in the equation for the stochastic condensate field.

5.4. Stochastic Averages for Quantum Correlation Functions

The quantum averages of symmetrically ordered products of the condensate field operators $\{\widehat{\Psi}_C^\dagger(\mathbf{r}_1)\widehat{\Psi}_C^\dagger(\mathbf{r}_2)....\widehat{\Psi}_C^\dagger(\mathbf{r}_p)\widehat{\Psi}_C(\mathbf{s}_q)..\widehat{\Psi}_C(\mathbf{s}_1)\}$ and normally ordered products of the non-condensate field operators
$\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_1)\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_2)....\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_r)\widehat{\Psi}_{NC}(\mathbf{v}_s)..\widehat{\Psi}_{NC}(\mathbf{v}_1)$ are now given by *stochastic averages*. These replace the *functional integrals* involving *quasi distribution functional* given above in (108). We have

$$Tr[\widehat{\rho}\{\widehat{\Psi}_C^\dagger(\mathbf{r}_1)..\widehat{\Psi}_C^\dagger(\mathbf{r}_p)\widehat{\Psi}_C(\mathbf{s}_q)..\widehat{\Psi}_C(\mathbf{s}_1)\}$$
$$\times\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_1)..\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_r)\widehat{\Psi}_{NC}(\mathbf{v}_s)..\widehat{\Psi}_{NC}(\mathbf{v}_1)]$$

$$= \overline{\begin{array}{c}\psi_C^+(\mathbf{r}_1)..\psi_C^+(\mathbf{r}_p)\,\psi_C(\mathbf{s}_q)..\psi_C(\mathbf{s}_1)\times\\ \times\,\psi_{NC}^+(\mathbf{u}_1)..\psi_{NC}^+(\mathbf{u}_r)\,\psi_{NC}(\mathbf{v}_s)..\psi_{NC}(\mathbf{v}_1)\end{array}} \qquad (251)$$

where the bar denotes a stochastic average.

6. Summary

The present paper sets up a general approach for treating both dephasing and decoherence effects due to collisions in interferometry experiments using single component Bose-Einstein condensates in double well situations, where two condensate modes may be involved. The treatment starts from a description of dephasing and fragmentation effects in two mode condensates in which the two modes satisfy generalised coupled Gross-Pitaevskii equations, and the amplitudes describing the fragmentation of the condensate into the two modes satisfy matrix equations. The two sets of equations, which are coupled and self-consistent, are derived from the Dirac-Frenkel variational principle. The treatment of decoherence effects requires the consideration of non-condensate modes and a full phase space method involving a distribution functional is used, where the highly occupied condensate modes are described via a truncated Wigner representation (since the bosons in condensate modes behave like a classical mean field), whilst the basically unoccupied non-condensate modes are described via a positive P representation (these bosons should exhibit quantum effects). The functional Fokker-Planck equation is derived using the correspondence rules and then Ito equations for the stochastic fields associated with the condensate and non-condensate field annihilation and creation field operators are determined. The Ito stochastic field equations contain a deterministic term which is obtained from the drift term in the functional Fokker-Planck equation, and a noise field term whose stochastic properties are obtained from the diffusion term in the functional Fokker-Planck equation. The link with interferometry experiments is via the quantum correlation functions, which are shown to be equal to phase space functional integrals of products of field functions with the distribution functional. These phase space functional integrals are then shown to be determined by stochastic averages of products of the stochastic fields, and in the present approach the quantum correlation functions would be evaluated numerically via such stochastic averages. Clearly, the general approach presented here is rather complex, so in order that the reader can understand what is involved this paper contains a full coverage of all the important steps in the derivations of the key expressions obtained for the quantum correlation functions, correspondence rules, functional Fokker-Planck equations and Ito stochastic field equations. These are not covered in any of the standard textbooks and previous papers only provide a brief outline of how such results are obtained.

For the condensate field, the first line in the Ito stochastic field equation reads like a time-dependent Gross-Pitaevskii equation if the condensate field is regarded as the order function. The first line in the non-condensate equation is a term coupling in the condensate field. The results for the two mode condensate have unusual features such as the Ito stochastic field equations being integro-differential equations and the diffusion matrix being non-local. These features are not found in the situation where there is only one condensate mode, where the Ito equations are differential equations and the diffusion matrix is local. The stochastic properties of the noise field terms are determined and are similar to those for Gaussian-Markov processes in that the stochastic averages of odd

numbers of noise fields are zero and those for even numbers of noise field terms are the sums of products of stochastic averages associated with pairs of noise fields. However each pair is represented by an element of the diffusion matrix rather than products of the noise fields themselves, as in the case of Gaussian-Markov processes. Hence it is only stochastic averages involving diffusion matrix elements that determine all the stochastic properties. Results for both two mode condensates and the simpler single mode condensate case are presented here.

The Ito stochastic field equations for single mode condensate have been compared to similar equations in the recent paper by Krachmalnicoff et al. [71]. We see that their equations are an approximate version for large N of those presented here, the approximation involving the neglect of noise terms and higher order terms in the condensate stochastic field equations - which requires ignoring off-diagonal terms in the diffusion matrix. In this approximation the condensate fields are non-stochastic and given by the $\sqrt{N}$ times the normalised solution to the single mode Gross-Pitaevskii equation, or its complex conjugate. The non-condensate fields are stochastic and involves two Gaussian-Markoff delta correlated stochastic fields.

Numerical applications to a range of actual and potential interferometry experiments with Bose-Einstein condensates are planned. These include the Heisenberg-limited interferometry experiment proposed by Dunningham and Burnett [6], where the existing theory is based on the Josephson Hamiltonian in which the two mode functions are unchanged during each stage of the process. A more comprehensive analysis of this potentially important experiment by a theory that allows for changes to the two mode functions and decoherence effects would be of interest. Future theoretical work will involve the extension of the present theory to two component condensates in single wells, where there are also two spatial mode functions involved, and where interferometry experiments of the Ramsey type have already been performed [39]. However, the current theoretical treatment [39] ignores decoherence and is based on a single mode theory. A theory along the lines of that presented here for single component condensates would enable both decoherence effects and the possibility of fragmentation effects to be studied.

7. Acknowlegements

8. References

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Appendix A. - Amplitude and Mode Equations for Two-Mode Theory

Appendix A.1. Angular Momentum Quantities

In the two-mode approximation the N boson system behaves like a giant spin system with spin quantum number $j = N/2$ and which can be described via angular momentum eigenstates $\left| \frac{N}{2}, k \right\rangle$, where $k = -N/2, .., +N/2$ is a magnetic quantum number which describes fragmented states of the bosonic system with $(\frac{N}{2} - k)$ bosons in mode $\phi_1(r,t)$ and $(\frac{N}{2} + k)$ bosons in mode $\phi_2(r,t)$. Details of the spin operator treatment for two mode theory are given in [17]. It is therefore not surprising that the basic equations will involve expressions arising from angular momentum theory. These are the quantities X_{kl}^{ij} and $Y_{kl}^{im\,jn}$ which are defined as

$$
\begin{aligned}
X_{kl}^{11} &= (\frac{N}{2}-k)\delta_{kl} \qquad X_{kl}^{12} = \{(\frac{N}{2}-k)(\frac{N}{2}+l)\}^{\frac{1}{2}}\delta_{k,l-1} \\
X_{kl}^{21} &= \{(\frac{N}{2}-l)(\frac{N}{2}+k)\}^{\frac{1}{2}}\delta_{l,k-1} \qquad X_{kl}^{22} = (\frac{N}{2}+k)\delta_{kl} \qquad (A.1) \\
Y_{kl}^{11\,11} &= (\frac{N}{2}-k)(\frac{N}{2}-k-1)\delta_{kl} \\
Y_{kl}^{22\,22} &= (\frac{N}{2}+k)(\frac{N}{2}+k-1)\delta_{kl} \\
Y_{kl}^{12\,12} &= Y_{kl}^{12\,21} = Y_{kl}^{21\,12} = Y_{kl}^{21\,21} = (\frac{N}{2}-k)(\frac{N}{2}+k)\delta_{kl} \\
Y_{kl}^{11\,12} &= Y_{kl}^{11\,21} = (\frac{N}{2}-l)\{(\frac{N}{2}-k)(\frac{N}{2}+l)\}^{\frac{1}{2}}\delta_{k,l-1} \\
Y_{kl}^{12\,22} &= Y_{kl}^{21\,22} = (\frac{N}{2}+k)\{(\frac{N}{2}-k)(\frac{N}{2}+l)\}^{\frac{1}{2}}\delta_{k,l-1} \\
Y_{kl}^{12\,11} &= Y_{kl}^{21\,11} = (\frac{N}{2}-k)\{(\frac{N}{2}-l)(\frac{N}{2}+k)\}^{\frac{1}{2}}\delta_{l,k-1} \\
Y_{kl}^{22\,12} &= Y_{kl}^{22\,21} = (\frac{N}{2}+l)\{(\frac{N}{2}-l)(\frac{N}{2}+k)\}^{\frac{1}{2}}\delta_{l,k-1} \\
Y_{kl}^{11\,22} &= \{(\frac{N}{2}-l+1)(\frac{N}{2}-k)(\frac{N}{2}+l)(\frac{N}{2}+k+1)\}^{\frac{1}{2}}\delta_{k,l-2} \\
Y_{kl}^{22\,11} &= \{(\frac{N}{2}-k+1)(\frac{N}{2}-l)(\frac{N}{2}+k)(\frac{N}{2}+l+1)\}^{\frac{1}{2}}\delta_{l,k-2}. \qquad (A.2)
\end{aligned}
$$

These results would apply for the general two-mode theory before the localisation assumption is made.

Appendix A.2. Hamiltonian and Rotation Matrices

The Hamiltonian and rotation matrix elements H_{kl} and U_{kl} that occur in the amplitude equations (A.14) involve spatial integrals involving the mode functions ϕ_1 and ϕ_2. They are therefore functionals of the mode functions. The

expressions depend also on the spatial and time derivatives of the mode functions through the quantities $\widetilde{W}_{ij}(\mathbf{r},t)$, $\widetilde{V}_{im\,jn}(\mathbf{r},t)$ and $\widetilde{T}_{ij}(\mathbf{r},t)$, where $(i,j,m,n = 1,2)$, and which are defined by

$$\widetilde{W}_{ij}(\mathbf{r},t) = \frac{\hbar^2}{2m} \sum_{\mu=x,y,z} \partial_\mu \phi_i^* \, \partial_\mu \phi_j + \phi_i^* V \phi_j \tag{A.3}$$

$$\widetilde{V}_{im\,jn}(\mathbf{r},t) = \frac{g}{2} \phi_i^* \, \phi_m^* \, \phi_j \, \phi_n \tag{A.4}$$

$$\widetilde{T}_{ij}(\mathbf{r},t) = \frac{1}{2i} (\partial_t \phi_i^* \, \phi_j - \phi_i^* \, \partial_t \phi_j) \tag{A.5}$$

The rotation matrix elements U_{kl} $(-\frac{N}{2} \leq k,l \leq +\frac{N}{2})$ are given by

$$U_{kl} = \frac{1}{2i}[(\partial_t \left\langle \frac{N}{2},k \right|) \left| \frac{N}{2},l \right\rangle - \left\langle \frac{N}{2},k \right| (\partial_t \left| \frac{N}{2},l \right\rangle)] = U_{lk}^* \tag{A.6}$$

$$= \int d\mathbf{r}\, \widetilde{U}_{kl}(\phi_i,\phi_i^*,\partial_t\phi_i,\partial_t\phi_i^*). \tag{A.7}$$

In the expression (A.7) for the rotation matrix the quantity $\widetilde{U}_{kl}$ is

$$\widetilde{U}_{kl} = \sum_{ij} X_{kl}^{ij} \, \widetilde{T}_{ij}. \tag{A.8}$$

The result involves the angular momentum theory quantities X_{kl}^{ij}. Thus for the rotation matrix, space integrals of the mode functions and their time derivatives are involved.

The Hamiltonian matrix elements H_{kl} $(-\frac{N}{2} \leq k,l \leq +\frac{N}{2})$ are given by

$$H_{kl} = \left\langle \frac{N}{2},k \right| \widehat{H} \left| \frac{N}{2},l \right\rangle = H_{lk}^* \tag{A.9}$$

$$= \int d\mathbf{r}\, \widetilde{H}_{kl}(\phi_i,\phi_i^*,\partial_\mu\phi_i,\partial_\mu\phi_i^*). \tag{A.10}$$

In the expression (A.10) for the Hamiltonian matrix the quantity $\widetilde{H}_{kl}$ is a Hamiltonian density and is given by

$$\widetilde{H}_{kl} = \sum_{ij} X_{kl}^{ij} \, \widetilde{W}_{ij} + \sum_{ijmn} Y_{kl}^{im\,jn} \, \widetilde{V}_{im\,jn}. \tag{A.11}$$

This result involves the angular momentum theory quantities X_{kl}^{ij} and $Y_{kl}^{im\,jn}$. Thus for the Hamiltonian matrix, space integrals of the mode functions and their spatial derivatives are involved.

The coefficients X_{ij} and $Y_{im\,jn}$ $(i,j,m,n = 1,2)$ that occur in the generalized Gross-Pitaevskii equations (A.15) for the mode functions are quadratic functions of the amplitudes b_k $(-\frac{N}{2} \leq k,l \leq +\frac{N}{2})$

$$X_{ij} = \sum_{k,l} b_k^* X_{kl}^{ij} \, b_l = X_{ji}^* \sim N \tag{A.12}$$

$$Y_{im\,jn} = \sum_{k,l} b_k^* Y_{kl}^{im\,jn} \, b_l = Y_{jn\,im}^* \sim N^2 \tag{A.13}$$

Note the Hermitian properties of these quantities and the N dependence of their order of magnitude.

Appendix A.3. Supplementary Equations

Amplitude Equations

$$i\hbar\frac{\partial b_k}{\partial t} = \sum_l (H_{kl} - \hbar U_{kl})b_l \qquad (k = -N/2, .., N/2). \qquad \text{(A.14)}$$

Mode Equations

$$i\hbar \sum_j X_{ij}\, \frac{\partial}{\partial t}\phi_j \;=\; \sum_j X_{ij}\Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\phi_j$$

$$+ \sum_j \Big(g \sum_{mn} Y_{im\,jn}\, \phi_m^*\, \phi_n\Big)\phi_j \qquad\qquad (i = 1,2)\text{(A.15)}$$

Appendix B. - Functional Calculus

The basic ideas of functional calculus are outlined here for the case of c-number quantities. The two main processes of interest are *functional differentiation* and *functional integration*, but we begin by explaining what is meant by a functional.

Appendix B.1. Definition of Functional

A *functional* $F[\psi(x)]$ maps a c-number function $\psi(x)$ onto a c-number that depends on *all* the values of $\psi(x)$ over its entire range. The independent variable x could in some cases refer to a position coordinate, in other cases it may refer to time. If x does refer to position then $\psi(x)$ is refered to as a *field function*. Note that the functional is written with square brackets to distinguish it from a *function*, written with round brackets.

We will assume that c-number functions $\psi(x)$ can be expanded in terms of a suitable orthonormal set of mode functions with *c-number* expansion coefficients α_k

$$\psi(x) = \sum_k \alpha_k \, \phi_k(x) \tag{B.1}$$

where the orthonormality conditions are

$$\int dx \, \phi_k^*(x)\phi_l(x) = \delta_{kl} \tag{B.2}$$

This gives the well-known result for the expansion coefficients

$$\alpha_k = \int dx \, \phi_k^*(x)\psi(x) \tag{B.3}$$

and the completeness relationship is

$$\sum_k \phi_k(x)\phi_k^*(y) = \delta(x - y). \tag{B.4}$$

As the value of the function at any point in the range for x is determined uniquely by the expansion coefficients $\{\alpha_k\}$, then the functional $F[\psi(x)]$ must therefore also just depend on the expansion coefficients, and hence may also be viewed as a *function* $f(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n)$ of the *expansion coefficients*, a useful equivalence when functional differentiation and integration are considered.

$$F[\psi(x)] \equiv f(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n) \tag{B.5}$$

It is sometimes convenient to expand a field function in terms of the *complex conjugate* modes $\phi_k^*(x)$. Thus $\psi^+(x)$ given by

$$\psi^+(x) = \sum_k \phi_k^*(x)\alpha_k^+ \tag{B.6}$$

is also a field function, and if $\alpha_k^+ = \alpha_k^*$ then $\psi^+(x) = \psi^*(x)$, the complex conjugate field.

The idea of a functional can be extended to cases of the form $F[\psi(x_1, x_2, .., x_n)]$ where $\psi(x_1, x_2, .., x_n)$ is a function of several variables $x_1, x_2, .., x_n$. For 3D fields the situation $x_1 = x, x_2 = y, x_3 = z$ is such an application. In addition, cases $F[\widehat{\psi}(x)]$ where $\widehat{\psi}(x)$ is an operator function rather than a c-number function occur. For example, $\widehat{\psi}(x)$ may be a bosonic field operator. In this case $F[\widehat{\psi}(x)]$ maps the operator function onto an operator. Also functionals $F[\psi_1(x), \psi_2(x), .., \psi_i(x), ..\psi_n(x)]$ involving several functions $\psi_1(x), \psi_2(x), .., \psi_i(x), ..\psi_n(x)$ occur. For example, a bosonic field operator $\widehat{\psi}(x)$ may be associated with a field function $\psi_1(x) = \psi(x)$ and the field operator $\widehat{\psi}(x)^\dagger$ may be associated with a different field function $\psi_2(x) = \psi^+(x)$, so functionals of the form $F[\psi(x), \psi^+(x)]$ are involved. Of particular relevance are cases where the functional involves fields and their complex conjugates, such as $F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$. Functional derivatives and functional integrals can be defined for all of these cases.

Appendix B.2. Examples of Functionals

A typical example of a functional involves an integration process:

$$F[\psi(x)] = \int_a^b dx\, \phi(\psi(x)) \tag{B.7}$$

where $\phi(\psi(x))$ is some function of $\psi(x)$.

The *scalar product* of $\psi(x)$ with a fixed function $\chi(x)$ is a typical example of a functional (written $\chi[\psi(x)]$) since

$$\chi[\psi(x)] = \int dx\, \chi^*(x)\,\psi(x). \tag{B.8}$$

A functional $F[\psi(x)]$ may take the form of an integral of a function $\mathcal{F}(\psi(x), \partial_x \psi(x))$ involving the *spatial derivative* $\partial_x \psi(x)$ as well as $\psi(x)$

$$F[\psi(x)] = \int dx\, \mathcal{F}(\psi(x), \partial_x \psi(x)) \tag{B.9}$$

A *function* $\psi(y)$ may also be expressed as a functional $F_y[\psi(x)]$

$$\begin{aligned} F_y[\psi(x)] &= \int dx\, \delta(x-y)\,\psi(x). \\ &= \psi(y) \end{aligned} \tag{B.10}$$

Another example involves the *spatial derivative* $\nabla_y \psi(y)$ which may also be expressed as a functional $F_{\nabla y}[\psi(x)]$

$$\begin{aligned} F_{\nabla y}[\psi(x)] &= \int dx\, \delta(x-y)\,\nabla_x \psi(x) \\ &= \nabla_y \psi(y) \end{aligned} \tag{B.11}$$

A functional is said to be *linear* if

$$F[c_1\psi_1(x) + c_2\psi_2(x)] = c_1 F[\psi_1(x)] + c_2 F[\psi_2(x)] \qquad (B.12)$$

where c_1, c_2 are constants. The scalar product is a linear functional.

Appendix B.3. Functional Differentiation

The *functional derivative* $\frac{\delta F[\psi(x)]}{\delta\psi(x)}$ is defined by

$$F[\psi(x) + \delta\psi(x)] \doteq F[\psi(x)] + \int dx\,\delta\psi(x)\left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_x \qquad (B.13)$$

where $\delta\psi(x)$ is small. In this equation the left side is a functional of $\psi(x) + \delta\psi(x)$ and the first term on the right side is a functional of $\psi(x)$. The second term on the right side is a functional of $\delta\psi(x)$ and thus the functional derivative must be a function of x, hence the subscript x. In most situations this subscript will be left understood. If we write $\delta\psi(x) = \epsilon\delta(x-y)$ for small ϵ then an equivalent result for the functional derivative at $x = y$ is

$$\left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_{x=y} = \lim_{\epsilon\to 0}\left(\frac{F[\psi(x) + \epsilon\delta(x-y)] - F[\psi(x)]}{\epsilon}\right). \qquad (B.14)$$

This definition of a functional derivative can be extended to cases where $\psi(x_1, x_2, .., x_n)$ is a function of several variables or where $\widehat{\psi}(x)$ is an operator function rather than a c-number function. Also functionals $F[\psi_1(x), \psi_2(x), .., \psi_i(x), ..\psi_n(x)]$ involving several functions $\psi_1(x), \psi_2(x), .., \psi_i(x), ..\psi_n(x)$ occur, and functional derivatives with respect to any of these functions can be defined. For example, the functional $F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$ leads to functional derivatives with respect to all four fields defined via an obvious generalisation of (B.13), the conjugate fields $\psi(x), \psi^*(x)$ and $\psi^+(x), \psi^{+*}(x)$ being regarded as independent of each other.

Finally, *higher order functional derivatives* can be defined by applying the basic definitions to lower order functional derivatives.

Appendix B.4. Examples of Functional Derivatives

For the case of the functional $F_y[\psi(x)]$ in Eq.(B.10) that gives the function $\psi(y)$

$$
\begin{aligned}
\left(\frac{\delta F_y[\psi(x)]}{\delta\psi(x)}\right)_x &= \left(\frac{\delta\psi(y)}{\delta\psi(x)}\right)_x \\
&= \lim_{\epsilon\to 0}\left(\frac{\int dz\,\delta(z-y)\{\psi(z) + \epsilon\delta(z-x)\} - \int dz\,\delta(z-y)\,\psi(z)}{\epsilon}\right) \\
&= \delta(x-y) \qquad\qquad (B.15)
\end{aligned}
$$

so here the functional derivative is a delta function.

A similar situation applies to the case where the functional $F_{\nabla y}[\psi(x)]$ in Eq.(B.11) that gives the spatial derivative function $\nabla_y \psi(y)$. Using integration by parts

$$
\begin{aligned}
F_{\nabla y}[\psi(x) + \delta\psi(x)] &= \int dx\, \delta(x - y)\, \nabla_x(\psi(x) + \delta\psi(x)) \\
&= F_{\nabla y}[\psi(x)] + \int dx\, \delta(x - y)\, \nabla_x \delta\psi(x) \\
&= F_{\nabla y}[\psi(x)] - \int dx\, \nabla_x \delta(x - y)\, \delta\psi(x) \\
&= F_{\nabla y}[\psi(x)] + \int dx\, \nabla_y \delta(x - y)\, \delta\psi(x)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(\frac{\delta F_{\nabla y}[\psi(x)]}{\delta \psi(x)} \right)_x &= \left(\frac{\delta \nabla_y \psi(y)}{\delta \psi(x)} \right)_x \\
&= \nabla_y \delta(x - y) = -\nabla_x \delta(x - y)
\end{aligned}
$$

(B.16)

so here the functional derivative is the derivative of a delta function.

Appendix B.5. Functional Derivative and Mode Functions

If a mode expansion for $\psi(x)$ as in Eq.(B.113) etc. is used, then we can obtain an expression for the functional derivative in terms of mode functions. By writing

$$
\delta\psi(x) = \sum_k \delta\alpha_k\, \phi_k(x)
$$

we see that

$$
\begin{aligned}
F[\psi(x) + \delta\psi(x)] - F[\psi(x)] &\doteq \int dx\, \delta\psi(x) \left(\frac{\delta F[\psi(x)]}{\delta \psi(x)} \right)_x \\
&\doteq \sum_k \delta\alpha_k \int dx\, \phi_k(x) \left(\frac{\delta F[\psi(x)]}{\delta \psi(x)} \right)_x
\end{aligned}
$$

But the left side is the same as

$$
f(\alpha_1 + \delta\alpha_1, .., \alpha_k + \delta\alpha_k, ..) - f(\alpha_1, .., \alpha_k, ..) \doteq \sum_k \delta\alpha_k \frac{\partial f(\alpha_1, .., \alpha_k, ..)}{\partial \alpha_k}
$$

Equating the coefficients of the independent $\delta\alpha_k$ and then using the completeness relationship in Eq.(B.4) gives the key result

$$
\left(\frac{\delta F[\psi(x)]}{\delta \psi(x)} \right)_x = \sum_k \phi_k^*(x) \frac{\partial f(\alpha_1, .., \alpha_k, ..)}{\partial \alpha_k} \tag{B.17}
$$

$$
\frac{\partial f(\alpha_1, .., \alpha_k, ..)}{\partial \alpha_k} = \int dx\, \phi_k(x) \left(\frac{\delta F[\psi(x)]}{\delta \psi(x)} \right)_x \tag{B.18}
$$

These relate the functional derivative to the mode functions and to the ordinary partial derivatives of the function $f(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n)$ that was equivalent to the original functional $F[\psi(x)]$. Again, we see that the result for the functional derivative is a function of x.

For the case of the functional $F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$ whose equivalent function based on the expansions (B.1) and (B.6) is $f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})$, the generalisation of (B.17) is

$$
\begin{aligned}
\left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi(x)} \right)_x &= \sum_k \phi_k^*(x) \frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k} \\
\left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi^+(x)} \right)_x &= \sum_k \phi_k(x) \frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k^+} \\
\left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi^*(x)} \right)_x &= \sum_k \phi_k(x) \frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k^*} \\
\left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi^{+*}(x)} \right)_x &= \sum_k \phi_k^*(x) \frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k^{+*}}
\end{aligned}
\tag{B.19}
$$

and

$$
\begin{aligned}
\frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k} &= \int dx \, \phi_k(x) \left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi(x)} \right)_x \\
\frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k^+} &= \int dx \, \phi_k^*(x) \left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi^+(x)} \right)_x \\
\frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k^*} &= \int dx \, \phi_k^*(x) \left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi^*(x)} \right)_x \\
\frac{\partial f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})}{\partial \alpha_k^{+*}} &= \int dx \, \phi_k(x) \left(\frac{\delta F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]}{\delta \psi^{+*}(x)} \right)_x
\end{aligned}
\tag{B.20}
$$

which relate the functional derivatives and the derivatives with respect to the mode amplitudes.

Appendix B.6. Rules for Functional Derivatives

Rules can be established for the functional derivative of the *sum* of two functionals. It is easily shown that

$$
\left(\frac{\delta \{F[\psi(x)] + G[\psi(x)]\}}{\delta \psi(x)} \right)_{x=y} = \left(\frac{\delta \{F[\psi(x)]\}}{\delta \psi(x)} \right)_{x=y} + \left(\frac{\delta \{G[\psi(x)]\}}{\delta \psi(x)} \right)_{x=y}
\tag{B.21}
$$

Also, rules can be established for functional derivative of the *product* of two functionals. We will keep these in order to cover the case where the functionals

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are operators

$$\left(\frac{\delta\{F[\psi(x)]G[\psi(x)]\}}{\delta\psi(x)}\right)_{x=y} = \left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_{x=y} G[\psi(x)] + F[\psi(x)]\left(\frac{\delta G[\psi(x)]}{\delta\psi(x)}\right)_{x=y}$$
(B.22)

A *chain* rule for functional differentiation can also be derived for the case where a functional $G[\psi_y(x)]$ involves not just one function $\psi(x)$, but a set of functions each labelled by a variable y. Since $G[\psi_y(x)]$ maps $\psi_y(x)$ onto a c-number which depends on y, we can regard the functional $G[\psi_y(x)]$ also as a function $G(y)$ of the variable y. Now consider a second functional $F[G(y)]$ of this function $G(y)$, and we could determine the functional derivative $\left(\frac{\delta F[G(y)]}{\delta G(y)}\right)_y$. But $F[G(y)]$ is also a functional of the $\psi_y(x)$ via

$$F[G[\psi_y(x)]] \equiv F[G(y)]$$

We obtain the chain rule

$$\left(\frac{\delta F[G[\psi_y(x)]]}{\delta\psi_y(x)}\right)_x = \int dy \left(\frac{\delta F[G(y)]}{\delta G(y)}\right)_y \left(\frac{\delta F[G(y)]}{\delta G(y)}\right)_x$$
(B.23)

where we have left the order of the factors as they appeared in order to allow for operator cases.

We may also define the *spatial derivative* of the functional derivative. Thus

$$\partial_y \left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_{x=y} = \lim_{\Delta y \to 0} \left(\frac{\left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_{x=y+\Delta y} - \left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_{x=y}}{\Delta y}\right)$$

$$= -\int dx \left(\frac{\partial}{\partial x}\delta(x-y)\right)_{x=y} \left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_x$$
(B.24)

This expresses the spatial derivative as an integral involving the functional derivative and the spatial derivative of the delta function. The result will be a function of s.

A number of *other rules* may also be established.

(1) Power rule

$$F[\psi(x)] = \int dx\,\psi(x)^n$$

$$\frac{\delta F[\psi(x)]}{\delta\psi(x)} = n\psi(x)^{n-1}$$
(B.25)

(2) Function rule

$$F[\psi(x)] = \int dx\,\phi(\psi(x))$$

$$\frac{\delta F[\psi(x)]}{\delta\psi(x)} = \phi'(\psi(x))$$
(B.26)

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(3) Power derivative rule

$$F[\psi(x)] = \int dx \, (\frac{d\psi(x)}{dx})^n$$

$$\frac{\delta F[\psi(x)]}{\delta \psi(x)} = -n\frac{d}{dx}((\frac{d\psi(x)}{dx})^{n-1}) \qquad (B.27)$$

(4) Function derivative rule

$$F[\psi(x)] = \int dx \, \phi(\frac{d\psi(x)}{dx})$$

$$\frac{\delta F[\psi(x)]}{\delta \psi(x)} = -\frac{d}{dx}((\frac{d\phi}{d(\frac{d\psi}{dx})})) \qquad (B.28)$$

(5) Convolution rule

$$F_y[\psi(x)] = \int dx \, K(y,x)\,\psi(x)$$

$$\left(\frac{\delta F_y[\psi(x)]}{\delta \psi(x)}\right)_x = K(y,x) \qquad (B.29)$$

(6) Trivial rule

$$F_y[\psi(x)] = \psi(y)$$

$$\left(\frac{\delta F_y[\psi(x)]}{\delta \psi(x)}\right)_x = \left(\frac{\delta \psi(y)}{\delta \psi(x)}\right)_x$$

$$= \delta(x-y) \qquad (B.30)$$

This was proved above.

(7) Gradient rule

$$F_{\nabla y}[\psi(x)] = \nabla_y \psi(y)$$

$$\left(\frac{\delta F_{\nabla y}[\psi(x)]}{\delta \psi(x)}\right)_x = \nabla_y \delta(x-y) = -\nabla_x \delta(x-y) \qquad (B.31)$$

This was proved above.

(8) Exponential rule

$$F[\psi(x)] = \exp G[\psi(x)]$$

$$\frac{\delta F[\psi(x)]}{\delta \psi(x)} = \exp G[\psi(x)] \frac{\delta G[\psi(x)]}{\delta \psi(x)} \qquad (B.32)$$

The exponential rule only applies in this form if $F[\psi(x)]$ and $G[\psi(x)]$ are c-numbers.

All these rules have obvious generalisations for functionals involving several fields, such as $F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$.

Appendix B.7. Functional Integration

If the range for the function $\psi(x)$ is divided up into n small intervals $\Delta x_i = x_{i+1} - x_i$ (the ith interval), then we may specify the *value* ψ_i of the function $\psi(x)$ in the ith interval via the *average*

$$\psi_i = \frac{1}{\Delta x_i} \int_{\Delta x_i} dx\, \psi(x) \tag{B.33}$$

and then the functional $F[\psi(x)]$ may be regarded as a *function* $F(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$ of all the ψ_i.

Introducing a suitable *weight function* $w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$ we may then define the *functional integral* for the case of *real* functions as

$$\int D\psi\, F[\psi(x)] = \lim_{n\to\infty} \lim_{\epsilon\to 0} \int ... \int d\psi_1 d\psi_2 .. d\psi_i .. d\psi_n\, w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$$
$$\times F(\psi_1, \psi_2, .., \psi_i, .., \psi_n) \tag{B.34}$$

where $\epsilon > \Delta x_i$. Thus the symbol $D\psi$ stands for $d\psi_1 d\psi_2 .. d\psi_i .. d\psi_n\, w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$.

If the functions are *complex* then the functional integral is

$$\int D^2\psi\, F[\psi(x)] = \lim_{n\to\infty} \lim_{\epsilon\to 0} \int ... \int d^2\psi_1 d^2\psi_2 .. d^2\psi_i .. d^2\psi_n\, w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$$
$$\times F(\psi_1, \psi_2, .., \psi_i, .., \psi_n) \tag{B.35}$$

The symbol $D^2\psi$ stands for $d^2\psi_1 d^2\psi_2 .. d^2\psi_i .. d^2\psi_n\, w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$, where with $\psi_i = \psi_{ix} + i\psi_{iy}$ the quantity $d^2\psi_i$ means $d\psi_{ix} d\psi_{iy}$, involving integration over the real, imaginary parts of the complex function.

For cases involving several complex functions such as $F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$ the functional integrals are of the form

$$\int \int D^2\psi\, D^2\psi^+\, F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$$
$$= \lim_{n\to\infty} \lim_{\epsilon\to 0} \int ... \int d^2\psi_1 d^2\psi_2 .. d^2\psi_i .. d^2\psi_n \lim_{n\to\infty} \lim_{\epsilon\to 0} \int ... \int d^2\psi_1^+ d^2\psi_2^+ .. d^2\psi_i^+ .. d^2\psi_n^+$$
$$\times w(\psi_1, .., \psi_i, .., \psi_n, \psi_1^+, .., \psi_i^+, .., \psi_n^+, \psi_1^*, .., \psi_i^*, .., \psi_n^*, \psi_1^{+*}, .., \psi_i^{+*}, .., \psi_n^{+*})$$
$$\times F(\psi_1, .., \psi_i, .., \psi_n, \psi_1^+, .., \psi_i^+, .., \psi_n^+, \psi_1^*, .., \psi_i^*, .., \psi_n^*, \psi_1^{+*}, .., \psi_i^{+*}, .., \psi_n^{+*}) \tag{B.36}$$

where $D^2\psi\, D^2\psi^+$ stands for

$$d^2\psi_1 .. d^2\psi_i .. d^2\psi_n\, d^2\psi_1^+ .. d^2\psi_i^+ .. d^2\psi_n^+\, w(\psi_1, .., \psi_i, .., \psi_n, \psi_1^+, .., \psi_i^+, .., \psi_n^+, \psi_1^*, .., \psi_i^*, .., \psi_n^*, \psi_1^{+*}, .., \psi_i^{+*}, .., \psi_n^{+*})$$

and where with $\psi_i^+ = \psi_{ix}^+ + i\psi_{iy}^+$, the quantity $d^2\psi_i^+$ means $d\psi_{ix}^+ d\psi_{iy}^+$.

A functional integral of a functional of a c-number function gives a c-number. Unlike ordinary calculus, functional integration and differentiation are not related as inverse processes.

Appendix B.8. Functional Integrals and Phase Space Integrals

We first consider the case of a functional $F[\psi(x)]$ of a real function $\psi(x)$, which we expand in terms of real, orthogonal mode functions. The expansion coefficients in this case will be real also. If a mode expansion such as in Eq.(B.113) etc. is used then the value ϕ_{ki} of the mode function in the ith interval is also defined via the *average*

$$\phi_{ki} = \frac{1}{\Delta x_i} \int_{\Delta x_i} dx \, \phi_k(x) \tag{B.37}$$

and hence

$$\psi_i = \sum_k \alpha_k \, \phi_{ki}. \tag{B.38}$$

This shows that the values in the ith interval of the function ψ_i and the mode function ϕ_{ki} are related via the expansion coefficients α_k. For simplicity we will choose the *same number n* of intervals as mode functions. Using the expression Eq.(B.3) for the expansion coefficients we then obtain the inverse formula to Eq.(B.38)

$$\alpha_k = \sum_i \Delta x_i \, \phi_{ki}.\psi_i. \tag{B.39}$$

Note that this involves a sum over intervals i and the interval size Δx_i is also involved.

The relationship in Eq.(B.38) shows that the functions $F(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$ and $w(\psi_1, \psi_2, .., \psi_i, .., \psi_n)$ of all the interval values ψ_i can also be regarded as functions of the expansion coefficients α_k which we may write as

$$f(\alpha_1, .., \alpha_k, ..\alpha_n) \equiv F(\psi_1(\alpha_1, .., \alpha_k, ..\alpha_n), ..., \psi_i(\alpha_1, .., \alpha_k, ..\alpha_n), .., \psi_n)$$
$$\tag{B.40}$$

$$v(\alpha_1, .., \alpha_k, ..\alpha_n) \equiv w(\psi_1(\alpha_1, .., \alpha_k, ..\alpha_n), ..., \psi_i(\alpha_1, .., \alpha_k, ..\alpha_n), .., \psi_n)$$
$$\tag{B.41}$$

Thus the various *values* $\psi_1, \psi_2, .., \psi_1, \psi_2, .., \psi_i, .., \psi_n, .., \psi_n$ of that the function $\psi(x)$ takes on in the n intervals - and which are integrated over in the functional integration process - are *all* determined by the choice of the expansion coefficients $\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n$. Hence integration over all the ψ_i will be equivalent to integration over all the α_k.

This enables us to express the functional integral in Eq.(B.34) as a *phase space integral* over the expansion coefficients $\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n$. We have

$$\int D\psi \, F[\psi(x)] = \lim_{n\to\infty} \lim_{\epsilon\to0} \int ... \int d\alpha_1 d\alpha_2 .. d\alpha_k .. d\alpha_n \, ||J(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n)||$$
$$\times v(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n) \, f(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n) \tag{B.42}$$

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where the Jacobian is given by

$$||J(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n)|| = \begin{Vmatrix} \frac{\partial \psi_1}{\partial \alpha_1} & \frac{\partial \psi_1}{\partial \alpha_2} & \cdots & \frac{\partial \psi_1}{\partial \alpha_n} \\ \frac{\partial \psi_2}{\partial \alpha_1} & \frac{\partial \psi_2}{\partial \alpha_2} & \cdots & \frac{\partial \psi_2}{\partial \alpha_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi_n}{\partial \alpha_1} & \frac{\partial \psi_n}{\partial \alpha_2} & & \frac{\partial \psi_n}{\partial \alpha_n} \end{Vmatrix} \tag{B.43}$$

Now using Eq.(B.38)

$$\frac{\partial \psi_i}{\partial \alpha_k} = \phi_{ki} \tag{B.44}$$

and evaluating the Jacobian using after showing that $(JJ^T)_{ik} = \delta_{ik}/\Delta x_i$ using the completeness relationship in Eq.(B.4) we find that

$$||J(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n)|| = \prod_i \frac{1}{(\Delta x_i)^{1/2}} \tag{B.45}$$

and thus

$$\begin{aligned} \int D\psi \, F[\psi(x)] &= \lim_{n \to \infty} \lim_{\epsilon \to 0} \int \ldots \int d\alpha_1 d\alpha_2 .. d\alpha_k .. d\alpha_n \prod_i \frac{1}{(\Delta x_i)^{1/2}} \\ &\quad \times v(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n) \, f(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n) \end{aligned} \tag{B.46}$$

This key result expresses the original functional integral as a phase space integral over the expansion coefficients α_k of the function $\psi(x)$ in terms of the mode functions $\phi_k(x)$.

The general result can be simplified with a *special choice* of the weight function

$$w(\psi_1, \psi_2, .., \psi_i, .., \psi_n) = \prod_i (\Delta x_i)^{1/2} \tag{B.47}$$

and we then get a simple expression for the functional integral

$$\int D\psi \, F[\psi(x)] = \lim_{n \to \infty} \lim_{\epsilon \to 0} \int \ldots \int d\alpha_1 d\alpha_2 .. d\alpha_k .. d\alpha_n \, f(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n) \tag{B.48}$$

In this form of the functional integral the original functional $F[\psi(x)]$ has been replaced by the equivalent function $f(\alpha_1, \alpha_2, .., \alpha_k, ..\alpha_n)$ of the expansion coefficients α_k, and the functional integration is now replaced by a phase space integration over the expansion coefficients.

The relationship between the functional integral and the phase space integral can be generalised to cases involving several complex functions. For the case of the functional $F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$, where $\psi(x), \psi^+(x)$ are expanded in terms of complex mode functions as in (B.1), (B.6) and ψ_i, ψ_i^+ defined as in (B.33) we have

$$\begin{aligned} \psi_i &= \sum_k \alpha_k \, \phi_{ki}. & \alpha_k &= \sum_i \Delta x_i \, \phi_{ki}^* . \psi_i. \\ \psi_i^+ &= \sum_k \alpha_k^+ \, \phi_{ki}^*. & \alpha_k^+ &= \sum_i \Delta x_i \, \phi_{ki} . \psi_i^+. \end{aligned}$$

$$\tag{B.49}$$

For variety we will turn the phase space integral into a functional integral. We first have the transformation involving real quantities

$$\alpha_{kX} = \sum_i \Delta x_i \left(\phi_{kiX}.\psi_{iX} + \phi_{kiY}.\psi_{iY} \right)$$

$$\alpha_{kY} = \sum_i \Delta x_i \left(\phi_{kiX}.\psi_{iY} - \phi_{kiY}.\psi_{iX} \right)$$

$$\alpha_{kX}^+ = \sum_i \Delta x_i \left(\phi_{kiX}.\psi_{iX}^+ - \phi_{kiY}.\psi_{iY}^+ \right)$$

$$\alpha_{kY}^+ = \sum_i \Delta x_i \left(\phi_{kiX}.\psi_{iY}^+ + \phi_{kiY}.\psi_{iX}^+ \right) \qquad (B.50)$$

In the standard notation with $\alpha_k = \alpha_{kX} + i\alpha_{kY}$, $\alpha_k^+ = \alpha_{kX}^+ + i\alpha_{kY}^+$ and $d^2\alpha_k = d\alpha_{kX}d\alpha_{kY}$, $d^2\alpha_k^+ = d\alpha_{kX}^+ d\alpha_{kY}^+$ the phase space integral is of the form

$$\int\int d^2\alpha \, d^2\alpha^+ \, f(\alpha, \alpha^+, \alpha^*, \alpha^{+*})$$

$$= \int \dots \int d^2\alpha_1 d^2\alpha_2 .. d^2\alpha_k .. d^2\alpha_n \int \dots \int d^2\alpha_1^+ d^2\alpha_2^+ .. d^2\alpha_k^+ .. d^2\alpha_n^+ \, f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})$$

$$(B.51)$$

and after transforming to the new variables $\psi_{iX}, \psi_{iY}, \psi_{iX}^+, \psi_{iY}^+$ we get

$$\int \dots \int d^2\alpha_1 d^2\alpha_2 .. d^2\alpha_k .. d^2\alpha_n \int \dots \int d^2\alpha_1^+ d^2\alpha_2^+ .. d^2\alpha_k^+ .. d^2\alpha_n^+ \, f(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})$$

$$= \int \dots \int d^2\psi_1 d^2\psi_2 .. d^2\psi_i .. d^2\psi_n \int \dots \int d^2\psi_1^+ d^2\psi_2^+ .. d^2\psi_i^+ .. d^2\psi_n^+ \, ||J(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})||$$

$$\times F(\psi_1, .., \psi_i, .., \psi_n, \psi_1^+, .., \psi_i^+, .., \psi_n^+, \psi_1^*, .., \psi_i^*, .., \psi_n^*, \psi_1^{+*}, .., \psi_i^{+*}, .., \psi_n^{+*}) \qquad (B.52)$$

where the Jacobian can be written in terms of the notation $\alpha_{kX} \to \alpha_{k1}, \alpha_{kY} \to \alpha_{k2}, \alpha_{kX}^+ \to \alpha_{k3}, \alpha_{kY}^+ \to \alpha_{k4}$ and $\psi_{iX} \to \psi_{i1}, \psi_{iY} \to \psi_{i2}, \psi_{iX}^+ \to \psi_{i3}, \psi_{iY}^+ \to \psi_{i4}$ in which the Jacobian is the determinent of the matrix J where

$$J_{k\mu \, i\nu} = \frac{\partial \alpha_{k\mu}}{\partial \psi_{i\nu}} \qquad (k = 1, .., n; i = 1, .., n; \mu = 1, .., 4; \nu = 1, .., 4)$$

$$||J(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*})|| = ||J_{k\mu \, i\nu}|| \qquad (B.53)$$

The elements in the $4x4$ submatrix $J_{k \, i}$ are obtained from (B.50) and are

$$[J_{k \, i}] = \begin{bmatrix} \Delta x_i \, \phi_{kiX} & \Delta x_i \, \phi_{kiY} & 0 & 0 \\ -\Delta x_i \, \phi_{kiY} & \Delta x_i \, \phi_{kiX} & 0 & 0 \\ 0 & 0 & \Delta x_i \, \phi_{kiX} & -\Delta x_i \, \phi_{kiY} \\ 0 & 0 & \Delta x_i \, \phi_{kiY} & \Delta x_i \, \phi_{kiX} \end{bmatrix} \qquad (B.54)$$

The completeness relationship (B.4) can then be used to show that

$$\Delta x_i \Delta x_j \sum_k (\phi_{kiX} \, \phi_{kjX} + \phi_{kiY} \, \phi_{kjY}) = \Delta x_i \, \delta_{i \, j}$$

$$\Delta x_i \Delta x_j \sum_k (-\phi_{kiX} \, \phi_{kjY} + \phi_{kiY} \, \phi_{kjX}) = 0 \qquad (B.55)$$

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which is the same as

$$\sum_{k\mu} J_{k\mu\ i\nu} J_{k\mu\ j\xi} = \Delta x_i\,\delta_{i\ j}\delta_{\nu\ \xi}$$

$$\left[J^T J\right]_{i\nu\ j\xi} = \Delta x_i\,\delta_{i\ j}\delta_{\nu\ \xi} \tag{B.56}$$

Hence

$$\|J_{k\mu\ i\nu}\| = \prod_{i=1}^{n}(\Delta x_i)^2 \tag{B.57}$$

so that we have finally after letting $n \to \infty$ and $\Delta x_i \to 0$ and with $d^2\alpha = \prod_k d^2\alpha_k$, $d^2\alpha^+ = \prod_k d^2\alpha_k^+$

$$\int\int d^2\alpha\ d^2\alpha^+\ f(\alpha,\alpha^+,\alpha^*,\alpha^{+*}) \tag{B.58}$$

$$= \lim_{n\to\infty}\lim_{\epsilon\to0}\int\ldots\int d^2\alpha_1 d^2\alpha_2..d^2\alpha_k..d^2\alpha_n$$

$$\times\int\ldots\int d^2\alpha_1^+ d^2\alpha_2^+..d^2\alpha_k^+..d^2\alpha_n^+\ f(\alpha_k,\alpha_k^+,\alpha_k^*,\alpha_k^{+*})$$

$$= \lim_{n\to\infty}\lim_{\epsilon\to0}\int\ldots\int d^2\psi_1 d^2\psi_2..d^2\psi_i..d^2\psi_n\int\ldots\int d^2\psi_1^+ d^2\psi_2^+..d^2\psi_i^+..d^2\psi_n^+$$

$$\times w(\psi_1,..,\psi_i,..,\psi_n,\psi_1^+,..,\psi_i^+,..,\psi_n^+,\psi_1^*,..,\psi_i^*,..,\psi_n^*,\psi_1^{+*},..,\psi_i^{+*},..,\psi_n^{+*})$$

$$\times F(\psi_1,..,\psi_i,..,\psi_n,\psi_1^+,..,\psi_i^+,..,\psi_n^+,\psi_1^*,..,\psi_i^*,..,\psi_n^*,\psi_1^{+*},..,\psi_i^{+*},..,\psi_n^{+*})$$

$$= \int\int D^2\psi\ D^2\psi^+\ F[\psi(x),\psi^+(x),\psi^*(x),\psi^{+*}(x)] \tag{B.59}$$

where $D^2\psi D^2\psi^+ = \prod_i d^2\psi_i \prod_i d^2\psi_i\ w(\psi_1,..,\psi_n,\psi_1^+,..,\psi_n^+,\psi_1^*,..,\psi_n^*,\psi_1^{+*},..,\psi_n^{+*})$
and the weight function is

$$w(\psi_1,..,\psi_n,\psi_1^+,..,\psi_n^+,\psi_1^*,..,\psi_n^*,\psi_1^{+*},..,\psi_n^{+*}) = \prod_{i=1}^{n}(\Delta x_i)^2 \tag{B.60}$$

and is independent of the functions. The power law $(\Delta x_i)^2$ is consistent with there being four real functions involved instead of the single function as previously.

Appendix B.9. Functional Integration Rules

A useful *integration by parts* rule can often be established from Eq.(B.22). Consider the functional $H[\psi(x)] = F[\psi(x)]G[\psi(x)]$. Then

$$F[\psi(x)]\left(\frac{\delta G[\psi(x)]}{\delta\psi(x)}\right) = \left(\frac{\delta\{F[\psi(x)]G[\psi(x)]\}}{\delta\psi(x)}\right) - \left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)G[\psi(x)]$$

Then

$$\int D\psi\ F[\psi(x)]\left(\frac{\delta G[\psi(x)]}{\delta\psi(x)}\right) = \int D\psi\left(\frac{\delta H[\psi(x)]}{\delta\psi(x)}\right) - \int D\psi\left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)G[\psi(x)]$$

If we now introduce mode expansions and use Eq.(B.17) for the functional derivative of $H[\psi(x)]$ and Eq.(B.48) for the first of the two functional integrals on the right hand side of the last equation then

$$\int D\psi \left(\frac{\delta H[\psi(x)]}{\delta \psi(x)} \right) = \lim_{n\to\infty} \lim_{\epsilon\to 0} \int \dots \int d\alpha_1 d\alpha_2 .. d\alpha_k .. d\alpha_n \sum_k \phi_k^*(x) \frac{\partial h(\alpha_1, .., \alpha_k, ..)}{\partial \alpha_k}$$

$$= \lim_{n\to\infty} \lim_{\epsilon\to 0} \sum_k \phi_k^*(x) \int \dots \int d\alpha_1 d\alpha_2 ..$$

$$\times \{ h(\alpha_1, .., \alpha_k, ..)_{\alpha_k \to +\infty} - h(\alpha_1, .., \alpha_k, ..)_{\alpha_k \to -\infty} \} .. d\alpha_n$$

so that the functional integral of this term reduces to contributions on the boundaries of phase space. Hence if $h(\alpha_1, .., \alpha_k, ..) \to 0$ as all $\alpha_k \to \pm\infty$ then the functional integral involving the functional derivative of $H[\psi(x)]$ vanishes and we have the integration by parts result

$$\int D\psi \, F[\psi(x)] \left(\frac{\delta G[\psi(x)]}{\delta \psi(x)} \right) = - \int D\psi \left(\frac{\delta F[\psi(x)]}{\delta \psi(x)} \right) G[\psi(x)] \qquad (B.61)$$

All these rules have obvious generalisations for functionals such as $F[\psi(x), \psi^+(x), \psi^*(x), \psi^{+*}(x)]$ involving several fields.

Appendix B.10. Restricted Functions

It is necessary to also consider functionals involving c-number field functions $\psi^K(x)$ which are still based on an expansion in terms of orthonormal mode functions $\phi_k(x)$, but where there is some restriction on the modes that are included. Such functions will be referred to as *restricted functions*. Examples include the fields $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$ used for condensate and non-condensate modes in the theory of Bose condensates, where even the combined condensate and non-condensate modes are subject to a restriction, in that modes associated with a momentum greater than a cut-off value are excluded.

Thus we have

$$\psi^K(x) = \sum_k^K \beta_k \, \phi_k(x) \qquad (B.62)$$

where the specific restricted mode expansion for the *restricted set* K is signified by the symbol K. Other restricted sets involving different modes will be designated L, M etc., with expansion coefficients γ_k, δ_k etc.

Orthonormality conditions still apply to all modes

$$\int dx \, \phi_k^*(x)\phi_l(x) = \delta_{kl} \qquad (B.63)$$

and this gives the well-known result for the expansion coefficients

$$\beta_k = \int dx \, \phi_k^*(x)\psi^K(x) \qquad (B.64)$$

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However the completeness relationship is now

$$\sum_{k}^{K} \phi_k(y)\phi_k^*(x) = \delta_K(y, x). \tag{B.65}$$

which defines the *restricted delta function* $\delta_K(y, x)$ for the K set. This is a function of two variables x and y, and does not depend on $y - x$.

The restricted delta functions have some interesting properties

$$\iint dx\, dy\, \phi_l^*(y)\, \delta_K(y, x)\, \phi_m(x) \;\;=\;\; \delta_{l,m} \qquad (l, m \in K)$$
$$=\;\; 0 \qquad (l \notin K,\, m \notin K) \tag{B.66}$$

$$\int dx\, \delta_K(y, x).\delta_L(x, z) \;\;=\;\; \int dx\, \sum_{k}^{K} \phi_k(y)\phi_k^*(x) \sum_{l}^{L} \phi_l(x)\phi_l^*(z)$$
$$=\;\; \sum_{k}^{K} \phi_k(y)\, \delta_{k,l} \sum_{l}^{L} \phi_l^*(z)$$
$$=\;\; \delta_{K,L}\, \delta_K(y, z) \tag{B.67}$$

and

$$\int dx\, \delta_K(x, x) = N_K \tag{B.68}$$

where N_K is the number of mode functions in the set K. Unlike the normal delta function the restricted delta functions are non-singular and can be treated as standard c-number functions within expressions.

Appendix B.11. Functionals of Restricted Functions

As for general functions, a *functional* $F[\psi^K(x)]$ of restricted functions $\psi^K(x)$ maps the *c-number function* $\psi^K(x)$ onto a *c-number* that depends on *all* the values of $\psi^K(x)$ over its entire range.

The *restricted function* $\psi^K(y)$ can be expressed as a functional $F_y[\psi^K(x)]$ of the restricted function $\psi^K(x)$. In terms of the restricted delta function we have

$$\psi^K(y) \;\;=\;\; \int dx\, \delta_K(y, x).\psi^K(x)$$
$$=\;\; F_y[\psi^K(x)] \tag{B.69}$$

showing how $\psi^K(y)$ can still be written as a functional $F_y[\psi^K(x)]$ of $\psi^K(x)$, but now Eq. (B.69) applies which involves the restricted delta function $\delta_K(y, x)$ as a kernal, rather than Eq. (B.10) which involved the normal delta function and applied to functions $\psi(x)$ with unrestricted mode expansions.

The *spatial derivative* $\nabla_y \psi^K(y)$ of the restricted function $\psi^K(y)$ can also be expressed as a functional $F_{\nabla y}[\psi^K(x)]$ of $\psi^K(x)$. Using (B.69) we have

$$
\begin{aligned}
\nabla_y \psi^K(y) &= \int dx\, \nabla_y \delta_K(y,x).\psi^K(x) \\
&= F_{\nabla y}[\psi^K(x)] \quad\quad\quad\quad\quad\quad\quad\quad\quad (B.70)
\end{aligned}
$$

which now involves $\nabla_y \delta_K(y,x)$ as a kernal. We can confirm the validity of (B.70) by substituting for $\psi^K(x)$ from (B.62) which gives

$$
\begin{aligned}
\int dx\, \nabla_y \delta_K(y,x).\psi^K(x) &= \sum_k^K \beta_k \int dx\, \nabla_y \delta_K(y,x).\phi_k(x) \\
&= \sum_k^K \beta_k \sum_l^K \int dx\, \nabla_y \phi_l(y)\phi_l^*(x).\phi_k(x) \\
&= \sum_k^K \beta_k \nabla_y \phi_k(y) \\
&= \nabla_y \psi^K(y)
\end{aligned}
$$

as required.

As the value of the function at any point in the range for x is determined uniquely by the expansion coefficients $\{\beta_k\}$, then the functional $F[\psi^K(x)]$ must therefore also just depend on the c-number expansion coefficients, and hence may also be viewed as a *function* $g(\beta_1, \beta_2, .., \beta_k, ..\beta_n)$ of the *expansion coefficients*, a useful equivalence when functional differentiation and integration are considered.

$$
F[\psi^K(x)] \equiv g(\beta_1, \beta_2, .., \beta_k, ..\beta_n) \quad\quad\quad\quad\quad\quad (B.71)
$$

Appendix B.12. Related Restricted Function Sets

We may also consider restricted functions based on the *conjugate modes*. This set will be referred to as K^* or K^+. Thus the previous equations become

$$
\psi^{K+}(x) = \sum_k^K \phi_k^*(x)\beta_k^+ \quad\quad\quad\quad\quad\quad\quad\quad (B.72)
$$

$$
\beta_k^+ = \int dx\, \phi_k(x)\psi^{K+}(x) \quad\quad\quad\quad\quad\quad\quad (B.73)
$$

$$
\delta_{K+}(y,x) = \sum_k^K \phi_k^*(y)\phi_k(x) \quad\quad\quad\quad\quad\quad\quad (B.74)
$$

where the last equation defines the restricted delta function for the K^+ case. We note that the restricted delta function $\delta_{K+}(y,x)$ is related to the previous one via

$$
\delta_{K+}(y,x) = \delta_K(x,y). \quad\quad\quad\quad\quad\quad\quad\quad (B.75)
$$

We can again write the *restricted function* $\psi^{K+}(y)$ as a functional $F_y[\psi^{K+}(x)]$ via

$$
\begin{aligned}
\psi^{K+}(y) &= \int dx\, \delta_{K+}(y,x).\psi^{K+}(x) \\
&= \int dx\, \delta_K(x,y).\psi^{K+}(y) \\
&= F_y[\psi^{K+}(x)] \qquad\qquad\qquad \text{(B.76)}
\end{aligned}
$$

Similarly the *spatial derivative* $\nabla_y \psi^{K+}(y)$ of the restricted function is also a functional $F_{\nabla_y}[\psi^{K+}(x)]$ given by

$$
\begin{aligned}
\nabla_y \psi^{K+}(y) &= \int dx\, \nabla_y \delta_{K+}(y,x).\psi^{K+}(x) \\
&= \int dx\, \nabla_y \delta_K(x,y).\psi^{K+}(x) \\
&= F_{\nabla_y}[\psi^{K+}(x)] \qquad\qquad\qquad \text{(B.77)}
\end{aligned}
$$

Note that considered as a function of y, the restricted delta function $\delta_K(y,x)$ is a member of the K set of restricted functions $\psi^K(y)$ (the expansion coefficients are $\phi_k^*(x)$). On the other hand, considered as a function of x the restricted delta function $\delta_K(y,x)$ is a member of the *conjugate set* K^+ of mode functions $\phi_k^*(x)$ (the expansion coefficients are $\phi_k(y)$).

As the value of the function at any point in the range for x is determined uniquely by the expansion coefficients $\{\beta_k^+\}$, then the functional $F[\psi^{K+}(x)]$ must therefore also just depend on the c-number expansion coefficients, and hence may also be viewed as a *function* $g^+(\beta_1^+, \beta_2^+, .., \beta_k^+, ..\beta_n^+)$ of the *expansion coefficients*, a useful equivalence when functional differentiation and integration are considered.

$$
F[\psi^{K+}(x)] \equiv g^+(\beta_1^+, \beta_2^+, .., \beta_k^+, ..\beta_n^+) \qquad\qquad \text{(B.78)}
$$

A second related restricted set is the *complementary set* $\overline{K}$ which includes all the other orthonormal mode functions not included in the K set.

Clearly, *any* function can be expanded in terms of modes in the K and $\overline{K}$ restricted sets. Thus we now have

$$
\psi(x) = \sum_k^K \gamma_k\, \phi_k(x) + \sum_k^{\overline{K}} \gamma_k\, \phi_k(x) \qquad\qquad \text{(B.79)}
$$

$$
\gamma_k = \int dx\, \phi_k^*(x)\psi(x) \qquad k \in K, \overline{K} \qquad\qquad \text{(B.80)}
$$

$$
\delta_{\overline{K}}(y,x) = \sum_k^{\overline{K}} \phi_k(y)\phi_k^*(x) = \sum_{L \neq K} \delta_L(y,x) \qquad\qquad \text{(B.81)}
$$

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and now the *full Dirac delta function* is

$$\delta(y, x) = \sum_k^K \phi_k(y)\phi_k^*(x) + \sum_k^{\overline{K}} \phi_k(y)\phi_k^*(x) \tag{B.82}$$

The general function $\psi(y)$ may be written as a functional $F_y[\psi(x)]$ of $\psi(x)$ involving the full delta function

$$
\begin{aligned}
\psi(y) &= \int dx\, \delta(y, x).\psi(x) \\
&= F_y[\psi(x)]
\end{aligned}
\tag{B.83}
$$

Applying (B.67) we obtain the interesting result

$$\int dx\, \delta_K(y, x).\delta_{\overline{K}}(x, z) = 0 \tag{B.84}$$

Note that the full delta function is still written as a function of x and y. Because the total set of functions is still restricted it will have a narrow though finite width and can be treated like a normal function.

Appendix B.13. Functional Derivatives for Restricted Functions

The *functional derivative* $\frac{\delta F[\psi^K(x)]}{\delta \psi^K(x)}$ is defined by

$$F[\psi^K(x) + \delta\psi^K(x)] \doteq F[\psi^K(x)] + \int dx\, \delta\psi^K(x) \left(\frac{\delta F[\psi^K(x)]}{\delta \psi^K(x)} \right)_x \tag{B.85}$$

where $\delta\psi^K(x)$ is a small change in $\psi^K(x)$. Since as in (B.71) the functional is equivalent to a function of the expansion coefficients β_k, the only meaningful change to $\psi^K(x)$ would be associated with changes $\delta\beta_k$ in these expansion coefficients and thus $\delta\psi^K(x)$ will be within the K restricted function space. In this equation the left side is a functional of $\psi^K(x) + \delta\psi^K(x)$ and the first term on the right side is a functional of $\psi^K(x)$. The second term on the right side is a functional of $\delta\psi^K(x)$ and thus the functional derivative must be a function of x, hence the subscript x. In most situations this subscript will be left understood.

Thus the functional derivative will be defined in terms of changes to the restricted function of the form

$$\delta\psi^K(x) = \sum_k^K \delta\beta_k\, \phi_k(x) \tag{B.86}$$

As the functional derivative is just a function of x we can expand it in terms of *all* the *conjugate* modes (these also form a full basis set of orthogonal functions)

$$\left(\frac{\delta F[\psi^K(x)]}{\delta \psi^K(x)} \right)_x = \sum_l \eta_l\, \phi_l^*(x)$$

then we have

$$\int dx \, \delta\psi^K(x) \left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)} \right)_x = \sum_l \eta_l \int dx \, \delta\psi^K(x) \phi_l^*(x)$$

$$= \sum_k^K \eta_k \delta\beta_k$$

since the contributions from modes l not in the K^+ set will be zero using orthogonality. This shows that any contribution to the functional derivative from modes $\phi_l^*(x)$ outside the K^+ set cannot contribute to $F[\psi^K(x) + \delta\psi^K(x)] - F[\psi^K(x)]$, and hence can be arbitarily set to zero in determining the functional derivative with respect to restricted functions $\psi^K(x)$ in the K set. Thus we have

$$\left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)} \right)_x = \sum_k^K \eta_k \, \phi_k^*(x) \tag{B.87}$$

showing that the functional derivative is a function in the $K^+ \equiv K^*$ set.

Noting that the function $\delta_K(x,y)$ is within the restricted function space, we may obtain a useful expression for the functional derivative by applying (B.69) for a function in the K^* set. Since $\delta_{K^+}(y,x) = \delta_K(x,y)$ this shows that the functional derivative may be obtained by choosing $\delta\psi^K(x) = \epsilon\delta_K(x,y)$ for small ϵ in the definition (B.85)

$$\left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)} \right)_y = \int dx \, \delta_{K^+}(y,x) \left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)} \right)_x$$

$$= \lim_{\epsilon \to 0} \left(\frac{F[\psi^K(x) + \epsilon\delta_K(x,y)] - F[\psi^K(x)]}{\epsilon} \right).$$

To confirm that the right side of the last equation does in fact give $\left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)} \right)_y$ we substitute from (B.87)

$$\int dx \, \delta_K(x,y) \left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)} \right)_x = \sum_k^K \eta_k \int dx \, \delta_K(x,y) \phi_k^*(x)$$

$$= \sum_k^K \eta_k \sum_l^K \phi_l^*(y) \int dx \, \phi_l(x) \, \phi_k^*(x)$$

$$= \sum_k^K \eta_k \, \phi_k^*(y)$$

$$= \left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)} \right)_y$$

as required. Thus we have the useful expression for the functional derivative of restricted functions

$$\left(\frac{\delta F[\psi^K(x)]}{\delta \psi^K(x)}\right)_y = \lim_{\epsilon \to 0} \left(\frac{F[\psi^K(x) + \epsilon\delta_K(x,y)] - F[\psi^K(x)]}{\epsilon}\right) \qquad \text{(B.88)}$$

We may also have functionals $F[\psi^K(x), \psi^L(x)]$ that involve two functions $\psi^K(x)$, $\psi^L(x)$ in two different restricted sets K, L. The straight-forward generalisation of useful result (B.88) is

$$\left(\frac{\delta F[\psi^K(x), \psi^L(x)]}{\delta \psi^K(x)}\right)_y = \lim_{\epsilon \to 0} \left(\frac{F[\psi^K(x) + \epsilon\delta_K(x,y), \psi^L(x)] - F[\psi^K(x), \psi^L(x)]}{\epsilon}\right)$$

$$\text{(B.89)}$$

$$\left(\frac{\delta F[\psi^K(x), \psi^L(x)]}{\delta \psi^L(x)}\right)_y = \lim_{\epsilon \to 0} \left(\frac{F[\psi^K(x), \psi^L(x) + \epsilon\delta_L(x,y)] - F[\psi^K(x), \psi^L(x)]}{\epsilon}\right)$$

$$\text{(B.90)}$$

Similar results apply for the functional derivative $\frac{\delta F[\psi^{K+}(x)]}{\delta \psi^{K+}(x)}$ with respect to the restricted function $\psi^{K+}(x)$ in the $K^+ \equiv K^*$ set, which is defined by

$$F[\psi^{K+}(x) + \delta\psi^{K+}(x)] \doteq F[\psi^{K+}(x)] + \int dx\, \delta\psi^{K+}(x) \left(\frac{\delta F[\psi^{K+}(x)]}{\delta \psi^{K+}(x)}\right)_x \quad \text{(B.91)}$$

where $\delta\psi^{K+}(x)$ is a small change in $\psi^{K+}(x)$. The function $\delta\psi^{K+}(x)$ be associated with changes $\delta\beta_k^+$ in these expansion coefficients and thus $\delta\psi^{K+}(x)$ will be within the K^+ restricted function space. We then have

$$\left(\frac{\delta F[\psi^{K+}(x)]}{\delta \psi^{K+}(x)}\right)_x = \sum_k^K \eta_k^+ \, \phi_k(x) \qquad \text{(B.92)}$$

showing that the functional derivative is a function in the K set.

Also the function $\delta_{K+}(x,y)$ is within the restricted function space, we may obtain a useful expression for the functional derivative as

$$\left(\frac{\delta F[\psi^{K+}(x)]}{\delta \psi^{K+}(x)}\right)_y = \lim_{\epsilon \to 0} \left(\frac{F[\psi^{K+}(x) + \epsilon\delta_{K+}(x,y)] - F[\psi^{K+}(x)]}{\epsilon}\right)$$

$$= \lim_{\epsilon \to 0} \left(\frac{F[\psi^{K+}(x) + \epsilon\delta_K(y,x)] - F[\psi^{K+}(x)]}{\epsilon}\right) \text{(B.93)}$$

Appendix B.14. Examples of Restricted Functional Derivatives

To obtain the functional derivative of the *function* $\psi^K(y)$ with respect to $\psi^K(x)$, we note that this derivative exists as a function of x in the $K^+ \equiv K^*$

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restricted set since $\psi^K(y)$ is also a functional $\psi^K(y) = F_y[\psi^K(x)]$. We can thus use the expression (B.88)

$$
\begin{aligned}
\left(\frac{\delta}{\delta\psi^K(x)}\psi^K(y)\right)_x &= \lim_{\epsilon\to 0}\left(\frac{F_y[\psi^K(u)+\epsilon\delta_K(u,x)]-F_y[\psi^K(u)]}{\epsilon}\right)\\
&= \lim_{\epsilon\to 0}\left(\frac{\int du\,\delta_K(y,u)\{\psi^K(u)+\epsilon\delta_K(u,x)\}-\int du\,\delta_K(y,u)\psi^K(u)}{\epsilon}\right)\\
&= \lim_{\epsilon\to 0}\left(\int du\,\delta_K(y,u)\,\delta_K(u,x)\right)\\
&= \delta_K(y,x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B.94)}
\end{aligned}
$$

where (B.67) has been used. As noted before considered as a function of x, the derivative of $\psi^K(y)$ with respect to $\psi^K(x)$ is in the K^* restricted set, but is in the K set considered as a function of y. This result is the modification of (B.15) for restricted functions.

A further result can be derived for when the functional derivative is with respect to $\psi^L(x)$ is in a different L restricted set. Applying (B.90) we get

$$
\begin{aligned}
\left(\frac{\delta}{\delta\psi^L(x)}\psi^K(y)\right)_x &= \lim_{\epsilon\to 0}\left(\frac{F_y[\psi^K(u),\psi^L(u)+\epsilon\delta_L(u,x)]-F_y[\psi^K(u),\psi^L(u)]}{\epsilon}\right)\\
&= \lim_{\epsilon\to 0}\left(\frac{\int du\,\delta_K(y,u)\psi^K(u)-\int du\,\delta_K(y,u)\psi^K(u)}{\epsilon}\right)\\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B.95)}
\end{aligned}
$$

since the functional $\psi^K(y) = F_y[\psi^K(x),\psi^L(x)]$ does not involve $\psi^L(x)$ at all.

For the functional derivative of the *spatial derivative* $\nabla_y\psi^K(y)$ of the *function* $\psi^K(y)$ with respect to $\psi^K(x)$, we note that this derivative exists as a function of x in the $K^+ \equiv K^*$ restricted set since $\nabla_y\psi^K(y)$ is also a functional $\nabla_y\psi^K(y) = F_{\nabla_y}[\psi^K(x)]$. We can thus use the expression (B.88)

$$
\begin{aligned}
\left(\frac{\delta}{\delta\psi^K(x)}\nabla_y\psi^K(y)\right)_x &= \lim_{\epsilon\to 0}\left(\frac{F_y[\psi^K(u)+\epsilon\delta_K(u,x)]-F_y[\psi^K(u)]}{\epsilon}\right)\\
&= \lim_{\epsilon\to 0}\left(\frac{\int du\,\nabla_y\delta_K(y,u)\{\psi^K(u)+\epsilon\delta_K(u,x)\}-\int du\nabla_y\,\delta_K(y,u)\psi^K(u)}{\epsilon}\right)\\
&= \lim_{\epsilon\to 0}\left(\int du\,\nabla_y\delta_K(y,u)\,\delta_K(u,x)\right)\\
&= \nabla_y\left(\int du\,\delta_K(y,u)\,\delta_K(u,x)\right)\\
&= \nabla_y\delta_K(y,x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B.96)}
\end{aligned}
$$

showing that the functional derivative involves a spatial derivative of the restricted delta function with respect to y. As pointed out previously, considered as a function of x the functional derivative is in the K^+ set. Note also that

we see that the functional derivative and the spatial derivative processes can be carried out in either order

$$
\left(\frac{\delta}{\delta \psi^K(x)} \nabla_y \psi^K(y) \right)_x = \nabla_y \left(\frac{\delta}{\delta \psi^K(x)} \psi^K(y) \right)_x
$$
$$
= \nabla_y \delta_K(y, x) \tag{B.97}
$$

We also can obtain similar results for the *function* $\psi^{K+}(y)$ which is in the $K^+ \equiv K^*$ restricted set, and can be written as a functional $F_y[\psi^{K+}(x)] \equiv \psi^{K+}(y)$. Thus

$$
\left(\frac{\delta}{\delta \psi^{K+}(x)} \psi^{K+}(y) \right)_x = \lim_{\epsilon \to 0} \left(\frac{F_y[\psi^{K+}(u) + \epsilon \delta_{K+}(u, x)] - F_y[\psi^{K+}(u)]}{\epsilon} \right)
$$
$$
= \lim_{\epsilon \to 0} \left(\frac{\int du \, \delta_{K+}(y, u) \{ \psi^{K+}(u) + \epsilon \delta_{K+}(u, x) \} - \int du \, \delta_{K+}(y, u) \psi^{K+}(u)}{\epsilon} \right)
$$
$$
= \lim_{\epsilon \to 0} \left(\int du \, \delta_{K+}(y, u) \, \delta_{K+}(u, x) \right)
$$
$$
= \delta_{K+}(y, x)
$$
$$
= \delta_K(x, y) \tag{B.98}
$$

For the *spatial derivative* $\nabla_y \psi^{K+}(y)$ of the *function* $\psi^{K+}(y)$ we have

$$
\left(\frac{\delta}{\delta \psi^{K+}(x)} \nabla_y \psi^{K+}(y) \right)_x = \lim_{\epsilon \to 0} \left(\frac{F_{\nabla_y}[\psi^{K+}(u) + \epsilon \delta_{K+}(u, x)] - F_{\nabla_y}[\psi^{K+}(u)]}{\epsilon} \right)
$$
$$
= \lim_{\epsilon \to 0} \left(\frac{\int du \, \nabla_y \delta_{K+}(y, u) \{ \psi^{K+}(u) + \epsilon \delta_{K+}(u, x) \} - \int du \, \nabla_y \delta_{K+}(y, u) \psi^{K+}(u)}{\epsilon} \right)
$$
$$
= \lim_{\epsilon \to 0} \left(\int du \nabla_y \delta_{K+}(y, u) \, \delta_{K+}(u, x) \right)
$$
$$
= \left(\int du \nabla_y \delta_K(u, y) \, \delta_K(x, u) \right)
$$
$$
= \nabla_y \delta_K(x, y)
$$
$$
= \nabla_y \delta_{K+}(y, x) \tag{B.99}
$$

as expected. Note also

$$
\left(\frac{\delta}{\delta \psi^{K+}(x)} \nabla_y \psi^{K+}(y) \right)_x = \nabla_y \left(\frac{\delta}{\delta \psi^{K+}(x)} \psi^{K+}(y) \right)_x
$$
$$
= \nabla_y \delta_{K+}(y, x) \tag{B.100}
$$

Appendix B.15. Restricted Functional Derivatives and Mode Functions

We can obtain an expression for the functional derivative $\left(\frac{\delta F[\psi^K(x)]}{\delta \psi^K(x)} \right)_x$ with respect to restricted function $\psi^K(x)$ in terms of the ordinary derivatives of the function (B.71) that is equivalent to the functional.

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Substituting from (B.86) we see that

$$
\begin{aligned}
F[\psi^K(x) + \delta\psi^K(x)] - F[\psi^K(x)] \;&\dot{=}\; \int dx\, \delta\psi^K(x)\, \left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)}\right)_x \\
&\dot{=}\; \sum_k^K \delta\beta_k \int dx\, \phi_k(x) \left(\frac{\delta F[\psi(x)]}{\delta\psi(x)}\right)_x
\end{aligned}
$$

But the left side is the same as

$$
g(\beta_1 + \delta\beta_1, .., \beta_k + \delta\beta_k, ..) - g(\beta_1, .., \beta_k, ..) \;\dot{=}\; \sum_k^K \delta\beta_k \frac{\partial g(\beta_1, .., \beta_k, ..)}{\partial\beta_k}
$$

Equating the coefficients of the independent $\delta\alpha_k$ and then using the completeness relationship in Eq.(B.74) gives the key result

$$
\left(\frac{\delta F[\psi^K(x)]}{\delta\psi^K(x)}\right)_x = \sum_k^K \phi_k^*(x) \frac{\partial g(\beta_1, .., \beta_k, ..)}{\partial\beta_k} \tag{B.101}
$$

This relates the functional derivative to the mode functions and to the ordinary partial derivatives of the function $g(\beta_1, .., \beta_k, ..\beta_n)$ that was equivalent to the original functional $F[\psi^K(x)]$. Again, we see that the result is a function of x. Note that the functional derivative involves an expansion in terms of the conjugate mode functions $\phi_k^*(x)$ rather than the original modes $\phi_k(x)$.

The last result can be put in the form of a useful *operator identity*

$$
\left(\frac{\delta}{\delta\psi^K(x)}\right)_x = \sum_k^K \phi_k^*(x) \frac{\partial}{\partial\beta_k} \tag{B.102}
$$

where it is understood that the left side operates on an arbitary functional $F[\psi^K(x)]$ of the restricted function $\psi^K(x)$ and the right side operates on the equivalent function $g(\beta_1, .., \beta_k, ..)$.

We can obtain a similar expression for the functional derivative $\left(\frac{\delta F[\psi^{K+}(x)]}{\delta\psi^{K+}(x)}\right)_x$ with respect to restricted function $\psi^{K+}(x)$ in the K^+ set in terms of the ordinary derivatives of the function (B.78) that is equivalent to the functional.

$$
\left(\frac{\delta F[\psi^{K+}(x)]}{\delta\psi^{K+}(x)}\right)_x = \sum_k^K \phi_k(x) \frac{\partial g^+(\beta_1^+, .., \beta_k^+, ..)}{\partial\beta_k^+} \tag{B.103}
$$

The last result can be put in the form of a useful operator identity

$$
\left(\frac{\delta}{\delta\psi^{K+}(x)}\right)_x = \sum_k^K \phi_k(x) \frac{\partial}{\partial\beta_k^+} \tag{B.104}
$$

where it is understood that the left side operates on an arbitary functional $F[\psi^{K+}(x)]$ of the restricted function $\psi^{K+}(x)$ and the right side operates on the equivalent function $g^+(\beta_1^+, .., \beta_k^+, ..)$.

The *spatial derivative* of a *functional derivative* can be found from

$$\nabla_x \left(\frac{\delta F[\psi^K(x)]}{\delta \psi^K(x)} \right)_x = \sum_k^K \{\nabla_x \phi_k^*(x)\} \frac{\partial g(\beta_1, .., \beta_k, ..)}{\partial \beta_k} \qquad (B.105)$$

$$\nabla_x \left(\frac{\delta F[\psi^{K+}(x)]}{\delta \psi^{K+}(x)} \right)_x = \sum_k^K \{\nabla_x \phi_k(x)\} \frac{\partial g^+(\beta_1^+, .., \beta_k^+, ..)}{\partial \beta_k^+} \quad (B.106)$$

in the two cases of functionals of $\psi^K(x)$ or $\psi^{K+}(x)$. Clearly the spatial derivative acts *only* on either the $\phi_k^*(x)$ or the $\phi_k(x)$.

The last results can be put in the form of *operator identities*

$$\nabla_x \left(\frac{\delta}{\delta \psi^K(x)} \right)_x = \sum_k^K \{\nabla_x \phi_k^*(x)\} \frac{\partial}{\partial \beta_k} \qquad (B.107)$$

$$\nabla_x \left(\frac{\delta}{\delta \psi^{K+}(x)} \right)_x = \sum_k^K \{\nabla_x \phi_k(x)\} \frac{\partial}{\partial \beta_k^+} \qquad (B.108)$$

where it is understood that the left side operates on an arbitary functional $F[\psi^K(x)]$ or $F[\psi^{K+}(x)]$ of the restricted function $\psi^K(x)$ or $\psi^{K+}(x)$ respectively, and the right side operates on the equivalent function $g(\beta_1, .., \beta_k, ..)$ or $g^+(\beta_1^+, .., \beta_k^+, ..)$. These operator forms are useful in deriving results for applying functional derivatives in succession.

As an example of applying these operator identities consider the case of the functionals $F_y[\psi^K(x)] \equiv \psi^K(y) = \sum_k \beta_k \phi_k(y)$ and $F_y[\psi^{K+}(x)] \equiv \psi^{K+}(y) = \sum_k \phi_k^*(y) \beta_k^+$. Since in these cases

$$\frac{\partial g(\beta_1, .., \beta_k, ..)}{\partial \beta_k} = \phi_k(y) \qquad \frac{\partial g^+(\beta_1^+, .., \beta_k^+, ..)}{\partial \beta_k^+} = \phi_k^*(y)$$

we have

$$\left(\frac{\delta \psi^K(y)}{\delta \psi^K(x)} \right)_x = \sum_k^K \phi_k^*(x) \, \phi_k(y) = \delta_K(y, x)$$

$$\left(\frac{\delta \psi^{K+}(y)}{\delta \psi^{K+}(x)} \right)_x = \sum_k^K \phi_k(x) \, \phi_k^*(y) = \delta_{K+}(y, x)$$

for the functional derivatives as before, and

$$\nabla_x \left(\frac{\delta \psi^K(y)}{\delta \psi^K(x)} \right)_x = \sum_k^K \{\nabla_x \phi_k^*(x)\} \phi_k(y) = \nabla_x \delta_K(y, x) \qquad (B.109)$$

$$\nabla_x \left(\frac{\delta \psi^{K+}(y)}{\delta \psi^{K+}(x)} \right)_x = \sum_k^K \{\nabla_x \phi_k(x)\} \phi_k^*(y) = \nabla_x \delta_{K+}(y, x) \quad (B.110)$$

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for the spatial derivatives of the functional derivatives.

Similarly for the spatial derivative functionals $F_{\nabla y}[\psi^K(x)] \equiv \nabla_y \psi^K(y) = \sum_k \beta_k \nabla_y \phi_k(y)$ and $F_{\nabla y}[\psi^{K+}(x)] \equiv \nabla_y \psi^{K+}(y) = \sum_k \nabla_y \phi_k^*(y) \beta_k^+$. Since in these cases

$$\frac{\partial g(\beta_1, .., \beta_k, ..)}{\partial \beta_k} = \nabla_y \phi_k(y) \qquad \frac{\partial g^+(\beta_1^+, .., \beta_k^+, ..)}{\partial \beta_k^+} = \nabla_y \phi_k^*(y)$$

we have

$$\left(\frac{\delta \nabla_y \psi^K(y)}{\delta \psi^K(x)} \right)_x = \sum_k^K \phi_k^*(x) \nabla_y \phi_k(y) = \nabla_y \delta_K(y, x)$$

$$\left(\frac{\delta \nabla_y \psi^{K+}(y)}{\delta \psi^{K+}(x)} \right)_x = \sum_k^K \phi_k(x) \nabla_y \phi_k^*(y) = \nabla_y \delta_{K+}(y, x)$$

which are the same results as before. Note the distinction between $\left(\frac{\delta \nabla_y \psi^K(y)}{\delta \psi^K(x)} \right)_x$ and $\nabla_x \left(\frac{\delta \psi^K(y)}{\delta \psi^K(x)} \right)_x$ - the first being the functional derivative of the y spatial derivative $\nabla_y \psi^K(y)$ with respect to $\psi^K(x)$, the second being the x spatial derivative of the functional derivative of $\psi^K(y)$ with respect to $\psi^K(x)$.

Appendix B.16. Functional Derivatives in Theory of Bose-Einstein Condensates

The theory of Bose-Einstein condensates (BEC) often requires separate consideration of certain highly occupied modes - the condensate modes, and other sparsely occupied modes - the non-condensate modes. In phase space distribution functional methods these two types of modes can be used in defining condensate fields and non-condensate fields as restricted functions, and the treatment presented in this section can then be used in evaluating the various functional derivatives.

In applying these rules to the BEC problem, the following functional derivative results can be obtained as straightforward generalisations of (B.94) and (B.95). The general functions $\psi(\mathbf{r})$ and $\psi^+(\mathbf{r})$ each will be used to cover the results for condensate and non-condensate modes. For the case where $\psi(\mathbf{r}) \equiv \psi_C(\mathbf{r})$ the restricted set K refers to two modes $\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$, and for the non-condensate case where $\psi(\mathbf{r}) \equiv \psi_{NC}(\mathbf{r})$ the restricted set $\overline{K}$ refers to the remaining modes $\phi_k(\mathbf{r})$. For the case where $\psi^+(\mathbf{r}) \equiv \psi_C^+(\mathbf{r})$ the restricted set K^+ $\equiv K^*$ refers to two conjugate modes $\phi_1^*(\mathbf{r})$, $\phi_2^*(\mathbf{r})$, and for the non-condensate case where $\psi^+(\mathbf{r}) \equiv \psi_{NC}^+(\mathbf{r})$ the restricted set $\overline{K}^+$ refers to the remaining conjugate modes $\phi_k^*(\mathbf{r})$. Because the coefficients are unrelated we are dealing with functionals such as the distribution functional $P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$ which involve

eight independent functions, namely $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})$ plus the complex conjugates $\psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})$.

$$
\begin{aligned}
\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{r}) &= \delta_K(\mathbf{r},\mathbf{s}) & \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{r}) = \delta_{K+}(\mathbf{r},\mathbf{s}) = \delta_K(\mathbf{s},\mathbf{r}) \\
\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{r}) &= 0 & \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{r}) = 0
\end{aligned}
\tag{B.111}
$$

Note the reverse order of $\mathbf{r}, \mathbf{s}$ in the second result, due to (B.95). The functional $\psi(\mathbf{r})$ is not a functional of $\psi^+(\mathbf{s})$ and vice-versa, the other two functional derivatives are zero. Similarly the functional derivatives of condensate fiels with respect to non-condensate fields are zero, and vice-versa. Thus

$$
\begin{aligned}
\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_{NC}(\mathbf{r}) &= 0 & \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_{NC}^+(\mathbf{r}) = 0 \\
\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_{NC}^+(\mathbf{r}) &= 0 & \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_{NC}(\mathbf{r}) = 0
\end{aligned}
\tag{B.112}
$$

with four other results obtained by interchanging C and NC.

Appendix B.17. Supplementary Equations

Field Expansions

$$
\begin{aligned}
\psi_C(\mathbf{r}) &= \alpha_1\,\phi_1(\mathbf{r}) + \alpha_2\,\phi_2(\mathbf{r}) & \text{(B.113)} \\
\psi_C^+(\mathbf{r}) &= \phi_1^*(\mathbf{r})\,\alpha_1^+ + \phi_2^*(\mathbf{r})\,\alpha_2^+ & \text{(B.114)} \\
\psi_{NC}(\mathbf{r}) &= \sum_{k\neq 1,2}^{K} \alpha_k\,\phi_k(\mathbf{r}) & \text{(B.115)} \\
\psi_{NC}^+(\mathbf{r}) &= \sum_{k\neq 1,2}^{K} \phi_k^*(\mathbf{r})\alpha_k^+ & \text{(B.116)}
\end{aligned}
$$

Appendix C. - Quantum Averages

To prove Eq.(C.1) it will be convenient to treat the condensate operators first ignoring the non-condensate operators and with the quasidistribution functional being purely of the Wigner type. Following that we then reverse the process by treating the non-condensate operators with the quasidistribution functional being of the positive P type.

Appendix C.1. The Condensate Averages

The functional derivative of the symmetrically ordered characteristic functional with respect to say, $\xi(\mathbf{r})$ is defined by

$$\left(\frac{\delta \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta \xi(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{r}_1} = \lim_{\epsilon \to 0}\left(\frac{\chi^W[\xi(\mathbf{r}) + \epsilon\delta(\mathbf{r} - \mathbf{r}_1), \xi^+(\mathbf{r})] - \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\epsilon}\right).$$

It is not difficult to see that

$$
\begin{aligned}
&\chi^W[\xi(\mathbf{r}) + \epsilon\delta(\mathbf{r} - \mathbf{r}_1), \xi^+(\mathbf{r})] \\
=\ & \iiiint D^2\psi\, D^2\psi^+\, W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
& \times \exp i \int d\mathbf{r}\, \{(\xi(\mathbf{r}) + \epsilon\delta(\mathbf{r} - \mathbf{r}_1))\psi^+(\mathbf{r}) + \psi(\mathbf{r})\xi^+(\mathbf{r})\} \\
=\ & \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})] \\
& + i\epsilon \iiiint D^2\psi\, D^2\psi^+\, W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]\, \psi^+(\mathbf{r}_1) \exp i \int d\mathbf{r}\, \{\xi(\mathbf{r})\psi^+(\mathbf{r}) + \psi(\mathbf{r})\xi^+(\mathbf{r})\}
\end{aligned}
$$

Thus the functional derivative is

$$
\begin{aligned}
\left(\frac{\delta \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta \xi(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{r}_1} =\ & \iiiint D^2\psi\, D^2\psi^+\, W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
& \times i\psi^+(\mathbf{r}_1) \exp i \int d\mathbf{r}\, \{\xi(\mathbf{r})\psi^+(\mathbf{r}) + \psi(\mathbf{r})\xi^+(\mathbf{r})\}.
\end{aligned}
$$

Note that the field function at position $\mathbf{r}_1$ is still subject to the functional integration.

Similarly

$$
\begin{aligned}
\left(\frac{\delta \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta \xi^+(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{r}_1} =\ & \iiiint D^2\psi\, D^2\psi^+\, W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
& \times i\psi(\mathbf{r}_1) \exp i \int d\mathbf{r}\, \{\xi(\mathbf{r})\psi^+(\mathbf{r}) + \psi(\mathbf{r})\xi^+(\mathbf{r})\}.
\end{aligned}
$$

Thus we see that these functional derivatives are in the form of expressions for characteristic functionals in which $W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$ is replaced by $i\psi^+(\mathbf{r}_1)W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$ or $i\psi(\mathbf{r}_1)W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$.

Continuing in this way we may establish a result for higher order functional derivatives

$$\left(\frac{\delta^{p+q} \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta^p \xi(\mathbf{r}) \, \delta^q \xi^+(\mathbf{r})} \right)_{\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_p; \mathbf{s}_q, .., \mathbf{s}_2, \mathbf{s}_1;}$$

$$= \iiiint D^2\psi \, D^2\psi^+ \, W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\times i^{p+q} \, \psi^+(\mathbf{r}_1) \, \psi^+(\mathbf{r}_2) .. \psi^+(\mathbf{r}_p) \, \psi(\mathbf{s}_q) .. \psi(\mathbf{s}_2) \, \psi(\mathbf{s}_1)$$

$$\times \exp i \int d\mathbf{r} \, \{\xi(\mathbf{r})\psi^+(\mathbf{r}) + \psi(\mathbf{r})\xi^+(\mathbf{r})\}.$$

where for bosonic systems the functional differentiation can be carried out in any order but with the differentiation with respect to $\xi(\mathbf{r})$ involving positions $\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_p$ and the $\xi^+(\mathbf{r})$ differentiation involving positions $\mathbf{s}_q, .., \mathbf{s}_2, \mathbf{s}_1$.

Evaluating the functional derivatives and then letting $\xi(\mathbf{r}), \xi^+(\mathbf{r})$ all approach zero (symbolically $\xi \longrightarrow \mathbf{0}$), we have for bosonic systems

$$\left(\frac{\delta^{p+q} \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta^p \xi(\mathbf{r}) \, \delta^q \xi^+(\mathbf{r})} \right)^{\xi \to \mathbf{0}}_{\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_p; \mathbf{s}_q, .., \mathbf{s}_2, \mathbf{s}_1;}$$

$$= \iiiint D^2\psi \, D^2\psi^+ \, W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\times i^{p+q} \, \psi^+(\mathbf{r}_1) \, \psi^+(\mathbf{r}_2) .. \psi^+(\mathbf{r}_p) \, \psi(\mathbf{s}_q) .. \psi(\mathbf{s}_2). \psi(\mathbf{s}_1)$$

We then apply the same process to the definition of the characteristic functional

$$\chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})] = Tr(\widehat{\rho} \exp \int d\mathbf{r} \, i\{\xi(\mathbf{r})\widehat{\Psi}^\dagger(\mathbf{r}) + \widehat{\Psi}(\mathbf{r})\xi^+(\mathbf{r})\}$$

$$= \sum_n \frac{1}{n!} Tr(\widehat{\rho} \left(\int d\mathbf{r} \, i\xi(\mathbf{r}) \, \widehat{\Psi}^\dagger(\mathbf{r}) + \int d\mathbf{r} \, \widehat{\Psi}(\mathbf{r}) \, i\xi^+(\mathbf{r}) \right)^n.$$

Now with $\widehat{A}[\xi(\mathbf{r})] = \int d\mathbf{r} \, i\xi(\mathbf{r}) \, \widehat{\Psi}^\dagger(\mathbf{r})$ and $\widehat{B}[\xi^+(\mathbf{r})] = \int d\mathbf{r} \, \widehat{\Psi}(\mathbf{r}) \, i\xi^+(\mathbf{r})$ there are $N(p, q) = (p+q)!/p!q!$ ways that the operator $\widehat{A}$ appears p times and the operator $\widehat{B}$ appears q times when we expand $(\widehat{A} + \widehat{B})^n$ (where $n = p + q$) and each order of these operators appears once. We can introduce the symbol $\{(\widehat{A})^p \, (\widehat{B})^q\}$ to denote the average of these $N(p, q)$ ordered products

$$\{(\widehat{A})^p \, (\widehat{B})^q\} = \frac{1}{N(p, q)} \left((\widehat{A})^p \, (\widehat{B})^q + (\widehat{A})^{p-1} \, (\widehat{B})^q \, (\widehat{A}) + ... + (\widehat{B})^q \, (\widehat{A})^p \right)$$

and write

$$\chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})] = \sum_{p,q} \frac{1}{p!q!} Tr(\widehat{\rho} \, \{(\widehat{A})^p \, (\widehat{B})^q\}).$$

In this form it is convenient to calculate the functional derivatives, since $\widehat{A}[\xi(\mathbf{r})]$ and $\widehat{B}[\xi^+(\mathbf{r})]$ are functionals only of $\xi(\mathbf{r})$ and $\xi^+(\mathbf{r})$.respectively, so their functional derivatives with respect to the other function will be zero. Then

$$
\begin{aligned}
\left(\frac{\delta \widehat{A}[\xi(\mathbf{r})]}{\delta \xi(\mathbf{r})} \right)_{\mathbf{r}=\mathbf{r}_1} &= \lim_{\epsilon \to 0} \left(\frac{\widehat{A}[\xi(\mathbf{r}) + \epsilon \delta(\mathbf{r} - \mathbf{r}_1)] - \widehat{A}[\xi(\mathbf{r})]}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{i\epsilon \widehat{\Psi}^\dagger(\mathbf{r}_1)}{\epsilon} \right) \\
&= i\widehat{\Psi}^\dagger(\mathbf{r}_1)
\end{aligned}
$$

Similarly

$$
\left(\frac{\delta \widehat{B}[\xi^+(\mathbf{r})]}{\delta \xi^+(\mathbf{r})} \right)_{\mathbf{r}=\mathbf{s}_1} = i\widehat{\Psi}(\mathbf{s}_1)
$$

We see that each time that either $\widehat{A}[\xi(\mathbf{r})]$ is differentiated with respect to $\xi(\mathbf{r})$ or $\widehat{B}[\xi^+(\mathbf{r})]$ is differentiated with respect to $\xi^+(\mathbf{r})$ an operator results, and therefore no further functional differentiation can occur. We also note that as $\xi \longrightarrow \mathbf{0}$ both $\widehat{A}[\xi(\mathbf{r})]$ and $\widehat{B}[\xi^+(\mathbf{r})]$ become zero. To proceed further we need to calculate functional derivatives of products of $\widehat{A}[\xi(\mathbf{r})]$ and $\widehat{B}[\xi^+(\mathbf{r})]$. This can be carried out by applying the general rule for functional derivatives of products of functionals. Consider the term $\{(\widehat{A})^p (\widehat{B})^q\}$ (which is the average of the $N(p,q)$ ordered products where $\widehat{A}[\xi(\mathbf{r})]$ appears p times and $\widehat{B}[\xi^+(\mathbf{r})]$ appears q times).If each of the terms in $\{(\widehat{A})^p (\widehat{B})^q\}$ is differentiated *less* than p times with respect to $\xi(\mathbf{r})$ then there will be at least one factor $\widehat{A}[\xi(\mathbf{r})]$ still remaining and thus as $\xi \longrightarrow \mathbf{0}$ the result of the differentiation will be zero. Similar conclusions apply if the term in $\{(\widehat{A})^p (\widehat{B})^q\}$ is differentiated less than q times with respect to $\xi^+(\mathbf{r})$. On the other hand if ach of the terms in $\{(\widehat{A})^p (\widehat{B})^q\}$ is differentiated *more* than p times with respect to $\xi(\mathbf{r})$ then the result of the differentiation must be zero because after the pth differentiation all of the $\widehat{A}[\xi(\mathbf{r})]$ will have been replaced by a factor $i\widehat{\Psi}^\dagger(\mathbf{r}_i)$ and therefore further functional differentiation with respect to $\xi(\mathbf{r})$ will give zero. Similar conclusions apply if if the term in $\{(\widehat{A})^p (\widehat{B})^q\}$ is differentiated more than q times with respect to $\xi^+(\mathbf{r})$. Hence only the p,q term in the last expression for $\chi[\xi(\mathbf{r}), \xi^+(\mathbf{r})]$ contributes in the required result for

$$
\left(\frac{\delta^{p+q} \chi^W[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta^p \xi(\mathbf{r}) \, \delta^q \xi^+(\mathbf{r})} \right)_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1;}^{\xi \to \mathbf{0}}
$$

$$
= \frac{1}{p!q!} Tr \left(\widehat{\rho} \left(\frac{\delta^{p+q} \{(\widehat{A})^p (\widehat{B})^q\}}{\delta^p \xi(\mathbf{r}) \, \delta^q \xi^+(\mathbf{r})} \right)_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1;}^{\xi \to \mathbf{0}} \right)
$$

Now consider the pth functional derivative of any term in $\{(\widehat{A})^p\,(\widehat{B})^q\}$ with respect to the $\xi(\mathbf{r})$, where the q factors $\widehat{B}[\xi^+(\mathbf{r})]$ are just represented by dots. The result will be the sum of products of factors $i\widehat{\Psi}^\dagger(\mathbf{r}_1), i\widehat{\Psi}^\dagger(\mathbf{r}_2), .., i\widehat{\Psi}^\dagger(\mathbf{r}_p)$ in all $p!$ orders

$$
\left(\frac{\delta^p(\widehat{A}[\xi(\mathbf{r})]..\widehat{A}[\xi(\mathbf{r})]..\widehat{A}[\xi(\mathbf{r})]..\widehat{A}[\xi(\mathbf{r})])^{p,q}}{\delta^p\xi(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p}
$$
$$
= i^p\sum_P(\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_1})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_2})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_i})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_p}))
$$

where the sum is over all permutations $P = \uparrow\left(\frac{\mu_1}{1}\frac{\mu_2}{2}..\frac{\mu_i}{i}..\frac{\mu_p}{p}\right)$ of $1,2,..,i,..p$. Considering also the qth functional derivative of the same term in $\{(\widehat{A})^p\,(\widehat{B})^q\}$ with respect to the $\xi^+(\mathbf{r})$ we get overall

$$
\left(\frac{\delta^{p+q}(\widehat{A}[\xi(\mathbf{r})]..\widehat{B}[\xi^+(\mathbf{r})]..\widehat{A}[\xi(\mathbf{r})]..\widehat{B}[\xi^+(\mathbf{r})]..\widehat{A}[\xi(\mathbf{r})..\widehat{B}[\xi^+(\mathbf{r})]])^{p,q}}{\delta^p\xi(\mathbf{r})\,\delta^q\xi^+(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1}
$$
$$
=\ i^{p+q}\sum_{P,Q}(\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_1})..\widehat{\Psi}(\mathbf{s}_{\lambda_q})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_2})..\widehat{\Psi}(\mathbf{s}_{\lambda_2})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_p})..\widehat{\Psi}(\mathbf{s}_{\lambda_1}))
$$

where the second sum is over all permutations $Q = \uparrow\left(\frac{\lambda_q}{q}..\frac{\lambda j}{j}..\frac{\lambda_2}{2}\frac{\lambda_1}{1}\right)$ of $q,..,i,..,2,1$. Now within each of the $N(p,q) = (p+q)!/p!q!$ orderings of products of $\widehat{A}$ and $\widehat{B}$ where $\widehat{A}$ appears p times and $\widehat{B}$ appears q times, there are $p!$ orderings of the $\widehat{\Psi}^\dagger$ operators and $q!$ orderings of the $\widehat{\Psi}$ operators, giving a total of $M(p,q) = N(p,q)\,p!q! = (p+q)!$ different orderings of the p operators $\widehat{\Psi}^\dagger$ and the q operators $\widehat{\Psi}$, and all possible orderings are present in view of the sum over the permutations P,Q. Taking into account the factor $1/p!q!$ we see that the when the differentiation is applied to the quantity $\{(\widehat{A})^p\,(\widehat{B})^q\}$ itself we see that we just get $i^{p+q}\{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_1)..\widehat{\Psi}(\mathbf{s}_q)\}$. Thus

$$
\left(\frac{\delta^{p+q}\chi^W[\xi(\mathbf{r}),\xi^+(\mathbf{r})]}{\delta^p\xi(\mathbf{r})\,\delta^q\xi^+(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1;}
$$
$$
=i^{p+q}Tr\left(\widehat{\rho}\{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\}\right)
$$

where the symmetric ordering symbol is given by

$$
\{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\}
$$
$$
=\frac{1}{(p+q)!}\sum_R\Re(\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1))
$$

In the the sum over R is over all $(p+q)!$ orderings $\Re$ of the factors $\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)$.

Hence we obtain the following key result for the quantum average of the symmetrically ordered product $\{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\}$ of the field operators

$$
\left\langle \{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\} \right\rangle
$$

$$
= Tr\left(\widehat{\rho}\{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\}\right)
$$

$$
= \iiiint D^2\psi\, D^2\psi^+\, W[\psi(\mathbf{r}),\psi^+(\mathbf{r})]
$$

$$
\times \psi^+(\mathbf{r}_1)\,\psi^+(\mathbf{r}_2)\,..\psi^+(\mathbf{r}_p)\,\psi(\mathbf{s}_q)\,..\psi(\mathbf{s}_2)\,\psi(\mathbf{s}_1)
$$

This result gives the required synmmetrically ordered average as a functional integral involving the quasi-distribution functional $W[\psi(\mathbf{r}),\psi^+(\mathbf{r})]$ times the product of the field functions, with the field operator $\widehat{\Psi}^\dagger(\mathbf{r}_i)$ being replaced by $\psi^+(\mathbf{r}_i)$ and $\widehat{\Psi}(\mathbf{s}_j)$ being replaced by $\psi(\mathbf{s}_j)$.

Appendix C.2. The Non-Condensate Averages

The functional derivative of a normally ordered characteristic functional with respect to say, $\xi(\mathbf{r})$ is defined by

$$
\left(\frac{\delta\chi_N[\xi(\mathbf{r}),\xi^+(\mathbf{r})]}{\delta\xi(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{r}_1} = \lim_{\epsilon\to 0}\left(\frac{\chi_N[\xi(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{r}_1),\xi^+(\mathbf{r})]-\chi_N[\xi(\mathbf{r}),\xi^+(\mathbf{r})]}{\epsilon}\right).
$$

It is not difficult to see that

$$
\chi_N[\xi(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{r}_1),\xi^+(\mathbf{r})]
$$

$$
= \iiiint D^2\psi\, D^2\psi^+\, P^+[\psi(\mathbf{r}),\psi^+(\mathbf{r})]
$$

$$
\times \exp i\int d\mathbf{r}\,\{(\xi(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{r}_1))\psi^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi(\mathbf{r})\xi^+(\mathbf{r})\}
$$

$$
= \chi_N[\xi(\mathbf{r}),\xi^+(\mathbf{r})]
$$

$$
+i\epsilon\iiiint D^2\psi\, D^2\psi^+\, P^+[\psi(\mathbf{r}),\psi^+(\mathbf{r})]\,\psi^+(\mathbf{r}_1)\exp i\int d\mathbf{r}\,\{\xi(\mathbf{r})\psi^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi(\mathbf{r})\xi^+(\mathbf{r})\}
$$

Thus the functional derivative is

$$
\left(\frac{\delta\chi_N[\xi(\mathbf{r}),\xi^+(\mathbf{r})]}{\delta\xi(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{r}_1} = \iiiint D^2\psi\, D^2\psi^+\, P^+[\psi(\mathbf{r}),\psi^+(\mathbf{r})]
$$

$$
\times i\psi^+(\mathbf{r}_1)\exp i\int d\mathbf{r}\,\{\xi(\mathbf{r})\psi^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi(\mathbf{r})\xi^+(\mathbf{r})\}.
$$

Note that the field function at position $\mathbf{r}_1$ is still subject to the functional integration.

Similarly

$$
\left(\frac{\delta \chi_N[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta \xi^+(\mathbf{r})} \right)_{\mathbf{r}=\mathbf{r}_1} = \iiiint D^2\psi \, D^2\psi^+ \, P^+[\psi(\mathbf{r}), \psi^+(\mathbf{r})]
$$

$$
\times i\psi(\mathbf{r}_1) \exp i \int d\mathbf{r} \left\{ \xi(\mathbf{r})\psi^+(\mathbf{r}) \exp i \int d\mathbf{r} \left\{ \psi(\mathbf{r})\xi^+(\mathbf{r}) \right\} \right\}.
$$

Thus we see that these functional derivatives are in the form of expressions for characteristic functionals in which $P^+[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$ is replaced by $i\psi^+(\mathbf{r}_1)P^+[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$ or $i\psi(\mathbf{r}_1)P^+[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$.

Continuing in this way we may establish a result for higher order functional derviatives

$$
\left(\frac{\delta^{p+q} \chi_N[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta^p \xi(\mathbf{r}) \, \delta^q \xi^+(\mathbf{r})} \right)_{\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_p; \mathbf{s}_q, .., \mathbf{s}_2, \mathbf{s}_1;}
$$

$$
= \iiiint D^2\psi \, D^2\psi^+ \, P^+[\psi(\mathbf{r}), \psi^+(\mathbf{r})]
$$

$$
\times i^{p+q} \psi^+(\mathbf{r}_1) \, \psi^+(\mathbf{r}_2) .. \psi^+(\mathbf{r}_p) \, \psi(\mathbf{s}_q) .. \psi(\mathbf{s}_2) \, \psi(\mathbf{s}_1)
$$

$$
\times \exp i \int d\mathbf{r} \left\{ \xi(\mathbf{r})\psi^+(\mathbf{r}) \exp i \int d\mathbf{r} \left\{ \psi(\mathbf{r})\xi^+(\mathbf{r}) \right\} \right\}.
$$

where for bosonic systems the functional differentiation can be carried out in any order but with the differentiation with respect to $\xi(\mathbf{r})$ involving positions $\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_p$ and the $\xi^+(\mathbf{r})$ differentiation involving positions $\mathbf{s}_q, .., \mathbf{s}_2, \mathbf{s}_1$.

Evaluating the functional derivatives and then letting $\xi(\mathbf{r}), \xi^+(\mathbf{r})$ all approach zero (symbolically $\xi \longrightarrow \mathbf{0}$), we have for bosonic systems

$$
\left(\frac{\delta^{p+q} \chi_N[\xi(\mathbf{r}), \xi^+(\mathbf{r})]}{\delta^p \xi(\mathbf{r}) \, \delta^q \xi^+(\mathbf{r})} \right)^{\xi \to \mathbf{0}}_{\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_p; \mathbf{s}_q, .., \mathbf{s}_2, \mathbf{s}_1;}
$$

$$
= \iiiint D^2\psi \, D^2\psi^+ \, P^+[\psi(\mathbf{r}), \psi^+(\mathbf{r})]
$$

$$
\times i^{p+q} \psi^+(\mathbf{r}_1) \, \psi^+(\mathbf{r}_2) .. \psi^+(\mathbf{r}_p) \, \psi(\mathbf{s}_q) .. \psi(\mathbf{s}_2) \, \psi(\mathbf{s}_1)
$$

We then apply the same process to the definition of the characteristic functional

$$
\chi_N[\xi(\mathbf{r}), \xi^+(\mathbf{r})]
$$

$$
= Tr(\widehat{\rho} \exp \int d\mathbf{r} \, i\{\xi(\mathbf{r})\widehat{\Psi}^\dagger(\mathbf{r})\} \exp \int d\mathbf{r} \, i\{\widehat{\Psi}(\mathbf{r})\xi^+(\mathbf{r})\}
$$

$$
= \sum_{p,q} \frac{1}{p!q!} Tr(\widehat{\rho} \left(\int d\mathbf{r} \, i\xi(\mathbf{r}) \, \widehat{\Psi}^\dagger(\mathbf{r}) \right)^p \left(\int d\mathbf{r} \, \widehat{\Psi}(\mathbf{r}) \, i\xi^+(\mathbf{r}) \right)^q).
$$

Now with $\widehat{A}[\xi(\mathbf{r})] = \int d\mathbf{r}\, i\xi(\mathbf{r})\, \widehat{\Psi}^\dagger(\mathbf{r})$ and $\widehat{B}[\xi^+(\mathbf{r})] = \int d\mathbf{r}\, \widehat{\Psi}(\mathbf{r})\, i\xi^+(\mathbf{r})$ we see that

$$\chi_N[\xi(\mathbf{r}), \xi^+(\mathbf{r})] = \sum_{p,q} \frac{1}{p!q!} Tr(\widehat{\rho}\,(\widehat{A})^p\,(\widehat{B})^q).$$

keeping strictly to the operator order.

In this form it is convenient to calculate the functional derivatives, since $\widehat{A}[\xi(\mathbf{r})]$ and $\widehat{B}[\xi^+(\mathbf{r})]$ are functionals only of $\xi(\mathbf{r})$ and $\xi^+(\mathbf{r})$.respectively, so their functional derivatives with respect to the other function will be zero. Then

$$\left(\frac{\delta\widehat{A}[\xi(\mathbf{r})]}{\delta\xi(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{r}_1} = \lim_{\epsilon\to 0}\left(\frac{\widehat{A}[\xi(\mathbf{r}) + \epsilon\delta(\mathbf{r}-\mathbf{r}_1)] - \widehat{A}[\xi(\mathbf{r})]}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{i\epsilon\widehat{\Psi}^\dagger(\mathbf{r}_1)}{\epsilon}\right)$$

$$= i\widehat{\Psi}^\dagger(\mathbf{r}_1)$$

Similarly

$$\left(\frac{\delta\widehat{B}[\xi^+(\mathbf{r})]}{\delta\xi^+(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{s}_1} = i\widehat{\Psi}(\mathbf{s}_1)$$

We see that each time that either $\widehat{A}[\xi(\mathbf{r})]$ is differentiated with respect to $\xi(\mathbf{r})$ or $\widehat{B}[\xi^+(\mathbf{r})]$ is differentiated with respect to $\xi^+(\mathbf{r})$ an operator results, and therefore no further functional differentiation can occur. We also note that as $\xi \longrightarrow \mathbf{0}$ both $\widehat{A}[\xi(\mathbf{r})]$ and $\widehat{B}[\xi^+(\mathbf{r})]$ become zero. To proceed further we need to calculate functional derivatives of powers of $\widehat{A}[\xi(\mathbf{r})]$ and $\widehat{B}[\xi^+(\mathbf{r})]$. This can be carried out by applying the general rule for functional derivatives of products of functionals. Consider the term $(\widehat{A})^p\,(\widehat{B})^q$ where $\widehat{A}[\xi(\mathbf{r})]$ appears p times and $\widehat{B}[\xi^+(\mathbf{r})]$ appears q times.If each of the terms in $(\widehat{A})^p\,(\widehat{B})^q$ is differentiated *less* than p times with respect to $\xi(\mathbf{r})$ then there will be at least one factor $\widehat{A}[\xi(\mathbf{r})]$ still remaining and thus as $\xi \longrightarrow \mathbf{0}$ the result of the differentiation will be zero. Similar conclusions apply if the term $(\widehat{A})^p\,(\widehat{B})^q$ is differentiated less than q times with respect to $\xi^+(\mathbf{r})$. On the other hand if each of the terms in $(\widehat{A})^p\,(\widehat{B})^q$ is differentiated *more* than p times with respect to $\xi(\mathbf{r})$ then the result of the differentiation must be zero because after the pth differentiation all of the $\widehat{A}[\xi(\mathbf{r})]$ will have been replaced by a factor $i\widehat{\Psi}^\dagger(\mathbf{r}_i)$ and therefore further functional differentiation with respect to $\xi(\mathbf{r})$ will give zero. Similar conclusions apply if $(\widehat{A})^p\,(\widehat{B})^q$ is differentiated more than q times with respect to $\xi^+(\mathbf{r})$. Hence only the p, q term in the last expression for $\chi_N[\xi(\mathbf{r}), \xi^+(\mathbf{r})]$ contributes in the required result for

$$\left(\frac{\delta^{p+q}\chi_N[\xi(\mathbf{r}),\xi^+(\mathbf{r})]}{\delta^p\xi(\mathbf{r})\,\delta^q\xi^+(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1;}$$

$$=\frac{1}{p!q!}Tr\left(\widehat{\rho}\left(\frac{\delta^{p+q}((\widehat{A})^p\,(\widehat{B})^q)}{\delta^p\xi(\mathbf{r})\,\delta^q\xi^+(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1;}\right)$$

Now consider the pth functional derivative of $(\widehat{A})^p\,(\widehat{B})^q$ with respect to the $\xi(\mathbf{r})$, where the q factors $\widehat{B}[\xi^+(\mathbf{r})]$ are always to the right of the $\widehat{A}[\xi(\mathbf{r})]$. The result will be the sum of products of factors $i\widehat{\Psi}^\dagger(\mathbf{r}_1),i\widehat{\Psi}^\dagger(\mathbf{r}_2),..,i\widehat{\Psi}^\dagger(\mathbf{r}_p)$ in all $p!$ orders

$$\left(\frac{\delta^p(\widehat{A}[\xi(\mathbf{r})]\,\widehat{A}[\xi(\mathbf{r})]...\widehat{A}][\xi(\mathbf{r})]\,)^p\widehat{B}[\xi^+(\mathbf{r})]^q}{\delta^p\xi(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p}$$

$$=\quad i^p\sum_P(\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_1})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_2})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_i})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_p})\widehat{B}[\xi^+(\mathbf{r})]^q)$$

where the sum is over all permutations $P=\uparrow\left(\frac{\mu_1}{1}\frac{\mu_2}{2}..\frac{\mu_i}{i}..\frac{\mu_p}{p}\right)$ of $1,2,..,i,..p$. Considering also the qth functional derivative of $(\widehat{A})^p\,(\widehat{B})^q$ with respect to the $\xi^+(\mathbf{r})$ we get overall

$$\left(\frac{\delta^{p+q}(\widehat{A}[\xi(\mathbf{r})]..\widehat{A}[\xi(\mathbf{r})]...\widehat{A}[\xi(\mathbf{r})].\widehat{B}[\xi^+(\mathbf{r})]..\widehat{B}[\xi^+(\mathbf{r})]..\widehat{B}[\xi^+(\mathbf{r})]])^{p,q}}{\delta^p\xi(\mathbf{r})\,\delta^q\xi^+(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1}$$

$$=\quad i^{p+q}\sum_{P,Q}(\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_1})..\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_2})...\widehat{\Psi}^\dagger(\mathbf{r}_{\mu_p}).\widehat{\Psi}(\mathbf{s}_{\lambda_q})...\widehat{\Psi}(\mathbf{s}_{\lambda_2})\,\widehat{\Psi}(\mathbf{s}_{\lambda_1}))$$

where the second sum is over all permutations $Q=\uparrow\left(\frac{\lambda_q}{q}..\frac{\lambda_j}{j}..\frac{\lambda_2}{2}\frac{\lambda_1}{1}\right)$ of $q,..,i,..,2,1$.

Now all of the $p!$ products of the $\widehat{\Psi}^\dagger$ operators commute with each other and can therefore be set out in the order $\widehat{\Psi}^\dagger(\mathbf{r}_1).\widehat{\Psi}^\dagger(\mathbf{r}_2)...\widehat{\Psi}^\dagger(\mathbf{r}_p)$. Similarly, all of the $q!$ products of the $\widehat{\Psi}$ operators commute with each other and can therefore be set out in the order $\widehat{\Psi}(\mathbf{s}_q)...\widehat{\Psi}(\mathbf{s}_2)\,\widehat{\Psi}(\mathbf{s}_1)$. Thus the sum over the permutations P,Q just cancells out the $1/p!q!$ factor and we just get $i^{p+q}\{\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\}$. Thus

$$\left(\frac{\delta^{p+q}\chi_N[\xi(\mathbf{r}),\xi^+(\mathbf{r})]}{\delta^p\xi(\mathbf{r})\,\delta^q\xi^+(\mathbf{r})}\right)^{\xi\to\mathbf{0}}_{\mathbf{r}_1,\mathbf{r}_2,..,\mathbf{r}_p;\mathbf{s}_q,..,\mathbf{s}_2,\mathbf{s}_1;}$$

$$=i^{p+q}Tr\left(\widehat{\rho}\,\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)\right)$$

Hence we obtain the following key result for the quantum average of the normally ordered product $\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1)$ of the field operators

$$\left\langle \widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1) \right\rangle$$

$$= \; Tr\left(\widehat{\rho}\,\widehat{\Psi}^\dagger(\mathbf{r}_1)\widehat{\Psi}^\dagger(\mathbf{r}_2)....\widehat{\Psi}^\dagger(\mathbf{r}_p)\widehat{\Psi}(\mathbf{s}_q)..\widehat{\Psi}(\mathbf{s}_1) \right)$$

$$= \; \iiiint D^2\psi\, D^2\psi^+\, P^+[\psi(\mathbf{r}),\psi^+(\mathbf{r})]$$

$$\times \psi^+(\mathbf{r}_1)\,\psi^+(\mathbf{r}_2)\,..\psi^+(\mathbf{r}_p)\,\psi(\mathbf{s}_q)\,..\psi(\mathbf{s}_2)\,\psi(\mathbf{s}_1)$$

This result gives the required synmmetrically ordered average as a functional integral involving the quasi-distribution functional $P^+[\psi(\mathbf{r}),\psi^+(\mathbf{r})]$ times the product of the field functions, with the field operator $\widehat{\Psi}^\dagger(\mathbf{r}_i)$ being replaced by $\psi^+(\mathbf{r}_i)$ and $\widehat{\Psi}(\mathbf{s}_j)$ being replaced by $\psi(\mathbf{s}_j)$.

Appendix C.3. Supplementary Equations

Quantum Correlation Function

$$\left\langle \{\widehat{\Psi}_C^\dagger(\mathbf{r}_1)....\widehat{\Psi}_C^\dagger(\mathbf{r}_p)\widehat{\Psi}_C(\mathbf{s}_q)..\widehat{\Psi}_C(\mathbf{s}_1)\}\, \widehat{\Psi}_{NC}^\dagger(\mathbf{u}_1)....\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_r)\widehat{\Psi}_{NC}(\mathbf{v}_s)..\widehat{\Psi}_{NC}(\mathbf{v}_1) \right\rangle$$

$$= \; Tr\left(\widehat{\rho}\,\{\widehat{\Psi}_C^\dagger(\mathbf{r}_1)....\widehat{\Psi}_C^\dagger(\mathbf{r}_p)\widehat{\Psi}_C(\mathbf{s}_q)..\widehat{\Psi}_C(\mathbf{s}_1)\}\, \widehat{\Psi}_{NC}^\dagger(\mathbf{u}_1)....\widehat{\Psi}_{NC}^\dagger(\mathbf{u}_r)\widehat{\Psi}_{NC}(\mathbf{v}_s)..\widehat{\Psi}_{NC}(\mathbf{v}_1) \right)$$

$$= \; \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+$$

$$\times P[\psi_C(\mathbf{r}),\psi_C^+(\mathbf{r}),\psi_{NC}(\mathbf{r}),\psi_{NC}^+(\mathbf{r}),\psi_C^*(\mathbf{r}),\psi_C^{+*}(\mathbf{r}),\psi_{NC}^*(\mathbf{r}),\psi_{NC}^{+*}(\mathbf{r})]$$

$$\times \psi_C^+(\mathbf{r}_1)\,\psi_C^+(\mathbf{r}_2)\,..\psi_C^+(\mathbf{r}_p)\,\psi_C(\mathbf{s}_q)\,..\psi_C(\mathbf{s}_2).\psi_C(\mathbf{s}_1)$$

$$\times \psi_{NC}^+(\mathbf{u}_1)\,\psi_{NC}^+(\mathbf{u}_2)\,..\psi_{NC}^+(\mathbf{u}_r)\,\psi_{NC}(\mathbf{v}_s)\,..\psi_{NC}(\mathbf{v}_2)\,\psi_{NC}(\mathbf{v}_1) \qquad \text{(C.1)}$$

Appendix D. - Correspondence Rules

As the expressions can get cumbersome we find it convenient at times to use the following notation:

$$\underset{\rightarrow}{\xi}(\mathbf{r}) \equiv \{\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r}), \xi_{NC}(\mathbf{r}), \xi_{NC}^+(\mathbf{r})\} \tag{D.1}$$

$$\underset{\rightarrow}{\xi^C}(\mathbf{r}) \equiv \{\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})\} \qquad \underset{\rightarrow}{\xi}^{NC}(\mathbf{r}) \equiv \{\xi_{NC}(\mathbf{r}), \xi_{NC}^+(\mathbf{r})\} \tag{D.2}$$

$$\chi[\underset{\rightarrow}{\xi}(\mathbf{r})] \equiv \chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \tag{D.3}$$

$$\underset{\rightarrow}{\psi}(\mathbf{r}) \equiv \{\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})\} \tag{D.4}$$

$$\underset{\rightarrow}{\psi^*}(\mathbf{r}) \equiv \{\psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})\} \tag{D.5}$$

$$P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \equiv P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})] \tag{D.6}$$

Appendix D.1. Functional Derivative Rules - Condensate Operators

To proceed further we need to establish some rules for functional derivatives of operator expressions.Consider

$$\widehat{\Omega}_C[\xi_C, \xi_C^+] = \exp \widehat{G}[\xi_C, \xi_C^+]$$

$$\widehat{G}[\xi_C, \xi_C^+] = \int d\mathbf{r}\, i\{\xi_C(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})\xi_C^+(\mathbf{r})\}$$

(1) We first establish a result for $\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\widehat{\Psi}_C^\dagger(\mathbf{s})$. Now

$$\left(\frac{\delta\widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta\xi_C}\right)_{\mathbf{r}=\mathbf{s}} = \lim_{\epsilon \to 0}\left(\frac{\exp\widehat{G}[\xi_C(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{s}), \xi_C^+(\mathbf{r})] - \exp\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]}{\epsilon}\right)$$

$$= \lim_{\epsilon \to 0}\left(\frac{\exp\{\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})\} - \exp\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]}{\epsilon}\right)$$

Now we can use the Baker-Haussdorf theorem which is that $\exp(\widehat{A} + \widehat{B}) = \exp(\widehat{A})\exp(\widehat{B})\exp\{-\frac{1}{2}[\widehat{A}, \widehat{B}]\}$, if the commutator commutes with $\widehat{A}$ and $\widehat{B}$, so with $\widehat{A} = \widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]$ and $\widehat{B} = \epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})$ we have

$$\exp\{\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})\} = \exp\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]\exp \epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})\exp\frac{1}{2}\epsilon\xi_C^+(\mathbf{s})$$

$$\doteq \exp\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]\{1 + \epsilon(i\widehat{\Psi}_C^\dagger(\mathbf{s}) + \frac{1}{2}\xi_C^+(\mathbf{s}))\}$$

since using Eqs.(19, 22)

$$
\begin{aligned}
[\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})], \epsilon i \widehat{\Psi}_C^\dagger(\mathbf{s})\,] &= \epsilon i \int d\mathbf{r}\, i[\{\xi_C(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})\xi_C^+(\mathbf{r})\}, \widehat{\Psi}_C^\dagger(\mathbf{s})] \\
&= \epsilon i^2 \int d\mathbf{r}\, \xi_C^+(\mathbf{r})[\widehat{\Psi}_C(\mathbf{r}), \widehat{\Psi}_C^\dagger(\mathbf{s})] \\
&= \epsilon i^2 \int d\mathbf{r}\, \xi_C^+(\mathbf{r})\delta_C(\mathbf{r}, \mathbf{s}) \\
&= -\epsilon \xi_C^+(\mathbf{s})
\end{aligned}
$$

noting that the $\xi_C^+(\mathbf{r})$ only involve complex conjugates of condensate modes. Hence

$$
\begin{aligned}
\left(\frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C}\right)_{\mathbf{r}=\mathbf{s}} &= \widehat{\Omega}_C[\xi_C, \xi_C^+]\,(i\widehat{\Psi}_C^\dagger(\mathbf{s}) + \tfrac{1}{2}\xi_C^+(\mathbf{s})) \\
\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\widehat{\Psi}_C^\dagger(\mathbf{s}) &= \frac{1}{i}\left(\frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C}\right)_{\mathbf{r}=\mathbf{s}} - \frac{1}{i}\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\frac{1}{2}\xi_C^+(\mathbf{s})
\end{aligned}
$$

$$
\text{(D.7)}
$$

(2) We next establish a result for $\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\widehat{\Psi}_C(\mathbf{s})$. Similarly

$$
\begin{aligned}
\left(\frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C^+}\right)_{\mathbf{r}=\mathbf{s}} &= \lim_{\epsilon \to 0}\left(\frac{\exp \widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{s})] - \exp \widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]}{\epsilon}\right) \\
&= \lim_{\epsilon \to 0}\left(\frac{\exp\{\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C(\mathbf{s})\} - \exp \widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]}{\epsilon}\right)
\end{aligned}
$$

Using the Baker-Haussdorf theorem again but now with $\widehat{A} = \widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]$ and $\widehat{B} = \epsilon i \widehat{\Psi}_C(\mathbf{s})$ we have

$$
\begin{aligned}
\exp\{\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C(\mathbf{s})\} &= \exp \widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]\,\exp \epsilon i\widehat{\Psi}_C(\mathbf{s})\,\exp -\frac{1}{2}\epsilon \xi_C(\mathbf{s}) \\
&\doteq \exp \widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})]\,\{1 + \epsilon(i\widehat{\Psi}_C(\mathbf{s}) - \frac{1}{2}\xi_C(\mathbf{s}))\}
\end{aligned}
$$

since using Eqs.(19, 22)

$$
\begin{aligned}
[\widehat{G}[\xi_C(\mathbf{r}), \xi_C^+(\mathbf{r})], \epsilon i \widehat{\Psi}_C(\mathbf{s})\,] &= \epsilon i \int d\mathbf{r}\, i[\{\xi_C(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})\xi_C^+(\mathbf{r})\}, \widehat{\Psi}_C(\mathbf{s})] \\
&= \epsilon i^2 \int d\mathbf{r}\, \xi_C(\mathbf{r})[\widehat{\Psi}_C^\dagger(\mathbf{r}), \widehat{\Psi}_C(\mathbf{s})] \\
&= \epsilon \int d\mathbf{r}\, \xi_C(\mathbf{r})\delta_C(\mathbf{s}, \mathbf{r}) \\
&= \epsilon \xi_C(\mathbf{s})
\end{aligned}
$$

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noting that the $\xi_C(\mathbf{r})$ only involve condensate modes. Hence

$$\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)_{\mathbf{r=s}} = \widehat{\Omega}_C[\xi_C,\xi_C^+]\,(i\widehat{\Psi}_C(\mathbf{s}) - \frac{1}{2}\xi_C(\mathbf{s}))$$

$$\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\widehat{\Psi}_C(\mathbf{s}) = \frac{1}{i}\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)_{\mathbf{r=s}} + \frac{1}{i}\widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\xi_C(\mathbf{s})$$

$$(\text{D.8})$$

(3) We next establish a result for $\widehat{\Psi}_C^\dagger(\mathbf{s})\,\widehat{\Omega}_C[\xi_C,\xi_C^+]$. From above

$$\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C}\right)_{\mathbf{r=s}} = \lim_{\epsilon\to 0}\left(\frac{\exp\{\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})\} - \exp\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]}{\epsilon}\right)$$

Now we use the Baker-Haussdorf theorem with $\widehat{A} = \epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})$ and $\widehat{B} = \widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]$ we have

$$\exp\{\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})\} = \exp\epsilon i\widehat{\Psi}_C^\dagger(\mathbf{s})\,\exp\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]\,\exp-\frac{1}{2}\epsilon\xi_C^+(\mathbf{s})$$

$$\doteq \{1 + \epsilon(i\widehat{\Psi}_C^\dagger(\mathbf{s}) - \frac{1}{2}\xi_C^+(\mathbf{s}))\}\exp\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]$$

using the commutation result derived earlier

Hence

$$\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C}\right)_{\mathbf{r=s}} = (i\widehat{\Psi}_C^\dagger(\mathbf{s}) - \frac{1}{2}\xi_C^+(\mathbf{s}))\,\widehat{\Omega}_C[\xi_C,\xi_C^+]$$

$$\widehat{\Psi}_C^\dagger(\mathbf{s})\,\widehat{\Omega}_C[\xi_C,\xi_C^+] = \frac{1}{i}\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C}\right)_{\mathbf{r=s}} + \frac{1}{i}\frac{1}{2}\xi_C^+(\mathbf{s})\,\widehat{\Omega}_C[\xi_C,\xi_C^+]$$

$$(\text{D.9})$$

(4) We next establish a result for $\widehat{\Psi}_C(\mathbf{s})\,\widehat{\Omega}_C[\xi_C,\xi_C^+]$. From above

$$\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)_{\mathbf{r=s}} = \lim_{\epsilon\to 0}\left(\frac{\exp\{\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C(\mathbf{s})\} - \exp\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]}{\epsilon}\right)$$

Using the Baker-Haussdorf theorem again but now with $\widehat{A} = \epsilon i\widehat{\Psi}_C(\mathbf{s})$ and $\widehat{B} = \widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]$ we have

$$\exp\{\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})] + \epsilon i\widehat{\Psi}_C(\mathbf{s})\} = \exp\epsilon i\widehat{\Psi}_C(\mathbf{s})\,\exp\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]\,\exp+\frac{1}{2}\epsilon\xi_C(\mathbf{s})$$

$$\doteq \{1 + \epsilon(i\widehat{\Psi}_C(\mathbf{s}) + \frac{1}{2}\xi_C(\mathbf{s}))\}\exp\widehat{G}[\xi_C(\mathbf{r}),\xi_C^+(\mathbf{r})]$$

using the commutation rule derived earlier.

Hence

$$\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)_{\mathbf{r=s}} = (i\widehat{\Psi}_C(\mathbf{s}) + \frac{1}{2}\xi_C(\mathbf{s}))\,\widehat{\Omega}_C[\xi_C,\xi_C^+]$$

$$\widehat{\Psi}_C(\mathbf{s})\,\widehat{\Omega}_C[\xi_C,\xi_C^+] = \frac{1}{i}\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)_{\mathbf{r=s}} - \frac{1}{i}\frac{1}{2}\xi_C(\mathbf{s})\,\widehat{\Omega}_C[\xi_C,\xi_C^+]$$

$$\text{(D.10)}$$

(5) To establish a result for $\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\partial_\mu\widehat{\Psi}_C^\dagger(\mathbf{s})$ we start with

$$\partial_\mu\widehat{\Psi}_C^\dagger(\mathbf{s}) = \lim_{\Delta s_\mu\to 0}\left(\frac{\widehat{\Psi}_C^\dagger(\mathbf{s}+\boldsymbol{\Delta s}_\mu) - \widehat{\Psi}_C^\dagger(\mathbf{s})}{\boldsymbol{\Delta s}_\mu}\right)$$

so we can use previous results in Eq.(D.7) for $\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\widehat{\Psi}_C^\dagger(\mathbf{s})$. Using the previous results we have

$$\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\partial_\mu\widehat{\Psi}_C^\dagger(\mathbf{s}) = \lim_{\Delta s_\mu\to 0}\frac{1}{i}\frac{1}{\boldsymbol{\Delta s}_\mu}\left(\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C}\right)_{\mathbf{r=s}+\boldsymbol{\Delta s}_\mu} - \left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C}\right)_{\mathbf{r=s}}\right)$$

$$- \lim_{\Delta s_\mu\to 0}\frac{1}{i}\widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\left(\frac{1}{\boldsymbol{\Delta s}_\mu}\left(\xi_C^+(\mathbf{s}+\boldsymbol{\Delta s}_\mu) - \xi_C^+(\mathbf{s})\right)\right)$$

$$= \frac{1}{i}\left(\partial_\mu\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C}\right)\right)_{\mathbf{r=s}} - \frac{1}{i}\widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\left(\partial_\mu\xi_C^+\right)_{\mathbf{r=s}}$$

so that from the definition of the spatial derivative we obtain the result

$$\widehat{\Omega}_C[\xi_C,\xi_C^+]\,(\partial_\mu\widehat{\Psi}_C^\dagger(\mathbf{s})) = \frac{1}{i}\left(\partial_\mu\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C}\right)_{\mathbf{r=s}} - \frac{1}{i}\widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\partial_\mu\xi_C^+(\mathbf{s})$$

$$\text{(D.11)}$$

(6) To establish a result for $\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\partial_\mu\widehat{\Psi}_C(\mathbf{s})$ we start with

$$\partial_\mu\widehat{\Psi}_C(\mathbf{s}) = \lim_{\Delta s_\mu\to 0}\left(\frac{\widehat{\Psi}_C(\mathbf{s}+\boldsymbol{\Delta s}_\mu) - \widehat{\Psi}_C(\mathbf{s})}{\boldsymbol{\Delta s}_\mu}\right)$$

so we can use previous results in Eq.(D.8) for $\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\widehat{\Psi}_C(\mathbf{s})$. Using the previous results we have

$$\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\partial_\mu\widehat{\Psi}_C(\mathbf{s}) = \lim_{\Delta s_\mu\to 0}\frac{1}{i}\frac{1}{\boldsymbol{\Delta s}_\mu}\left(\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)_{\mathbf{r=s}+\boldsymbol{\Delta s}_\mu} - \left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)_{\mathbf{r=s}}\right)$$

$$+ \lim_{\Delta s_\mu\to 0}\frac{1}{i}\widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\left(\frac{1}{\boldsymbol{\Delta s}_\mu}\left(\xi_C(\mathbf{s}+\boldsymbol{\Delta s}_\mu) - \xi_C(\mathbf{s})\right)\right)$$

$$= \frac{1}{i}\left(\partial_\mu\left(\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+}\right)\right)_{\mathbf{r=s}} + \frac{1}{i}\widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\left(\partial_\mu\xi_C\right)_{\mathbf{r=s}}$$

so that from the definition of the spatial derivative we obtain the result

$$\widehat{\Omega}_C[\xi_C, \xi_C^+] \, (\partial_\mu \widehat{\Psi}_C(\mathbf{s})) = \frac{1}{i} \left(\partial_\mu \frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C^+} \right)_{\mathbf{r}=\mathbf{s}} + \frac{1}{i} \widehat{\Omega}_C[\xi_C, \xi_C^+] \frac{1}{2} \partial_\mu \xi_C(\mathbf{s})$$

(D.12)

(7) To establish a result for $\partial_\mu \widehat{\Psi}_C^\dagger(\mathbf{s}) \, \widehat{\Omega}_C[\xi_C, \xi_C^+]$ we can use Eq.(D.9) and follow the previous procedure to obtain the result

$$\partial_\mu \widehat{\Psi}_C^\dagger(\mathbf{s}) \, \widehat{\Omega}_C[\xi_C, \xi_C^+] = \frac{1}{i} \left(\partial_\mu \frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C} \right)_{\mathbf{r}=\mathbf{s}} + \frac{1}{i} \frac{1}{2} \partial_\mu \xi_C^+(\mathbf{s}) \, \widehat{\Omega}_C[\xi_C, \xi_C^+]$$

(D.13)

(8) To establish a result for $\partial_\mu \widehat{\Psi}_C(\mathbf{s}) \, \widehat{\Omega}_C[\xi_C, \xi_C^+]$ we can use Eq.(D.10) and follow the previous procedure to obtain the result

$$\partial_\mu \widehat{\Psi}_C(\mathbf{s}) \, \widehat{\Omega}_C[\xi_C, \xi_C^+] = \frac{1}{i} \left(\partial_\mu \frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C^+} \right)_{\mathbf{r}=\mathbf{s}} - \frac{1}{i} \frac{1}{2} \partial_\mu \xi_C(\mathbf{s}) \, \widehat{\Omega}_C[\xi_C, \xi_C^+]$$

(D.14)

Appendix D.2. Condensate Operators

(1) If $\widehat{\rho}$ is replaced by $\widehat{\Psi}_C(\mathbf{s})\widehat{\rho}$ then the characteristic function becomes

$$\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] &\rightarrow Tr(\widehat{\Psi}_C(\mathbf{s})\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \\
&= Tr(\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]\widehat{\Psi}_C(\mathbf{s})) \\
&= Tr(\widehat{\rho}\,\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\widehat{\Psi}_C(\mathbf{s})\,\widehat{\Omega}_{NC})
\end{aligned}$$

using the cyclic property of the trace and the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.8)

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow Tr(\widehat{\rho}\,\frac{1}{i} \left(\frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C^+(\mathbf{s})} + \widehat{\Omega}_C[\xi_C, \xi_C^+] \frac{1}{2} \xi_C(\mathbf{s}) \right) \widehat{\Omega}_{NC})$$

Hence

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow \frac{1}{i} \left(\frac{\delta}{\delta \xi_C^+(\mathbf{s})} + \frac{1}{2} \xi_C(\mathbf{s}) \right) \chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]$$

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Then using the relationship to the distribution functional we see that

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \rightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_C^+(\mathbf{s})} + \frac{1}{2}\xi_C(\mathbf{s})\right) \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]
$$

$$
\times \exp i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}
$$

$$
\times \exp i \int d\mathbf{r}\, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r}\, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
$$

$$
= \quad \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]
$$

$$
\times [\left(\psi_C(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right) \exp i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}]
$$

$$
\times \exp i \int d\mathbf{r}\, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r}\, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
$$

since from the functional differentiation rules with $G[\psi_C, \xi_C^+, \xi_C, \psi_C^+] = i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}$

$$
\frac{1}{i}\frac{\delta}{\delta\xi_C^+(\mathbf{s})} \exp i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}
$$

$$
= \quad \frac{1}{i}\exp G[\psi_C, \xi_C^+, \xi_C, \psi_C^+] \frac{\delta G[\psi_C, \xi_C^+, \xi_C, \psi_C^+]}{\delta\xi_C^+(\mathbf{s})}
$$

$$
= \quad \frac{1}{i}\exp G[\psi_C, \xi_C^+, \xi_C, \psi_C^+]\, i\psi_C(\mathbf{s})
$$

$$
= \quad \psi_C(\mathbf{s})\, \exp G[\psi_C, \xi_C^+, \xi_C, \psi_C^+] \tag{D.15}
$$

$$
-\frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})} \exp i \int d\mathbf{r}\, \{\xi_\mathbf{C}(\mathbf{r})\psi_\mathbf{C}^+(\mathbf{r}) + \psi_\mathbf{C}(\mathbf{r})\xi_\mathbf{C}^+(\mathbf{r})\}
$$

$$
= \quad -\frac{1}{2}\exp G[\psi_C, \xi_C^+, \xi_C, \psi_C^+] \frac{\delta G[\psi_C, \xi_C^+, \xi_C, \psi_C^+]}{\delta\psi_C^+(\mathbf{s})}
$$

$$
= \quad -\frac{1}{2}\exp G[\psi_C, \xi_C^+, \xi_C, \psi_C^+]\, i\xi_C(\mathbf{s}) \tag{D.16}
$$

$$
\frac{1}{i}\frac{1}{2}\xi_C(\mathbf{s})\, \exp G[\psi_C, \xi_C^+, \xi_C, \psi_C^+]
$$

$$
= \quad -\frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})} \exp i \int d\mathbf{r}\, \{\xi_\mathbf{C}(\mathbf{r})\psi_\mathbf{C}^+(\mathbf{r}) + \psi_\mathbf{C}(\mathbf{r})\xi_\mathbf{C}^+(\mathbf{r})\} \tag{D.17}
$$

To proceed further we need to replace the functional differentiation of the exponential functional with a functional differentiation of the quasi distribution functional itself. This can be accomplished using a functional integration by parts result, which requires the condition that the mode expansion form of the product functional $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \exp i \int d\mathbf{r}\, \{\xi_\mathbf{C}(\mathbf{r})\psi_\mathbf{C}^+(\mathbf{r}) + \psi_\mathbf{C}(\mathbf{r})\xi_\mathbf{C}^+(\mathbf{r})\}$ goes to zero as the expansion coefficients become large (note that there is no normalisation condition on the $\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r})$ that bounds the expansion coefficients).

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Using this integration by parts result we then find that

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \rightarrow \quad \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+
$$

$$
\times \{ \left(\psi_C(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})} \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \}
$$

$$
\times \exp i \int d\mathbf{r} \, \{\xi_\mathbf{C}(\mathbf{r})\psi_\mathbf{C}^+(\mathbf{r}) + \psi_\mathbf{C}(\mathbf{r})\xi_\mathbf{C}^+(\mathbf{r})\}]
$$

$$
\times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\Psi}_C(\mathbf{s})\widehat{\rho}$ is equivalent to then the quasi distribution functional is replaced as follows

$$
P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow \left(\psi_C(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})} \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \tag{D.18}
$$

Thus $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$ is both multiplied by $\psi_C(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_C(\mathbf{s})$ is mapped onto and functionally differentiated with respect to $\psi_C^+(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_C^\dagger(\mathbf{s})$ is mapped onto.

(2) If $\widehat{\rho}$ is replaced by $\widehat{\Psi}_C^\dagger(\mathbf{s})\widehat{\rho}$ then the characteristic function becomes

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad &\rightarrow \quad Tr(\widehat{\Psi}_C^\dagger(\mathbf{s})\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \\
&= \quad Tr(\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]\widehat{\Psi}_C^\dagger(\mathbf{s})) \\
&= \quad Tr(\widehat{\rho}\,\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\widehat{\Psi}_C^\dagger(\mathbf{s})\,\widehat{\Omega}_{NC})
\end{aligned}
$$

using the cyclic property of the trace and the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.7)

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow Tr(\widehat{\rho}\,\frac{1}{i} \left(\frac{\delta\widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta\xi_C(\mathbf{s})} - \widehat{\Omega}_C[\xi_C, \xi_C^+]\frac{1}{2}\xi_C^+(\mathbf{s}) \right) \widehat{\Omega}_{NC})
$$

Hence

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow \frac{1}{i} \left(\frac{\delta}{\delta\xi_C(\mathbf{s})} - \frac{1}{2}\xi_C^+(\mathbf{s}) \right) \chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]
$$

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Then using the relationship to the distribution functional we see that

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_C(\mathbf{s})} - \frac{1}{2}\xi_C^+(\mathbf{s})\right) \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})]
$$

$$
\times \exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}
$$

$$
\times \exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
$$

$$
= \quad \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})]
$$

$$
\times[\left(\psi_C^+(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) \exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}]
$$

$$
\times \exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
$$

where the proof of the second step is similar to that in (1).

To proceed further we use integration by parts result we then find that

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+
$$

$$
\times\{\left(\psi_C^+(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})]\}
$$

$$
\times \exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}
$$

$$
\times \exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\Psi}_C^\dagger(\mathbf{s})\widehat{\rho}$ is equivalent to then the quasi distribution functional is replaced as follows

$$
P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \to \left(\psi_C^+(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \tag{D.19}
$$

Thus $P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})]$ is both multiplied by $\psi_C^+(\mathbf{r})$, the field function that the operator $\widehat{\Psi}_C^\dagger(\mathbf{r})$ is mapped onto and functionally differentiated with respect to $\psi_C(\mathbf{r})$, the field function that the operator $\widehat{\Psi}_C(\mathbf{r})$ is mapped onto.

(3) If $\widehat{\rho}$ is replaced by $\widehat{\rho}\,\widehat{\Psi}_C(\mathbf{s})$ then the characteristic function becomes

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad Tr(\widehat{\rho}\,\widehat{\Psi}_C(\mathbf{s})\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+])
$$

$$
= \quad Tr(\widehat{\rho}\,\widehat{\Psi}_C(\mathbf{s})\,\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\widehat{\Omega}_{NC})
$$

using the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.10)

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \to Tr(\widehat{\rho}\,\frac{1}{i}\left(\frac{\delta\widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta\xi_C^+(\mathbf{s})} - \frac{1}{2}\xi_C(\mathbf{s})\,\widehat{\Omega}_C[\xi_C, \xi_C^+]\right)\widehat{\Omega}_{NC})
$$

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Hence

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \to \frac{1}{i} \left(\frac{\delta}{\delta \xi_C^+(\mathbf{s})} - \frac{1}{2} \xi_C(\mathbf{s}) \right) \chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]$$

Then using the relationship to the distribution functional we see that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad & \frac{1}{i} \left(\frac{\delta}{\delta \xi_C^+(\mathbf{s})} - \frac{1}{2} \xi_C(\mathbf{s}) \right) \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \\
& \times \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\} \\
= \quad & \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \\
& \times [\left(\psi_C(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \right) \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}] \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

To proceed further we use the integration by parts result and then find that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad & \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \\
& \times \{\left(\psi_C(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \right) P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})]\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\rho}\,\widehat{\Psi}_C(\mathbf{s})$ is equivalent to then the quasi distribution functional is replaced as follows

$$P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \to \left(\psi_C(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \right) P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \qquad \text{(D.20)}$$

Thus $P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})]$ is both multiplied by $\psi_C(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_C(\mathbf{s})$ is mapped onto and functionally differentiated with respect to $\psi_C^+(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_C^\dagger(\mathbf{s})$ is mapped onto.

(4) If $\widehat{\rho}$ is replaced by $\widehat{\rho}\,\widehat{\Psi}_C^\dagger(\mathbf{s})$ then the characteristic function becomes

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad & Tr(\widehat{\rho}\widehat{\Psi}_C^\dagger(\mathbf{s})\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \\
= \quad & Tr(\widehat{\rho}\widehat{\Psi}_C^\dagger(\mathbf{s})\,\widehat{\Omega}_C[\xi_C, \xi_C^+]\,\widehat{\Omega}_{NC})
\end{aligned}
$$

using the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.9)

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \to Tr(\widehat{\rho} \frac{1}{i} \left(\frac{\delta \widehat{\Omega}_C[\xi_C, \xi_C^+]}{\delta \xi_C(\mathbf{s})} + \frac{1}{2} \xi_C^+(\mathbf{s}) \widehat{\Omega}_C[\xi_C, \xi_C^+] \right) \widehat{\Omega}_{NC})$$

Hence

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \to \frac{1}{i} \left(\frac{\delta}{\delta \xi_C(\mathbf{s})} + \frac{1}{2} \xi_C^+(\mathbf{s}) \right) \chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]$$

Then using the relationship to the distribution functional we see that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad & \frac{1}{i} \left(\frac{\delta}{\delta \xi_C(\mathbf{s})} + \frac{1}{2} \xi_C^+(\mathbf{s}) \right) \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \\
& \times \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\} \\
= \quad & \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \\
& \times [\left(\psi_C^+(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta \psi_C(\mathbf{s})} \right) \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}] \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

To proceed further we use the integration by parts result and then find that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \to \quad & \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \\
& \times \{\left(\psi_C^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta \psi_C(\mathbf{s})} \right) P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\rho} \, \widehat{\Psi}_C^\ddagger(\mathbf{s})$ is equivalent to then the quasi distribution functional is replaced as follows

$$P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \to \left(\psi_C^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta \psi_C(\mathbf{s})} \right) P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \qquad \text{(D.21)}$$

Thus $P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]$ is both multiplied by $\psi_C^+(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_C^\ddagger(\mathbf{s})$ is mapped onto and functionally differentiated with respect to $\psi_C(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_C(\mathbf{s})$ is mapped onto.

(5) A summary of these key results is as follows:

$$\widehat{\rho} \;\to\; \widehat{\Psi}_C(\mathbf{s})\widehat{\rho} \qquad P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})] \to \left(\psi_C(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right) P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})]$$

$$\widehat{\rho} \;\to\; \widehat{\Psi}_C^{\dagger}(\mathbf{s})\widehat{\rho} \qquad P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})] \to \left(\psi_C^+(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})]$$

$$\widehat{\rho} \;\to\; \widehat{\rho}\,\widehat{\Psi}_C(\mathbf{s}) \qquad P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})] \to \left(\psi_C(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right) P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})]$$

$$\widehat{\rho} \;\to\; \widehat{\rho}\,\widehat{\Psi}_C^{\dagger}(\mathbf{s}) \qquad P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})] \to \left(\psi_C^+(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) P[\underset{\to}{\psi}(\mathbf{r}),\underset{\to}{\psi^*}(\mathbf{r})]$$

(6) If $\widehat{\rho}$ is replaced by $\partial_\mu\widehat{\Psi}_C(\mathbf{s})\widehat{\rho}$ then the characteristic function becomes

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \;\to\; Tr(\partial_\mu\widehat{\Psi}_C(\mathbf{s})\widehat{\rho}\,\widehat{\Omega}[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+])$$
$$= \; Tr(\widehat{\rho}\,\widehat{\Omega}_C[\xi_C,\xi_C^+]\,\partial_\mu\widehat{\Psi}_C(\mathbf{s})\,\widehat{\Omega}_{NC})$$

using the cyclic property of the trace and the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.12)

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \to Tr(\widehat{\rho}\,\frac{1}{i}\left(\left(\partial_\mu\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{s}} + \widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\partial_\mu\xi_C(\mathbf{s})\right)\widehat{\Omega}_{NC})$$

Hence

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \to \frac{1}{i}\left(\left(\partial_\mu\frac{\delta\widehat{\Omega}_C[\xi_C,\xi_C^+]}{\delta\xi_C^+(\mathbf{r})}\right)_{\mathbf{r}=\mathbf{s}} + \widehat{\Omega}_C[\xi_C,\xi_C^+]\frac{1}{2}\partial_\mu\xi_C(\mathbf{s})\right)\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+]$$

Appendix D.3. Functional Derivative Rules - Non-Condensate Operators

To proceed further we need to establish some rules for functional derivatives of operator expressions.Consider

$$\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+] \;=\; \exp\widehat{F}[\xi_{NC}]\,\exp\widehat{H}[\xi_{NC}^+]$$
$$\widehat{F}[\xi_{NC}] \;=\; \int d\mathbf{r}\,i\{\xi_{NC}(\mathbf{r})\widehat{\Psi}_{NC}^{\dagger}(\mathbf{r})\} \qquad \widehat{H}[\xi_{NC}^+] = \int d\mathbf{r}\,i\{\widehat{\Psi}_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}$$

(1) We first establish a result for $\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]\,\widehat{\Psi}_{NC}^{\dagger}(\mathbf{s})$. Now using the product rule and noting that $\widehat{H}[\xi_{NC}^+]$ is not a functional of ξ_{NC}

$$\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]}{\delta\xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} = \lim_{\epsilon\to 0}\left(\frac{\exp\widehat{F}[\xi_{NC}(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{s})] - \exp\widehat{F}[\xi_{NC}(\mathbf{r})]}{\epsilon}\right)\exp\widehat{H}[\xi_{NC}^+]$$

$$= \lim_{\epsilon\to 0}\left(\frac{\exp\{\widehat{F}[\xi_{NC}(\mathbf{r})] + \epsilon i\widehat{\Psi}_{NC}^{\dagger}(\mathbf{s})\} - \exp\widehat{F}[\xi_{NC}(\mathbf{r})]}{\epsilon}\right)\exp\widehat{H}[\xi_{NC}^+]$$

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Now we can use the Baker-Haussdorf theorem which is that $\exp(\widehat{A} + \widehat{B}) = \exp(\widehat{A})\exp(\widehat{B})\exp\{-\frac{1}{2}[\widehat{A},\widehat{B}]\}$, if the commutator commutes with $\widehat{A}$ and $\widehat{B}$, so with $\widehat{A} = \widehat{F}[\xi_{NC}(\mathbf{r})]$ and $\widehat{B} = \epsilon i \widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})$ we have

$$
\begin{aligned}
\exp\{\widehat{F}[\xi_{NC}(\mathbf{r})] + \epsilon i \widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})\} &= \exp \widehat{F}[\xi_{NC}(\mathbf{r})] \exp \epsilon i \widehat{\Psi}^{\dagger}_{NC}(\mathbf{s}) \\
&\doteq \exp \widehat{F}[\xi_{NC}(\mathbf{r})]\left\{1 + \epsilon(i\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s}))\right\}
\end{aligned}
$$

since using Eqs.(D.32, E.318)

$$
\begin{aligned}
[\widehat{F}[\xi_{NC}(\mathbf{r})], \epsilon i \widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})] &= \epsilon i \int d\mathbf{r}\, i[\{\xi_{NC}(\mathbf{r})\widehat{\Psi}^{\dagger}_{NC}(\mathbf{r})\}, \widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})] \\
&= 0
\end{aligned}
$$

Hence

$$
\left(\frac{\delta \widehat{\Omega}_{NC}[\xi_{NC}, \xi_C^+]}{\delta \xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} = \exp \widehat{F}[\xi_C]\,(i\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s}))\, \exp \widehat{H}[\xi_{NC}^+]
$$

But although $i\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})$ does not commute with $\exp \widehat{H}[\xi_{NC}^+]$ we can use the identity $\widehat{\Xi}\exp \widehat{S} = \exp \widehat{S}\,\{\widehat{\Xi} - [\widehat{S},\widehat{\Xi}] + \frac{1}{2!}[\widehat{S},[\widehat{S},\widehat{\Xi}]] - \frac{1}{3!}[\widehat{S},[\widehat{S},[\widehat{S},\widehat{\Xi}]]] + ..\}$ to place the exponential on the left. Here we have $\widehat{S} = \widehat{H}[\xi_{NC}^+]$ and $\widehat{\Xi} = i\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})$. Using Eqs.(D.32, E.318) we have on noting that $\xi_{NC}^+(\mathbf{r})$ only involves the complex conjugates of non-condensate modes

$$
\begin{aligned}
[\widehat{H}[\xi_{NC}^+], i\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})] &= i^2 \int d\mathbf{r}\, [\widehat{\Psi}_{NC}(\mathbf{r}), \widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})]\, \xi_{NC}^+(\mathbf{r}) \\
&= -\int d\mathbf{r}\, \delta_{NC}(\mathbf{r},\mathbf{s})\, \xi_{NC}^+(\mathbf{r}) \\
&= -\xi_{NC}^+(\mathbf{s})
\end{aligned}
$$

Thus we see that the series terminates after the second term giving

$$
\begin{aligned}
\left(\frac{\delta \widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]}{\delta \xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} &= \exp \widehat{F}[\xi_{NC}] \exp \widehat{H}[\xi_{NC}^+]\,\{i\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})) + \xi_{NC}^+(\mathbf{s})\} \\
&= \widehat{\Omega}_{NC}[\xi_{NC}, \xi_C^+]\,\{i\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})) + \xi_{NC}^+(\mathbf{s})\} \\
\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]\,\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})) &= \frac{1}{i}\left(\frac{\delta \widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]}{\delta \xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} - \frac{1}{i}\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]\,\xi_{NC}^+(\mathbf{s}) \\
&= \frac{1}{i}\left(\frac{\delta \widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]}{\delta \xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} - \frac{1}{i}\xi_{NC}^+(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]
\end{aligned}
$$

$$\tag{D.22}$$

(2) We next establish a result for $\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]\,\widehat{\Psi}_{NC}(\mathbf{s})$. Similarly using the product rule and noting that $\widehat{F}[\xi_{NC}]$ is not a functional of ξ_{NC}^+

$$\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]}{\delta\xi_{NC}^+}\right)_{\mathbf{r}=\mathbf{s}} = \exp\widehat{F}[\xi_{NC}]\lim_{\epsilon\to0}\left(\frac{\exp\widehat{H}[\xi_{NC}^+(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{s})]-\exp\widehat{H}[\xi_{NC}^+(\mathbf{r})]}{\epsilon}\right)$$

$$= \exp\widehat{F}[\xi_{NC}]\lim_{\epsilon\to0}\left(\frac{\exp\{\widehat{H}[\xi_{NC}^+(\mathbf{r})]+\epsilon i\widehat{\Psi}_{NC}(\mathbf{s})\}-\exp\widehat{H}[\xi_{NC}^+(\mathbf{r})]}{\epsilon}\right)$$

Using the Baker-Haussdorf theorem again but now with $\widehat{A}=\widehat{H}[\xi_{NC}^+(\mathbf{r})]$ and $\widehat{B}=\epsilon i\widehat{\Psi}_{NC}(\mathbf{s})$ we have

$$\exp\{\widehat{H}[\xi_{NC}^+(\mathbf{r})]+\epsilon i\widehat{\Psi}_{NC}(\mathbf{s})\} = \exp\widehat{H}[\xi_{NC}^+(\mathbf{r})]\exp\epsilon i\widehat{\Psi}_{NC}(\mathbf{s})$$

$$= \exp\widehat{H}[\xi_{NC}^+(\mathbf{r})]\{1+\epsilon i\widehat{\Psi}_{NC}(\mathbf{s})\}$$

since from Eqs.(D.32, E.318)

$$[\widehat{H}[\xi_{NC}^+(\mathbf{r})],\epsilon i\widehat{\Psi}_{NC}(\mathbf{s})] = \epsilon i\int d\mathbf{r}\, i[\{\widehat{\Psi}_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\},\widehat{\Psi}_{NC}(\mathbf{s})]$$

$$= 0$$

Hence

$$\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]}{\delta\xi_{NC}^+}\right)_{\mathbf{r}=\mathbf{s}} = \exp\widehat{F}[\xi_{NC}]\exp\widehat{H}[\xi_{NC}^+(\mathbf{r})]\,(i\widehat{\Psi}_{NC}(\mathbf{s}))$$

$$= \widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]\,(i\widehat{\Psi}_{NC}(\mathbf{s}))$$

$$\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]\,\widehat{\Psi}_{NC}(\mathbf{s}) = \frac{1}{i}\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]}{\delta\xi_{NC}^+}\right)_{\mathbf{r}=\mathbf{s}} \tag{D.23}$$

(3) We next establish a result for $\widehat{\Psi}_{NC}^\dagger(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]$. From above and now using the result that $i\widehat{\Psi}_{NC}^\dagger(\mathbf{s})$ commutes with $\widehat{F}[\xi_C]$

$$\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_C^+]}{\delta\xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} = \exp\widehat{F}[\xi_C]\,(i\widehat{\Psi}_{NC}^\dagger(\mathbf{s}))\exp\widehat{H}[\xi_{NC}^+]$$

$$= (i\widehat{\Psi}_{NC}^\dagger(\mathbf{s}))\exp\widehat{F}[\xi_C]\exp\widehat{H}[\xi_{NC}^+]$$

$$= (i\widehat{\Psi}_{NC}^\dagger(\mathbf{s}))\,\widehat{\Omega}_{NC}[\xi_{NC},\xi_C^+]$$

$$\widehat{\Psi}_{NC}^\dagger(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC},\xi_C^+] = \frac{1}{i}\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_C^+]}{\delta\xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} \tag{D.24}$$

(4) We next establish a result for $\widehat{\Psi}_{NC}(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]$. From above and now using the result that $i\widehat{\Psi}_{NC}(\mathbf{s})$ commutes with $\widehat{H}[\xi_C^+]$

$$\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+]}{\delta\xi_{NC}^+}\right)_{\mathbf{r}=\mathbf{s}} = \exp\widehat{F}[\xi_{NC}]\exp\widehat{H}[\xi_{NC}^+(\mathbf{r})]\,(i\widehat{\Psi}_{NC}(\mathbf{s}))$$

$$= \exp\widehat{F}[\xi_{NC}]\,(i\widehat{\Psi}_{NC}(\mathbf{s}))\exp\widehat{H}[\xi_{NC}^+(\mathbf{r})]$$

But although $i\widehat{\Psi}_{NC}(\mathbf{s})$ does not commute with $\exp\widehat{F}[\xi_{NC}]$ we can use the identity $\exp\widehat{S}\,\widehat{\Xi} = \{\widehat{\Xi} + [\widehat{S}, \widehat{\Xi}] + \frac{1}{2!}[\widehat{S}, [\widehat{S}, \widehat{\Xi}]] + \frac{1}{3!}[\widehat{S}, [\widehat{S}, [\widehat{S}, \widehat{\Xi}]]] + ..\}\exp\widehat{S}$ to place the exponential on the right. Here we have $\widehat{S} = \widehat{F}[\xi_{NC}]$ and $\widehat{\Xi} = i\widehat{\Psi}_{NC}(\mathbf{s})$. Using Eqs.(D.32, E.318) we have on noting that $\xi_{NC}(\mathbf{r})$ only involves non-condensate modes

$$
\begin{aligned}
[\widehat{F}[\xi_{NC}], i\widehat{\Psi}_{NC}(\mathbf{s})] &= i^2 \int d\mathbf{r}\,[\xi_{NC}(\mathbf{r})\widehat{\Psi}^{\dagger}_{NC}(\mathbf{r}), \widehat{\Psi}_{NC}(\mathbf{s})] \\
&= \int d\mathbf{r}\,\xi_{NC}(\mathbf{r})\delta_{NC}(\mathbf{s}, \mathbf{r}) \\
&= +\xi_{NC}(\mathbf{s})
\end{aligned}
$$

Thus we see that the series terminates after the second term giving

Hence

$$
\begin{aligned}
\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC}, \xi^{+}_{NC}]}{\delta\xi^{+}_{NC}}\right)_{\mathbf{r}=\mathbf{s}} &= (i\widehat{\Psi}_{NC}(\mathbf{s}) + \xi_{NC}(\mathbf{s}))\,\exp\widehat{F}[\xi_{NC}]\,\exp\widehat{H}[\xi^{+}_{NC}(\mathbf{r})] \\
&= (i\widehat{\Psi}_{NC}(\mathbf{s}) + \xi_{NC}(\mathbf{s}))\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi^{+}_{NC}] \\
\widehat{\Psi}_{NC}(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi^{+}_{NC}] &= \frac{1}{i}\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC}, \xi^{+}_{NC}]}{\delta\xi^{+}_{NC}}\right)_{\mathbf{r}=\mathbf{s}} - \frac{1}{i}\xi_{NC}(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi^{+}_{NC}]
\end{aligned}
$$

(D.25)

Appendix D.4. Non-Condensate Operators

(1) If $\widehat{\rho}$ is replaced by $\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\rho}$ then the characteristic function becomes

$$
\begin{aligned}
\chi[\xi_C, \xi^{+}_C, \xi_{NC}, \xi^{+}_{NC}] &\rightarrow Tr(\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi^{+}_C, \xi_{NC}, \xi^{+}_{NC}]) \\
&= Tr(\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi^{+}_C, \xi_{NC}, \xi^{+}_{NC}]\widehat{\Psi}_{NC}(\mathbf{s})) \\
&= Tr(\widehat{\rho}\,\widehat{\Omega}_C\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi^{+}_{NC}]\,\widehat{\Psi}_{NC}(\mathbf{s}))
\end{aligned}
$$

using the cyclic property of the trace and the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.23)

$$
\chi[\xi_C, \xi^{+}_C, \xi_{NC}, \xi^{+}_{NC}] \rightarrow Tr(\widehat{\rho}\,\widehat{\Omega}_C\,\frac{1}{i}\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC}, \xi^{+}_{NC}]}{\delta\xi^{+}_{NC}}\right)_{\mathbf{r}=\mathbf{s}})
$$

Hence

$$
\chi[\xi_C, \xi^{+}_C, \xi_{NC}, \xi^{+}_{NC}] \rightarrow \frac{1}{i}\left(\frac{\delta}{\delta\xi^{+}_{NC}(\mathbf{s})}\right)\chi[\xi_C, \xi^{+}_C, \xi_{NC}, \xi^{+}_{NC}]
$$

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Then using the relationship to the distribution functional we see that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad &\rightarrow \quad \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}^+(\mathbf{s})}\right) \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \\
&\quad \times \exp i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
&\quad \times \exp i \int d\mathbf{r}\, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r}\, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\} \\
&= \quad \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \\
&\quad \times [(\psi_{NC}(\mathbf{s}))\exp i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}] \\
&\quad \times \exp i \int d\mathbf{r}\, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r}\, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

To proceed further we only need to place the multiplicative term $\psi_{NC}(\mathbf{s})$ next to the quasi distribution functional itself. We then find that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad &\rightarrow \quad \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+ \\
&\quad \times \{(\psi_{NC}(\mathbf{s}))\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\} \\
&\quad \times \exp i \int d\mathbf{r}\, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
&\quad \times \exp i \int d\mathbf{r}\, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r}\, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\rho}$ is equivalent to then the quasi distribution functional is replaced as follows

$$
P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow (\psi_{NC}(\mathbf{s}))\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \tag{D.26}
$$

Thus $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$ is multiplied by $\psi_{NC}(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_{NC}(\mathbf{s})$ is mapped onto.

(2) If $\widehat{\rho}$ is replaced by $\widehat{\Psi}_{NC}^\dagger(\mathbf{s})\widehat{\rho}$ then the characteristic function becomes

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad &\rightarrow \quad Tr(\widehat{\Psi}_{NC}^\dagger(\mathbf{s})\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \\
&= \quad Tr(\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]\widehat{\Psi}_{NC}^\dagger(\mathbf{s})) \\
&= \quad Tr(\widehat{\rho}\,\widehat{\Omega}_C\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]\,\widehat{\Psi}_{NC}^\dagger(\mathbf{s}))
\end{aligned}
$$

using the cyclic property of the trace and the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.22)

$$
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow Tr(\widehat{\rho}\,\widehat{\Omega}_C\, \frac{1}{i}\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]}{\delta\xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}} - \frac{1}{i}\xi_{NC}^+(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+])
$$

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Hence

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}(\mathbf{s})} - \xi_{NC}^+(\mathbf{s})\right)\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]$$

Then using the relationship to the distribution functional we see that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \rightarrow \quad & \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}(\mathbf{s})} - \xi_{NC}^+(\mathbf{s})\right)\iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \\
& \times \exp i \int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i \int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\} \\
= \quad & \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+\, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \\
& \times[\left(\psi_{NC}^+(\mathbf{s}) + \frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\exp i \int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i \int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}] \\
& \times \exp i \int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}
\end{aligned}
$$

To proceed further we use the integration by parts result we then find that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \rightarrow \quad & \iiiint D^2\psi_C\, D^2\psi_C^+\, D^2\psi_{NC}\, D^2\psi_{NC}^+ \\
& \times\{\left(\psi_{NC}^+(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\} \\
& \times \exp i \int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i \int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\Psi}_{NC}^\dagger(\mathbf{s})\widehat{\rho}$ is equivalent to then the quasi distribution functional is replaced as follows

$$P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow \left(\psi_{NC}^+(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \qquad \text{(D.27)}$$

Thus $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$ is both multiplied by $\psi_{NC}^+(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_{NC}^\dagger(\mathbf{s})$ is mapped onto and functionally differentiated with respect to $\psi_{NC}(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_{NC}(\mathbf{s})$ is mapped onto.

(3) If $\widehat{\rho}$ is replaced by $\widehat{\rho}\,\widehat{\Psi}_{NC}(\mathbf{s})$ then the characteristic function becomes

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \quad \rightarrow \quad & Tr(\widehat{\rho}\,\widehat{\Psi}_{NC}(\mathbf{s})\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \\
= \quad & Tr(\widehat{\rho}\,\widehat{\Omega}_C\,\widehat{\Psi}_{NC}(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+])
\end{aligned}
$$

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using the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.25)

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow Tr(\widehat{\rho}\,\widehat{\Omega}_C\,\frac{1}{i}\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+]}{\delta\xi_{NC}^+}\right)_{\mathbf{r=s}} - \frac{1}{i}\xi_{NC}(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC}, \xi_{NC}^+])$$

Hence

$$\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \rightarrow \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}^+(\mathbf{s})} - \xi_{NC}(\mathbf{s})\right)\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]$$

Then using the relationship to the distribution functional we see that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \;\rightarrow\; & \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}^+(\mathbf{s})} - \xi_{NC}(\mathbf{s})\right)\iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+\,P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \\
& \times \exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\} \\
=\; & \iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+\,P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \\
& \times [\left(\psi_{NC}(\mathbf{s}) + \frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}] \\
& \times \exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}
\end{aligned}
$$

To proceed further we use the integration by parts result and then find that

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \;\rightarrow\; & \iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+ \\
& \times \{\left(\psi_{NC}(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]\} \\
& \times \exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\rho}\,\widehat{\Psi}_{NC}(\mathbf{s})$ is equivalent to then the quasi distribution functional is replaced as follows

$$P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \rightarrow \left(\psi_{NC}(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})] \qquad (D.28)$$

Thus $P[\underrightarrow{\psi}(\mathbf{r}), \underrightarrow{\psi^*}(\mathbf{r})]$ is both multiplied by $\psi_{NC}(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_{NC}(\mathbf{s})$ is mapped onto and functionally differentiated with respect to $\psi_{NC}^+(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_{NC}^\dagger(\mathbf{s})$ is mapped onto.

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(4) If $\widehat{\rho}$ is replaced by $\widehat{\rho}\,\widehat{\Psi}^\dagger_{NC}(\mathbf{s})$ then the characteristic function becomes

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \;\to\; Tr(\widehat{\rho}\,\widehat{\Psi}^\dagger_{NC}(\mathbf{s})\,\widehat{\Omega}[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+])$$
$$= \; Tr(\widehat{\rho}\,\widehat{\Omega}_C\,\widehat{\Psi}^\dagger_{NC}(\mathbf{s})\,\widehat{\Omega}_{NC}[\xi_{NC},\xi_{NC}^+])$$

using the feature that condensate operators commute with non-condensate operators.

Hence from Eq.(D.24)

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \to Tr(\widehat{\rho}\,\widehat{\Omega}_C\,\frac{1}{i}\left(\frac{\delta\widehat{\Omega}_{NC}[\xi_{NC},\xi_C^+]}{\delta\xi_{NC}}\right)_{\mathbf{r}=\mathbf{s}})$$

Hence

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \to \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}(\mathbf{s})}\right)\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+]$$

Then using the relationship to the distribution functional we see that

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \;\to\; \frac{1}{i}\left(\frac{\delta}{\delta\xi_{NC}(\mathbf{s})}\right)\iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+\,P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]$$
$$\times\exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r})+\psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}$$
$$\times\exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}$$
$$= \;\iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+\,P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]$$
$$\times[(\psi_{NC}^+(\mathbf{s}))\,\exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}]$$
$$\times\exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r})+\psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}$$

To proceed further we only need to place the multiplicative term $\psi_{NC}^+(\mathbf{s})$ next to the quasi distribution functional itself. We then find that

$$\chi[\xi_C,\xi_C^+,\xi_{NC},\xi_{NC}^+] \;\to\; \iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+$$
$$\times\{(\psi_{NC}^+(\mathbf{s}))\,P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\}$$
$$\times\exp i\int d\mathbf{r}\,\{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r})+\psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\}$$
$$\times\exp i\int d\mathbf{r}\,\{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\}\exp i\int d\mathbf{r}\,\{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}$$

Hence the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\widehat{\rho}\,\widehat{\Psi}^\dagger_{NC}(\mathbf{s})$ is equivalent to then the quasi distribution functional is replaced as follows

$$P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})] \to (\psi_{NC}^+(\mathbf{s}))\,P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})] \tag{D.29}$$

Thus $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$ is multiplied by $\psi_{NC}^+(\mathbf{s})$, the field function that the operator $\widehat{\Psi}_{NC}^\dagger(\mathbf{s})$ is mapped onto.

(5) A summary of these key results is as follows:

$$\widehat{\rho} \;\rightarrow\; \widehat{\Psi}_{NC}(\mathbf{s})\widehat{\rho} \qquad P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow (\psi_{NC}(\mathbf{s})) \, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$$

$$\widehat{\rho} \;\rightarrow\; \widehat{\Psi}_{NC}^\dagger(\mathbf{s})\widehat{\rho} \qquad P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow \left(\psi_{NC}^+(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$$

$$\widehat{\rho} \;\rightarrow\; \widehat{\rho}\,\widehat{\Psi}_{NC}(\mathbf{s}) \qquad P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow \left(\psi_{NC}(\mathbf{s}) - \frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$$

$$\widehat{\rho} \;\rightarrow\; \widehat{\rho}\,\widehat{\Psi}_{NC}^\dagger(\mathbf{s}) \qquad P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow (\psi_{NC}^+(\mathbf{s})) \, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$$

Appendix D.5. Time Derivative

If $\widehat{\rho}$ is replaced by $\frac{\partial\widehat{\rho}}{\partial t}$ then

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \;\rightarrow\;& Tr(\frac{\partial}{\partial t}\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \\
=\;& \frac{\partial}{\partial t}Tr(\widehat{\rho}\,\widehat{\Omega}[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]) \\
=\;& \frac{\partial}{\partial t}\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+]
\end{aligned}
$$

Hence

$$
\begin{aligned}
\chi[\xi_C, \xi_C^+, \xi_{NC}, \xi_{NC}^+] \;\rightarrow\;& \frac{\partial}{\partial t}\iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \\
& \times \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\} \\
=\;& \iiiint D^2\psi_C \, D^2\psi_C^+ \, D^2\psi_{NC} \, D^2\psi_{NC}^+ \, \frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \\
& \times \exp i \int d\mathbf{r} \, \{\xi_C(\mathbf{r})\psi_C^+(\mathbf{r}) + \psi_C(\mathbf{r})\xi_C^+(\mathbf{r})\} \\
& \times \exp i \int d\mathbf{r} \, \{\xi_{NC}(\mathbf{r})\psi_{NC}^+(\mathbf{r})\} \exp i \int d\mathbf{r} \, \{\psi_{NC}(\mathbf{r})\xi_{NC}^+(\mathbf{r})\}
\end{aligned}
$$

Thus the change to the characteristic functional if $\widehat{\rho}$ is replaced by $\frac{\partial\widehat{\rho}}{\partial t}$ is equivalent to then the quasi distribution functional is replaced as follows

$$P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \rightarrow \frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \qquad\qquad \text{(D.30)}$$

Thus $P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]$ is replaced by its time derivative.

Appendix D.6. Supplementary Equations

Commutation Rules and Delta Functions

$$
\begin{aligned}
[\widehat{\Psi}_C(\mathbf{r}), \widehat{\Psi}_{NC}^\dagger(\mathbf{r})] &= 0 \\
[\widehat{\Psi}_C(\mathbf{r}), \widehat{\Psi}_C^\dagger(\mathbf{r}')] &= \phi_1(\mathbf{r})\phi_1^*(\mathbf{r}') + \phi_2(\mathbf{r})\phi_2^*(\mathbf{r}') \\
&= \delta_C(\mathbf{r}, \mathbf{r}') \quad\quad\quad\quad (D.31) \\
[\widehat{\Psi}_{NC}(\mathbf{r}), \widehat{\Psi}_{NC}^\dagger(\mathbf{r}')] &= \sum_{k \neq 1,2} \phi_k(\mathbf{r})\phi_k^*(\mathbf{r}') \\
&= \delta_{NC}(\mathbf{r}, \mathbf{r}') \quad\quad\quad\quad (D.32)
\end{aligned}
$$

Field Expansions and Delta Functions

$$
\begin{aligned}
\psi_C(\mathbf{r}) &= \alpha_1\phi_1(\mathbf{r}) + \alpha_2\phi_2(\mathbf{r}) &\quad \psi_C^+(\mathbf{r}) &= \phi_1^*(\mathbf{r})\alpha_1^+ + \phi_2^*(\mathbf{r})\alpha_2^+ &\quad (D.33) \\
\psi_{NC}(\mathbf{r}) &= \sum_{k \neq 1,2} \alpha_k\phi_k(\mathbf{r}) &\quad \psi_{NC}^+(\mathbf{r}) &= \sum_{k \neq 1,2} \phi_k^*(\mathbf{r})\alpha_k^+ &\quad (D.34) \\
\psi_C(\mathbf{r}) &= \int d\mathbf{r}'\, \delta_C(\mathbf{r}, \mathbf{r}')\psi_C(\mathbf{r}') &\quad \psi_C^+(\mathbf{r}) &= \int d\mathbf{r}'\, \psi_C^+(\mathbf{r}')\delta_C(\mathbf{r}', \mathbf{r}) & \\
\psi_{NC}(\mathbf{r}) &= \int d\mathbf{r}'\, \delta_{NC}(\mathbf{r}, \mathbf{r}')\psi_{NC}(\mathbf{r}') &\quad \psi(\mathbf{r}) &= \int d\mathbf{r}'\, \psi_C^+(\mathbf{r}')\delta_C(\mathbf{r}', \mathbf{r}) &\quad (D.35)
\end{aligned}
$$

Appendix E. - Functional Fokker-Planck Equation

In this Appendix we derive the Functional Fokker-Planck equation. We will derive it based on the full Hamiltonian including the $\widehat{H}_4$ and $\widehat{H}_5$ terms. This gives the exact equation. We can then write down the corresponding FFPE for the case of the Bogoliubov Hamiltonian (E.314) by discarding terms for the exact FFPE - which would be needed for the stong interaction regime. For this derivation it will be convenient to write the Hamiltonian in the form

$$\widehat{H} = \widehat{H}_C + \widehat{H}_{NC} + \widehat{V} \tag{E.1}$$

where

$$
\begin{aligned}
\widehat{H}_C &= \int d\mathbf{r}(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_C(\mathbf{r})^\dagger \cdot \nabla\widehat{\Psi}_C(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})^\dagger V\widehat{\Psi}_C(\mathbf{r}) \\
&\quad + \frac{g}{2}\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) \tag{E.2} \\
\widehat{H}_{NC} &= \int d\mathbf{r}(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_{NC}(\mathbf{r})^\dagger \cdot \nabla\widehat{\Psi}_{NC}(\mathbf{r}) + \widehat{\Psi}_{NC}(\mathbf{r})^\dagger V\widehat{\Psi}_{NC}(\mathbf{r}) \\
&\quad + \frac{g}{2}\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})) \tag{E.3}
\end{aligned}
$$

are Hamiltonians for the *condensate* and *non-condensate*. The *interaction* between condensate and non-condensate is written as the sum of three contributions which are linear, quadratic and cubic in the non-condensate operators

$$
\begin{aligned}
\widehat{V} &= \widehat{V}_1 + \widehat{V}_2 + \widehat{V}_3 \tag{E.4} \\
\widehat{V}_1 &= \int d\mathbf{r}(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_{NC}(\mathbf{r})^\dagger \cdot \nabla\widehat{\Psi}_C(\mathbf{r}) + \frac{\hbar^2}{2m}\nabla\widehat{\Psi}_C(\mathbf{r})^\dagger \cdot \nabla\widehat{\Psi}_{NC}(\mathbf{r}) \\
&\quad + \widehat{\Psi}_{NC}(\mathbf{r})^\dagger V\widehat{\Psi}_C(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})^\dagger V\widehat{\Psi}_{NC}(\mathbf{r}) \\
&\quad + g\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) + g\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})) \tag{E.5} \\
\widehat{V}_2 &= \int d\mathbf{r}(\frac{g}{2}\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) + \frac{g}{2}\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r}) \\
&\quad + 2g\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) \tag{E.6} \\
\widehat{V}_3 &= \int d\mathbf{r}(g\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) + g\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})) \tag{E.7}
\end{aligned}
$$

However, using the coupled generalised Gross-Pitaevskii equation we can make the simplifications

$$
\begin{aligned}
\int d\mathbf{r}\,\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\left\{\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\widehat{\Psi}_C(\mathbf{r})\right\} &= -\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\,\widehat{\Psi}_C(\mathbf{s}) \\
\int d\mathbf{r}\left\{\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\widehat{\Psi}_C^\dagger(\mathbf{r})\right\}\widehat{\Psi}_{NC}(\mathbf{r}) &= -\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F^*(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{s}) \tag{E.8}
\end{aligned}
$$

to write $\widehat{V}_1$ in a form

$$\widehat{V}_1 \;=\; \int d\mathbf{r}\,\widehat{\Psi}_{NC}^{\dagger}(\mathbf{r})g\{\widehat{\Psi}_{C}^{\dagger}(\mathbf{r})\,\widehat{\Psi}_{C}(\mathbf{r})\}\widehat{\Psi}_{C}(\mathbf{r}) - \int\int d\mathbf{r}\,d\mathbf{s}\,gF(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\widehat{\Psi}_{C}(\mathbf{s})$$
$$+ \int d\mathbf{r}\,\widehat{\Psi}_{C}^{\dagger}(\mathbf{r})g\{\widehat{\Psi}_{C}^{\dagger}(\mathbf{r})\,\widehat{\Psi}_{C}(\mathbf{r})\}\widehat{\Psi}_{NC}(\mathbf{r}) - \int\int d\mathbf{r}\,d\mathbf{s}\,gF^{*}(\mathbf{s},\mathbf{r})\widehat{\Psi}_{C}(\mathbf{r})^{\dagger}\widehat{\Psi}_{NC}(\mathbf{s})$$

$$\text{(E.9)}$$

We see that $\widehat{V}_1$ is the sum of term $\widehat{V}_{14}$ which is fourth order in the field operators and a term $\widehat{V}_{12}$ which is second order.

$$\widehat{V}_1 \;=\; \widehat{V}_{14} + \widehat{V}_{12} \tag{E.10}$$

$$\widehat{V}_{14} \;=\; g\int d\mathbf{r}\,(\widehat{\Psi}_{NC}^{\dagger}(\mathbf{r})\widehat{\Psi}_{C}^{\dagger}(\mathbf{r})\,\widehat{\Psi}_{C}(\mathbf{r})\widehat{\Psi}_{C}(\mathbf{r})) + g\int d\mathbf{r}\,(\widehat{\Psi}_{C}^{\dagger}(\mathbf{r})\widehat{\Psi}_{C}^{\dagger}(\mathbf{r})\,\widehat{\Psi}_{C}(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r}))$$

$$\text{(E.11)}$$

$$\widehat{V}_{12} \;=\; -g\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\widehat{\Psi}_{C}(\mathbf{s}) - g\int\int d\mathbf{r}\,d\mathbf{s}\,F^{*}(\mathbf{s},\mathbf{r})\widehat{\Psi}_{C}(\mathbf{r})^{\dagger}\widehat{\Psi}_{NC}(\mathbf{s})$$

$$\text{(E.12)}$$

Thus we see that $\widehat{V}$ is now associated only with *boson-boson* interaction terms.

From Eqs. (E.1), (E.2), (E.3) and (E.4) we see that there are a total of seventeen distinct contributions to the Hamiltonian to be considered, ranging from the kinetic energy contribution to the condensate Hamiltonian to an interaction term between the condensate and non-condensate fields which is third order in the non-condensate field. For the Bogoliubov Hamiltonian for which we derive the functional Fokker-Planck equation the terms $\widehat{V}_3$ and the boson-boson interaction in the non-condensate Hamiltonian $\widehat{H}_{NC}$ are discarded.

In order to avoid using too many superscripts and subscripts, in considering each term a simplified notation will be used, which is as follows. For terms which only involve condensate fields or only non-condensate fields we will use $\psi(\mathbf{s})$ and $\psi^{+}(\mathbf{s})$ for $\psi_C(\mathbf{s})$ and $\psi_C^{+}(\mathbf{s})$ or $\psi_{NC}(\mathbf{s})$ and $\psi_{NC}^{+}(\mathbf{s})$. We will write $W[\psi(\mathbf{r}),\psi^{+}(\mathbf{r})]$ or $P[\psi(\mathbf{r}),\psi^{+}(\mathbf{r})]$ (and sometimes just W or P in large expressions) instead of the complete expression
$P[\psi_C(\mathbf{r}),\psi_C^{+}(\mathbf{r}),\psi_{NC}(\mathbf{r}),\psi_{NC}^{+}(\mathbf{r}),\psi_C^{*}(\mathbf{r}),\psi_C^{+*}(\mathbf{r}),\psi_{NC}^{*}(\mathbf{r}),\psi_{NC}^{+*}(\mathbf{r})] \equiv P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^{*}}(\mathbf{r})]$
in the pure condensate and pure non-condensate cases. For the interaction between condensate and non-condensate we will write $WP[\psi(\mathbf{r}),\psi^{+}(\mathbf{r}),\phi(\mathbf{r}),\phi^{+}(\mathbf{r})]$ (and sometimes just WP in large expressions) instead of the complete expression
$P[\psi_C(\mathbf{r}),\psi_C^{+}(\mathbf{r}),\psi_{NC}(\mathbf{r}),\psi_{NC}^{+}(\mathbf{r}),\psi_C^{*}(\mathbf{r}),\psi_C^{+*}(\mathbf{r}),\psi_{NC}^{*}(\mathbf{r}),\psi_{NC}^{+*}(\mathbf{r})] \equiv P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^{*}}(\mathbf{r})]$,
using $\psi(\mathbf{s})$, $\psi^{+}(\mathbf{s})$ for $\psi_C(\mathbf{s})$, $\psi_C^{+}(\mathbf{s})$, and $\phi(\mathbf{s})$, $\phi^{+}(\mathbf{s})$ for $\psi_{NC}(\mathbf{s})$, $\psi_{NC}^{+}(\mathbf{s})$ in the expressions, since both condensate and non-condensate fields will be present and must be distinguished. In this notation the fact that the distribution functionals also depend on the complex conjugate fields $\psi^{*}(\mathbf{s})$, $\psi^{+*}(\mathbf{s})$ and $\phi^{*}(\mathbf{s})$, $\phi^{+*}(\mathbf{s})$ has been ignored. This is because functional derivatives or functions involving these complex conjugate fields are *not* involved in the derivation as a consequence of their absence from the correspondence rules. As in Appendix D

the general notation that applies is

$$\underset{\rightarrow}{\psi}(\mathbf{r}) \equiv \{\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r})\} \tag{E.13}$$

$$\underset{\rightarrow}{\psi^*}(\mathbf{r}) \equiv \{\psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})\} \tag{E.14}$$

$$P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \equiv P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$$
$$\tag{E.15}$$

$$\underset{\rightarrow}{\alpha} \equiv \{\alpha_k, \alpha_k^+\} \tag{E.16}$$

$$\underset{\rightarrow}{\alpha}^* \equiv \{\alpha_k^*, \alpha_k^{+*}\} \tag{E.17}$$

$$P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha}^*) \equiv P_b(\alpha_k, \alpha_k^+, \alpha_k^*, \alpha_k^{+*}) \equiv P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \tag{E.18}$$

At the completion of the determination of the contribution to the Fokker-Planck equation, the original notation
$$P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \equiv P[\psi_C(\mathbf{r}), \psi_C^+(\mathbf{r}), \psi_{NC}(\mathbf{r}), \psi_{NC}^+(\mathbf{r}), \psi_C^*(\mathbf{r}), \psi_C^{+*}(\mathbf{r}), \psi_{NC}^*(\mathbf{r}), \psi_{NC}^{+*}(\mathbf{r})]$$
for the distribution functional will be reintroduced.

Also, to avoid too many nested brackets we will adopt the convention that a functional derivative will operate on *everything* to the right of it unless otherwise indicated. Note that spatial derivatives do not operate on functionals, only on functions.

The terms in the Bogoliubov Hamiltonian that we need to consider are

$$\widehat{H}_1 = \int d\mathbf{r}(\frac{\hbar^2}{2m}\nabla\widehat{\Psi}_C^\dagger(\mathbf{r}) \cdot \nabla\widehat{\Psi}_C(\mathbf{r}) + \widehat{\Psi}_C^\dagger(\mathbf{r})V\widehat{\Psi}_C(\mathbf{r})$$
$$+ \frac{g_N}{2N}\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) \tag{E.19}$$

The term $\widehat{H}_1$ is the sum of the condensate kinetic energy, condensate trap potential energy and condensate boson-boson interaction.

$$\widehat{H}_2 = \int d\mathbf{r}(\widehat{\Psi}_{NC}(\mathbf{r})^\dagger \left\{ -\frac{\hbar^2}{2m}\nabla^2\widehat{\Psi}_C(\mathbf{r}) + V\widehat{\Psi}_C(\mathbf{r}) + \frac{g_N}{N}\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) \right\}$$
$$+ \left\{ -\frac{\hbar^2}{2m}\nabla^2\widehat{\Psi}_C^\dagger(\mathbf{r}) + \widehat{\Psi}_C(\mathbf{r})^\dagger V + \frac{g_N}{N}\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) \right\}\widehat{\Psi}_{NC}(\mathbf{r}))$$
$$\tag{E.20}$$

The term $\widehat{H}_2$ is the coupling between the condensate and non-condensate fields that is linear in the non-condensate field. It can be put into different forms not involving the spatial derivatives. Thus

$$\widehat{H}_2 = \widehat{H}_{2U4} + \widehat{H}_{2U2} \tag{E.21}$$

$$\widehat{H}_{2U4} = \frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}_{NC}^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r})) + \frac{g_N}{N}\int d\mathbf{r}\,(\widehat{\Psi}_C^\dagger(\mathbf{r})\widehat{\Psi}_C^\dagger(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r}))$$
$$\tag{E.22}$$

$$\widehat{H}_{2U2} = -\frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\,\widehat{\Psi}_C(\mathbf{s}) - \frac{g_N}{N}\int\int d\mathbf{r}\,d\mathbf{s}\,F^*(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})$$
$$\tag{E.23}$$

$$\widehat{H}_3 = \int d\mathbf{r} \left\{ \frac{\hbar^2}{2m} \nabla \widehat{\Psi}^\dagger_{NC}(\mathbf{r}) \cdot \nabla \widehat{\Psi}_{NC}(\mathbf{r}) + \widehat{\Psi}^\dagger_{NC}(\mathbf{r}) V \widehat{\Psi}_{NC}(\mathbf{r}) \right\}$$

$$+ \frac{g_N}{2N} \int d\mathbf{r} \left\{ \widehat{\Psi}^\dagger_{NC}(\mathbf{r}) \widehat{\Psi}^\dagger_{NC}(\mathbf{r}) \widehat{\Psi}_C(\mathbf{r}) \widehat{\Psi}_C(\mathbf{r}) + \widehat{\Psi}^\dagger_C(\mathbf{r}) \widehat{\Psi}^\dagger_C(\mathbf{r}) \widehat{\Psi}_{NC} \widehat{\Psi}_{NC} \right\}$$

$$+ \frac{g_N}{2N} \int d\mathbf{r} \left\{ 4 \widehat{\Psi}^\dagger_{NC}(\mathbf{r}) \widehat{\Psi}^\dagger_C(\mathbf{r}) \widehat{\Psi}_{NC}(\mathbf{r}) \widehat{\Psi}_C(\mathbf{r}) \right\} \tag{E.24}$$

Appendix E.1. Condensate Kinetic Energy Terms

We write the kinetic energy as

$$\widehat{T} = \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s}\, \partial_\mu \widehat{\Psi}(\mathbf{s})^\dagger \, \partial_\mu \widehat{\Psi}(\mathbf{s}) \tag{E.25}$$

Now if

$$\widehat{\rho} \to \widehat{T}\widehat{\rho} = \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s}\, (\partial_\mu \widehat{\Psi}(\mathbf{s})^\dagger \, \partial_\mu \widehat{\Psi}(\mathbf{s})) \widehat{\rho} \tag{E.26}$$

then

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \psi^+(\mathbf{s}) - \frac{1}{2}\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\partial_\mu \psi(\mathbf{s}) + \frac{1}{2}\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$
$$\tag{E.27}$$

After expanding we find that

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ (\partial_\mu \psi^+(\mathbf{s})) (\partial_\mu \psi(\mathbf{s})) \right\} W[\psi, \psi^+]$$
$$+ \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ (\partial_\mu \psi^+(\mathbf{s})) \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$
$$- \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\partial_\mu \psi(\mathbf{s})) \right\} W[\psi, \psi^+]$$
$$- \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{4} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$
$$\tag{E.28}$$

Now the standard approach to space integration gives the result

$$\int d\mathbf{s} \left\{ \partial_\mu C(\mathbf{s}) \right\} = 0 \tag{E.29}$$

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for functions $C(\mathbf{s})$ that become zero on the boundary. This then leads to the useful result involving product functions $C(\mathbf{s}) = A(\mathbf{s})B(\mathbf{s})$ enabling the spatial derivative to be applied to either $A(\mathbf{s})$ or $B(\mathbf{s})$

$$\int d\mathbf{s}\,\{\partial_\mu A(\mathbf{s})\,\}B(\mathbf{s}) = -\int d\mathbf{s}\,A(\mathbf{s})\,\{\partial_\mu B(\mathbf{s})\,\} \qquad \text{(E.30)}$$

We can assume that the $\psi(\mathbf{s})$ and $\psi^+(\mathbf{s})$ become zero on the boundary, since they both involve condensate mode functions or their conjugates that are localised due to the trap potential. Also the functional derivatives produce linear combinations of either the condensate mode functions or their conjugates (see (B.87), (B.92)) so the various $C(\mathbf{s})$ that will be involved should become zero on the boundary.

For the first term, the product of the spatial functions can be written in opposite order so that

$$\int d\mathbf{s}\,\{\left(\partial_\mu \psi^+(\mathbf{s})\right)\left(\partial_\mu \psi(\mathbf{s})\right)\}\,W[\psi,\psi^+]$$

$$= \int d\mathbf{s}\,\{\left(\partial_\mu \psi(\mathbf{s})\right)\left(\partial_\mu \psi^+(\mathbf{s})\right)\}\,W[\psi,\psi^+] \qquad \text{(E.31)}$$

We can then use (E.30) together with the explicit forms (E.315) for the functional derivatives and their spatial derivatives to modify the terms in the new $W[\psi,\psi^+]$, which is equivalent to the function $w(\alpha_k,\alpha_k^+)$ if $\psi(\mathbf{s})$ and $\psi^+(\mathbf{s})$ are expanded in terms of modes $\phi_k(\mathbf{s})$ or $\phi_k^*(\mathbf{s})$, as in (E.316) and (E.317) with expansion coefficients α_k and α_k^*.

In the second term, the spatial derivative of the functional derivative can be removed and applied to the spatial function

$$\int d\mathbf{s}\,\left\{\left(\partial_\mu \psi^+(\mathbf{s})\right)\left(\partial_\mu \frac{\delta}{\delta \psi^+(\mathbf{s})}\right)\right\}\,W[\psi,\psi^+]$$

$$= \int d\mathbf{s}\,\sum_{k=1,2}\{\partial_\mu \phi_k^*(\mathbf{s})\}\alpha_k^+ \sum_{l=1,2}\{\partial_\mu \phi_l(\mathbf{s})\}\frac{\partial}{\partial \alpha_l^+}w(\alpha_k,\alpha_k^+)$$

$$= -\int d\mathbf{s}\,\sum_{k=1,2}\{\partial_\mu^2 \phi_k^*(\mathbf{s})\}\alpha_k^+ \sum_{l=1,2}\{\phi_l(\mathbf{s})\}\frac{\partial}{\partial \alpha_l^+}w(\alpha_k,\alpha_k^+)$$

$$= -\int d\mathbf{s}\,\left\{\left(\partial_\mu^2 \psi^+(\mathbf{s})\right)\left(\frac{\delta}{\delta \psi^+(\mathbf{s})}\right)\right\}\,W[\psi,\psi^+]$$

$$\text{(E.32)}$$

Applying the product rule (E.319) to the product of $\left(\partial_\mu^2 \psi^+(\mathbf{s})\right)$ with the distri-

bution functional gives

$$
\left(\partial_\mu^2 \psi^+(\mathbf{s})\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\right)
$$

$$
= \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\left(\partial_\mu^2\psi^+(\mathbf{s})\right)W[\psi,\psi^+]\right) - \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\left(\partial_\mu^2\psi^+(\mathbf{s})\right)\right)W[\psi,\psi^+]
$$

$$
= \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\left(\partial_\mu^2\psi^+(\mathbf{s})\right)W[\psi,\psi^+]\right) - \left(\chi(\mathbf{s})\right)W[\psi,\psi^+]
$$

$$\tag{E.33}$$

using

$$
\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\left(\partial_\mu^2\psi^+(\mathbf{s})\right)\right) = \sum_{l=1,2}\{\phi_l(\mathbf{s})\}\frac{\partial}{\partial\alpha_l^+}\sum_{k=1,2}\{\partial_\mu^2\phi_k^*(\mathbf{s})\}\alpha_k^+
$$

$$
= \sum_{k=1,2}\{\phi_k(\mathbf{s})\}\{\partial_\mu^2\phi_k^*(\mathbf{s})\}
$$

$$
\equiv \omega_C(\mathbf{s}) \tag{E.34}
$$

Note that the function $\omega_C(\mathbf{s})$ just defined only depends on condensate mode functions. Thus the second term becomes

$$
\int d\mathbf{s}\left\{\left(\partial_\mu\psi^+(\mathbf{s})\right)\left(\partial_\mu\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\right\}W[\psi,\psi^+]
$$

$$
= -\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\left(\partial_\mu^2\psi^+(\mathbf{s})\right)\right\}W[\psi,\psi^+] + \int d\mathbf{s}\left\{\omega_C(\mathbf{s})\right\}W[\psi,\psi^+]
$$

$$\tag{E.35}$$

In the third term

$$
\int d\mathbf{s}\left\{\left(\partial_\mu\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\partial_\mu\psi(\mathbf{s})\right)\right\}W[\psi,\psi^+]
$$

$$
= \int d\mathbf{s}\sum_{k=1,2}\{\partial_\mu\phi_k^*(\mathbf{s})\}\frac{\partial}{\partial\alpha_k}\sum_{l=1,2}\alpha_l\{\partial_\mu\phi_l(\mathbf{s})\}w(\alpha_k,\alpha_k^+)
$$

$$
= -\int d\mathbf{s}\sum_{k=1,2}\{\phi_k^*(\mathbf{s})\}\frac{\partial}{\partial\alpha_k}\sum_{l=1,2}\alpha_l\{\partial_\mu^2\phi_l(\mathbf{s})\}w(\alpha_k,\alpha_k^+)
$$

$$
= -\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\partial_\mu^2\psi(\mathbf{s})\right)\right\}W[\psi,\psi^+] \tag{E.36}
$$

For the fourth term, the double functional derivative term can be written in

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the opposite order

$$\int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$

$$= \int d\mathbf{s} \sum_{k=1,2} \{\partial_\mu \phi_k^*(\mathbf{s})\} \frac{\partial}{\partial\alpha_k} \sum_{l=1,2} \{\partial_\mu \phi_l(\mathbf{s})\} \frac{\partial}{\partial\alpha_l^+} w(\alpha_k, \alpha_k^+)$$

$$= \int d\mathbf{s} \sum_{l=1,2} \{\partial_\mu \phi_l(\mathbf{s})\} \frac{\partial}{\partial\alpha_l^+} \sum_{k=1,2} \{\partial_\mu \phi_k^*(\mathbf{s})\} \frac{\partial}{\partial\alpha_k} w(\alpha_k, \alpha_k^+)$$

$$= \int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} W[\psi, \psi^+] \qquad \text{(E.37)}$$

Using results (E.31), (E.35), (E.36) and (E.37) we find that

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ (\partial_\mu \psi(\mathbf{s})) (\partial_\mu \psi^+(\mathbf{s})) \right\} W[\psi, \psi^+]$$

$$- \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \left(\partial_\mu^2 \psi^+(\mathbf{s}) \right) \right\} W[\psi, \psi^+]$$

$$+ \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \omega_C(\mathbf{s}) \right\} W[\psi, \psi^+]$$

$$+ \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\partial_\mu^2 \psi(\mathbf{s}) \right) \right\} W[\psi, \psi^+]$$

$$- \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{4} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} W[\psi, \psi^+] \quad \text{(E.38)}$$

Now if

$$\widehat{\rho} \rightarrow \widehat{\rho T} = \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s}\, \widehat{\rho}(\partial_\mu \widehat{\Psi}(\mathbf{s})^\dagger\, \partial_\mu \widehat{\Psi}(\mathbf{s})) \qquad \text{(E.39)}$$

then

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \psi(\mathbf{s}) - \frac{1}{2}\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\partial_\mu \psi^+(\mathbf{s}) + \frac{1}{2}\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$

$$\text{(E.40)}$$

After expanding we find that

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ (\partial_\mu \psi(\mathbf{s})) \left(\partial_\mu \psi^+(\mathbf{s}) \right) \right\} W[\psi, \psi^+]$$

$$+ \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ (\partial_\mu \psi(\mathbf{s})) \left(\partial_\mu \frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$

$$- \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \left(\partial_\mu \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\partial_\mu \psi^+(\mathbf{s})) \right\} W[\psi, \psi^+]$$

$$- \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{4} \left\{ \left(\partial_\mu \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\partial_\mu \frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$

$$\tag{E.41}$$

Applying the same approach as above we find that the second term becomes

$$\int d\mathbf{s} \left\{ (\partial_\mu \psi(\mathbf{s})) \left(\partial_\mu \frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$

$$= \quad - \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi(\mathbf{s})} (\partial_\mu^2 \psi(\mathbf{s})) \right\} W[\psi, \psi^+] + \int d\mathbf{s} \left\{ \omega_C(\mathbf{s})^* \right\} W[\psi, \psi^+]$$

$$\tag{E.42}$$

and the third term is given by

$$\int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\partial_\mu \psi^+(\mathbf{s})) \right\} W[\psi, \psi^+]$$

$$= \quad - \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\partial_\mu^2 \psi^+(\mathbf{s})) \right\} W[\psi, \psi^+] \tag{E.43}$$

Using results (E.42) and (E.43) and using the result obtained from integration by parts

$$\int d\mathbf{s} \left\{ \omega_C(\mathbf{s})^* \right\} = \int d\mathbf{s} \left\{ \omega_C(\mathbf{s}) \right\} \tag{E.44}$$

we find that

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ (\partial_\mu \psi(\mathbf{s})) \left(\partial_\mu \psi^+(\mathbf{s}) \right) \right\} W[\psi, \psi^+]$$

$$-\frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \left(\partial_\mu^2 \psi(\mathbf{s}) \right) \right\} W[\psi, \psi^+]$$

$$+\frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \omega_C(\mathbf{s}) \right\} W[\psi, \psi^+]$$

$$+\frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\partial_\mu^2 \psi^+(\mathbf{s}) \right) \right\} W[\psi, \psi^+]$$

$$-\frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \frac{1}{4} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} W[\psi, \psi^+]$$

$$\tag{E.45}$$

We now combine the contributions so that when

$$\widehat{\rho} \rightarrow [\widehat{T}, \widehat{\rho}] = [\frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} (\partial_\mu \widehat{\Psi}(\mathbf{s})^\dagger \, \partial_\mu \widehat{\Psi}(\mathbf{s})), \widehat{\rho}] \tag{E.46}$$

then

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \rightarrow W^1 \tag{E.47}$$

where

$$W^1 = -\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi^+(\mathbf{s}) \right) W[\psi, \psi^+] \right\}$$

$$+ \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi(\mathbf{s}) \right) W[\psi, \psi^+] \right\}$$

$$\tag{E.48}$$

where the $(\omega_C(\mathbf{s}))$, the $(\partial_\mu \psi(\mathbf{s})) \left(\partial_\mu \psi^+(\mathbf{s}) \right)$ and the $\left(\partial_\mu \frac{\delta}{\delta\psi^+(s)} \right) \left(\partial_\mu \frac{\delta}{\delta\psi(s)} \right)$ terms cancel and the first order functional derivative terms combine to remove the $\frac{1}{2}$ factors. Thus only a first order functional derivative term occurs.

Overall, the contribution to the functional Fokker-Planck equation from the

kinetic energy term is given by

$$
\left(\frac{\partial}{\partial t} W[\psi, \psi^+] \right)_K
$$

$$
= \frac{-i}{\hbar} \left\{ - \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi^+(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi^+(\mathbf{s}) \right) W[\psi, \psi^+] \right\} \right\}
$$

$$
+ \frac{-i}{\hbar} \left\{ + \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi(\mathbf{s}) \right) W[\psi, \psi^+] \right) \right\} \quad \text{(E.49)}
$$

Reverting to the original notation, the contribution to the functional Fokker-Planck equation from the kinetic energy term is given by

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_K
$$

$$
= \frac{+i}{\hbar} \left\{ \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_C^+(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi_C^+(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \right\}
$$

$$
- \frac{i}{\hbar} \left\{ \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_C(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi_C(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right) \right\}
$$

$$
\text{(E.50)}
$$

Appendix E.2. Condensate Trap Potential Terms

We write the trap potential as

$$
\widehat{V} = \int d\mathbf{s} (\widehat{\Psi}(\mathbf{s})^\dagger V \widehat{\Psi}(\mathbf{s})) \quad \text{(E.51)}
$$

Now if

$$
\widehat{\rho} \to \widehat{V}\widehat{\rho} = \int d\mathbf{s} (\widehat{\Psi}(\mathbf{s})^\dagger V \widehat{\Psi}(\mathbf{s})) \widehat{\rho} \quad \text{(E.52)}
$$

then

$$
W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]
$$

$$
\to \int d\mathbf{s} \left\{ \left(\psi^+(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) V(\mathbf{s}) \left(\psi(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} W[\psi, \psi^+]
$$

$$
\text{(E.53)}
$$

After expanding we find that

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\rightarrow \int d\mathbf{s}\left\{\psi^+(\mathbf{s})V(\mathbf{s})\psi(\mathbf{s})\right\}W[\psi,\psi^+]$$
$$+ \int d\mathbf{s}\frac{1}{2}\left\{\psi^+(\mathbf{s})V(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W[\psi,\psi^+]$$
$$- \int d\mathbf{s}\frac{1}{2}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}V(\mathbf{s})\psi(\mathbf{s})\right\}W[\psi,\psi^+]$$
$$- \int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}V(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W[\psi,\psi^+]$$

$$(\text{E.54})$$

We can now use the product rule for functional derivatives (E.319) together with (E.320) and (E.321) to place all the derivatives on the left of the expression and obtain

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\rightarrow \int d\mathbf{s}\left\{\psi^+(\mathbf{s})V(\mathbf{s})\psi(\mathbf{s})\right\}W[\psi,\psi^+] \quad T1$$
$$+ \int d\mathbf{s}\frac{1}{2}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})V(\mathbf{s})\} - \delta_C(\mathbf{s},\mathbf{s})V(\mathbf{s})\right\}W[\psi,\psi^+] \quad T22, T21$$
$$- \int d\mathbf{s}\frac{1}{2}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}V(\mathbf{s})\psi(\mathbf{s})\right\}W[\psi,\psi^+] \quad T3$$
$$- \int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}V(\mathbf{s})\right\}W[\psi,\psi^+] \quad T4$$

$$(\text{E.55})$$

Details are
T2

$$\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})V(\mathbf{s})W\}$$
$$= \{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}V(\mathbf{s})W + \psi^+(\mathbf{s})V(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$
$$= \delta_K(0)V(\mathbf{s})W + \psi^+(\mathbf{s})V(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$
$$\psi^+(\mathbf{s})V(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}$$
$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})V(\mathbf{s})W[\psi,\psi^+]\} - \delta_C(\mathbf{s},\mathbf{s})V(\mathbf{s})W[\psi,\psi^+]$$

Now if

$$\widehat{\rho} \rightarrow \widehat{\rho V} = \int d\mathbf{s}\widehat{\rho}\widehat{\Psi}(\mathbf{s})^\dagger V\widehat{\Psi}(\mathbf{s}) \qquad (\text{E.56})$$

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then

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\rightarrow \quad \int d\mathbf{s} \left(\psi(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) V(\mathbf{s}) \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) W[\psi, \psi^+]$$

$$(E.57)$$

After expanding we have

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\rightarrow \quad \int d\mathbf{s} \{ \psi(\mathbf{s}) V(\mathbf{s}) \psi^+(\mathbf{s}) \} W[\psi, \psi^+]$$
$$+ \int d\mathbf{s} \frac{1}{2} \left\{ \psi(\mathbf{s}) V(\mathbf{s}) \frac{\delta}{\delta\psi(\mathbf{s})} \right\} W[\psi, \psi^+]$$
$$- \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s}) \psi^+(\mathbf{s}) \right\} W[\psi, \psi^+]$$
$$- \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s}) \frac{\delta}{\delta\psi(\mathbf{s})} \right\} W[\psi, \psi^+]$$

$$(E.58)$$

We can now use the product rule for functional derivatives (E.319) together with (E.320) and (E.321) to place all the derivatives on the left of the expression. However the results can more easily be obtained by noticing that the $\widehat{\rho}\widehat{V}$ is the same as the $\widehat{V}\widehat{\rho}$ if we interchange $\psi(\mathbf{s})$ and $\psi^+(\mathbf{s})$ everywhere. Hence

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\rightarrow \quad \int d\mathbf{s} \left\{ \psi(\mathbf{s}) V(\mathbf{s}) \psi^+(\mathbf{s}) \right\} W[\psi, \psi^+] \qquad T1$$
$$+ \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \{ \psi(\mathbf{s}) V(\mathbf{s}) \} - \delta_C(\mathbf{s}, \mathbf{s}) V(\mathbf{s}) \right\} W[\psi, \psi^+] \qquad T22, T21$$
$$- \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s}) \psi^+(\mathbf{s}) \right\} W[\psi, \psi^+] \qquad T3$$
$$- \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} V(\mathbf{s}) \right\} W[\psi, \psi^+] \qquad T4$$

$$(E.59)$$

We now combine the contributions so that when

$$\widehat{\rho} \rightarrow [\widehat{V}, \widehat{\rho}] = [\int d\mathbf{s} \ \widehat{\Psi}(\mathbf{s})^\dagger V(\mathbf{s}) \widehat{\Psi}(\mathbf{s}), \widehat{\rho}] \qquad (E.60)$$

then

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \rightarrow W^0 + W^1 + W^2 \qquad (E.61)$$

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where the terms are listed via the order of derivatives that occur

$$W^0$$

$$= \int d\mathbf{s} \left\{ \psi^+(\mathbf{s}) V(\mathbf{s}) \psi(\mathbf{s}) \right\} W[\psi, \psi^+] - \int d\mathbf{s} \left\{ \psi(\mathbf{s}) V(\mathbf{s}) \psi^+(\mathbf{s}) \right\} W[\psi, \psi^+]$$

$$+ \int d\mathbf{s} \frac{1}{2} \left\{ -\delta_C(\mathbf{s}, \mathbf{s}) V(\mathbf{s}) \right\} W[\psi, \psi^+] - \int d\mathbf{s} \frac{1}{2} \left\{ -\delta_C(\mathbf{s}, \mathbf{s}) V(\mathbf{s}) \right\} W[\psi, \psi^+]$$

$$= 0$$

$$\text{(E.62)}$$

$$W^1$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \{\psi^+(\mathbf{s}) V(\mathbf{s})\} \right\} W[\psi, \psi^+] - \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \{\psi(\mathbf{s}) V(\mathbf{s})\} \right\} W[\psi, \psi^+]$$

$$- \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} V(\mathbf{s}) \psi(\mathbf{s}) \right\} W[\psi, \psi^+] + \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s}) \psi^+(\mathbf{s}) \right\} W[\psi, \psi^+]$$

$$= - \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \{V(\mathbf{s}) \psi(\mathbf{s})\} \right\} W[\psi, \psi^+] + \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s}) \psi^+(\mathbf{s}) \right\} W[\psi, \psi^+]$$

$$\text{(E.63)}$$

$$W^2$$

$$= - \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s}) \right\} W[\psi, \psi^+] + \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} V(\mathbf{s}) \right\} W[\psi, \psi^+]$$

$$= 0$$

$$\text{(E.64)}$$

Overall, the contribution to the functional Fokker-Planck equation from the trap potential term is given by

$$\left(\frac{\partial}{\partial t} W[\psi, \psi^+] \right)_V$$

$$= \frac{-i}{\hbar} \left\{ - \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \{V(\mathbf{s}) \psi(\mathbf{s})\} \right\} W[\psi, \psi^+] + \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s}) \psi^+(\mathbf{s}) \right\} W[\psi, \psi^+] \right\}$$

$$\text{(E.65)}$$

which only involves first order functional derivatives.

Reverting to the original notation, the contribution to the functional Fokker-Planck equation from the trap potential term is given by

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_V$$

$$= \frac{-i}{\hbar} \left\{ - \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi_C(\mathbf{s})} \{V(\mathbf{s}) \psi_C(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$+ \frac{-i}{\hbar} \left\{ + \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi_C^+(\mathbf{s})} V(\mathbf{s}) \psi_C^+(\mathbf{s}) \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\text{(E.66)}$$

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Appendix E.3. Condensate Boson-Boson Interaction Terms

We write the boson-boson interaction potential as

$$\widehat{U} = \frac{g}{2} \int d\mathbf{s}\, \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s}) \widehat{\Psi}(\mathbf{s}) \tag{E.67}$$

Now if

$$\widehat{\rho} \to \widehat{U}\widehat{\rho} = \frac{g}{2} \int d\mathbf{s}\, \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s}) \widehat{\Psi}(\mathbf{s}) \widehat{\rho} \tag{E.68}$$

$$
\begin{aligned}
&W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
&\to \frac{g}{2} \int d\mathbf{s} \left(\psi^+(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\psi(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) W
\end{aligned}
\tag{E.69}
$$

After expanding we get

$$
\begin{aligned}
&W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
&\to \frac{g}{2}\int d\mathbf{s}\left\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{2}\left\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&+\frac{g}{2}\int d\mathbf{s}\frac{1}{2}\left\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&-\frac{g}{2}\int d\mathbf{s}\frac{1}{2}\left\{\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&-\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&-\frac{g}{2}\int d\mathbf{s}\frac{1}{2}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&-\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&+\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&+\frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{16}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W
\end{aligned}
\tag{E.70}
$$

We can now use the product rule for functional derivatives (E.319) together with (E.320) and (E.321) to place all the derivatives on the left of the expression

and obtain

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T1$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) - 2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T22, T21$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) - 2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T32, T31$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})}4\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s}) + 2\delta_K(0)^2 \right\} W \qquad T43, T42, T41$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T5$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T62, T61$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T72, T71$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s}) \right\} W \qquad T82, T81$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T9$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T10.2, T10.1$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T11.2, T11.1$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s}) \right\} W \qquad T12.2, T12.1$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s}) \right\} W \qquad T13$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s}) \right\} W \qquad T14$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s}) \right\} W \qquad T15$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{16} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})} \right\} W \qquad T16$$

$$(E.71)$$

Details include

T2

$$\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi^+(\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi(\mathbf{s})W$$

$$+\psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\}W + \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$= \{\delta_C(\mathbf{s},\mathbf{s})\}\psi^+(\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\{\delta_C(\mathbf{s},\mathbf{s})\}\psi(\mathbf{s})W$$

$$+\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$= 2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\} - 2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]$$

T3

$$\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi^+(\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi(\mathbf{s})W\}$$

$$+\psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}$$

$$= \delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W\} + \psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}$$

$$\psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\} - 2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]$$

T4

$$\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\left(\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi^+(\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}W\right)$$

$$+\frac{\delta}{\delta\psi^+(\mathbf{s})}\left(\psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right)$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\left(2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})W + \psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right)$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\{2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})W\} +$$

$$+\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$+\psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\{2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})W\} +$$

$$+2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\} + \psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\{2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})W + \psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$+\frac{\delta}{\delta\psi^+(\mathbf{s})}\{2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})W\} - 2\delta_C(\mathbf{s},\mathbf{s})^2W$$

$$\psi^+(\mathbf{s})\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}$$

$$= \frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})W[\psi,\psi^+]\} - \frac{\delta}{\delta\psi^+(\mathbf{s})}\{2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})W[\psi,\psi^+]\}$$

$$-\frac{\delta}{\delta\psi^+(\mathbf{s})}\{2\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})W[\psi,\psi^+]\} + 2\delta_C(\mathbf{s},\mathbf{s})^2W[\psi,\psi^+]$$

T5

$$\frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\}\psi(\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})W\}$$

$$\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\}$$

T6

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi(\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\}W\right\}$$

$$+ \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\psi^+(\mathbf{s})\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right\}$$

$$= \{\frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W\} + \{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\}\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$+ \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}$$

$$= \{\frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}$$

$$\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\} - \{\frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\}$$

T7

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi(\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}\right\}$$

$$= \{\frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W\} + \{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}$$

$$+\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W\}\}$$

$$\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})W[\psi,\psi^+]\}\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\} - \frac{\delta}{\delta\psi(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\}$$

T8

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})W + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})W\} + \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})W\}$$
$$+ \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})W\}$$
$$+ \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})W\} + \frac{\delta}{\delta\psi(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$
$$+ \{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}$$
$$+ \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})W\} + \frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})W\}$$
$$+ \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\}$$
$$\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}\}\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})W[\psi,\psi^+]\} - \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})W[\psi,\psi^+]\}$$
$$- \frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})W[\psi,\psi^+]\}$$

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T10

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi(\mathbf{s})W + \psi^+(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}W\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}W\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W\} + \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}$$

$$\frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\} - \frac{\delta}{\delta\psi(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\}$$

T11

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\psi(\mathbf{s})W + \psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W + \psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W\} + \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})W\}\}$$

$$\frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})W[\psi,\psi^+]\}\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\} - \frac{\delta}{\delta\psi(\mathbf{s})}\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})W[\psi,\psi^+]\}$$

T12

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})W\}$$

$$+ \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\delta_C(\mathbf{s},\mathbf{s})W\}$$

$$+ \frac{\delta}{\delta\psi(\mathbf{s})}\left\{\delta_C(\mathbf{s},\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\} + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s})W\} + \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W\}\}\}$$

$$\frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}\}\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})W[\psi,\psi^+]\} - \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s})W[\psi,\psi^+]\}$$

T14

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})W\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})W\}\right\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\left\{\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}W\right\}$$

$$\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\{\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}W[\psi,\psi^+]\}$$

$$= \frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})W[\psi,\psi^+]\}$$

Now if

$$\widehat{\rho} \rightarrow \widehat{\rho}\widehat{U} = \frac{g}{2}\int d\mathbf{s}\widehat{\rho}\widehat{\Psi}(\mathbf{s})^\dagger\widehat{\Psi}(\mathbf{s})^\dagger\widehat{\Psi}(\mathbf{s})\widehat{\Psi}(\mathbf{s}) \qquad (E.72)$$

then

$$W[\psi(\mathbf{r}),\psi^+(\mathbf{r})]$$

$$\rightarrow \frac{g}{2}\int d\mathbf{s}\left(\psi(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\psi(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\psi^+(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\psi^+(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)W$$

$$(E.73)$$

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After expanding we obtain

$$
W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]
$$

$$
\rightarrow \quad \frac{g}{2}\int ds\left\{\psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}W[\psi,\psi^+] + \frac{g}{2}\int ds\frac{1}{2}\left\{\psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$
+\frac{g}{2}\int ds\frac{1}{2}\left\{\psi(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\right\}W + \frac{g}{2}\int ds\frac{1}{4}\left\{\psi(\mathbf{s})\psi(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$
-\frac{g}{2}\int ds\frac{1}{2}\left\{\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}W - \frac{g}{2}\int ds\frac{1}{4}\left\{\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$
-\frac{g}{2}\int ds\frac{1}{4}\left\{\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\right\}W - \frac{g}{2}\int ds\frac{1}{8}\left\{\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$
-\frac{g}{2}\int ds\frac{1}{2}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}W - \frac{g}{2}\int ds\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$
-\frac{g}{2}\int ds\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s}_1)\right\}W - \frac{g}{2}\int ds\frac{1}{8}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$
+\frac{g}{2}\int ds\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}W + \frac{g}{2}\int ds\frac{1}{8}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$
+\frac{g}{2}\int ds\frac{1}{8}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\right\}W + \frac{g}{2}\int ds\frac{1}{16}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W
$$

$$\text{(E.74)}$$

We can now use the product rule for functional derivatives (E.319) together with (E.320) and (E.321) to place all the derivatives on the left of the expression. However the results can more easily be obtained by noticing that the $\widehat{\rho}\widehat{U}$ is the same as the $\widehat{U}\widehat{\rho}$ if we interchange $\psi(\mathbf{s})$ and $\psi^+(\mathbf{s})$ everywhere. Hence

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T1$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) - 2\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T22, T21$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) - 2\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T32, T31$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})} 4\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s}) + 2\delta_C(\mathbf{s},\mathbf{s})^2 \right\} W \qquad T43, T42, T41$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T5$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T62, T61$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T72$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} 2\delta_C(\mathbf{s},\mathbf{s}) \right\} W \qquad T82, T81$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T9$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T10.2, T10.1$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T11.2, T11.1$$

$$- \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} 2\delta_C(\mathbf{s},\mathbf{s}) \right\} W \qquad T12.2, T12.1$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{4} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})} \psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \qquad T13$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi^+(\mathbf{s}) \right\} W \qquad T14$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{8} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi^+(\mathbf{s}) \right\} W \qquad T15$$

$$+ \frac{g}{2} \int d\mathbf{s} \frac{1}{16} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \right\} W \qquad T16$$

$$\text{(E.75)}$$

We now combine the contributions so that when

$$\widehat{\rho} \rightarrow [\widehat{U}, \widehat{\rho}] = [\frac{g}{2} \int d\mathbf{s} \ \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s}) \widehat{\Psi}(\mathbf{s}), \widehat{\rho}] \qquad \text{(E.76)}$$

then

$$W[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \rightarrow W^0 + W^1 + W^2 + W^3 + W^4$$

(E.77)

where the terms are listed via the order of derivatives that occur

W^0

$$
\begin{aligned}
&= \frac{g}{2} \int ds \left\{ \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} W - \frac{g}{2} \int ds \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&+ \frac{g}{2} \int ds \frac{1}{2} \left\{ -2\delta_C(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} W - \frac{g}{2} \int ds \frac{1}{2} \left\{ -2\delta_C(\mathbf{s}, \mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&+ \frac{g}{2} \int ds \frac{1}{2} \left\{ -2\delta_C(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} W - \frac{g}{2} \int ds \frac{1}{2} \left\{ -2\delta_C(\mathbf{s}, \mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&+ \frac{g}{2} \int ds \frac{1}{4} \left\{ +2\delta_C(\mathbf{s}, \mathbf{s})^2 \right\} W - \frac{g}{2} \int ds \frac{1}{4} \left\{ +2\delta_C(\mathbf{s}, \mathbf{s})^2 \right\} W \\
&= 0
\end{aligned}
$$

(E.78)

W^1

$$
\begin{aligned}
&= +\frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} W - \frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&+ \frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} W - \frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&+ \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi^+(\mathbf{s})} 4\delta_C(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s}) \right\} W - \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi(\mathbf{s})} 4\delta_C(\mathbf{s}, \mathbf{s})\psi(\mathbf{s}) \right\} W \\
&- \frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} W + \frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&- \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi(\mathbf{s}) \right\} W + \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&- \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi(\mathbf{s}) \right\} W + \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&- \frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} W + \frac{g}{2} \int ds \frac{1}{2} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&- \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi(\mathbf{s}) \right\} W + \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&- \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi(\mathbf{s}) \right\} W + \frac{g}{2} \int ds \frac{1}{4} \left\{ -\frac{\delta}{\delta\psi^+(\mathbf{s})} \delta_C(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s}) \right\} W \\
&= -g \int ds \frac{\delta}{\delta\psi(\mathbf{s})} \left\{ (\psi^+(\mathbf{s})\psi(\mathbf{s}) - \delta_C(\mathbf{s}, \mathbf{s}))\psi(\mathbf{s}) \right\} W[\psi, \psi^+] \\
&+ g \int ds \frac{\delta}{\delta\psi^+(\mathbf{s})} \left\{ (\psi^+(\mathbf{s})\psi(\mathbf{s}) - \delta_C(\mathbf{s}, \mathbf{s}))\psi^+(\mathbf{s}) \right\} W[\psi, \psi^+]
\end{aligned}
$$

(E.79)

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$$W^2$$

$$
\begin{aligned}
= & \ \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})\right\}W \\
& -\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\right\}W \\
& -\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\right\}W \\
& -\frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{-\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{-\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s})\right\}W \\
& -\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\right\}W \\
& -\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\right\}W \\
& -\frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{-\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{-\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}2\delta_C(\mathbf{s},\mathbf{s})\right\}W \\
& +\frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{4}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}W \\
= & \ 0
\end{aligned}
$$

$$(\text{E.80})$$

$$W^3$$

$$
\begin{aligned}
= & \ -\frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\right\}W \\
& -\frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\right\}W + \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\right\}W \\
& +\frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\right\}W \\
& +\frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\right\}W - \frac{g}{2}\int d\mathbf{s}\frac{1}{8}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\right\}W \\
= & \ g\int d\mathbf{s}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\frac{1}{4}\psi(\mathbf{s})\}W[\psi,\psi^+] \\
& -g\int d\mathbf{s}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\{\frac{1}{4}\psi^+(\mathbf{s})\}W[\psi,\psi^+]
\end{aligned}
$$

$$(\text{E.81})$$

$$W^4$$

$$
\begin{aligned}
&= +\frac{g}{2}\int d\mathbf{s}\frac{1}{16}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right\}W \\
&\quad -\frac{g}{2}\int d\mathbf{s}\frac{1}{16}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\right\}W \\
&= 0
\end{aligned}
$$

$$\text{(E.82)}$$

Overall, the contribution to the functional Fokker-Planck equation from the boson-boson interaction is given by

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}W[\psi,\psi^+]\right)_U \\
&= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi(\mathbf{s})}\left\{(\psi^+(\mathbf{s})\psi(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s}))\psi(\mathbf{s})\right\}W[\psi,\psi^+]\right\} \\
&\quad \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\frac{\delta}{\delta\psi^+(\mathbf{s})}\left\{(\psi^+(\mathbf{s})\psi(\mathbf{s})--\delta_C(\mathbf{s},\mathbf{s}))\psi^+(\mathbf{s})\right\}W[\psi,\psi^+]\right\} \\
&\quad \frac{-i}{\hbar}\left\{g\int d\mathbf{s}\frac{1}{4}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi(\mathbf{s})\}W[\psi,\psi^+]\right\} \\
&\quad \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{1}{4}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\}W[\psi,\psi^+]\right\}
\end{aligned}
$$

$$\text{(E.83)}$$

which involves first order and third order functional derivatives. We have replaced $\delta_K(0)$ by its full form $\delta_K(\mathbf{s},\mathbf{s})$.

Reverting to the original notation we have

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_U \\
&= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi_C(\mathbf{s})}\left\{(\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s}))\psi_C(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
&\quad +\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\left\{(\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s}))\psi_C^+(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
&\quad +\frac{-i}{\hbar}\left\{g\int d\mathbf{s}\frac{\delta}{\delta\psi_C(\mathbf{s})}\frac{\delta}{\delta\psi_C(\mathbf{s})}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\{\frac{1}{4}\psi_C(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
&\quad +\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\frac{\delta}{\delta\psi_C(\mathbf{s})}\{\frac{1}{4}\psi_C^+(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
\end{aligned}
$$

$$\text{(E.84)}$$

Appendix E.4. Non-Condensate Kinetic Energy Terms

We write the kinetic energy as

$$\widehat{T}=\frac{\hbar^2}{2m}\sum_\mu\int d\mathbf{s}\,\partial_\mu\widehat{\Psi}(\mathbf{s})^\dagger\,\partial_\mu\widehat{\Psi}(\mathbf{s}) \tag{E.85}$$

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Now if

$$\widehat{\rho} \to \widehat{T}\widehat{\rho} = \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} (\partial_\mu \widehat{\Psi}(\mathbf{s})^\dagger \, \partial_\mu \widehat{\Psi}(\mathbf{s}))\widehat{\rho} \qquad (E.86)$$

then

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \quad \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \psi^+(\mathbf{s}) - \partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\partial_\mu \psi(\mathbf{s})) \right\} P[\psi, \psi^+]$$

$$\qquad (E.87)$$

After expanding we find that

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \quad \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ (\partial_\mu \psi^+(\mathbf{s})) (\partial_\mu \psi(\mathbf{s})) \right\} P[\psi, \psi^+]$$
$$\quad - \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\partial_\mu \psi(\mathbf{s})) \right\} P[\psi, \psi^+]$$

$$\qquad (E.88)$$

For the first term, the product of the spatial functions can be written in opposite order so that

$$\int d\mathbf{s} \left\{ (\partial_\mu \psi^+(\mathbf{s})) (\partial_\mu \psi(\mathbf{s})) \right\} P[\psi, \psi^+]$$
$$= \quad \int d\mathbf{s} \left\{ (\partial_\mu \psi(\mathbf{s})) (\partial_\mu \psi^+(\mathbf{s})) \right\} P[\psi, \psi^+] \qquad (E.89)$$

We can use (E.30) together with the explicit forms (E.315) and for the functional derivatives to modify the terms in the new $P[\psi, \psi^+]$, which is equivalent to the function $p(\alpha_k, \alpha_k^+)$ if $\psi(\mathbf{s})$ and $\psi^+(\mathbf{s})$ are expanded in terms of modes $\phi_k(\mathbf{s})$ or $\phi_k^*(\mathbf{s})$, as in (E.316) and (E.317) with expansion coefficients α_k and α_k^*.

In the second term we use (E.30) to apply the spatial derivative to the $\psi(\mathbf{s})$ factor

$$\int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\partial_\mu \psi(\mathbf{s})) \right\} P[\psi, \psi^+]$$
$$= \quad \int d\mathbf{s} \sum_{k \neq 1,2}^K \{\partial_\mu \phi_k^*(\mathbf{s})\} \frac{\partial}{\partial \alpha_k} \sum_{l \neq 1,2}^K \alpha_l \{\partial_\mu \phi_l(\mathbf{s})\} p(\alpha_k, \alpha_k^+)$$
$$= \quad -\int d\mathbf{s} \sum_{k \neq 1,2}^K \{\phi_k^*(\mathbf{s})\} \frac{\partial}{\partial \alpha_k} \sum_{l \neq 1,2}^K \alpha_l \{\partial_\mu^2 \phi_l(\mathbf{s})\} p(\alpha_k, \alpha_k^+)$$
$$= \quad -\int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\partial_\mu^2 \psi(\mathbf{s})) \right\} P[\psi, \psi^+] \qquad (E.90)$$

Using results (E.89) and (E.90) we find that

$$
\begin{aligned}
&P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
\rightarrow\quad & \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \psi(\mathbf{s})\right) \left(\partial_\mu \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+] \\
& + \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\partial_\mu^2 \psi(\mathbf{s})\right) \right\} P[\psi, \psi^+]
\end{aligned}
\tag{E.91}
$$

Now if

$$
\widehat{\rho} \rightarrow \widehat{\rho}\widehat{T} = \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s}\, \widehat{\rho}(\partial_\mu \widehat{\Psi}(\mathbf{s})^\dagger \, \partial_\mu \widehat{\Psi}(\mathbf{s}))
\tag{E.92}
$$

then

$$
\begin{aligned}
&P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
\rightarrow\quad & \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \psi(\mathbf{s}) - \partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\partial_\mu \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+]
\end{aligned}
\tag{E.93}
$$

After expanding we find that

$$
\begin{aligned}
&P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
\rightarrow\quad & \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \psi(\mathbf{s})\right) \left(\partial_\mu \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+] \\
& - \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\partial_\mu \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+]
\end{aligned}
\tag{E.94}
$$

Applying the same approach as before the second term is given by

$$
\begin{aligned}
& \int d\mathbf{s} \left\{ \left(\partial_\mu \frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\partial_\mu \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+] \\
=\quad & -\int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\partial_\mu^2 \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+]
\end{aligned}
\tag{E.95}
$$

Using the result (E.95) we find that

$$
\begin{aligned}
&P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \\
\rightarrow\quad & \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\partial_\mu \psi(\mathbf{s})\right) \left(\partial_\mu \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+] \\
& + \frac{\hbar^2}{2m} \sum_\mu \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\partial_\mu^2 \psi^+(\mathbf{s})\right) \right\} P[\psi, \psi^+]
\end{aligned}
\tag{E.96}
$$

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We now combine the contributions so that when

$$\widehat{\rho} \rightarrow [\widehat{T}, \widehat{\rho}] = [\frac{\hbar^2}{2m} \sum_{\mu} \int d\mathbf{s} (\partial_{\mu} \widehat{\Psi}(\mathbf{s})^{\dagger} \partial_{\mu} \widehat{\Psi}(\mathbf{s})), \widehat{\rho}] \qquad \text{(E.97)}$$

then

$$P[\psi(\mathbf{r}), \psi^{+}(\mathbf{r})] \rightarrow P^{1} \qquad \text{(E.98)}$$

where

$$
\begin{aligned}
P^{1} &= -\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^{+}(\mathbf{s})} \left(\sum_{\mu} \frac{\hbar^2}{2m} \partial_{\mu}^{2} \psi^{+}(\mathbf{s}) \right) P[\psi, \psi^{+}] \right\} \\
&\quad + \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \left(\sum_{\mu} \frac{\hbar^2}{2m} \partial_{\mu}^{2} \psi(\mathbf{s}) \right) P[\psi, \psi^{+}]) \right\}
\end{aligned}
$$

$$\text{(E.99)}$$

where the $(\partial_{\mu}\psi(\mathbf{s}))(\partial_{\mu}\psi^{+}(\mathbf{s}))$ terms cancel. Thus only a first order functional derivative term occurs.

Overall, the contribution to the functional Fokker-Planck equation from the kinetic energy term is given by

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} P[\psi, \psi^{+}] \right)_{K} \\
&= \frac{-i}{\hbar} \left\{ -\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^{+}(\mathbf{s})} \left(\sum_{\mu} \frac{\hbar^2}{2m} \partial_{\mu}^{2} \psi^{+}(\mathbf{s}) \right) P[\psi, \psi^{+}] \right\} \right\} \\
&\quad + \frac{-i}{\hbar} \left\{ +\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \left(\sum_{\mu} \frac{\hbar^2}{2m} \partial_{\mu}^{2} \psi(\mathbf{s}) \right) P[\psi, \psi^{+}] \right\} \right\}
\end{aligned}
$$

$$\text{(E.100)}$$

Reverting to the original notation, the contribution to the functional Fokker-Planck equation from the non-condensate kinetic energy term is given by

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^{*}}(\mathbf{r})] \right)_{K} \\
&= \frac{-i}{\hbar} \left\{ -\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi_{NC}^{+}(\mathbf{s})} \left(\sum_{\mu} \frac{\hbar^2}{2m} \partial_{\mu}^{2} \psi_{NC}^{+}(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^{*}}(\mathbf{r})] \right\} \right\} \\
&\quad + \frac{-i}{\hbar} \left\{ +\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi_{NC}(\mathbf{s})} \left(\sum_{\mu} \frac{\hbar^2}{2m} \partial_{\mu}^{2} \psi_{NC}(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^{*}}(\mathbf{r})] \right\} \right\}
\end{aligned}
$$

$$\text{(E.101)}$$

Appendix E.5. Non-Condensate Trap Potential Terms

We write the trap potential as

$$\widehat{V} = \int d\mathbf{s}(\widehat{\Psi}(\mathbf{s})^\dagger V \widehat{\Psi}(\mathbf{s})) \tag{E.102}$$

Now if

$$\widehat{\rho} \to \widehat{V}\widehat{\rho} = \int d\mathbf{s}(\widehat{\Psi}(\mathbf{s})^\dagger V \widehat{\Psi}(\mathbf{s}))\widehat{\rho} \tag{E.103}$$

then

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \int d\mathbf{s}\left\{\left(\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})}\right) V(\mathbf{s})\,(\psi(\mathbf{s}))\right\} P[\psi, \psi^+] \tag{E.104}$$

After expanding we find that

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \int d\mathbf{s}\left\{\psi(\mathbf{s})V(\mathbf{s})\psi^+(\mathbf{s})\right\} P[\psi, \psi^+] - \int d\mathbf{s}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}V(\mathbf{s})\psi(\mathbf{s})\right\} P[\psi, \psi^+] \tag{E.105}$$

where we have re-ordered the $\psi^+(\mathbf{s})V(\mathbf{s})\psi(\mathbf{s})$ factor in the first term.

Now if

$$\widehat{\rho} \to \widehat{\rho}\widehat{V} = \int d\mathbf{s}\widehat{\rho}\widehat{\Psi}(\mathbf{s})^\dagger V \widehat{\Psi}(\mathbf{s}) \tag{E.106}$$

then

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \int d\mathbf{s}\left(\psi(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})}\right) V(\mathbf{s})\,(\psi^+(\mathbf{s}))\, P[\psi, \psi^+] \tag{E.107}$$

After expanding we have

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$
$$\to \int d\mathbf{s}\{\psi(\mathbf{s})V(\mathbf{s})\psi^+(\mathbf{s})\}P[\psi, \psi^+] - \int d\mathbf{s}\left\{\frac{\delta}{\delta\psi^+(\mathbf{s})}V(\mathbf{s})\psi^+(\mathbf{s})\right\} P[\psi, \psi^+] \tag{E.108}$$

We now combine the contributions so that when

$$\widehat{\rho} \to [\widehat{V}, \widehat{\rho}] = [\int d\mathbf{s}\,\widehat{\Psi}(\mathbf{s})^\dagger V(\mathbf{s})\widehat{\Psi}(\mathbf{s}), \widehat{\rho}] \tag{E.109}$$

then

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$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \to P^1 \qquad (E.110)$$

which only involves a first order derivative since the zero order terms cancel.

$$
\begin{aligned}
&P^1 \\
&= \ -\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \{V(\mathbf{s})\psi(\mathbf{s})\} \right\} P[\psi, \psi^+] + \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]
\end{aligned}
$$
$$(E.111)$$

Overall, the contribution to the functional Fokker-Planck equation from the trap potential term is given by

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} P[\psi, \psi^+] \right)_V \\
&= \ \frac{-i}{\hbar} \left\{ -\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \{V(\mathbf{s})\psi(\mathbf{s})\} \right\} P[\psi, \psi^+] + \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} V(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+] \right\}
\end{aligned}
$$
$$(E.112)$$

which only involves first order functional derivatives.

Reverting to the original notation, the contribution to the functional Fokker-Planck equation from the non-condensate trap potential term is given by

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \right)_V \\
&= \ \frac{-i}{\hbar} \left\{ -\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi_{NC}(\mathbf{s})} \{V(\mathbf{s})\psi_{NC}(\mathbf{s})\} \right\} P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \right\} \\
&\quad \frac{-i}{\hbar} \left\{ +\int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})} V(\mathbf{s})\psi_{NC}^+(\mathbf{s}) \right\} P[\underset{\to}{\psi}(\mathbf{r}), \underset{\to}{\psi^*}(\mathbf{r})] \right\} \quad (E.113)
\end{aligned}
$$

Appendix E.6. Non-Condensate Boson-Boson Interaction Terms

For the Bogoliubov Hamiltonian for which we derive the functional Fokker-Planck equation this boson-boson interaction in the non-condensate Hamiltonian $\widehat{H}_{NC}$ is discarded, but for completeness we treat it here. We write the boson-boson interaction potential as

$$\widehat{U} = \frac{g}{2} \int d\mathbf{s}\, \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s}) \widehat{\Psi}(\mathbf{s})$$

$$(E.114)$$

Now if

$$\widehat{\rho} \to \widehat{U}\widehat{\rho} = \frac{g}{2} \int d\mathbf{s}\, \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s}) \widehat{\Psi}(\mathbf{s}) \widehat{\rho}$$

$$(E.115)$$

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left(\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) - \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\psi(\mathbf{s})) P[\psi, \psi^+]$$

$$= \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ -\psi^+(\mathbf{s}) \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ -\frac{\delta}{\delta\psi(\mathbf{s})} \psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$\text{(E.116)}$$

After expanding we get

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \quad \rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- \frac{g}{2} \int d\mathbf{s} \left\{ \psi^+(\mathbf{s}) \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})} \psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$\text{(E.117)}$$

We can now use the product rule for functional derivatives (E.319) together with (E.320) and (E.321) to place all the derivatives on the left of the expression

and obtain

$$
\begin{aligned}
&P[\psi(\mathbf{r}),\psi^+(\mathbf{r})] \\
\rightarrow \quad & \frac{g}{2}\int d\mathbf{s}\left\{\psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}P[\psi,\psi^+] \\
& -\frac{g}{2}\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\right\}P[\psi,\psi^+] \qquad T2 \\
& -\frac{g}{2}\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\right\}P[\psi,\psi^+] \\
& +\frac{g}{2}\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})\right\}W[\psi,\psi^+] \\
= \quad & \frac{g}{2}\int d\mathbf{s}\left\{\psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}P[\psi,\psi^+] \\
& -g\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\right\}P[\psi,\psi^+] \\
& +\frac{g}{2}\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})\right\}W[\psi,\psi^+]
\end{aligned}
$$

$$(E.118)$$

where we have also rearranged the order of the factors in $\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})$.

Details include

T2

$$
\begin{aligned}
& \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})P\} \\
= \quad & \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\}\psi(\mathbf{s})\psi(\mathbf{s})P + \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})P\} \\
= \quad & \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})P\} \\
& \psi^+(\mathbf{s})\{\frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})P[\psi,\psi^+]\} \\
= \quad & \frac{\delta}{\delta\psi(\mathbf{s})}\{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})P[\psi,\psi^+]\}
\end{aligned}
$$

Now if

$$
\widehat{\rho}\rightarrow\widehat{\rho}\widehat{U}=\frac{g}{2}\int d\mathbf{s}\,\widehat{\rho}\,\widehat{\Psi}(\mathbf{s})^\dagger\widehat{\Psi}(\mathbf{s})^\dagger\widehat{\Psi}(\mathbf{s})\widehat{\Psi}(\mathbf{s})
$$

$$(E.119)$$

then

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left(\psi(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) - \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) \right) \left(\psi^+(\mathbf{s}) \right) P[\psi, \psi^+]$$

$$= \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ -\psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ -\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$(E.120)$$

After expanding we obtain

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \quad \rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- \frac{g}{2} \int d\mathbf{s} \left\{ \psi(\mathbf{s})\frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$(E.121)$$

We can now use the product rule for functional derivatives (E.319) together with (E.320) and (E.321) to place all the derivatives on the left of the expression. However the results can more easily be obtained by noticing that the $\widehat{\rho U}$ is the same as the $\widehat{U}\widehat{\rho}$ if we interchange $\psi(\mathbf{s})$ and $\psi^+(\mathbf{s})$ everywhere. Hence

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$= \quad \frac{g}{2} \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- g \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$\text{(E.122)}$$

We now combine the contributions so that when

$$\widehat{\rho} \rightarrow [\widehat{U}, \widehat{\rho}] = [\frac{g}{2} \int d\mathbf{s} \; \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s})^\dagger \widehat{\Psi}(\mathbf{s})\widehat{\Psi}(\mathbf{s}), \widehat{\rho}]$$

$$\text{(E.123)}$$

then

$$P[\psi(\mathbf{r}), \psi^+(\mathbf{r})] \rightarrow P^1 + P^2$$

$$\text{(E.124)}$$

where the terms are listed via the order of derivatives that occur

$$P^1$$
$$= \quad +g \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$- g \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$\text{(E.125)}$$

$$P^2$$
$$= \quad -\frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi^+(\mathbf{s})} \frac{\delta}{\delta\psi^+(\mathbf{s})}\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$+ \frac{g}{2} \int d\mathbf{s} \left\{ \frac{\delta}{\delta\psi(\mathbf{s})} \frac{\delta}{\delta\psi(\mathbf{s})}\psi(\mathbf{s})\psi(\mathbf{s}) \right\} P[\psi, \psi^+]$$

$$\text{(E.126)}$$

Overall, the contribution to the functional Fokker-Planck equation from the boson-boson interaction is given by

$$
\left(\frac{\partial}{\partial t} P[\psi, \psi^+]\right)_U
$$
$$
= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi(\mathbf{s})}\left\{(\psi^+(\mathbf{s})\psi(\mathbf{s})\}\psi(\mathbf{s})\right\}P[\psi,\psi^+]\right\}
$$
$$
+\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\frac{\delta}{\delta\psi^+(\mathbf{s})}\left\{(\psi^+(\mathbf{s})\psi(\mathbf{s}))\psi^+(\mathbf{s})\right\}P[\psi,\psi^+]\right\}
$$
$$
+\frac{-i}{\hbar}\left\{\frac{g}{2}\int d\mathbf{s}\frac{\delta}{\delta\psi(\mathbf{s})}\frac{\delta}{\delta\psi(\mathbf{s})}\{\psi(\mathbf{s})\psi(\mathbf{s})\}P[\psi,\psi^+]\right\}
$$
$$
+\frac{-i}{\hbar}\left\{-\frac{g}{2}\int d\mathbf{s}\frac{\delta}{\delta\psi^+(\mathbf{s})}\frac{\delta}{\delta\psi^+(\mathbf{s})}\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}P[\psi,\psi^+]\right\}
$$
(E.127)

which involves first order and second order functional derivatives.

Reverting to the original notation the contribution to the functional Fokker-Planck equation from the non-condensate boson-boson interaction term is given by

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_U
$$
$$
= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\left\{(\psi^+_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s}))\psi_{NC}(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$
$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\left\{(\psi^+_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s}))\psi^+_{NC}(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$
$$
\frac{-i}{\hbar}\left\{g\int d\mathbf{s}\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\{\frac{1}{2}\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$
$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\{\frac{1}{2}\psi^+_{NC}(\mathbf{s})\psi^+_{NC}(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$
(E.128)

Note that this term is not included in the final functional Fokker-Planck equation for the Bogoliubov Hamiltonian.

Similar expressions for the functional Fokker-Planck equation in the case of a pure P representation (but not involving a doubled phase space) are given in the paper by Steel et al [[55]] (see Eq. (17)). Comparisons can be made by substituting $\psi^+_{NC}(\mathbf{s})$ with $\psi^*_{NC}(\mathbf{s})$. As in the present result, no restricted delta function $\delta_C(\mathbf{s},\mathbf{s})$ term in the interaction contribution appears in a P representation approach.

Appendix E.7. Condensate - Non-Condensate Interaction - First Order in Non-Condensate

The first order term in the interaction between the condensate and the non-condensate is

$$
\begin{aligned}
\widehat{V}_1 \;=\;& g\int d\mathbf{r}\,\widehat{\Psi}^{\dagger}_{NC}(\mathbf{r})\{\widehat{\Psi}^{\dagger}_C(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\}\widehat{\Psi}_C(\mathbf{r}) - g\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\widehat{\Psi}_C(\mathbf{s}) \\
&+ g\int d\mathbf{r}\,\widehat{\Psi}^{\dagger}_C(\mathbf{r})\{\widehat{\Psi}^{\dagger}_C(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\}\widehat{\Psi}_{NC}(\mathbf{r}) - g\int\int d\mathbf{r}\,d\mathbf{s}\,F^{*}(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^{\dagger}\widehat{\Psi}_{NC}(\mathbf{s})
\end{aligned}
$$

$$(E.129)$$

This is the sum of two terms, one fourth order in the field operators, the other second order.

$$
\widehat{V}_{14} \;=\; g\int d\mathbf{r}\,\widehat{\Psi}^{\dagger}_{NC}(\mathbf{r})\widehat{\Psi}^{\dagger}_C(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_C(\mathbf{r}) + g\int d\mathbf{r}\,\widehat{\Psi}^{\dagger}_C(\mathbf{r})\widehat{\Psi}^{\dagger}_C(\mathbf{r})\,\widehat{\Psi}_C(\mathbf{r})\widehat{\Psi}_{NC}(\mathbf{r})
$$

$$(E.130)$$

$$
\widehat{V}_{12} \;=\; -g\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^{\dagger}\,\widehat{\Psi}_C(\mathbf{s}) - g\int\int d\mathbf{r}\,d\mathbf{s}\,F^{*}(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^{\dagger}\widehat{\Psi}_{NC}(\mathbf{s})
$$

$$(E.131)$$

Appendix E.7.1. Fourth Order Term

Now if

$$
\widehat{\rho}\to\widehat{V}_{14}\,\widehat{\rho} = g\int d\mathbf{s}\,(\widehat{\Psi}^{\dagger}_{NC}(\mathbf{s})\widehat{\Psi}^{\dagger}_C(\mathbf{s})\,\widehat{\Psi}_C(\mathbf{s})\widehat{\Psi}_C(\mathbf{s})) + \widehat{\Psi}^{\dagger}_C(\mathbf{s})\widehat{\Psi}^{\dagger}_C(\mathbf{s})\,\widehat{\Psi}_C(\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{s}))\widehat{\rho}
$$

$$(E.132)$$

then

$$
\begin{aligned}
& WP[\psi(\mathbf{r}),\psi^{+}(\mathbf{r}),\phi(\mathbf{r}),\phi^{+}(\mathbf{r})] \\
\to\;& g\int d\mathbf{s}\,\left(\phi^{+}(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\psi^{+}(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\psi(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi^{+}(\mathbf{s})}\right) \\
& \times\left(\psi(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi^{+}(\mathbf{s})}\right)WP \\
& + g\int d\mathbf{s}\,\left(\psi^{+}(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\psi^{+}(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\psi(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi^{+}(\mathbf{s})}\right) \\
& \times(\phi(\mathbf{s}))\,WP[\psi,\psi^{+},\phi,\phi^{+}]
\end{aligned}
$$

$$(E.133)$$

Expanding out the terms gives

$$
\begin{aligned}
& WP[\psi(\mathbf{r}),\psi^{+}(\mathbf{r}),\phi(\mathbf{r}),\phi^{+}(\mathbf{r})] \\
\to\;& WP[\psi(\mathbf{r}),\psi^{+}(\mathbf{r}),\phi(\mathbf{r}),\phi^{+}(\mathbf{r})]_{1-16} + WP[\psi(\mathbf{r}),\psi^{+}(\mathbf{r}),\phi(\mathbf{r}),\phi^{+}(\mathbf{r})]_{17-24}
\end{aligned}
$$

$$(E.134)$$

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where

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-16}$$

$$= g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\psi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\psi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\psi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\psi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s})\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$+ g\int d\mathbf{s} \left\{ \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

and

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{17-24}$$

$$= +g \int d\mathbf{s} \left\{ (\psi^+(\mathbf{s})) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ (\psi^+(\mathbf{s})) (\psi^+(\mathbf{s})) \left(\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ (\psi^+(\mathbf{s})) \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ (\psi^+(\mathbf{s})) \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(-\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\mathbf{1}}{\mathbf{2}} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

or on further simplification

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-16}$$

$$= g \int d\mathbf{s} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\psi^+(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T1$$

$$+ g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\psi^+(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T2$$

$$+ g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\psi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T3$$

$$+ g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\psi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T4$$

$$- g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T5$$

$$- g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T6$$

$$- g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T7$$

$$- g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\phi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T8$$

$$- g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T9$$

$$- g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T10$$

$$- g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T11$$

$$- g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T12$$

$$+ g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T13$$

$$+ g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T14$$

$$+ g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \right\} WP \qquad T15$$

$$+ g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad T16$$

$$(\text{E.135})$$

and

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{17-24}$$

$$= +g \int d\mathbf{s} \left\{ (\psi^+(\mathbf{s})) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP \qquad T17$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ (\psi^+(\mathbf{s})) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP \qquad T18$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP \qquad T19$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP \qquad T20$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP \qquad T21$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP \qquad T22$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP \qquad T23$$

$$+g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP \qquad T24$$

$$(\text{E.136})$$

These terms can be simplified in terms of placing all the functional derivatives on the left by using the product rule (E.319) together with (E.320) and (E.321) for functional differentiation and noting that many functional derivatives are zero.

For the T2 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ (\phi^+(\mathbf{s})) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \} - \{ \delta_C(\mathbf{s}, \mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s}) \} \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \{ \delta_C(\mathbf{s}, \mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s}) \} WP$$

For the T3 term

$$\int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\psi^+(\mathbf{s})\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\} - \{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s}\, \frac{1}{2} \{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\} WP$$

For the T4 term

$$\int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\psi^+(\mathbf{s})\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\} - \{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s}\, \frac{1}{4} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right) WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP$$

For the T5 term

$$\int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

For the T6 term

$$\int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

For the T7 term

$$\int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\phi^+(\mathbf{s})\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

For the T8 term

$$\int d\mathbf{s} \frac{1}{8} \left\{ (\phi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP$$

For the T9 term

$$\int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) (\psi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

For the T10 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

For the T11 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

For the T12 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left\{ \delta_C(\mathbf{s},\mathbf{s}) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP$$

For the T13 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\psi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

For the T14 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP$$

For the T15 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP$$

For the T18 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ (\psi^+(\mathbf{s})) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \left\{ \mathbf{2}\delta_\mathbf{C}(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s}) \right\} WP$$

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For the T19 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s}) \} \right\} WP$$

For the T20 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\phi(\mathbf{s}) \} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi(\mathbf{s}) \} \right\} WP$$

For the T21 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s}) \} \right\} WP$$

For the T22 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\phi(\mathbf{s}) \} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi(\mathbf{s}) \} \right\} WP$$

For the T23 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP$$

For the T24 term

$$\int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s}) \} \right\} WP$$

Hence substituting these results we have

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$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-16}$$

$$= g \int d\mathbf{s} \left\{ \phi^+(\mathbf{s}) \psi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \right\} WP \qquad F1$$

$$+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \psi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F2.1$$

$$- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \delta_C(\mathbf{s}, \mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} WP \qquad F2.2$$

$$+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \psi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F3.1$$

$$- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \delta_C(\mathbf{s}, \mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} WP \qquad F3.2$$

$$+ g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \psi^+(\mathbf{s}) \right\} \right\} WP \qquad F4.1$$

$$- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \delta_C(\mathbf{s}, \mathbf{s}) \right\} \right\} WP \qquad F4.2$$

$$- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F5$$

$$- g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F6$$

$$- g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F7$$

$$- g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \right\} \right\} WP \qquad F8$$

$$- g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left\{ \psi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F9$$

$$- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \psi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F10.1$$

$$+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left\{ \delta_C(\mathbf{s}, \mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F10.2$$

$$- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \psi^+(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F11.1$$

$$+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left\{ \delta_C(\mathbf{s}, \mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F11.2$$

$$- g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \psi^+(\mathbf{s}) \right\} \right\} WP \qquad F12.1$$

$$+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \delta_C(\mathbf{s}, \mathbf{s}) \right\} \right\} WP \qquad F12.2$$

$$+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left\{ \psi(\mathbf{s}) \psi(\mathbf{s}) \right\} \right\} WP \qquad F13$$

$$+ g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \psi(\mathbf{s}) \right\} \right\} WP \qquad F14$$

$$+ g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left\{ \psi(\mathbf{s}) \right\} \right\} WP \qquad F15$$

$$+ g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad F16$$

$$(E.137)$$

and

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{17-24}$$

$$= +g \int d\mathbf{s} \left\{ \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s}) \right\} WP \qquad F17$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F18.1$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ 2\delta_{\mathbf{C}}(\mathbf{s}, \mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s}) \right\} WP \qquad F18.2$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F19$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F20.1$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s}, \mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F20.2$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F21$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F22.1$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s}, \mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F22.2$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) (\phi(\mathbf{s})) \right\} WP \qquad F23$$

$$+g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP \qquad F24$$

$$(E.138)$$

Collecting terms with the same order of functional derivatives we have

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$
$$\rightarrow \quad WP^0 + WP^1 + WP^2 + WP^3 + WP^4$$

$$(E.139)$$

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where we have used upper subscripts for the $\widehat{V}_{14}\widehat{\rho}$ contributions and

$$
\begin{aligned}
WP^0 & \\
= \quad & g\int d\mathbf{s}\,\{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\}\,WP \qquad F1 \\
& +g\int d\mathbf{s}\,\frac{1}{2}\,\{-\{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\}\}\,WP \qquad F2.2 \\
& +g\int d\mathbf{s}\,\frac{1}{2}\,\{-\{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\}\}\,WP \qquad F3.2 \\
& +g\int d\mathbf{s}\,\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\}\,WP \qquad F17 \\
& -g\int d\mathbf{s}\,\frac{1}{2}\,\{2\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\}\,WP \qquad F18 \\
= \quad & g\int d\mathbf{s}\,\{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\}\,WP[\psi,\psi^+,\phi,\phi^+] \\
& +g\int d\mathbf{s}\,\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\}\,WP[\psi,\psi^+,\phi,\phi^+] \\
& -g\int d\mathbf{s}\,\{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\}WP[\psi,\psi^+,\phi,\phi^+] \\
& -g\int d\mathbf{s}\,\{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\}WP[\psi,\psi^+,\phi,\phi^+]
\end{aligned}
$$

$$(\text{E.140})$$

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$$WP^1$$

$$
= \quad +g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP \qquad F2.1
$$

$$
+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP \qquad F3.1
$$

$$
-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP \qquad F4.2
$$

$$
-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP \qquad F5
$$

$$
-g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP \qquad F9
$$

$$
+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\} \right\} WP \qquad F10.2
$$

$$
+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\} \right\} WP \qquad F11.2
$$

$$
+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F18.1
$$

$$
-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F19
$$

$$
+g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F20.2
$$

$$
-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F21
$$

$$
+g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad F22.2
$$

$$
= \quad +g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$
+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$
-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$
-g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$
-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$
+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$
-g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$
+g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad (E.141)
$$

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$$WP^2$$

$$= +g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad F4.1$$

$$-g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad F6$$

$$-g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad F7$$

$$-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad F10.1$$

$$-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad F11.1$$

$$+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP \qquad F12.2$$

$$+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad F13$$

$$-g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi(\mathbf{s})\}\right\}WP \qquad F20.1$$

$$-g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi(\mathbf{s})\}\right\}WP \qquad F22.1$$

$$+g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi(\mathbf{s}))\,(\phi(\mathbf{s}))\right\}WP \qquad F23$$

$$= +g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$-g\int d\mathbf{s}\,\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$+g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\phi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$(\text{E.142})$$

$$WP^3$$

$$= -g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP \qquad F8$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP \qquad F12.1$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP \qquad F14$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP \qquad F15$$

$$+g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP \qquad F24$$

$$= -g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$+g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$\text{(E.143)}$$

$$WP^4$$

$$= +g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \right\} WP \qquad F16$$

$$= +g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$\text{(E.144)}$$

Now if

$$\widehat{\rho} \to \widehat{\rho} \widehat{V}_{14} = g \int d\mathbf{s}\, \widehat{\rho}\, (\widehat{\Psi}^\dagger_{NC}(\mathbf{s}) \widehat{\Psi}^\dagger_C(\mathbf{s})\, \widehat{\Psi}_C(\mathbf{s}) \widehat{\Psi}_C(\mathbf{s}))$$

$$+ g \int d\mathbf{s}\, \widehat{\rho}\, (\widehat{\Psi}^\dagger_C(\mathbf{s}) \widehat{\Psi}^\dagger_C(\mathbf{s})\, \widehat{\Psi}_C(\mathbf{s}) \widehat{\Psi}_{NC}(\mathbf{s}))$$

$$\text{(E.145)}$$

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then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow \quad g \int d\mathbf{s} \left(\psi(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right)$$

$$\times \left(\phi^+(\mathbf{s}) \right) WP$$

$$+ g \int d\mathbf{s} \left(\phi(\mathbf{s}) - \frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right)$$

$$\times \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) WP$$

$$(\text{E.146})$$

Expanding this expression gives

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow \quad WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-8} + WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{9-24}$$

$$(\text{E.147})$$

where

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-8}$$

$$= \quad g \int d\mathbf{s} \left\{ \left(\psi(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \left(\psi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(\psi(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(\psi(\mathbf{s}) \right) \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(\psi(\mathbf{s}) \right) \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \left(\psi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

$$+ g \int d\mathbf{s} \left\{ \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) \right\} WP$$

and

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{9-24}$$

$$= +g \int d\mathbf{s} \, (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \, (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+g \int d\mathbf{s} \, (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \, (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+g \int d\mathbf{s} \, (\phi(\mathbf{s})) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \, (\phi(\mathbf{s})) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+g \int d\mathbf{s} \, (\phi(\mathbf{s})) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \, (\phi(\mathbf{s})) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, WP$$

$$+g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

Collecting terms gives

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-8}$$

$$= g \int d\mathbf{s} \left\{ (\psi(\mathbf{s})) (\psi(\mathbf{s})) (\psi^+(\mathbf{s})) (\phi^+(\mathbf{s})) \right\} WP \qquad S1$$

$$+ g \int d\mathbf{s} \frac{1}{2} \left\{ (\psi(\mathbf{s})) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP \qquad S2$$

$$- g \int d\mathbf{s} \frac{1}{2} \left\{ (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\phi^+(\mathbf{s})) \right\} WP \qquad S3$$

$$- g \int d\mathbf{s} \frac{1}{4} \left\{ (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP \qquad \mathcal{S}4$$

$$- g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) (\psi^+(\mathbf{s})) (\phi^+(\mathbf{s})) \right\} WP \qquad S5$$

$$- g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP \qquad S6$$

$$+ g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\phi^+(\mathbf{s})) \right\} WP \qquad S7$$

$$+ g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP \qquad S8$$

$$\text{(E.148)}$$

and

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{9-24}$$

$$= +g \int d\mathbf{s} \left\{ (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \right\} WP \qquad S9$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S10$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \right\} WP \qquad S11$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S12$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ (\phi(\mathbf{s})) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \right\} WP \qquad S13$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ (\phi(\mathbf{s})) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S14$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ (\phi(\mathbf{s})) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \right\} WP \qquad S15$$

$$-g \int d\mathbf{s} \frac{1}{8} \left\{ (\phi(\mathbf{s})) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S16$$

$$-g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \right\} WP \qquad S17$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S18$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \right\} WP \qquad S19$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S20$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \right\} WP \qquad S21$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S22$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \right\} WP \qquad S23$$

$$+g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \right\} WP \qquad S24$$

$$(\text{E.149})$$

Again we use the product rule (E.319) together with (E.320) and (E.321) to place all the functional derivatives on the left.

For the $S2$T18 term

$$\int d\mathbf{s}\, \frac{1}{2} \left\{ (\psi(\mathbf{s}))\,(\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s}\, \frac{1}{2} \left\{ \mathbf{2}\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP$$

For the $S3$T19 term

$$\int d\mathbf{s}\, \frac{1}{2} \left\{ (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s}))\,(\phi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\phi\} \right\} WP$$

For the $S4$T20 term

$$\int d\mathbf{s}\, \frac{1}{4} \left\{ (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP$$

For the $S5$T21 term

$$\int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,(\phi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP$$

For the $S6$T22 term

$$\int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP$$

For the $S7$T23 term

$$\int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s}))\,(\phi^+(\mathbf{s})) \right\} WP$$

For the $S8$T24 term

$$\int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP$$

For the $S10$T2 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\} - \{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\} WP$$

For the $S11$T3 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\} - \{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\} WP$$

For the $S12$T4 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ (\phi(\mathbf{s})) \, (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\} - \{\phi(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \{\phi(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right) WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP$$

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For the $S13$T5 term

$$\int d\mathbf{s}\,\frac{1}{2}\left\{(\phi(\mathbf{s}))\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)(\psi^+(\mathbf{s}))(\psi^+(\mathbf{s}))\right\}WP$$

$$=\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

For the $S14$T6 term

$$\int d\mathbf{s}\,\frac{1}{4}\left\{(\phi(\mathbf{s}))\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\right\}WP$$

$$=\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

For the $S15$T7 term

$$\int d\mathbf{s}\,\frac{1}{4}\left\{(\phi(\mathbf{s}))\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\right\}WP$$

$$=\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

For the $S16$T8 term

$$\int d\mathbf{s}\,\frac{1}{8}\left\{(\phi(\mathbf{s}))\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\right\}WP$$

$$=\int d\mathbf{s}\,\frac{1}{8}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\}\right\}WP$$

For the $S17$T9 term

$$\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\psi(\mathbf{s}))(\psi^+(\mathbf{s}))(\psi^+(\mathbf{s}))\right\}WP$$

$$=\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

For the $S18$T10 term

$$\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\psi(\mathbf{s}))(\psi^+(\mathbf{s}))\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\right\}WP$$

$$=\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

$$-\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

For the $S19$T11 term

$$\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\psi(\mathbf{s}))\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\right\}WP$$

$$=\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

$$-\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP$$

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For the $S20$T12 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left\{ \delta_C(\mathbf{s}, \mathbf{s}) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_C(\mathbf{s}, \mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_C(\mathbf{s}, \mathbf{s})\} \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP$$

$$- \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_C(\mathbf{s}, \mathbf{s})\} \right\} WP$$

For the $S21$T13 term

$$\int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) (\psi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP$$

For the $S22$T14 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP$$

For the $S23$T15 term

$$\int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \right\} WP$$

$$= \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP$$

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Substituting these results gives

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-8}$$

$$= g \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP \qquad G1$$

$$+ g \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G2.1$$

$$- g \int d\mathbf{s}\, \frac{1}{2} \left\{ 2\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP \qquad G2.2$$

$$- g \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ \psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G3$$

$$- g \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G4.1$$

$$+ g \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G4.2$$

$$- g \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ \psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G5$$

$$- g \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G6.1$$

$$+ g \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G6.2$$

$$+ g \int d\mathbf{s}\, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ \psi^+(\mathbf{s})\phi^+(\mathbf{s}) \right\} \right\} WP \qquad G7$$

$$+ g \int d\mathbf{s}\, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s}) \right\} \right\} WP \qquad G8$$

$$(\text{E.150})$$

and

$$WP[\psi(\mathbf{r}),\psi^+(\mathbf{r}),\phi(\mathbf{r}),\phi^+(\mathbf{r})]_{9-24}$$

$$= +g\int d\mathbf{s}\left\{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}WP[\psi,\psi^+,\phi,\phi^+] \qquad G9$$

$$+g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G10.1$$

$$-g\int d\mathbf{s}\frac{1}{2}\{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad G10.2$$

$$+g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G11.1$$

$$-g\int d\mathbf{s}\frac{1}{2}\{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad G11.2$$

$$+g\int d\mathbf{s}\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad G12.1$$

$$-g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP \qquad G12.2$$

$$-g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G13$$

$$-g\int d\mathbf{s}\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G14$$

$$-g\int d\mathbf{s}\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G15$$

$$-g\int d\mathbf{s}\frac{1}{8}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\}\right\}WP \qquad G16$$

$$-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G17$$

$$-g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G18.1$$

$$+g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G18.2$$

$$-g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G19.1$$

$$+g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G19.2$$

$$-g\int d\mathbf{s}\frac{1}{4}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\}\right\}WP \qquad G20.1$$

$$+g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP \qquad G20.2$$

$$+g\int d\mathbf{s}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad G21$$

$$+g\int d\mathbf{s}\frac{1}{4}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\}\right\}WP \qquad G22$$

$$+g\int d\mathbf{s}\frac{1}{4}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\}\right\}WP \qquad G23$$

$$+g\int d\mathbf{s}\frac{1}{8}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\right\}WP \qquad G24$$

$$(E.151)$$

Collecting terms with the same order of functional derivatives we have

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$
$$\rightarrow \quad WP_0 + WP_1 + WP_2 + WP_3 + WP_4 \qquad (E.152)$$

where we have used lower subscripts for the $\widehat{\rho}\,\widehat{V}_{14}$ contributions and

$$
\begin{aligned}
&WP_0 \\
= \quad & g \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP \qquad G1 \\
& -g \int d\mathbf{s}\, \frac{1}{2} \left\{ \mathbf{2}\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP \qquad G2.2 \\
& +g \int d\mathbf{s} \left\{ \phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} WP \qquad G9 \\
& -g \int d\mathbf{s}\, \frac{1}{2} \{ \delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s}) \} WP \qquad G10.2 \\
& -g \int d\mathbf{s}\, \frac{1}{2} \{ \delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s}) \} WP \qquad G11.2 \\
= \quad & g \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP[\psi, \psi^+, \phi, \phi^+] \\
& +g \int d\mathbf{s} \left\{ \phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} WP[\psi, \psi^+, \phi, \phi^+] \\
& -g \int d\mathbf{s} \left\{ \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP[\psi, \psi^+, \phi, \phi^+] \\
& -g \int d\mathbf{s} \left\{ \delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s}) \right\} WP[\psi, \psi^+, \phi, \phi^+] \qquad (E.153)
\end{aligned}
$$

$$WP_1$$

$$= +g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G2.1$$

$$-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G3$$

$$+g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G4.2$$

$$-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G5$$

$$+g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G6.2$$

$$+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G10.1$$

$$+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G11.1$$

$$-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP \qquad G12.2$$

$$-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G13$$

$$-g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G17$$

$$+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G18.2$$

$$+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G19.2$$

$$= +g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G10.1, G11.1$$

$$+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G2.1$$

$$-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G12.2$$

$$-g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G3, G5$$

$$-g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G13$$

$$+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G4.2, G6.2$$

$$-g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G17$$

$$+g \int d\mathbf{s} \, \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G18.2, G19.2$$

$$\tag{E.154}$$

WP_2

$$= -g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G4.1$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G6.1$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP \qquad G7$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP \qquad G12.1$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G14$$

$$-g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G15$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G18.1$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G19.1$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP \qquad G20.2$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP \qquad G21$$

$$= +g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G12.1$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G14,G15$$

$$-g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G18.1,19.1$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\delta_C(\mathbf{s},\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G20.2$$

$$+g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G21$$

$$-g \int d\mathbf{s} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G4.1,6.1$$

$$+g \int d\mathbf{s} \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G7$$

$$(E.155)$$

$$WP_3$$

$$
= \; +g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP \qquad G8
$$

$$
-g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP \qquad G16
$$

$$
-g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP \qquad G20.1
$$

$$
+g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP \qquad G22
$$

$$
+g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP \qquad G23
$$

$$
= \; -g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G16
$$

$$
-g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G20.1
$$

$$
+g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G22, G23
$$

$$
+g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP[\psi,\psi^+,\phi,\phi^+] \qquad G8
$$

$$\text{(E.156)}$$

$$WP_4$$

$$
= \; +g \int d\mathbf{s} \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \right\} WP[\psi,\psi^+,\phi,\phi^+]
$$

$$\text{(E.157)}$$

Now if

$$\widehat{\rho} \to [\widehat{V}_{14}, \widehat{\rho}]$$

$$
= [g \int d\mathbf{s} \, (\widehat{\Psi}^\dagger_{NC}(\mathbf{s}) \widehat{\Psi}^\dagger_C(\mathbf{s}) \, \widehat{\Psi}_C(\mathbf{s}) \widehat{\Psi}_C(\mathbf{s})) + \widehat{\Psi}^\dagger_C(\mathbf{s}) \widehat{\Psi}^\dagger_C(\mathbf{s}) \, \widehat{\Psi}_C(\mathbf{s}) \widehat{\Psi}_{NC}(\mathbf{s})) , \widehat{\rho}]
$$

$$\text{(E.158)}$$

then

$$WP[\psi(\mathbf{r}),\psi^+(\mathbf{r}),\phi(\mathbf{r}),\phi^+(\mathbf{r})]$$

$$
\to \quad WP^0_T + WP^1_T + WP^2_T + WP^3_T + WP^4_T \qquad \text{(E.159)}
$$

where the WP^n_T are obtained by subtracting the results for $\widehat{\rho} \widehat{V}_{14}$ from those for $\widehat{V}_{14} \widehat{\rho}$

Collecting terms gives

$$
\begin{aligned}
& W P_T^0 \\
=\ & g \int d\mathbf{s} \left\{ \phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} WP + g \int d\mathbf{s} \left\{ \psi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s}) \right\} WP \\
& -g \int d\mathbf{s} \left\{ \delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s}) \right\} WP - g \int d\mathbf{s} \left\{ \delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s}) \right\} WP \\
& -g \int d\mathbf{s} \left\{ \psi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP - g \int d\mathbf{s} \left\{ \phi(\mathbf{s})\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} WP \\
& +g \int d\mathbf{s} \left\{ \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s}) \right\} WP + g \int d\mathbf{s} \left\{ \delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s}) \right\} WP \\
=\ & 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{E.160})
\end{aligned}
$$

$$
\begin{aligned}
& W P_T^1 \\
=\ & +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ 2\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s}) \right\} \right\} WP \\
& +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ \psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s}) \right\} \right\} WP \\
& -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s}) \right\} \right\} WP \\
& -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ 2\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s}) \right\} \right\} WP \\
& -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} \right\} WP \\
& +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi(\mathbf{s}) \right\} \right\} WP \\
& -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left\{ \psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \right\} \right\} WP \\
& +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left\{ \delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s}) \right\} \right\} WP \\
& +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left\{ \psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \right\} \right\} WP \\
& -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left\{ \delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s}) \right\} \right\} WP
\end{aligned}
$$

$$ (\text{E.161}) $$

$$WP_T^2$$

$$
\begin{aligned}
= \quad &+ g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\psi^+(\mathbf{s}) \} \right\} WP \\
&- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\psi(\mathbf{s}) \} \right\} WP \\
&- g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\psi(\mathbf{s}) \} \right\} WP \\
&+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \delta_C(\mathbf{s},\mathbf{s}) \} \right\} WP \\
&+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \psi(\mathbf{s})\psi(\mathbf{s}) \} \right\} WP \\
&- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\phi(\mathbf{s}) \} \right\} WP \\
&+ g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \psi(\mathbf{s})\phi(\mathbf{s}) \} \right\} WP \\
&- g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \phi(\mathbf{s})\psi(\mathbf{s}) \} \right\} WP \\
&+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s})\psi^+(\mathbf{s}) \} \right\} WP \\
&+ g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \psi(\mathbf{s})\psi^+(\mathbf{s}) \} \right\} WP \\
&- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \delta_C(\mathbf{s},\mathbf{s}) \} \right\} WP \\
&- g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\psi^+(\mathbf{s}) \} \right\} WP \\
&+ g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \psi(\mathbf{s})\phi^+(\mathbf{s}) \} \right\} WP \\
&- g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \psi^+(\mathbf{s})\phi^+(\mathbf{s}) \} \right\} WP
\end{aligned}
$$

Collecting all the terms gives

$$WP_T^2$$

$$
\begin{aligned}
= \quad &+g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad Cancel\\
&-g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}\right\}WP \qquad Cancel\\
&-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad Cancel\\
&+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\phi^+(\mathbf{s})\}\right\}WP \qquad Cancel\\
&-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP\\
&+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP\\
&+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP\\
&-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP\\
&+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP\\
&-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP\\
&-g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi(\mathbf{s})\}\right\}WP \qquad Cancel\\
&+g\int d\mathbf{s}\,\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP \qquad Cancel\\
&+g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi(\mathbf{s})\phi(\mathbf{s})\}\right\}WP \qquad Cancel\\
&-g\int d\mathbf{s}\,\frac{1}{4}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP \qquad Cancel\\
= \quad &-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP\\
&+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP\\
&+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left\{\frac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\right\}\right\}WP\\
&-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left\{\frac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\right\}\right\}WP\\
&+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left\{\frac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\right\}\right\}WP\\
&-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left\{\frac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\right\}\right\}WP
\end{aligned}
$$

$$(E.162)$$

$$WP_T^3$$

$$
\begin{aligned}
= \quad & -g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP \\
& -g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP \\
& +g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP \\
& +g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP \\
& +g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP \\
& +g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP \\
& -g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP \\
& -g \int d\mathbf{s} \, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP
\end{aligned}
$$

Collecting the terms gives

$$WP_T^3$$

$$
\begin{aligned}
= \quad & -g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\} \right\} WP \\
& -g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP \\
& +g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP \\
& +g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{\psi(\mathbf{s})\} \right\} WP \\
& -g \int d\mathbf{s} \, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi^+(\mathbf{s})\} \right\} WP \\
& +g \int d\mathbf{s} \, \frac{1}{4} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\} \right\} WP
\end{aligned}
$$

$$\text{(E.163)}$$

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$$WP_T^4$$

$$
= \ +g \int d\mathbf{s}\, \frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \right\} WP
$$

$$
-g \int d\mathbf{s}\frac{1}{8} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \right\} WP
$$

$$(\text{E.164})$$

Thus we see that the $\widehat{V}_{14}$ term produces functional derivatives of orders one, two, three and four. We may write the contributions to the functional Fokker-Planck equation in the form

$$
\left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V14}
$$

$$
= \ \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V14}^{1} + \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V14}^{2}
$$

$$
+ \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V14}^{3} + \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V14}^{4}
$$

$$(\text{E.165})$$

where

$$\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)^1_{V14}$$

$$= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{2\phi^+(\mathbf{s})\psi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{2\psi^+(\mathbf{s})\psi(\mathbf{s})\phi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$(\text{E.166})$$

$$\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)^2_{V14}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\frac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\frac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$(E.167)$$

$$\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)^3_{V14}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\frac{1}{4}\phi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{4}\phi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{4}\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\frac{1}{4}\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\psi(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\frac{1}{2}\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$(E.168)$$

$$\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)^4_{V14}$$

$$= \frac{-i}{\hbar}\left\{g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$(E.169)$$

Reverting to the original notation we have

$$
\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right)^{1}_{V14}
$$

$$
= \quad \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^{+}_{C}(\mathbf{s})}\right)\{[2\psi^{+}_{C}(\mathbf{s})\psi_{C}(\mathbf{s})-\delta_{C}(\mathbf{s},\mathbf{s})]\psi^{+}_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^{+}_{\mathbf{C}}(\mathbf{s})}\right)\{[\psi^{+}_{C}(\mathbf{s})\psi^{+}_{C}(\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{\mathbf{C}}(\mathbf{s})}\right)\{[2\psi_{\mathbf{C}}(\mathbf{s})\psi^{+}_{C}(\mathbf{s})-\delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\{[\psi_{C}(\mathbf{s})\psi_{C}(\mathbf{s})]\psi^{+}_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{[\psi^{+}_{C}(\mathbf{s})\psi_{C}(\mathbf{s})-\delta_{C}(\mathbf{s},\mathbf{s})]\psi_{C}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^{+}_{NC}(\mathbf{s})}\right)\{[\psi_{C}(\mathbf{s})\psi^{+}_{C}(\mathbf{s})-\delta_{C}(\mathbf{s},\mathbf{s})]\psi^{+}_{C}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\text{(E.170)}
$$

$$
\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right)^{2}_{V14}
$$

$$
= \quad \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^{+}_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi^{+}_{C}(\mathbf{s})\psi_{C}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^{+}_{NC}(\mathbf{s})}\right)\{\psi_{C}(\mathbf{s})\psi^{+}_{C}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^{+}_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\delta_{C}(\mathbf{s},\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^{+}_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\delta_{C}(\mathbf{s},\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_{C}(\mathbf{s})\psi_{C}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^{+}_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^{+}_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^{+}_{C}(\mathbf{s})\psi^{+}_{C}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^{*}}(\mathbf{r})]\right\}
$$

$$
\text{(E.171)}
$$

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{V14}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{4}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.172)}$$

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^4_{V14}$$

$$= \frac{-i}{\hbar}\left\{g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.173)}$$

Appendix E.7.2. Second Order Term

$$\widehat{V}_{12} = -g\int\int d\mathbf{r}\,d\mathbf{s}\,F(\mathbf{r},\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{r})^\dagger\,\widehat{\Psi}_C(\mathbf{s}) - g\int\int d\mathbf{r}\,d\mathbf{s}\,F^*(\mathbf{s},\mathbf{r})\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})$$

$$\text{(E.174)}$$

Now if

$$\widehat{\rho} \rightarrow \widehat{V}_{12}\,\widehat{\rho} = \int\int d\mathbf{s}\,d\mathbf{u}\,(\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\triangle V(\mathbf{s},\mathbf{u})\widehat{\Psi}_C(\mathbf{u}) + \widehat{\Psi}_C(\mathbf{s})^\dagger\triangle V(\mathbf{u},\mathbf{s})^*\widehat{\Psi}_{NC}(\mathbf{u}))\,\widehat{\rho}$$

$$\text{(E.175)}$$

where we write $\triangle V(\mathbf{s}, \mathbf{u}) = -gF(\mathbf{s}, \mathbf{u})$ for short, then

$$
WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]
$$
$$
\rightarrow \quad \int \int d\mathbf{s}\, d\mathbf{u} \left\{ \left(\phi^+(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})} \right) \triangle V(\mathbf{s}, \mathbf{u}) \left(\psi(\mathbf{u}) + \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{u})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
+ \int \int d\mathbf{s}\, d\mathbf{u} \left\{ \left(\psi^+(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \triangle V(\mathbf{u}, \mathbf{s})^* (\phi(\mathbf{u})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$

$$(E.176)$$

Expanding we get

$$
WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]
$$
$$
\rightarrow \quad \int \int d\mathbf{s}\, d\mathbf{u} \left\{ (\phi^+(\mathbf{s})) \triangle V(\mathbf{s}, \mathbf{u}) (\psi(\mathbf{u})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
+ \int \int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ (\phi^+(\mathbf{s})) \triangle V(\mathbf{s}, \mathbf{u}) \left(\frac{\delta}{\delta\psi^+(\mathbf{u})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
- \int \int d\mathbf{s}\, d\mathbf{u} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \triangle V(\mathbf{s}, \mathbf{u}) (\psi(\mathbf{u})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
- \int \int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \triangle V(\mathbf{s}, \mathbf{u}) \left(\frac{\delta}{\delta\psi^+(\mathbf{u})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
+ \int \int d\mathbf{s}\, d\mathbf{u} \left\{ (\psi^+(\mathbf{s})) \triangle V(\mathbf{u}, \mathbf{s})^* (\phi(\mathbf{u})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
- \int \int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \triangle V(\mathbf{u}, \mathbf{s})^* (\phi(\mathbf{u})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$

$$(E.177)$$

The first term is

$$
\int \int d\mathbf{s}\, d\mathbf{u} \left\{ (\phi^+(\mathbf{s})) \triangle V(\mathbf{s}, \mathbf{u}) (\psi(\mathbf{u})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
= \quad \int \int d\mathbf{s}\, d\mathbf{u} \left\{ (\psi(\mathbf{u})) \triangle V(\mathbf{s}, \mathbf{u}) (\phi^+(\mathbf{s})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$

$$(E.178)$$

Using the product rule and the second term becomes

$$
\int \int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ (\phi^+(\mathbf{s})) \triangle V(\mathbf{s}, \mathbf{u}) \left(\frac{\delta}{\delta\psi^+(\mathbf{u})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$
$$
= \quad \int \int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{u})} \right) (\triangle V(\mathbf{s}, \mathbf{u})\phi^+(\mathbf{s})) \right\} WP[\psi, \psi^+, \phi, \phi^+]
$$

$$(E.179)$$

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The third term is

$$
\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\triangle V(\mathbf{s},\mathbf{u})\left(\psi(\mathbf{u})\right)\right\}WP(\psi,\psi^{+},\phi,\phi^{+}]
$$
$$
=\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)(\triangle V(\mathbf{s},\mathbf{u})\psi(\mathbf{u}))\right\}WP(\psi,\psi^{+},\phi,\phi^{+}]
$$

$$(\text{E.180})$$

Using the result that the functional derivatives can be performed in either order the fourth term is

$$
\int\int ds\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\triangle V(\mathbf{s},\mathbf{u})\left(\frac{\delta}{\delta\psi^{+}(\mathbf{u})}\right)\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$
$$
=\int\int ds\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^{+}(\mathbf{u})}\right)\triangle V(\mathbf{s},\mathbf{u})\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$

$$(\text{E.181})$$

Combining these results we find that

$$
WP[\psi(\mathbf{r}),\psi^{+}(\mathbf{r}),\phi(\mathbf{r}),\phi^{+}(\mathbf{r})]
$$
$$
\rightarrow\int\int ds\,d\mathbf{u}\left\{\phi^{+}(\mathbf{s})\triangle V(\mathbf{s},\mathbf{u})\,\psi(\mathbf{u})\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$
$$
+\int\int ds\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^{+}(\mathbf{u})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\phi^{+}(\mathbf{s})\}\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$
$$
-\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\psi(\mathbf{u})\}\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$
$$
-\int\int ds\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^{+}(\mathbf{u})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\}\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$
$$
+\int\int ds\,d\mathbf{u}\left\{\psi^{+}(\mathbf{s})\triangle V(\mathbf{u},\mathbf{s})^{*}\,\phi(\mathbf{u})\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$
$$
-\int\int ds\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\triangle V(\mathbf{u},\mathbf{s})^{*}\,\phi(\mathbf{u})\}\right\}WP[\psi,\psi^{+},\phi,\phi^{+}]
$$

$$(\text{E.182})$$

Now if

$$
\widehat{\rho}\rightarrow\widehat{\rho}\widehat{V}_{12}
$$
$$
=\int\int ds\,d\mathbf{u}\,\widehat{\rho}(\widehat{\boldsymbol{\Psi}}_{\mathbf{NC}}(\mathbf{s})^{\dagger}\triangle\mathbf{V}(\mathbf{s},\mathbf{u})\widehat{\boldsymbol{\Psi}}_{\mathbf{C}}(\mathbf{u})+\widehat{\boldsymbol{\Psi}}_{\mathbf{C}}(\mathbf{s})^{\dagger}\triangle\mathbf{V}(\mathbf{u},\mathbf{s})^{*}\widehat{\boldsymbol{\Psi}}_{\mathbf{NC}}(\mathbf{u}))
$$

$$(\text{E.183})$$

then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow \int d\mathbf{s} \left\{ \left(\psi(\mathbf{u}) - \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{u})} \right) \triangle\mathbf{V}(\mathbf{s}, \mathbf{u}) \left(\phi^+(\mathbf{s}) \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$+ \int d\mathbf{s} \left\{ \left(\phi(\mathbf{u}) - \frac{\delta}{\delta\phi^+(\mathbf{u})} \right) \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^* \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$\text{(E.184)}$$

Expanding out gives

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow \int\int d\mathbf{s}\, d\mathbf{u} \left\{ (\psi(\mathbf{u})) \triangle\mathbf{V}(\mathbf{s}, \mathbf{u}) \left(\phi^+(\mathbf{s}) \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$- \int\int d\mathbf{s}\, d\mathbf{u} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{u})} \right) \triangle\mathbf{V}(\mathbf{s}, \mathbf{u}) \left(\phi^+(\mathbf{s}) \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$+ \int\int d\mathbf{s}\, d\mathbf{u} \left\{ (\phi(\mathbf{u})) \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^* \left(\psi^+(\mathbf{s}) \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$+ \int\int d\mathbf{s}\, d\mathbf{u} \frac{1}{2} \left\{ (\phi(\mathbf{u})) \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^* \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$- \int\int d\mathbf{s}\, d\mathbf{u} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{u})} \right) \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^* \left(\psi^+(\mathbf{s}) \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$- \int\int d\mathbf{s}\, d\mathbf{u} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{u})} \right) \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^* \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

Using a similar approach to that above we find that

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow \int\int d\mathbf{s}\, d\mathbf{u} \left\{ \psi(\mathbf{u}) \triangle\mathbf{V}(\mathbf{s}, \mathbf{u})\, \phi^+(\mathbf{s}) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$- \int\int d\mathbf{s}\, d\mathbf{u} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{u})} \right) \left\{ \triangle\mathbf{V}(\mathbf{s}, \mathbf{u})\phi^+(\mathbf{s}) \right\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$+ \int\int d\mathbf{s}\, d\mathbf{u} \left\{ \phi(\mathbf{u}) \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^*\, \psi^+(\mathbf{s}) \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$+ \int\int d\mathbf{s}\, d\mathbf{u} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \phi(\mathbf{u}) \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^* \right\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$- \int\int d\mathbf{s}\, d\mathbf{u} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{u})} \right) \left\{ \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^*\, \psi^+(\mathbf{s}) \right\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$- \int\int d\mathbf{s}\, d\mathbf{u} \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\phi^+(\mathbf{u})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left\{ \triangle\mathbf{V}(\mathbf{u}, \mathbf{s})^* \right\} \right\} WP[\psi, \psi^+, \phi, \phi^+]$$

$$\text{(E.185)}$$

Now if

$$\widehat{\rho} \to [\widehat{V}_{12}, \widehat{\rho}]$$
$$=[\int\int d\mathbf{s}\,d\mathbf{u}\,(\widehat{\mathbf{\Psi}}_{\mathbf{NC}}(\mathbf{s})^\dagger \triangle\mathbf{V}(\mathbf{s},\mathbf{u})\widehat{\mathbf{\Psi}}_{\mathbf{C}}(\mathbf{u}) + \widehat{\mathbf{\Psi}}_{\mathbf{C}}(\mathbf{s})^\dagger \triangle\mathbf{V}(\mathbf{u},\mathbf{s})^*\widehat{\mathbf{\Psi}}_{\mathbf{NC}}(\mathbf{u})),\widehat{\rho}]$$

(E.186)

then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$
$$\to\ \int\int d\mathbf{s}\,d\mathbf{u}\left\{\phi^+(\mathbf{s})\triangle V(\mathbf{s},\mathbf{u})\,\psi(\mathbf{u})\right\}WP$$
$$-\int\int d\mathbf{s}\,d\mathbf{u}\left\{\psi(\mathbf{u})\triangle\mathbf{V}(\mathbf{s},\mathbf{u})\,\phi^+(\mathbf{s})\right\}WP$$
$$+\int\int d\mathbf{s}\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{u})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\phi^+(\mathbf{s})\}\right\}WP$$
$$+\int\int d\mathbf{s}\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{u})}\right)\{\triangle\mathbf{V}(\mathbf{s},\mathbf{u})\phi^+(\mathbf{s})\}\right\}WP$$
$$-\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\psi(\mathbf{u})\}\right\}WP$$
$$+\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{u})}\right)\{\triangle\mathbf{V}(\mathbf{u},\mathbf{s})^*\,\psi^+(\mathbf{s})\}\right\}WP$$
$$-\int\int d\mathbf{s}\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{u})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\}\right\}WP$$
$$+\int\int d\mathbf{s}\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{u})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\triangle\mathbf{V}(\mathbf{u},\mathbf{s})^*\}\right\}WP$$
$$+\int\int d\mathbf{s}\,d\mathbf{u}\left\{\psi^+(\mathbf{s})\triangle V(\mathbf{u},\mathbf{s})^*\,\phi(\mathbf{u})\right\}WP$$
$$-\int\int d\mathbf{s}\,d\mathbf{u}\left\{\phi(\mathbf{u})\triangle\mathbf{V}(\mathbf{u},\mathbf{s})^*\,\psi^+(\mathbf{s})\right\}WP$$
$$-\int\int d\mathbf{s}\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\triangle V(\mathbf{u},\mathbf{s})^*\,\phi(\mathbf{u})\}\right\}WP$$
$$-\int\int d\mathbf{s}\,d\mathbf{u}\frac{1}{2}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{u})\triangle\mathbf{V}(\mathbf{u},\mathbf{s})^*\}\right\}WP$$

(E.187)

$$WP[\psi(\mathbf{r}),\psi^+(\mathbf{r}),\phi(\mathbf{r}),\phi^+(\mathbf{r})]$$

$$\rightarrow \quad \int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{u})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\phi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$-\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\triangle V(\mathbf{u},\mathbf{s})^*\,\phi(\mathbf{u})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$-\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\psi(\mathbf{u})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$+\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{u})}\right)\{\triangle \mathbf{V}(\mathbf{u},\mathbf{s})^*\,\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$-\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{u})}\right)\{\tfrac{1}{2}\triangle V(\mathbf{s},\mathbf{u})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$+\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{u})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\tfrac{1}{2}\triangle \mathbf{V}(\mathbf{u},\mathbf{s})^*\}\right\}WP[\psi,\psi^+,\phi,\phi^+]$$

$$\text{(E.188)}$$

Thus we see that the $\widehat{V}_{12}$ term produces functional derivatives of orders one and two. We may write the contributions to the functional Fokker-Planck equation in the form

$$\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)_{V12}$$

$$=\quad\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)^1_{V12}+\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)^2_{V12}$$

$$\text{(E.189)}$$

where

$$\left(\frac{\partial}{\partial t}WP[\psi,\psi^+,\phi,\phi^+]\right)^1_{V12}$$

$$=\quad\frac{-i}{\hbar}\left\{\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi^+(\mathbf{u})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\phi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\triangle V(\mathbf{u},\mathbf{s})^*\,\phi(\mathbf{u})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$+\frac{-i}{\hbar}\left\{\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi^+(\mathbf{u})}\right)\{\triangle \mathbf{V}(\mathbf{u},\mathbf{s})^*\,\psi^+(\mathbf{s})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$+\frac{-i}{\hbar}\left\{-\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\triangle V(\mathbf{s},\mathbf{u})\,\psi(\mathbf{u})\}\right\}WP[\psi,\psi^+,\phi,\phi^+]\right\}$$

$$\text{(E.190)}$$

$$\left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)^2_{V12}$$

$$= \frac{-i}{\hbar} \left\{ - \int \int ds \, d\mathbf{u} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{u})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \tfrac{1}{2} \triangle V(\mathbf{s}, \mathbf{u}) \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$+ \frac{-i}{\hbar} \left\{ \int \int ds \, d\mathbf{u} \left\{ \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{u})} \right) \{ \tfrac{1}{2} \triangle \mathbf{V}(\mathbf{u}, \mathbf{s})^* \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$\text{(E.191)}$$

For the single condensate mode case the result is simpler and can be obtained via the substitution $\triangle V(\mathbf{s}, \mathbf{u}) = \triangle V(\mathbf{s}) \delta(\mathbf{u} - \mathbf{s})$ with $\triangle V(\mathbf{s}) = -g \left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle$ and is given by

$$\left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)^1_{V12}$$

$$= \frac{-i}{\hbar} \left\{ + \int ds \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \triangle V \, \phi^+(\mathbf{s}) \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$\frac{-i}{\hbar} \left\{ - \int ds \left\{ \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \triangle V \, \phi(\mathbf{s}) \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$\frac{-i}{\hbar} \left\{ + \int ds \left\{ \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \triangle V \, \psi^+(\mathbf{s}) \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$\frac{-i}{\hbar} \left\{ - \int ds \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \triangle V \, \psi(\mathbf{s}) \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$\text{(E.192)}$$

$$\left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)^2_{V12}$$

$$= \frac{-i}{\hbar} \left\{ - \int ds \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \tfrac{1}{2} \triangle V \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$\frac{-i}{\hbar} \left\{ + \int ds \left\{ \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \tfrac{1}{2} \triangle V \} \right\} WP[\psi, \psi^+, \phi, \phi^+] \right\}$$

$$\text{(E.193)}$$

Reverting to the original notation and replacing $\triangle V(\mathbf{s}, \mathbf{u}) = -gF(\mathbf{s}, \mathbf{u})$ we

have for the two mode condensate case

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{1}_{V12}
$$

$$
= \frac{-i}{\hbar}\left\{-g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{u})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi_{NC}^{+}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\{F(\mathbf{u},\mathbf{s})^{*}\,\psi_{NC}(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{-g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{NC}^{+}(\mathbf{u})}\right)\{F(\mathbf{u},\mathbf{s})^{*}\,\psi_{C}^{+}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi_{C}(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$\text{(E.194)}$$

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{2}_{V12}
$$

$$
= \frac{-i}{\hbar}\left\{+g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{u})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}F(\mathbf{s},\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{-g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^{+}(\mathbf{u})}\right)\{\frac{1}{2}\mathbf{F}(\mathbf{u},\mathbf{s})^{*}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$\text{(E.195)}$$

For the case of the single mode condensate with $\triangle V(\mathbf{s}) = -g\left\langle\widehat{\Psi}_{C}(\mathbf{s})^{\dagger}\widehat{\Psi}_{C}(\mathbf{s})\right\rangle$

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right)^{1}_{V12}
$$

$$
= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}^{+}(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_{C}(\mathbf{s})^{\dagger}\widehat{\Psi}_{C}(\mathbf{s})\right\rangle\psi_{NC}^{+}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{C}(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_{C}(\mathbf{s})^{\dagger}\widehat{\Psi}_{C}(\mathbf{s})\right\rangle\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^{+}(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_{C}(\mathbf{s})^{\dagger}\widehat{\Psi}_{C}(\mathbf{s})\right\rangle\psi_{C}^{+}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_{C}(\mathbf{s})^{\dagger}\widehat{\Psi}_{C}(\mathbf{s})\right\rangle\psi_{C}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^{*}(\mathbf{r})]\right\}
$$

$$\text{(E.196)}$$

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V12}$$

$$= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\frac{1}{2}\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.197)$$

We can show using the special form of $F(\mathbf{r},\mathbf{s})$ for a single mode condensate, that the Fokker-Planck equation terms for $\widehat{V}_{12}$ can be obtained from the two mode case. We have

$$F(\mathbf{r},\mathbf{s}) = (N-1)\phi_1^*(\mathbf{r})\phi_1(\mathbf{r})\phi_1(\mathbf{r})\,\phi_1^*(\mathbf{s}) \qquad (E.198)$$

we can use the forms (E.315) for the functional derivatives involving the expansion coefficients

$$\frac{\delta}{\delta\psi_C(\mathbf{s})} \equiv \phi_1^*(\mathbf{s})\frac{\partial}{\partial\alpha_1} \qquad \frac{\delta}{\delta\psi_C^+(\mathbf{s})} \equiv \phi_1(\mathbf{s})\frac{\partial}{\partial\alpha_1^+}$$

$$\frac{\delta}{\delta\psi_{NC}(\mathbf{s})} \equiv \sum_{k\neq 1}\phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k} \qquad \frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})} \equiv \sum_{k\neq 1}\phi_k(\mathbf{s})\frac{\partial}{\partial\alpha_k^+}$$

$$P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})] \equiv P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*}) \qquad (E.199)$$

to show that this is the case.

Considering the first order functional derivative terms we see that

$$
\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V12}
$$

$$
= \frac{-i}{\hbar}\left\{-g\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{u})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{F(\mathbf{u},\mathbf{s})^*\,\psi_{NC}(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{-g\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{u})}\right)\{F(\mathbf{u},\mathbf{s})^*\,\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int\int ds\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi_C(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$

$$
= \frac{-i}{\hbar}\left\{-g\int\int ds\,d\mathbf{u}\left\{\left(\phi_1(\mathbf{u})\frac{\partial}{\partial\alpha_1^+}\right)\{(N-1)\phi_1^*(\mathbf{s})\phi_1(\mathbf{s})\phi_1(\mathbf{s})\,\phi_1^*(\mathbf{u})\,\psi_{NC}^+(\mathbf{s})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int\int ds\,d\mathbf{u}\left\{\left(\phi_1^*(\mathbf{s})\frac{\partial}{\partial\alpha_1}\right)\{(N-1)\phi_1(\mathbf{u})\phi_1^*(\mathbf{u})\phi_1^*(\mathbf{u})\,\phi_1(\mathbf{s})\,\psi_{NC}(\mathbf{u})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
+\frac{-i}{\hbar}\left\{-g\int\int ds\,d\mathbf{u}\left\{\left(\sum_{k\neq1}\phi_k(\mathbf{u})\frac{\partial}{\partial\alpha_k^+}\right)\{(N-1)\phi_1(\mathbf{u})\phi_1^*(\mathbf{u})\phi_1^*(\mathbf{u})\,\phi_1(\mathbf{s})\,\psi_C^+(\mathbf{s})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int\int ds\,d\mathbf{u}\left\{\left(\sum_{k\neq1}\phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k}\right)\{(N-1)\phi_1^*(\mathbf{s})\phi_1(\mathbf{s})\phi_1(\mathbf{s})\,\phi_1^*(\mathbf{u})\,\psi_C(\mathbf{u})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
= \frac{-i}{\hbar}\left\{-g\int ds\left\{\left(\phi_1(\mathbf{s})\frac{\partial}{\partial\alpha_1^+}\right)\{(N-1)\phi_1^*(\mathbf{s})\phi_1(\mathbf{s})\,\psi_{NC}^+(\mathbf{s})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int d\mathbf{u}\left\{\left(\phi_1^*(\mathbf{u})\frac{\partial}{\partial\alpha_1}\right)\{(N-1)\phi_1(\mathbf{u})\phi_1^*(\mathbf{u})\,\psi_{NC}(\mathbf{u})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
+\frac{-i}{\hbar}\left\{-g\int d\mathbf{u}\left\{\left(\sum_{k\neq1}\phi_k(\mathbf{u})\frac{\partial}{\partial\alpha_k^+}\right)\{(N-1)\phi_1(\mathbf{u})\phi_1^*(\mathbf{u})\,\alpha_1^+\phi_1^*(\mathbf{u})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int ds\left\{\left(\sum_{k\neq1}\phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k}\right)\{(N-1)\phi_1^*(\mathbf{s})\phi_1(\mathbf{s})\,\alpha_1\phi_1(\mathbf{s})\}\right\}P_b(\underset{\rightarrow}{\alpha},\underset{\rightarrow}{\alpha^*})\right\}
$$

$$
= \frac{-i}{\hbar}\left\{-g\int ds\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int ds\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
$$

$$
+\frac{-i}{\hbar}\left\{+g\int ds\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \tag{E.200}
$$

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which is the same as the single mode condensate result. We have used the results

$$\int d\mathbf{u}\, \phi_1(\mathbf{u})\phi_1^*(\mathbf{u}) = \int d\mathbf{s}\, \phi_1(\mathbf{s})\phi_1^*(\mathbf{s}) = 1$$

in the first and second terms and

$$\int d\mathbf{s}\, \phi_1(\mathbf{s})\, \psi_C^+(\mathbf{s}) = \alpha_1^+$$

$$\int d\mathbf{u}\, \phi_1^*(\mathbf{u})\, \psi_C(\mathbf{u}) = \alpha_1$$

in the third and fourth terms, changed dummies of integration and recalled the notation $\left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle = (N-1)\,|\phi_1(\mathbf{s})|^2$.

For the second order functional derivative terms

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)^2_{V12}$$

$$= \frac{-i}{\hbar} \left\{ +g \int\!\!\int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi_C^+(\mathbf{u})} \right) \left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})} \right) \{F(\mathbf{s},\mathbf{u})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$+ \frac{-i}{\hbar} \left\{ -g \int\!\!\int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi_C(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{u})} \right) \{\mathbf{F}(\mathbf{u},\mathbf{s})^*\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$= \frac{-i}{\hbar} \left\{ +g \int\!\!\int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ \left(\phi_1(\mathbf{u})\frac{\partial}{\partial\alpha_1^+} \right) \left(\sum_{k\neq 1} \phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k} \right) \{(N-1)\phi_1^*(\mathbf{s})\phi_1(\mathbf{s})\phi_1(\mathbf{s})\phi_1^*(\mathbf{u})\} \right\} P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha^*}) \right\}$$

$$\frac{-i}{\hbar} \left\{ -g \int\!\!\int d\mathbf{s}\, d\mathbf{u}\, \frac{1}{2} \left\{ \left(\phi_1^*(\mathbf{s})\frac{\partial}{\partial\alpha_1} \right) \left(\sum_{k\neq 1} \phi_k(\mathbf{u})\frac{\partial}{\partial\alpha_k^+} \right) \{(N-1)\phi_1(\mathbf{u})\phi_1^*(\mathbf{u})\phi_1^*(\mathbf{u})\phi_1(\mathbf{s})\} \right\} P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha^*}) \right\}$$

$$= \frac{-i}{\hbar} \left\{ +g \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\phi_1(\mathbf{s})\frac{\partial}{\partial\alpha_1^+} \right) \left(\sum_{k\neq 1} \phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k} \right) \{(N-1)\phi_1^*(\mathbf{s})\phi_1(\mathbf{s})\} \right\} P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha^*}) \right\}$$

$$+ \frac{-i}{\hbar} \left\{ -g \int d\mathbf{u}\, \frac{1}{2} \left\{ \left(\phi_1^*(\mathbf{u})\frac{\partial}{\partial\alpha_1} \right) \left(\sum_{k\neq 1} \phi_k(\mathbf{u})\frac{\partial}{\partial\alpha_k^+} \right) \{(N-1)\phi_1(\mathbf{u})\phi_1^*(\mathbf{u})\} \right\} P_b(\underset{\rightarrow}{\alpha}, \underset{\rightarrow}{\alpha^*}) \right\}$$

$$= \frac{-i}{\hbar} \left\{ +g \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})} \right) \{\left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$+ \frac{-i}{\hbar} \left\{ -g \int d\mathbf{s}\, \frac{1}{2} \left\{ \left(\frac{\delta}{\delta\psi_C(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})} \right) \{\left\langle \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \right\rangle\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \qquad \text{(E.201)}$$

which is the same as the single condensate result. Again the results $\int d\mathbf{u}\, \phi_1(\mathbf{u})\phi_1^*(\mathbf{u}) = \int d\mathbf{s}\, \phi_1(\mathbf{s})\phi_1^*(\mathbf{s}) = 1$ are used.

The second order term in the interaction between the condensate and the non-condensate is

$$
\widehat{V}_2 = \frac{g}{2} \int d\mathbf{s} \{ \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \widehat{\Psi}_C(\mathbf{s}) + \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s}) \widehat{\Psi}_{NC}(\mathbf{s})
$$

$$
+ 4 \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s}) \widehat{\Psi}_C(\mathbf{s}) \}
$$

$$(E.202)$$

This term is due to the boson-boson interaction.

Now if

$$
\widehat{\rho} \rightarrow \widehat{V}_2 \, \widehat{\rho}
$$

$$
= \frac{g}{2} \int d\mathbf{s} (\widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s}) \widehat{\Psi}_C(\mathbf{s}) + \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s}) \widehat{\Psi}_{NC}(\mathbf{s})) \widehat{\rho}
$$

$$
+ \frac{g}{2} \int d\mathbf{s} (4 \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s}) \widehat{\Psi}_C(\mathbf{s})) \widehat{\rho}
$$

$$(E.203)$$

then

$$
WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]
$$

$$
\rightarrow \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\psi(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right)
$$

$$
\times \left(\psi(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) WP[\psi, \psi^+, \phi, \phi^+]
$$

$$
+ \frac{g}{2} \int d\mathbf{s} \left(\psi^+(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi(\mathbf{s}))
$$

$$
\times (\phi(\mathbf{s})) \, WP[\psi, \psi^+, \phi, \phi^+]
$$

$$
+ 2g \int d\mathbf{s} \left(\phi^+(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi(\mathbf{s}))
$$

$$
\times \left(\psi(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) WP[\psi, \psi^+, \phi, \phi^+]
$$

$$(E.204)$$

Expanding gives

$$
WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]
$$

$$
= WP[\psi, \psi^+, \phi, \phi^+]_{1-16} + WP[\psi, \psi^+, \phi, \phi^+]_{17-20} + WP[\psi, \psi^+, \phi, \phi^+]_{21-28}
$$

$$(E.205)$$

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-16}$$

$$= \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(\phi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(\phi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(\phi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(\phi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\phi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\phi^+(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\phi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\phi^+(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\psi(\mathbf{s})\right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP$$

$$\text{(E.206)}$$

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{17-20}$$

$$= +\frac{g}{2}\int d\mathbf{s}\,(\psi^+(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\,WP$$

$$+\frac{g}{2}\int d\mathbf{s}\,(\psi^+(\mathbf{s}))\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\,WP$$

$$+\frac{g}{2}\int d\mathbf{s}\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\,WP$$

$$+\frac{g}{2}\int d\mathbf{s}\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\,WP$$

$$\text{(E.207)}$$

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{21-28}$$

$$= +2g\int d\mathbf{s}\,(\phi^+(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,(\phi(\mathbf{s}))\,(\psi(\mathbf{s}))\,WP$$

$$+2g\int d\mathbf{s}\,(\phi^+(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)WP$$

$$+2g\int d\mathbf{s}\,(\phi^+(\mathbf{s}))\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\phi(\mathbf{s}))\,(\psi(\mathbf{s}))\,WP$$

$$+2g\int d\mathbf{s}\,(\phi^+(\mathbf{s}))\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)WP$$

$$+2g\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,(\phi(\mathbf{s}))\,(\psi(\mathbf{s}))\,WP$$

$$+2g\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)WP$$

$$+2g\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\phi(\mathbf{s}))\,(\psi(\mathbf{s}))\,WP$$

$$+2g\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)WP$$

$$\text{(E.208)}$$

The functional derivatives are now placed on the left using results in which the

functional derivatives of differing fields are zero (see (E.320) and (E.321)) giving

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-16}$$

$$= g \int d\mathbf{s} \{ \frac{1}{2} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T1$$

$$+ g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T2$$

$$+ g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T3$$

$$+ g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{8} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad T4$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \frac{1}{2} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T5$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T6$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T7$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{8} \phi^+(\mathbf{s}) \} WP \qquad T8$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \frac{1}{2} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T9$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T10$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T11$$

$$- g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{8} \phi^+(\mathbf{s}) \} WP \qquad T12$$

$$+ g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \frac{1}{2} \psi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T13$$

$$+ g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \psi(\mathbf{s}) \} WP \qquad T14$$

$$+ g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \psi(\mathbf{s}) \} WP \qquad T15$$

$$+ g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{8} \} WP \qquad T16$$

$$(E.209)$$

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{17-20}$$

$$= +g \int d\mathbf{s} \{\frac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T17$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T18$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T19$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{8}\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T20$$

$$(E.210)$$

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{21-28}$$

$$= +g \int d\mathbf{s} \{2\phi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T21$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad T22.1$$

$$-g \int d\mathbf{s} \left\{ \phi^+(\mathbf{s})\delta_C(\mathbf{s}, \mathbf{s})\phi(\mathbf{s}) \right\} WP \qquad T22.2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T23$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T24$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{2\psi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T25$$

$$-g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad T26.1$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\delta_C(\mathbf{s}, \mathbf{s})\phi(\mathbf{s})\} \right\} WP \qquad T26.2$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T27$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\}WP \qquad T28$$

$$(E.211)$$

The two terms that needed extra treatment are

$$2g \int d\mathbf{s} \left(\phi^+(\mathbf{s})\right) \left(\psi^+(\mathbf{s})\right) (\phi(\mathbf{s})) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP \qquad T22$$

$$= g \int d\mathbf{s} \left\{ (\phi^+(\mathbf{s})) \left[\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) (\psi^+(\mathbf{s})) (\phi(\mathbf{s})) - (\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})) \right] \right\} WP$$

$$= g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP$$

$$-g \int d\mathbf{s} \left\{ \phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s}) \right\} WP$$

and

$$-2g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\psi^+(\mathbf{s})\right) (\phi(\mathbf{s})) \left(\frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) WP \qquad T26$$

$$= -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left[\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) (\psi^+(\mathbf{s})) (\phi(\mathbf{s})) - \delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s}) \right] \right\} WP$$

$$= -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\psi^+(\mathbf{s})\phi(\mathbf{s})\} \right\} WP$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\} \right\} WP$$

Collecting terms with the same order of functional derivatives we have

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$
$$\rightarrow \quad WP^0 + WP^1 + WP^2 + WP^3 + WP^4 \qquad (E.212)$$

where we have used upper subscripts for the $\widehat{V}_2\widehat{\rho}$ contributions and

$$WP^0$$
$$= g \int d\mathbf{s}\{\frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T1$$

$$+g \int d\mathbf{s}\{\frac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T17$$

$$+g \int d\mathbf{s}\{2\phi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T21$$

$$-g \int d\mathbf{s} \left\{ \phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s}) \right\} WP \qquad T22.2$$

$$(E.213)$$

$$WP^1$$

$$= \; +g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T2$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T3$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \frac{1}{2} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T5$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \frac{1}{2} \phi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T9$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \psi^+(\mathbf{s}) \phi(\mathbf{s}) \phi(\mathbf{s}) \} WP \qquad T18$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \psi^+(\mathbf{s}) \phi(\mathbf{s}) \phi(\mathbf{s}) \} WP \qquad T19$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \phi^+(\mathbf{s}) \psi^+(\mathbf{s}) \phi(\mathbf{s}) \} \right\} WP \qquad T22.1$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s}) \phi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T23$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ 2 \psi^+(\mathbf{s}) \phi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T25$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \delta_C(\mathbf{s}, \mathbf{s}) \phi(\mathbf{s}) \} \right\} WP \qquad T26.2$$

$$= \; +g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{2} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T2, T3$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \phi^+(\mathbf{s}) \psi^+(\mathbf{s}) \phi(\mathbf{s}) \} \right\} WP \qquad T22.1$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{2} \psi^+(\mathbf{s}) \phi(\mathbf{s}) \phi(\mathbf{s}) \} WP \qquad T18, T19$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s}) \phi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T23$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ 2 \psi^+(\mathbf{s}) \phi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T25$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s}) \psi(\mathbf{s}) \psi(\mathbf{s}) \} WP \qquad T5, T9$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \delta_C(\mathbf{s}, \mathbf{s}) \phi(\mathbf{s}) \} \right\} WP \qquad T26.2$$

$$(E.214)$$

$$
\begin{aligned}
WP^2 & \\
= \ & +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{8}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad T4 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{4}\phi^+(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T6 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{4}\phi^+(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T7 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{4}\phi^+(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T10 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{4}\phi^+(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T11 \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T13 \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\frac{1}{8}\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T20 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T24 \\
& -g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi(\mathbf{s})\}\right\}WP \qquad T26.1 \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T27 \\
= \ & +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{8}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad T4 \qquad A \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T24 \qquad H \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\frac{1}{8}\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T20 \qquad G \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s})+\psi^+(\mathbf{s})\phi(\mathbf{s})\}WP \\
& \qquad T6,T7,T10,T11,T26.1 \qquad BCDEI \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T27 \qquad J \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T13 \qquad F
\end{aligned}
$$

$$
\text{(E.215)}
$$

$$WP^3$$

$$= -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{8}\phi^+(\mathbf{s})\}WP \qquad T8$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{8}\phi^+(\mathbf{s})\}WP \qquad T12$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{4}\psi(\mathbf{s})\}WP \qquad T14$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{4}\psi(\mathbf{s})\}WP \qquad T15$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{2}\phi(\mathbf{s})\}WP \qquad T28$$

$$= -g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{4}\phi^+(\mathbf{s})\}WP \qquad T8,T12$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\phi(\mathbf{s})\}WP \qquad T28$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\psi(\mathbf{s})\}WP \qquad T14,T15$$

$$\text{(E.216)}$$

$$WP^4$$

$$= +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{8}\}WP \qquad T16$$

$$\text{(E.217)}$$

Now if

$$\widehat{\rho} \rightarrow \widehat{\rho}\widehat{V}_2$$

$$= \frac{g}{2}\int d\mathbf{s}\,\widehat{\rho}\,(\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\widehat{\Psi}_C(\mathbf{s}) + \widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{s}))$$

$$+\frac{g}{2}\int d\mathbf{s}\,\widehat{\rho}\,(4\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_C(\mathbf{s})) \qquad \text{(E.218)}$$

then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow \quad \frac{g}{2} \int d\mathbf{s} \left(\psi(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right)$$

$$\times \left(\phi^+(\mathbf{s}) \right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\phi(\mathbf{s}) - \frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\phi(\mathbf{s}) - \frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right)$$

$$\times \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) WP$$

$$+ 2g \int d\mathbf{s} \left(\psi(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\phi(\mathbf{s}) - \frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\psi^+(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right)$$

$$\times \left(\phi^+(\mathbf{s}) \right) WP[\psi, \psi^+, \phi, \phi^+] \tag{E.219}$$

Expanding this result gives

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$= \quad WP[\psi, \psi^+, \phi, \phi^+]_{1-4} + WP[\psi, \psi^+, \phi, \phi^+]_{5-20} + WP[\psi, \psi^+, \phi, \phi^+]_{21-28} \tag{E.220}$$

where

$$WP[\psi, \psi^+, \phi, \phi^+]_{1-4}$$

$$= \quad \frac{g}{2} \int d\mathbf{s} \left(\psi(\mathbf{s}) \right) \left(\psi(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(\psi(\mathbf{s}) \right) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\psi(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) WP$$

$$+ \frac{g}{2} \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) WP \tag{E.221}$$

$$WP[\psi, \psi^+, \phi, \phi^+]_{5-20}$$

$$
\begin{aligned}
= \quad & +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\,(\psi^+(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP \\
& +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\,(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP \\
& +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP \\
& +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\,(\phi(\mathbf{s}))\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\phi(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\phi(\mathbf{s}))\,(\psi^+(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\phi(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)(\psi^+(\mathbf{s}))\,WP \\
& +\frac{g}{2}\int d\mathbf{s}\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right)WP
\end{aligned}
$$

$$(\text{E.222})$$

$$WP[\psi, \psi^+, \phi, \phi^+]_{21-28}$$

$$
\begin{aligned}
= \quad & +2g \int d\mathbf{s} \, (\psi(\mathbf{s})) \, (\phi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP \\
& +2g \int d\mathbf{s} \, (\psi(\mathbf{s})) \, (\phi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, WP \\
& +2g \int d\mathbf{s} \, (\psi(\mathbf{s})) \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP \\
& +2g \int d\mathbf{s} \, (\psi(\mathbf{s})) \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, WP \\
& +2g \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP \\
& +2g \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, WP \\
& +2g \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\psi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP \\
& +2g \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, WP
\end{aligned}
$$

$$(\text{E.223})$$

The functional derivatives are now placed on the left using results in which the functional derivatives of differing fields are zero (see (E.320) and (E.321)) giving

$$WP[\psi, \psi^+, \phi, \phi^+]_{1-4}$$

$$
\begin{aligned}
= \quad & g \int d\mathbf{s} \{ \frac{1}{2} \psi(\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad && U1 \\
& -g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad && U2 \\
& -g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad && U3 \\
& +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \frac{1}{8} \phi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad && U4
\end{aligned}
$$

$$(\text{E.224})$$

$$WP[\psi, \psi^+, \phi, \phi^+]_{5-20}$$

$$= +g \int d\mathbf{s} \{\frac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U5$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\phi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U6$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\phi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U7$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{8}\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad U8$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U9$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U10$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U11$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{8}\phi(\mathbf{s})\}WP \qquad U12$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U13$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U14$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\phi(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U15$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{8}\phi(\mathbf{s})\}WP \qquad U16$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U17$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\psi^+(\mathbf{s})\}WP \qquad U18$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\psi^+(\mathbf{s})\}WP \qquad U19$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{8}\}WP \qquad U20$$

$$(\text{E.225})$$

$$WP[\psi, \psi^+, \phi, \phi^+]_{21-28}$$

$$= +g \int d\mathbf{s}\{2\psi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U21$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\psi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U22.1$$

$$-g \int d\mathbf{s}\{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U22.2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{2\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U23$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U24.1$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U24.2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\phi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U25$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U26$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U27$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\}WP \qquad U28$$

$$(E.226)$$

The two terms that needed extra treatment are

$$2g \int d\mathbf{s}\, (\psi(\mathbf{s}))\, (\phi(\mathbf{s})) \left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) (\phi^+(\mathbf{s}))\, WP \qquad U22$$

$$= g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) [\psi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})] - \delta_C(\mathbf{s},\mathbf{s}) [\phi(\mathbf{s})\phi^+(\mathbf{s})] \right\} WP$$

$$= g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\psi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP - g \int d\mathbf{s}\{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP$$

and

$$2g \int d\mathbf{s}\, (\psi(\mathbf{s})) \left(-\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) (\phi^+(\mathbf{s}))\, WP \qquad U24$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left\{ \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) [\psi(\mathbf{s})\phi^+(\mathbf{s})] - \delta_C(\mathbf{s},\mathbf{s})[\phi^+(\mathbf{s})] \right\} WP$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\}WP + g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\}WP$$

Collecting terms with the same order of functional derivatives we have

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$
$$\rightarrow WP_0 + WP_1 + WP_2 + WP_3 + WP_4 \qquad (E.227)$$

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where we have used lower subscripts for the $\widehat{\rho}\,\widehat{V}_2$contributions and

$$
\begin{aligned}
& WP_0 \\
= \;\; & g\int ds\{\tfrac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP && U1 \\
& +g\int ds\{\tfrac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP && U5 \\
& +g\int ds\{2\psi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP && U21 \\
& -g\int ds\{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP && U22.2
\end{aligned}
$$

$$(E.228)$$

$$WP_1$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \psi(\mathbf{s}) \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{4} \psi(\mathbf{s}) \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U3$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \phi(\mathbf{s}) \phi(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U6$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \phi(\mathbf{s}) \phi(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U7$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \frac{1}{2} \phi(\mathbf{s}) \psi^+(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U9$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \frac{1}{2} \phi(\mathbf{s}) \psi^+(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U13$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \psi(\mathbf{s}) \phi(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U22.1$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ 2\psi(\mathbf{s}) \psi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U23$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \delta_C(\mathbf{s},\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U24.2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s}) \psi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U25$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{2} \psi(\mathbf{s}) \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U2, U3$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s}) \psi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U25$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{2} \phi(\mathbf{s}) \phi(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U6, U7$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \psi(\mathbf{s}) \phi(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U22.1$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ 2\psi(\mathbf{s}) \psi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U23$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s}) \psi^+(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U9, U13$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \delta_C(\mathbf{s},\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U24.2$$

$$(E.229)$$

$$WP_2$$

$$= \ +g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{8} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U4$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{8} \phi(\mathbf{s}) \phi(\mathbf{s}) \} WP \qquad U8$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \phi(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U10$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \phi(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U11$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \phi(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U14$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{4} \phi(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U15$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \frac{1}{2} \psi^+(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U17$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \psi(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U24.1$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{2} \phi(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U26$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \psi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U27$$

$$= \ +g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{8} \phi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U4$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \{ \frac{1}{2} \phi(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U26$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \{ \frac{1}{8} \phi(\mathbf{s}) \phi(\mathbf{s}) \} WP \qquad U8$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s}) \psi^+(\mathbf{s}) + \psi(\mathbf{s}) \phi^+(\mathbf{s}) \} WP$$
$$U10, U11, U14, U15, U24.1$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \psi^+(\mathbf{s}) \phi^+(\mathbf{s}) \} WP \qquad U27$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \frac{1}{2} \psi^+(\mathbf{s}) \psi^+(\mathbf{s}) \} WP \qquad U17$$

$$(E.230)$$

$$WP_3$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{8}\phi(\mathbf{s})\}WP \qquad U12$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{8}\phi(\mathbf{s})\}WP \qquad U16$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\psi^+(\mathbf{s})\}WP \qquad U18$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{4}\psi^+(\mathbf{s})\}WP \qquad U19$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\}WP \qquad U28$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{4}\phi(\mathbf{s})\}WP \qquad U12, U16$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\}WP \qquad U28$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\psi^+(\mathbf{s})\}WP \qquad U18, U19$$

$$\text{(E.231)}$$

$$WP_4$$

$$= +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{8}\}WP \qquad U20$$

$$\text{(E.232)}$$

Now if

$$\widehat{\rho} \rightarrow [\widehat{V}_2, \widehat{\rho}]$$

$$= [\frac{g}{2} \int d\mathbf{s}(\widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s})\widehat{\Psi}_C(\mathbf{s}) + \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{s})), \widehat{\rho}]$$

$$+[\frac{g}{2} \int d\mathbf{s}(4\widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_C(\mathbf{s})), \widehat{\rho}]$$

$$\text{(E.233)}$$

then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$
$$\rightarrow WP_T^0 + WP_T^1 + WP_T^2 + WP_T^3 + WP_T^4 \qquad \text{(E.234)}$$

where the WP_T^n are obtained by subtracting the results for $\widehat{\rho}\widehat{V}_2$ from those for

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$\widehat{V}_2\,\widehat{\rho}$. We find that

$$
\begin{aligned}
& WP_T^0 \\
=\ & g\int d\mathbf{s}\{\tfrac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\}WP && T1 \\
& -g\int d\mathbf{s}\{\tfrac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP && U1 \\
& +g\int d\mathbf{s}\{\tfrac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP && T17 \\
& -g\int d\mathbf{s}\{\tfrac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP && U5 \\
& +g\int d\mathbf{s}\{2\phi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP && T21 \\
& -g\int d\mathbf{s}\{2\psi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP && U21 \\
& -g\int d\mathbf{s}\left\{\phi^+(\mathbf{s})\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\right\}WP && T22.2 \\
& +g\int d\mathbf{s}\{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP && U22.2 \\
=\ & 0 && \text{(E.235)}
\end{aligned}
$$

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$$WP_T^1$$

$$= \quad +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s}) \} WP \qquad T2, T3$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \frac{1}{2}\psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad U2, U3$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s}) \} \right\} WP \qquad T22.1$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad U25$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \frac{1}{2}\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad T18, T19$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \frac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\psi^+(\mathbf{s}) \} WP \qquad U6, U7$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s}) \} WP \qquad T23$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{ \psi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad U22.1$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ 2\psi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s}) \} WP \qquad T25$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ 2\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad U23$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s}) \} WP \qquad T5, T9$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s}) \} WP \qquad U9, U13$$

$$+g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s}) \} \right\} WP \qquad T26.2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad U24.2$$

After collecting together similar terms we get

$$
WP_T^1
$$

$$
\begin{aligned}
= \ &+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\phi^+(\mathbf{s})\psi(\mathbf{s})\}WP & T2,T3,U2,U3\\[4pt]
&+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{2\phi^+(\mathbf{s})\psi^+(\mathbf{s})\phi(\mathbf{s})\}WP & T22.1,U25\\[4pt]
&-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP & T18,T19,U6,U7\\[4pt]
&-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{2\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP & T23,U22.1\\[4pt]
&+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{2\psi(\mathbf{s})\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP & U23\\[4pt]
&+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP & U9,U13\\[4pt]
&-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\phi^+(\mathbf{s})\}WP & U24.2\\[4pt]
&-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{2\psi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP & T25\\[4pt]
&-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s})\psi(\mathbf{s})\}WP & T5,T9\\[4pt]
&+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\delta_C(\mathbf{s},\mathbf{s})\phi(\mathbf{s})\}WP & T26.2\\[8pt]
= \ &+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{[\phi^+(\mathbf{s})\psi(\mathbf{s})+2\psi^+(\mathbf{s})\phi(\mathbf{s})]\phi^+(\mathbf{s})\}WP\\[4pt]
&-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{[\phi(\mathbf{s})\psi^+(\mathbf{s})+2\psi(\mathbf{s})\phi^+(\mathbf{s})]\phi(\mathbf{s})\}WP\\[4pt]
&+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{[2\psi(\mathbf{s})\psi^+(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\phi^+(\mathbf{s})+[\psi^+(\mathbf{s})\psi^+(\mathbf{s})]\phi(\mathbf{s})\}WP\\[4pt]
&-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{[2\psi^+(\mathbf{s})\psi(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\phi(\mathbf{s})+[\psi(\mathbf{s})\psi(\mathbf{s})]\phi^+(\mathbf{s})\}WP
\end{aligned}
$$

$$(\text{E.236})$$

$$WP_T^2$$

$$= +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{8}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad T4Can1$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{8}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U4Can1$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T24Can2$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\frac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U26Can2$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\frac{1}{8}\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad T20Can3$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\frac{1}{8}\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad U8Can3$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s}) + \psi^+(\mathbf{s})\phi(\mathbf{s})\}WP$$
$$T6,T7,T10,T11,T26.1$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s}) + \psi(\mathbf{s})\phi^+(\mathbf{s})\}WP$$
$$U10,U11,U14,U15,U24.1$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T27$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad U27$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad T13$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\frac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad U17$$

$$= -g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi(\mathbf{s}) + \psi^+(\mathbf{s})\phi(\mathbf{s})\}WP$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi^+(\mathbf{s}) + \psi(\mathbf{s})\phi^+(\mathbf{s})\}WP$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\phi(\mathbf{s})\psi(\mathbf{s})\}WP$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{\frac{1}{2}\psi(\mathbf{s})\psi(\mathbf{s})\}WP$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\frac{1}{2}\psi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP$$

$$(E.237)$$

$$WP_T^3$$

$$
= -g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \tfrac{1}{4}\phi^+(\mathbf{s}) \} WP \qquad T8, T12
$$

$$
+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \tfrac{1}{4}\phi(\mathbf{s}) \} WP \qquad U12, U16
$$

$$
+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \tfrac{1}{2}\phi(\mathbf{s}) \} WP \qquad T28
$$

$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \tfrac{1}{2}\phi^+(\mathbf{s}) \} WP \qquad U28
$$

$$
+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \tfrac{1}{2}\psi(\mathbf{s}) \} WP \qquad T14, T15
$$

$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \tfrac{1}{2}\psi^+(\mathbf{s}) \} WP \qquad U18, U19
$$

$$\text{(E.238)}$$

$$WP_T^4$$

$$
= +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \tfrac{1}{8} \} WP \qquad T16
$$

$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{ \tfrac{1}{8} \} WP \qquad U20
$$

$$\text{(E.239)}$$

Thus we see that the $\widehat{V}_2$ term produces functional derivatives of orders one, two, three and four. We may write the contributions to the functional Fokker-Planck equation in the form

$$
\left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V2}
$$

$$
= \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V2}^1 + \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V2}^2
$$

$$
+ \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V2}^3 + \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V2}^4 \quad \text{(E.240)}
$$

where on reverting to the original notation

$$
\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V2}
$$

$$
\begin{aligned}
= \quad & \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\{[\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s})+2\psi_C^+(\mathbf{s})\psi_{NC}(\mathbf{s})]\psi_{NC}^+(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{[\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s})+2\psi_C(\mathbf{s})\psi_{NC}^+(\mathbf{s})]\psi_{NC}(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{[2\psi_C(\mathbf{s})\psi_C^+(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{[2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
\end{aligned}
$$

$$\text{(E.241)}$$

$$
\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V2}
$$

$$
\begin{aligned}
= \quad & \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s})+\psi_C^+(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s})+\psi_C(\mathbf{s})\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_C^+(\mathbf{s})\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\} \\
& \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\frac{1}{2}\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}
\end{aligned}
$$

$$\text{(E.242)}$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{V2}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{4}\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{4}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.243)$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^4_{V2}$$

$$= \frac{-i}{\hbar}\left\{g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.244)$$

Appendix E.9. Condensate - Non-Condensate Interaction - Third Order in Non-Condendate

For the Bogoliubov Hamiltonian for which we derive the functional Fokker-Planck equation the term $\widehat{V}_3$ is discarded, but for completeness we treat it here. The third order term in the interaction between the condensate and the non-condensate is

$$\widehat{V}_3 = g\int d\mathbf{s}(\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_C(\mathbf{s})) + \widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{s}))$$

$$(E.245)$$

This term is due to the boson-boson interaction.

Now if

$$\widehat{\rho} \rightarrow \widehat{V}_3\widehat{\rho}$$

$$= g\int d\mathbf{s}(\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_C(\mathbf{s}) + \widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})^\dagger\widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{s}))\widehat{\rho}$$

$$(E.246)$$

245

then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow \quad g \int d\mathbf{s} \left(\phi^+(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})} \right) (\phi(\mathbf{s}))$$

$$\times \left(\psi(\mathbf{s}) + \frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) WP[\psi, \psi^+, \phi, \phi^+]$$

$$+ g \int d\mathbf{s} \left(\psi^+(\mathbf{s}) - \frac{1}{2} \frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) - \frac{\delta}{\delta\phi(\mathbf{s})} \right) (\phi(\mathbf{s}))$$

$$\times (\phi(\mathbf{s})) WP[\psi, \psi^+, \phi, \phi^+]$$

$$(E.247)$$

Expanding gives

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$= \quad WP[\psi, \psi^+, \phi, \phi^+]_{1-8} + WP[\psi, \psi^+, \phi, \phi^+]_{9-12} \qquad (E.248)$$

where

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-8}$$

$$= \quad g \int d\mathbf{s} \left(\phi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) (\phi(\mathbf{s})) (\psi(\mathbf{s})) WP$$

$$+ g \int d\mathbf{s} \left(\phi^+(\mathbf{s}) \right) \left(\phi^+(\mathbf{s}) \right) (\phi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) WP$$

$$+ g \int d\mathbf{s} \left(\phi^+(\mathbf{s}) \right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\phi(\mathbf{s})) (\psi(\mathbf{s})) WP$$

$$+ g \int d\mathbf{s} \left(\phi^+(\mathbf{s}) \right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\phi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) (\phi(\mathbf{s})) (\psi(\mathbf{s})) WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\phi^+(\mathbf{s}) \right) (\phi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\phi(\mathbf{s})) (\psi(\mathbf{s})) WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})} \right) (\phi(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta\psi^+(\mathbf{s})} \right) WP$$

$$(E.249)$$

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{9-12}$$

$$= \ +g \int d\mathbf{s} \left(\psi^+(\mathbf{s})\right) \left(\phi^+(\mathbf{s})\right) \left(\phi(\mathbf{s})\right) \left(\phi(\mathbf{s})\right) WP$$

$$+g \int d\mathbf{s} \left(\psi^+(\mathbf{s})\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\phi(\mathbf{s})\right) \left(\phi(\mathbf{s})\right) WP$$

$$+g \int d\mathbf{s} \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\phi^+(\mathbf{s})\right) \left(\phi(\mathbf{s})\right) \left(\phi(\mathbf{s})\right) WP$$

$$+g \int d\mathbf{s} \left(-\frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(-\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\phi(\mathbf{s})\right) \left(\phi(\mathbf{s})\right) WP$$

$$\text{(E.250)}$$

The functional derivatives are now placed on the left using results in which the functional derivatives of differing fields are zero (see (E.320) and (E.321)) giving

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{1-8}$$

$$= \ g \int d\mathbf{s} \{\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\} WP \qquad V1$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\} WP \qquad V3$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V4$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\} WP \qquad V5$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V6$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\phi(\mathbf{s})\psi(\mathbf{s})\} WP \qquad V7$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\} WP \qquad V8$$

$$\text{(E.251)}$$

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]_{9-12}$$

$$= +g \int d\mathbf{s} \{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V9$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V10$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V11$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V12$$

$$(E.252)$$

Collecting terms with the same order of functional derivatives we have

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$
$$\rightarrow \quad WP^0 + WP^1 + WP^2 + WP^3 \qquad (E.253)$$

where we have used upper subscripts for the $\widehat{V}_3\widehat{\rho}$ contributions and

$$WP^0$$

$$= \quad g \int d\mathbf{s} \{\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\} WP \qquad V1$$

$$+g \int d\mathbf{s} \{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V9$$

$$(E.254)$$

$$WP^1$$

$$= \quad +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\} WP \qquad V3$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\} WP \qquad V5$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{\psi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V10$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V11$$

$$= \quad +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V2$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\} WP \qquad V11$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})}\right) \{[2\phi^+(\mathbf{s})\psi(\mathbf{s}) + \psi^+(\mathbf{s})\phi(\mathbf{s})]\phi(\mathbf{s})\} WP \qquad V3, V5, V10$$

$$(E.255)$$

$$WP^2$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \tfrac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad V4$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{ \tfrac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad V6$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \phi(\mathbf{s})\psi(\mathbf{s}) \} WP \qquad V7$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \tfrac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad V12$$

$$= -g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad V4, V6$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \tfrac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad V12$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \phi(\mathbf{s})\psi(\mathbf{s}) \} WP \qquad V7$$

$$(E.256)$$

$$WP^3$$

$$= +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi(\mathbf{s})} \right) \{ \tfrac{1}{2}\phi(\mathbf{s}) \} WP \qquad V8$$

$$(E.257)$$

Now if

$$\widehat{\rho} \rightarrow \widehat{\rho}\widehat{V}_3$$

$$= g \int d\mathbf{s}\, \widehat{\rho}\, (\widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s}) \widehat{\Psi}_C(\mathbf{s}) + \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s}) \widehat{\Psi}_{NC}(\mathbf{s}))$$

$$(E.258)$$

then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow g \int d\mathbf{s} \left(\psi(\mathbf{s}) - \frac{1}{2}\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\phi(\mathbf{s}) - \frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s}))$$

$$\times \left(\phi^+(\mathbf{s}) \right) WP[\psi, \psi^+, \phi, \phi^+]$$

$$+g \int d\mathbf{s} \left(\phi(\mathbf{s}) - \frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\phi(\mathbf{s}) - \frac{\delta}{\delta\phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s}))$$

$$\times \left(\psi^+(\mathbf{s}) + \frac{1}{2}\frac{\delta}{\delta\psi(\mathbf{s})} \right) WP[\psi, \psi^+, \phi, \phi^+]$$

$$(E.259)$$

Expanding gives

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$= WP[\psi, \psi^+, \phi, \phi^+]_{1-4} + WP[\psi, \psi^+, \phi, \phi^+]_{5-12} \qquad (E.260)$$

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where

$$WP[\psi, \psi^+, \phi, \phi^+]_{1-4}$$

$$= g \int d\mathbf{s} \, (\psi(\mathbf{s})) \, (\phi(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP$$

$$+ g \int d\mathbf{s} \, (\psi(\mathbf{s})) \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP$$

$$+ g \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP$$

$$+ g \int d\mathbf{s} \left(-\frac{1}{2} \frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, WP$$

$$(\text{E.261})$$

$$WP[\psi, \psi^+, \phi, \phi^+]_{5-12}$$

$$= + g \int d\mathbf{s} \, (\phi(\mathbf{s})) \, (\phi(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+ g \int d\mathbf{s} \, (\phi(\mathbf{s})) \, (\phi(\mathbf{s})) \, (\phi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) WP$$

$$+ g \int d\mathbf{s} \, (\phi(\mathbf{s})) \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+ g \int d\mathbf{s} \, (\phi(\mathbf{s})) \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \, (\phi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi(\mathbf{s})) \, (\phi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \, (\psi^+(\mathbf{s})) \, WP$$

$$+ g \int d\mathbf{s} \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(-\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) (\phi^+(\mathbf{s})) \left(\frac{1}{2} \frac{\delta}{\delta \psi(\mathbf{s})} \right) WP$$

$$(\text{E.262})$$

The functional derivatives are now placed on the left using results in which the

functional derivatives of differing fields are zero (see (E.320) and (E.321)) giving

$$
WP[\psi, \psi^+, \phi, \phi^+]_{1-4}
$$
$$
= \quad g \int d\mathbf{s}\{\psi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W1
$$
$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W2
$$
$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \{\frac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W3
$$
$$
+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W4
$$
$$
\text{(E.263)}
$$

and

$$
WP[\psi, \psi^+, \phi, \phi^+]_{5-12}
$$
$$
= \quad +g \int d\mathbf{s}\{\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W5
$$
$$
+g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\frac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W6
$$
$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W7
$$
$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\frac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W8
$$
$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\phi(\mathbf{s})\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W9
$$
$$
-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\frac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W10
$$
$$
+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W11
$$
$$
+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta\psi(\mathbf{s})} \right) \{\frac{1}{2}\phi^+(\mathbf{s})\}WP \qquad W12
$$
$$
\text{(E.264)}
$$

Collecting terms with the same order of functional derivatives we have

$$
WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]
$$
$$
\rightarrow \quad WP_0 + WP_1 + WP_2 + WP_3 \qquad \text{(E.265)}
$$

where we have used lower subscripts for the $\widehat{\rho}\,\widehat{V}_3$ contributions and

$$
\begin{aligned}
WP_0 \\
= \quad & g\int d\mathbf{s}\{\psi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W1 \\
& +g\int d\mathbf{s}\{\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W5
\end{aligned}
$$

$$(E.266)$$

$$
\begin{aligned}
WP_1 \\
= \quad & -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\psi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W2 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\tfrac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W3 \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\tfrac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W6 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W7 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{\phi(\mathbf{s})\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W9 \\
= \quad & -g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\tfrac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W3 \\
& +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\tfrac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W6 \\
& -g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{[2\phi(\mathbf{s})\psi^+(\mathbf{s})+\psi(\mathbf{s})\phi^+(\mathbf{s})]\phi^+(\mathbf{s})\}WP \qquad W7, W9, W2
\end{aligned}
$$

$$(E.267)$$

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$$WP_2$$

$$= +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W4$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W8$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \{\frac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W10$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W11$$

$$= +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W4$$

$$-g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W8, W10$$

$$+g \int d\mathbf{s} \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W11$$

$$\text{(E.268)}$$

$$WP_3$$

$$= +g \int d\mathbf{s} \left(\frac{\delta}{\delta\psi(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right) \{\frac{1}{2}\phi^+(\mathbf{s})\}WP \qquad W12$$

$$\text{(E.269)}$$

Now if

$$\widehat{\rho} \rightarrow [\widehat{V}_3, \widehat{\rho}]$$

$$= [g \int d\mathbf{s}(\widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_C(\mathbf{s}) + \widehat{\Psi}_C(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})^\dagger \widehat{\Psi}_{NC}(\mathbf{s})\widehat{\Psi}_{NC}(\mathbf{s})), \widehat{\rho}]$$

$$\text{(E.270)}$$

then

$$WP[\psi(\mathbf{r}), \psi^+(\mathbf{r}), \phi(\mathbf{r}), \phi^+(\mathbf{r})]$$

$$\rightarrow WP_T^0 + WP_T^1 + WP_T^2 + WP_T^3 \qquad \text{(E.271)}$$

where the WP_T^n are obtained by subtracting the results for $\widehat{\rho}\widehat{V}_3$ from those for

$\widehat{V}_3\,\widehat{\rho}$. We find that

$$WP_T^0$$

$$= g\int d\mathbf{s}\{\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\psi(\mathbf{s})\}WP \qquad V1$$

$$-g\int d\mathbf{s}\{\psi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W1$$

$$+g\int d\mathbf{s}\{\psi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad V9$$

$$-g\int d\mathbf{s}\{\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\psi^+(\mathbf{s})\}WP \qquad W5$$

$$= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad (E.272)$$

$$WP_T^1$$

$$= +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\tfrac{1}{2}\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\}WP \qquad V2$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\tfrac{1}{2}\phi(\mathbf{s})\phi^+(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W3$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\tfrac{1}{2}\phi^+(\mathbf{s})\phi(\mathbf{s})\phi(\mathbf{s})\}WP \qquad V11$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\tfrac{1}{2}\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad W6$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{[2\phi^+(\mathbf{s})\psi(\mathbf{s})+\psi^+(\mathbf{s})\phi(\mathbf{s})]\phi(\mathbf{s})\}WP \qquad V3,V5,V10$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{[2\phi(\mathbf{s})\psi^+(\mathbf{s})+\psi(\mathbf{s})\phi^+(\mathbf{s})]\phi^+(\mathbf{s})\}WP \qquad W7,W9,W2$$

$$= +g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi^+(\mathbf{s})}\right)\{\phi^+(\mathbf{s})\phi^+(\mathbf{s})\phi(\mathbf{s})\}WP \qquad V2,W3$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi(\mathbf{s})}\right)\{\phi(\mathbf{s})\phi(\mathbf{s})\phi^+(\mathbf{s})\}WP \qquad V11,W6$$

$$+g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi^+(\mathbf{s})}\right)\{[2\phi(\mathbf{s})\psi^+(\mathbf{s})+\psi(\mathbf{s})\phi^+(\mathbf{s})]\phi^+(\mathbf{s})\}WP \qquad W7,W9,W2$$

$$-g\int d\mathbf{s}\left(\frac{\delta}{\delta\phi(\mathbf{s})}\right)\{[2\phi^+(\mathbf{s})\psi(\mathbf{s})+\psi^+(\mathbf{s})\phi(\mathbf{s})]\phi(\mathbf{s})\}WP \qquad V3,V5,V10$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (E.273)$$

$$WP_T^2$$

$$
\begin{aligned}
= \quad &-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \tfrac{1}{2} \phi^+(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad W4 \\
&+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \tfrac{1}{2} \phi(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad V12 \\
&-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\phi(\mathbf{s}) \} WP \qquad V4, V6 \\
&+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \phi(\mathbf{s})\phi^+(\mathbf{s}) \} WP \qquad W8, W10 \\
&-g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \phi^+(\mathbf{s})\psi^+(\mathbf{s}) \} WP \qquad W11 \\
&+g \int d\mathbf{s} \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \phi(\mathbf{s})\psi(\mathbf{s}) \} WP \qquad V7
\end{aligned}
$$

$$(E.274)$$

$$WP_T^3$$

$$
\begin{aligned}
= \quad &+g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi(\mathbf{s})} \right) \{ \tfrac{1}{2} \phi(\mathbf{s}) \} WP \qquad V8 \\
&-g \int d\mathbf{s} \left(\frac{\delta}{\delta \psi(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \left(\frac{\delta}{\delta \phi^+(\mathbf{s})} \right) \{ \tfrac{1}{2} \phi^+(\mathbf{s}) \} WP \qquad W12
\end{aligned}
$$

$$(E.275)$$

Thus we see that the $\widehat{V}_3$ term produces functional derivatives of orders one, two and three. We may write the contributions to the functional Fokker-Planck equation in the form

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V3} \\
= \quad &\left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V3}^1 + \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V3}^2 + \left(\frac{\partial}{\partial t} WP[\psi, \psi^+, \phi, \phi^+] \right)_{V3}^3
\end{aligned}
$$

$$(E.276)$$

where on reverting to the original notation

$$
\left(\frac{\partial}{\partial t}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right)^1_{V3}
$$

$$
= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\{\psi^+_{NC}(\mathbf{s})\psi^+_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\psi^+_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{[2\psi_{NC}(\mathbf{s})\psi^+_C(\mathbf{s})+\psi_C(\mathbf{s})\psi^+_{NC}(\mathbf{s})]\psi^+_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{[2\psi^+_{NC}(\mathbf{s})\psi_C(\mathbf{s})+\psi^+_C(\mathbf{s})\psi_{NC}(\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

(E.277)

$$
\left(\frac{\partial}{\partial t}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right)^2_{V3}
$$

$$
= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{1}{2}\psi^+_{NC}(\mathbf{s})\right)(\psi^+_{NC}(\mathbf{s}))\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi^+_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi^+_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\psi^+_{NC}(\mathbf{s})\psi^+_C(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

(E.278)

$$
\left(\frac{\partial}{\partial t}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right)^3_{V3}
$$

$$
= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

$$
\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_{NC}(\mathbf{s})\}\right\}P[\underrightarrow{\psi}(\mathbf{r}),\underrightarrow{\psi^*}(\mathbf{r})]\right\}
$$

(E.279)

Note that these third order terms are not included in the functional Fokker-Planck equation for the Bogoliubov Hamiltonian.

Appendix E.10. Summary of Results

The *functional Fokker-Planck equation* may be written in the form

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_C + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{NC}$$

$$+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_V \tag{E.280}$$

of the sum of terms from the condensate, non-condensate and interaction terms in the Hamiltonian.

Appendix E.10.1. Condensate Hamiltonian Terms

The contributions to the *functional Fokker-Planck equation* from the *condensate Hamiltonian* may be written in the form

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_C$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_K + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_V$$

$$+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_U \tag{E.281}$$

of the sum of terms from the kinetic energy, the trap potential and the boson-boson interaction. Derivations of the form for each term are given in Appendix E. Here and elsewhere ∂_μ is short for $\frac{\partial}{\partial s_\mu}$.

The contribution to the functional Fokker-Planck equation from the *kinetic energy* is given by

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_K$$

$$= \frac{-i}{\hbar}\left\{-\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\left(\sum_\mu\frac{\hbar^2}{2m}\partial_\mu^2\psi_C^+(\mathbf{s})\right)P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}\right\}$$

$$+ \frac{-i}{\hbar}\left\{+\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi_C(\mathbf{s})}\left(\sum_\mu\frac{\hbar^2}{2m}\partial_\mu^2\psi_C(\mathbf{s})\right)P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}\right\}$$

$$\tag{E.282}$$

The contribution to the functional Fokker-Planck equation from the *trap*

potential is given by

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_V$$

$$= \frac{-i}{\hbar}\left\{-\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi_C(\mathbf{s})}\{V(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+\int d\mathbf{s}\left\{\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\{V(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.283)}$$

The contribution to the functional Fokker-Planck equation from the *boson-boson interaction* is given by

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_U$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi_C(\mathbf{s})}\left\{[\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_C(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\left\{[\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_C^+(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{g\int d\mathbf{s}\frac{\delta}{\delta\psi_C(\mathbf{s})}\frac{\delta}{\delta\psi_C(\mathbf{s})}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\{\frac{1}{4}\psi_C(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\frac{\delta}{\delta\psi_C(\mathbf{s})}\{\frac{1}{4}\psi_C^+(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.284)}$$

which involves first order and third order functional derivatives. The quantity $\delta_C(\mathbf{s},\mathbf{s})$ is a diagonal element of the restricted delta function for condensate modes. We note that

$$\int d\mathbf{s}\,\delta_C(\mathbf{s},\mathbf{s})=1 \qquad\qquad \text{(E.285)}$$

corresponding to there being a single occupied condensate mode in this treatment. The total condensate number given by

$$N_C = \iiiint D^2\psi_C\,D^2\psi_C^+\,D^2\psi_{NC}\,D^2\psi_{NC}^+\int d\mathbf{s}(\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}))P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]$$

$$\text{(E.286)}$$

is depleted by one.

Appendix E.10.2. Non-Condensate Hamiltonian Terms

The contributions to the *functional Fokker-Planck equation* from the *non-condensate Hamiltonian* may be written in the form

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_{NC}$$

$$= \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_K + \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_V$$

$$+ \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_U \tag{E.287}$$

of the sum of terms from the kinetic energy, the trap potential and the boson-boson interaction. Derivations of the form for each term are given in Appendix E.

The contribution to the functional Fokker-Planck equation from the *kinetic energy* is given by

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_K$$

$$= \frac{-i}{\hbar} \left\{ - \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_{NC}^+(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi_{NC}^+(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \right\}$$

$$+ \frac{-i}{\hbar} \left\{ + \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_{NC}(\mathbf{s})} \left(\sum_\mu \frac{\hbar^2}{2m} \partial_\mu^2 \psi_{NC}(\mathbf{s}) \right) P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \right\} \tag{E.288}$$

The contribution to the functional Fokker-Planck equation from the *trap potential* is given by

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right)_V$$

$$= \frac{-i}{\hbar} \left\{ - \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_{NC}(\mathbf{s})} \{ V(\mathbf{s}) \psi_{NC}(\mathbf{s}) \} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\frac{-i}{\hbar} \left\{ + \int d\mathbf{s} \left\{ \frac{\delta}{\delta \psi_{NC}^+(\mathbf{s})} V(\mathbf{s}) \psi_{NC}^+(\mathbf{s}) \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\} \tag{E.289}$$

The contribution to the functional Fokker-Planck equation from the *boson-*

boson interaction is given by

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_U$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\left\{[\psi_{NC}^+(\mathbf{s})\psi_{NC}(\mathbf{s})]\psi_{NC}(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\left\{[\psi_{NC}^+(\mathbf{s})\psi_{NC}(\mathbf{s})]\psi_{NC}^+(\mathbf{s})\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\{\frac{1}{2}\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\{\frac{1}{2}\psi_{NC}^+(\mathbf{s})\psi_{NC}^+(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right\}$$

$$(\text{E.290})$$

This term is part of the interaction term $\widehat{H}_5$ and its contribution to the functional Fokker-Planck equation will be ignored.

Appendix E.10.3. Interaction between Condensate and Non-Condensate Terms
The contributions to the *functional Fokker-Planck equation* from the *interaction Hamiltonian* between the condensate and non-condensate may be written in the form

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_V$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{V1} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{V2}$$

$$+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{V3} \qquad (\text{E.291})$$

of the sum of first, second and third order terms in the non-condensate field operators. Derivations of the form for each term are given in Appendix E.

First Order Terms The contribution to the *functional Fokker-Planck equation* from the *first order term* in the *interaction Hamiltonian* between the condensate and non-condensate may be written in the form

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{V1}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{V14} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi}^*(\mathbf{r})]\right)_{V12} \qquad (\text{E.292})$$

These two contributions may be written as the sum of terms which are linear, quadratic, cubic and quartic in the number of functional derivatives. For the

$\widehat{V}_{14}$ term

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V14}$$

$$= \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V14}^{1} + \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V14}^{2}$$

$$+ \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V14}^{3} + \left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V14}^{4} \quad \text{(E.293)}$$

where

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V14}^{1}$$

$$= \frac{-i}{\hbar} \left\{ +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right) \{[2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - \delta_C(\mathbf{s},\mathbf{s})]\psi_{NC}^+(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\frac{-i}{\hbar} \left\{ +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi_{\mathbf{C}}^+(\mathbf{s})}\right) \{[\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})]\psi_{NC}(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\frac{-i}{\hbar} \left\{ -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi_{\mathbf{C}}(\mathbf{s})}\right) \{[2\psi_{\mathbf{C}}(\mathbf{s})\psi_C^+(\mathbf{s}) - \delta_{\mathbf{C}}(\mathbf{s},\mathbf{s})]\psi_{NC}(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\frac{-i}{\hbar} \left\{ -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right) \{[\psi_C(\mathbf{s})\psi_C(\mathbf{s})]\psi_{NC}^+(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\frac{-i}{\hbar} \left\{ -g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right) \{[\psi_C^+(\mathbf{s})\psi_C(\mathbf{s}) - \delta_C(\mathbf{s},\mathbf{s})]\psi_C(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\frac{-i}{\hbar} \left\{ +g \int d\mathbf{s} \left\{ \left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right) \{[\psi_C(\mathbf{s})\psi_C^+(\mathbf{s}) - \delta_C(\mathbf{s},\mathbf{s})]\psi_C^+(\mathbf{s})\} \right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})] \right\}$$

$$\text{(E.294)}$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V14}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi^+_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\psi_C(\mathbf{s})\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\delta_C(\mathbf{s},\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_C(\mathbf{s})\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(\text{E.295})$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{V14}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\frac{1}{4}\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\{\frac{1}{4}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{4}\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{4}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(\text{E.296})$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^4_{V14}$$

$$= \frac{-i}{\hbar}\left\{g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.297)}$$

For the $\widehat{V}_{12}$ term

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V12}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V12} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V12} \quad \text{(E.298)}$$

where for the two mode condensate case

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V12}$$

$$= \frac{-i}{\hbar}\left\{-g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{u})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+ \frac{-i}{\hbar}\left\{+g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{F(\mathbf{u},\mathbf{s})^*\,\psi_{NC}(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+ \frac{-i}{\hbar}\left\{-g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{u})}\right)\{F(\mathbf{u},\mathbf{s})^*\,\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+ \frac{-i}{\hbar}\left\{+g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{F(\mathbf{s},\mathbf{u})\,\psi_C(\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.299)}$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V12}$$

$$= \frac{-i}{\hbar}\left\{+g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{u})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}F(\mathbf{s},\mathbf{u})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+ \frac{-i}{\hbar}\left\{-g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{u})}\right)\{\frac{1}{2}\mathbf{F}(\mathbf{u},\mathbf{s})^*\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.300)}$$

These results may also be written as

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V12}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\{\int d\mathbf{u}\, F(\mathbf{u},\mathbf{s})\,\psi^+_{NC}(\mathbf{u})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\int d\mathbf{u}\, F(\mathbf{u},\mathbf{s})^*\,\psi_{NC}(\mathbf{u})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\int d\mathbf{u}\, F(\mathbf{s},\mathbf{u})^*\,\psi^+_C(\mathbf{u})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\int d\mathbf{u}\, F(\mathbf{s},\mathbf{u})\,\psi_C(\mathbf{u})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(\text{E.301})$$

so the quantity inside the inner brackets is just another functional. The quadratic term is left unchanged except for interchanging positions to make the expression more symmetrical

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V12}$$

$$= \frac{-i}{\hbar}\left\{+g\int\int d\mathbf{s}\, d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{u})}\right)\{\tfrac{1}{2}F(\mathbf{u},\mathbf{s})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$+\frac{-i}{\hbar}\left\{-g\int\int d\mathbf{s}\,d\mathbf{u}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{u})}\right)\{\tfrac{1}{2}\mathbf{F}(\mathbf{u},\mathbf{s})^*\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(\text{E.302})$$

For the single mode condensate case

$$\left(\frac{\partial}{\partial t} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V12}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi^+_{NC}(\mathbf{s})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi_{NC}(\mathbf{s})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi^+_C(\mathbf{s})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\left\langle\widehat{\Psi}_C(\mathbf{s})^\dagger\widehat{\Psi}_C(\mathbf{s})\right\rangle\psi_C(\mathbf{s})\}\right\} P[\underset{\rightarrow}{\psi}(\mathbf{r}), \underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(\text{E.303})$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V12}$$

$$= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\left\langle\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})\right\rangle\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\frac{1}{2}\left\langle\widehat{\Psi}_C(\mathbf{r})^\dagger\widehat{\Psi}_C(\mathbf{r})\right\rangle\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.304)$$

Second Order Terms The contribution to the *functional Fokker-Planck equation* from the *second order term* in the *interaction Hamiltonian* between the condensate and non-condensate may be written as the sum of terms which are linear, quadratic, cubic and quartic in the number of functional derivatives

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V2}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V2} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V2}$$

$$+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{V2} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^4_{V2} \qquad (E.305)$$

where

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V2}$$

$$= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\{[\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s})+2\psi_C^+(\mathbf{s})\psi_{NC}(\mathbf{s})]\psi_{NC}^+(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{[\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s})+2\psi_C(\mathbf{s})\psi_{NC}^+(\mathbf{s})]\psi_{NC}(\mathbf{s})\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{[2\psi_C(\mathbf{s})\psi_C^+(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{[2\psi_C^+(\mathbf{s})\psi_C(\mathbf{s})-\delta_C(\mathbf{s},\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.306)$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V2}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi^+_{NC}(\mathbf{s})\psi_C(\mathbf{s})+\psi^+_C(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi^+_C(\mathbf{s})+\psi_C(\mathbf{s})\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\psi^+_C(\mathbf{s})\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_C(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_C(\mathbf{s})\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.307)$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{V2}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{4}\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{4}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi^+_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.308)$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^4_{V2}$$

$$= \frac{-i}{\hbar}\left\{g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi^+_{NC}(\mathbf{s})}\right)\{\frac{1}{8}\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$(E.309)$$

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Third Order Terms The contribution to the *functional Fokker-Planck equation* from the *third order term* in the *interaction Hamiltonian* between the condensate and non-condensate may be written as the sum of terms which are linear, quadratic and cubic in the number of functional derivatives

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)_{V3}$$

$$= \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V3} + \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V3}$$

$$+ \left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{V3} \tag{E.310}$$

where

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^1_{V3}$$

$$= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_{NC}^+(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{[2\psi_{NC}(\mathbf{s})\psi_C^+(\mathbf{s})+\psi_C(\mathbf{s})\psi_{NC}^+(\mathbf{s})]\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{[2\psi_{NC}^+(\mathbf{s})\psi_C(\mathbf{s})+\psi_C^+(\mathbf{s})\psi_{NC}(\mathbf{s})]\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\tag{E.311}$$

$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^2_{V3}$$

$$= \frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\frac{1}{2}\psi_{NC}^+(\mathbf{s})\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\frac{1}{2}\psi_{NC}(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\psi_{NC}^+(\mathbf{s})\psi_C^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\psi_{NC}(\mathbf{s})\psi_C(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\tag{E.312}$$

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$$\left(\frac{\partial}{\partial t}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right)^3_{V3}$$

$$= \frac{-i}{\hbar}\left\{+g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_{NC}(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\frac{-i}{\hbar}\left\{-g\int d\mathbf{s}\left\{\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)\{\tfrac{1}{2}\psi_{NC}^+(\mathbf{s})\}\right\}P[\underset{\rightarrow}{\psi}(\mathbf{r}),\underset{\rightarrow}{\psi^*}(\mathbf{r})]\right\}$$

$$\text{(E.313)}$$

This term is part of the interaction term $\widehat{H}_4$ and its contribution to the functional Fokker-Planck equation will be ignored.

Appendix E.11. Supplementary Equations

Bogoliubov Hamiltonian

$$\widehat{H}_B = \widehat{H}_1 + \widehat{H}_2 + \widehat{H}_3 \tag{E.314}$$

Operator identities for various functional derivatives

$$\left(\frac{\delta}{\delta\psi_C(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k=1,2}\phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k} \qquad \left(\frac{\delta}{\delta\psi_{NC}(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k\neq1,2}^{K}\phi_k^*(\mathbf{s})\frac{\partial}{\partial\alpha_k}$$

$$\left(\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k=1,2}\phi_k(\mathbf{s})\frac{\partial}{\partial\alpha_k^+} \qquad \left(\frac{\delta}{\delta\psi_{NC}^+(\mathbf{s})}\right)_{\mathbf{s}} \equiv \sum_{k\neq1,2}^{K}\phi_k(\mathbf{s})\frac{\partial}{\partial\alpha_k^+}$$

$$\text{(E.315)}$$

Field Functions

$$\psi_C(\mathbf{r}) = \alpha_1\phi_1(\mathbf{r}) + \alpha_2\phi_2(\mathbf{r}) \qquad \psi_C^+(\mathbf{r}) = \phi_1^*(\mathbf{r})\alpha_1^+ + \phi_2^*(\mathbf{r})\alpha_2^+ \quad \text{(E.316)}$$

$$\psi_{NC}(\mathbf{r}) = \sum_{k\neq1,2}\alpha_k\phi_k(\mathbf{r}) \qquad \psi_{NC}^+(\mathbf{r}) = \sum_{k\neq1,2}\phi_k^*(\mathbf{r})\alpha_k^+ \qquad \text{(E.317)}$$

$$\psi_C(\mathbf{r}) = \int d\mathbf{r}'\,\delta_C(\mathbf{r},\mathbf{r}')\psi_C(\mathbf{r}') \qquad \psi_C^+(\mathbf{r}) = \int d\mathbf{r}'\,\psi_C^+(\mathbf{r}')\delta_C(\mathbf{r}',\mathbf{r})$$

$$\psi_{NC}(\mathbf{r}) = \int d\mathbf{r}'\,\delta_{NC}(\mathbf{r},\mathbf{r}')\psi_{NC}(\mathbf{r}') \qquad \psi(\mathbf{r}) = \int d\mathbf{r}'\,\psi_C^+(\mathbf{r}')\delta_C(\mathbf{r}',\mathbf{r}) \quad \text{(E.318)}$$

Product rule for functional derivatives

$$\frac{\delta}{\delta\psi(\mathbf{s})}(F[\psi(\mathbf{r}),\psi^+(\mathbf{r})]G[\psi(\mathbf{r}),\psi^+(\mathbf{r})])$$

$$= \ (\frac{\delta}{\delta\psi(\mathbf{s})}F[\psi(\mathbf{r}),\psi^+(\mathbf{r})])G[\psi(\mathbf{r}),\psi^+(\mathbf{r})] + F[\psi(\mathbf{r}),\psi^+(\mathbf{r})](\frac{\delta}{\delta\psi(\mathbf{s})}G[\psi(\mathbf{r}),\psi^+(\mathbf{r})])$$

$$\frac{\delta}{\delta\psi^+(\mathbf{s})}(F[\psi(\mathbf{r}),\psi^+(\mathbf{r})]G[\psi(\mathbf{r}),\psi^+(\mathbf{r})])$$

$$= \ (\frac{\delta}{\delta\psi^+(\mathbf{s})}F[\psi(\mathbf{r}),\psi^+(\mathbf{r})])G[\psi(\mathbf{r}),\psi^+(\mathbf{r})] + F[\psi(\mathbf{r}),\psi^+(\mathbf{r})](\frac{\delta}{\delta\psi^+(\mathbf{s})}G[\psi(\mathbf{r}),\psi^+(\mathbf{r})])$$

$$(E.319)$$

Functional Derivative Results

$$\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_C(\mathbf{r}) \ = \ \delta_C(\mathbf{r},\mathbf{s})$$

$$\frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_C^+(\mathbf{r}) \ = \ \delta_{C+}(\mathbf{r},\mathbf{s}) = \delta_C(\mathbf{s},\mathbf{r})$$

$$\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_C^+(\mathbf{r}) \ = \ 0 \qquad \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_C(\mathbf{r}) = 0 \qquad (E.320)$$

$$\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_{NC}(\mathbf{r}) \ = \ 0 \qquad \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_{NC}^+(\mathbf{r}) = 0$$

$$\frac{\delta}{\delta\psi_C(\mathbf{s})}\psi_{NC}^+(\mathbf{r}) \ = \ 0 \qquad \frac{\delta}{\delta\psi_C^+(\mathbf{s})}\psi_{NC}(\mathbf{r}) = 0 \qquad (E.321)$$

Appendix F. - Ito Stochastic Equations

The Ito stochastic equations are obtained after neglecting *third, fourth order* functional derivatives in the functional Fokker-Planck equation. The drift and diffusion terms are then identified from the remaining first and second order functional derivative terms that are left and the Ito stochastic equations for the stochastic fields can then be written down.

Appendix F.1. Symmetric Forms of Functional Fokker-Planck Equation

For the two mode case the diffusion term in (F.27) becomes

$$
\begin{aligned}
T_{Diff} \;=\;& \sum_{A \leq B} \int \int dx\, dy\, \frac{\delta}{\delta \psi_A(x)} \frac{\delta}{\delta \psi_B(y)} H_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y)\, P \\
=\;& \frac{1}{2} \sum_{A < B} \int \int dx\, dy\, \frac{\delta}{\delta \psi_A(x)} \frac{\delta}{\delta \psi_B(y)} H_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y)\, P \\
& + \frac{1}{2} \sum_{A < B} \int \int dx\, dy\, \frac{\delta}{\delta \psi_A(x)} \frac{\delta}{\delta \psi_B(y)} H_{AB}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y)\, P \\
& + \frac{1}{2} \sum_{A} \int \int dx\, dy\, \frac{\delta}{\delta \psi_A(x)} \frac{\delta}{\delta \psi_A(y)} H_{AA}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y)\, P \\
& + \frac{1}{2} \sum_{A} \int \int dx\, dy\, \frac{\delta}{\delta \psi_A(x)} \frac{\delta}{\delta \psi_A(y)} H_{AA}(\underset{\rightarrow}{\psi}(x), x, \underset{\rightarrow}{\psi}(y), y)\, P
\end{aligned}
$$

$$\text{(F.1)}$$

If we interchange A, B and x, y in the second term and just x, y in the fourth term, we find on using the result that double functional differentiation can be

carried out in either order that

$$
\begin{aligned}
T_{Diff} \;=\; & \frac{1}{2}\sum_{A<B}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_B(y)}H_{AB}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y)\,P \\
& +\frac{1}{2}\sum_{B<A}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_B(y)}H_{BA}(\underrightarrow{\psi}(y),y,\underrightarrow{\psi}(x),x)\,P \\
& +\frac{1}{2}\sum_{A}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_A(y)}H_{AA}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y)\,P \\
& +\frac{1}{2}\sum_{A}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_A(y)}H_{AA}(\underrightarrow{\psi}(y),y,\underrightarrow{\psi}(x),x)\,P \\
\;=\; & \frac{1}{2}\sum_{A<B}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_B(y)}H_{AB}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y)\,P \\
& +\frac{1}{2}\sum_{A>B}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_B(y)}H_{BA}(\underrightarrow{\psi}(y),y,\underrightarrow{\psi}(x),x)\,P \\
& +\frac{1}{2}\sum_{A}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_A(y)}(H_{AA}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y)+H_{AA}(\underrightarrow{\psi}(y),y,\underrightarrow{\psi}(x),x))\,P
\end{aligned}
$$
$$\text{(F.2)}$$

If we now define a new diffusion matrix such that

$$
\begin{aligned}
D_{AB}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y) \;&=\; H_{AB}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y) && A<B \\
D_{AB}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y) \;&=\; H_{BA}(\underrightarrow{\psi}(y),y,\underrightarrow{\psi}(x),x) && A>B \\
D_{AA}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y) \;&=\; H_{AA}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y)+H_{AA}(\underrightarrow{\psi}(y),y,\underrightarrow{\psi}(x),x) && A=B
\end{aligned}
$$
$$\text{(F.3)}$$

we see that the functional Fokker-Planck equation for the two mode case becomes

$$
\begin{aligned}
\frac{\partial P}{\partial t} \;=\; & \sum_{A}\int dx\,\frac{\delta}{\delta\psi_A(x)}A_A(\underrightarrow{\psi}(x),x)\,P \\
& +\frac{1}{2}\sum_{A,B}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_B(y)}D_{AB}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y)\,P
\end{aligned}
$$
$$\text{(F.4)}$$

The expressions have been defined so that D_{AB} is symmetric. For the two mode condensate case

$$
D_{AB}(\underrightarrow{\psi}(x),x,\underrightarrow{\psi}(y),y)=D_{BA}(\underrightarrow{\psi}(y),y,\underrightarrow{\psi}(x),x) \qquad \text{(F.5)}
$$

Appendix F.2. Complex Symmetric Matrices

We present a proof that any $n\times n$ complex symmetric matrix F can also be written in the form $F=\mathcal{B}\mathcal{B}^T$, where $\mathcal{B}$ is also complex and has dimension

$n \times 2n$. The proof is adapted from material in [[83]] and [[84]] (see Sect. 6.4.7). This result is less useful than the Takagi factorisation, where $\mathcal{B}$ has dimension $n \times n$, the same as F.

The matrix F is $n \times n$ and we have $F_{pq} = F_{qp}$.

We first write

$$F = F^x + iF^y \tag{F.6}$$

where F^x and F^y are real symmetric matrices, both $n \times n$ in size.

We then construct a $2n \times 2n$ matrix D using F^x and F^y as sub-matrices

$$D = \begin{bmatrix} \frac{1}{2}F^x & \frac{1}{2}F^y \\ \frac{1}{2}F^y & -\frac{1}{2}F^x \end{bmatrix} = \begin{bmatrix} D^{xx} & D^{xy} \\ D^{yx} & D^{yy} \end{bmatrix} \tag{F.7}$$

Clearly D is both symmetric and real. We use $D^{xx}, .., D^{yy}$ as an alternative notation for the $n \times n$ submatrices of D.

Hence we can find a real $2n \times 2n$ matrix B such that

$$D = B\,B^T \tag{F.8}$$

Such a matrix can be obtained by construction using the real eigenvalues λ and real, orthogonal eigenvectors X_λ of D. Thus with

$$\begin{aligned} D\,X_\lambda &= \lambda\,X_\lambda \qquad X_\lambda^T\,X_\mu = \delta_{\lambda\mu} \\ D &= \sum_\lambda \lambda\,X_\lambda X_\lambda^T \end{aligned} \tag{F.9}$$

we can choose

$$B = \sum_\lambda \sqrt{\lambda}\,X_\lambda X_\lambda^T \tag{F.10}$$

from which it is easy to show that $D = B\,B^T$. Note that B is complex unless D is positive semi-definite.

We now divide B into two $n \times 2n$ submatrices as

$$B = \begin{bmatrix} B^x \\ B^y \end{bmatrix} \tag{F.11}$$

Clearly as

$$\begin{aligned} B\,B^T &= D \\ &= \begin{bmatrix} B^x\,B^{xT} = D^{xx} & B^x\,B^{yT} = D^{xy} \\ B^y\,B^{xT} = D^{yx} & B^y\,B^{yT} = D^{yy} \end{bmatrix} \end{aligned} \tag{F.12}$$

we can express the submatrices of D in terms of B^x and B^y.

Now define the $n \times 2n$ complex matrix $\mathcal{B}$ as

$$\mathcal{B} = B^x + iB^y \tag{F.13}$$

Then

$$
\begin{aligned}
\mathcal{B}\mathcal{B}^T &= (B^x + iB^y)(B^{xT} + iB^{yT}) \\
&= B^x\,B^{xT} + iB^x\,B^{yT} + iB^y\,B^{xT} - B^y\,B^{yT} \\
&= D^{xx} - D^{yy} + i(D^{xy} + D^{yx}) \\
&= \frac{1}{2}F^x - (-\frac{1}{2}F^x) + i(\frac{1}{2}F^y + \frac{1}{2}F^y) \\
&= F^x + iF^y \\
&= F
\end{aligned}
\tag{F.14}
$$

showing that a $n \times 2n$ complex matrix $\mathcal{B}$ can be found such that $\mathcal{B}\mathcal{B}^T = F$, as required.

Appendix F.3. Properties of Noise Fields - Two Mode Case

We can use the results in (F.28) relating the $\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))$ to the non-local diffusion terms $D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1), x_1, \underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2), x_2)$ and the fundamental noise properties of the Gaussian-Markov noise variables Γ_k^D in (F.30), together with (F.31) to determine the stochastic properties of the noise fields. For a single noise field

$$
\begin{aligned}
&\overline{\left\{ \left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1), \underset{\rightarrow}{\Gamma}(t_{1+})) \right) \right\}} \\
&= \sum_{Dk} \overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\,\Gamma_k^D(t_1)} = 0
\end{aligned}
\tag{F.15}
$$

and for two noise fields.

$$
\begin{aligned}
&\overline{\left\{ \left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1), \underset{\rightarrow}{\Gamma}(t_{1+})) \right) \left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2), \underset{\rightarrow}{\Gamma}(t_{2+})) \right) \right\}} \\
&= \overline{\sum_{Dk} \eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\Gamma_k^D(t_{1+}) \sum_{El} \eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\Gamma_l^E(t_{2+})} \\
&= \sum_{Dk}\sum_{El} \overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\Gamma_k^D(t_{1+})\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\Gamma_l^E(t_{2+})} \\
&= \sum_{Dk}\sum_{El} \overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))}\;\overline{\Gamma_k^D(t_{1+})\Gamma_l^E(t_{2+})} \\
&= \sum_{Dk}\sum_{El} \overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))}\;\delta_{kl}\delta_{DE}\delta(t_1 - t_2) \\
&= \overline{\sum_{Dk} \eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,2}))\eta_k^{B;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{1,2}))}\;\delta(t_1 - t_2) \\
&= \overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,2}), x_1, \underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{1,2}), x_2)}\;\delta(t_1 - t_2)
\end{aligned}
\tag{F.16}
$$

Thus the stochastic average of the linear noise term is zero, and the stochastic average of the product of two linear noise terms is *delta function correlated in*

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time, but is not delta function correlated in space. Instead the spatial correlation is given by the *non-local diffusion term* in the original functional Fokker-Planck equation!

The noise terms do however satisfy the Gaussian-Markoff conditions that averages of products of odd numbers of noise terms are zero, however averages of products of even numbers of noise terms can be written as sums of stochastic averages of products of pairs of non-local diffusion terms, rather than pairs of noise terms. Thus

$$
\overline{\left\{ \left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1), \underset{\rightarrow}{\Gamma}(t_{1+}))\right) \left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2), \underset{\rightarrow}{\Gamma}(t_{2+}))\right) \left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3), \underset{\rightarrow}{\Gamma}(t_{3+}))\right) \right\}}
$$
$$
= \sum_{Dk}\sum_{El}\sum_{Fm} \overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\eta_m^{C;F}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3))}\ \overline{\Gamma_k^D(t_{1+})\Gamma_l^E(t_{2+})\Gamma_m^F(t_{3+})}
$$
$$
= 0 \tag{F.17}
$$

and

$$
\overline{\left\{ \left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1), \underset{\rightarrow}{\Gamma}(t_{1+}))\right) \left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2), \underset{\rightarrow}{\Gamma}(t_{2+}))\right) \atop \times \left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3), \underset{\rightarrow}{\Gamma}(t_{3+}))\right) \left(\frac{\partial}{\partial t}\widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4), \underset{\rightarrow}{\Gamma}(t_{4+}))\right) \right\}}
$$
$$
= \sum_{Hk}\sum_{El}\sum_{Fm}\sum_{Gn} \overline{\eta_k^{A;H}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\eta_m^{C;F}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3))\eta_n^{D;G}(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4))}
$$
$$
\times \overline{\Gamma_k^H(t_{1+})\Gamma_l^E(t_{2+})\Gamma_m^F(t_{3+})\Gamma_n^G(t_{4+})}
$$
$$
= \sum_{Hk}\sum_{El}\sum_{Fm}\sum_{Gn} \overline{\eta_k^{A;H}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\eta_m^{C;F}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3))\eta_n^{D;G}(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4))}
$$
$$
\times \left\{ {(\delta_{kl}\delta_{HE}\delta(t_1-t_2))(\delta_{mn}\delta_{FG}\delta(t_3-t_4)) + (\delta_{km}\delta_{HF}\delta(t_1-t_3))(\delta_{nl}\delta_{EG}\delta(t_2-t_4)) \atop +(\delta_{kn}\delta_{HG}\delta(t_1-t_4))(\delta_{ml}\delta_{EF}\delta(t_2-t_3))} \right\}
$$
$$
= {\left[\sum_{Hk}\eta_k^{A;H}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\sum_{El}\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\delta_{kl}\delta_{HE}\delta(t_1-t_2)\right] \atop \times \left[\sum_{Fm}\eta_m^{C;F}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3))\sum_{Gn}\eta_n^{D;G}(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4))\delta_{mn}\delta_{FG}\delta(t_3-t_4)\right]}
$$
$$
+ {\left[\sum_{Hk}\eta_k^{A;H}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\sum_{Fm}\eta_m^{C;F}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3))\delta_{km}\delta_{HF}\delta(t_1-t_3)\right] \atop \times \left[\sum_{El}\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\sum_{Gn}\eta_n^{D;G}(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4))\delta_{nl}\delta_{EG}\delta(t_2-t_4)\right]}
$$
$$
+ {\left[\sum_{Hk}\eta_k^{A;H}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\sum_{Gn}\eta_n^{D;G}(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4))\delta_{kn}\delta_{HG}\delta(t_1-t_4)\right] \atop \times \left[\sum_{El}\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\sum_{Fm}\eta_m^{C;F}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3))\delta_{lm}\delta_{EF}\delta(t_2-t_3)\right]}
$$
$$
= \left[\overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,2}),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{1,2}),x_2)}\right]\left[\overline{D_{CD}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_{3,4}),x_3,\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_{3,4}),x_4)}\right]\delta(t_1-t_2)\delta(t_3-t_4)
$$
$$
+ \left[\overline{D_{AC}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,3}),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_{1,3}),x_3)}\right]\left[\overline{D_{BD}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{2,4}),x_2,\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_{2,4}),x_4)}\right]\delta(t_1-t_3)\delta(t_2-t_4)
$$
$$
+ \left[\overline{D_{AD}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,4}),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_{1,4}),x_2)}\right]\left[\overline{D_{BC}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{2,3}),x_2,\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_{2,3}),x_3)}\right]\delta(t_1-t_4)\delta(t_2-t_3)
$$
$$
\tag{F.18}
$$

using the results (F.28). This is not quite the same as

$$\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\}}$$

$$\times\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4),\underset{\rightarrow}{\Gamma}(t_{4+}))\right)\}}$$

$$+\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\}}$$

$$\times\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4),\underset{\rightarrow}{\Gamma}(t_{4+}))\right)\}}$$

$$+\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4),\underset{\rightarrow}{\Gamma}(t_{4+}))\right)\}}$$

$$\times\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\}}$$

$$(F.19)$$

because in general

$$\neq \frac{\overline{\left[D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,2}),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{1,2}),x_2)\right]}\,\overline{\left[D_{CD}(\underset{\rightarrow}{\widetilde{\psi}}(x_{3,4},t_3),x_3,\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_{3,4}),x_4)\right]}}{\overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,2}),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{1,2}),x_2)}\times\overline{D_{CD}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_{3,4}),x_3,\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_{3,4}),x_4)}}$$

$$(F.20)$$

etc., so the noise terms are not themselves Gaussian-Markov processes, though there is some similarity.

Appendix F.4. Properties of Noise Fields - Single Mode Case

We can use the results in (F.29) relating the $\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x,t))$ to the local diffusion terms $D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x,t),x)$ and the fundamental noise properties of the Gaussian-Markov noise variables Γ_k^D in (F.30), together with (F.31) to determine the stochastic properties of the noise fields. For a single noise field

$$\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\}}$$

$$=\sum_{Dk}\overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\,\Gamma_k^D(t_1)}=0 \qquad (F.21)$$

and for two noise fields

$$\overline{\left\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\underline{\Gamma}}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\underline{\Gamma}}(t_{2+}))\right)\right\}}$$

$$= \overline{\sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\Gamma_k^D(t_{1+})\sum_{El}\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\Gamma_l^E(t_{2+})}$$

$$= \sum_{Dk}\sum_{El}\overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\Gamma_k^D(t_{1+})\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\Gamma_l^E(t_{2+})}$$

$$= \sum_{Dk}\sum_{El}\overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))}\;\overline{\Gamma_k^D(t_{1+})\Gamma_l^E(t_{2+})}$$

$$= \sum_{Dk}\sum_{El}\overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))}\;\delta_{kl}\delta_{DE}\delta(t_1-t_2)$$

$$= \overline{\sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_{1,2}))\eta_k^{B;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_{1,2}))}\;\delta(t_1-t_2)$$

$$= \overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t_{1,2}),x_{1,2})}\;\delta(x_1-x_2)\delta(t_1-t_2) \qquad\text{(F.22)}$$

Thus the stochastic average of the linear noise term is zero, and the stochastic average of the product of two linear noise terms is *delta function correlated in time*, and is also delta function correlated in space. The spatial correlation is given by the *local diffusion term* in the original functional Fokker-Planck equation!

The noise terms do however satisfy the Gaussian-Markoff conditions that averages of products of odd numbers of noise terms are zero, but averages of products of even numbers of noise terms can be written as sums of stochastic averages of products of pairs of non-local diffusion terms, rather than pairs of noise terms. Thus

$$\overline{\left\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\underline{\Gamma}}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\underline{\Gamma}}(t_{2+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3),\underset{\rightarrow}{\underline{\Gamma}}(t_{3+}))\right)\right\}}$$

$$= \sum_{Dk}\sum_{El}\sum_{Fm}\overline{\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2))\eta_m^{C;F}(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3))}\;\overline{\Gamma_k^D(t_{1+})\Gamma_l^E(t_{2+})\Gamma_m^F(t_{3+})}$$

$$= 0 \qquad\text{(F.23)}$$

and

$$
\begin{aligned}
&\overline{\left\{\left(\tfrac{\partial}{\partial t}\widetilde{G}_A(\underrightarrow{\widetilde{\psi}}(x_1,t_1),\underrightarrow{\Gamma}(t_{1+}))\right)\left(\tfrac{\partial}{\partial t}\widetilde{G}_B(\underrightarrow{\widetilde{\psi}}(x_2,t_2),\underrightarrow{\Gamma}(t_{2+}))\right)\right.}\\
&\quad\overline{\left.\times\left(\tfrac{\partial}{\partial t}\widetilde{G}_C(\underrightarrow{\widetilde{\psi}}(x_3,t_3),\underrightarrow{\Gamma}(t_{3+}))\right)\left(\tfrac{\partial}{\partial t}\widetilde{G}_D(\underrightarrow{\widetilde{\psi}}(x_4,t_4),\underrightarrow{\Gamma}(t_{4+}))\right)\right\}}
\end{aligned}
$$

$$
=\sum_{Hk}\sum_{El}\sum_{Fm}\sum_{Gn}\overline{\eta_k^{A;H}(\underrightarrow{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underrightarrow{\widetilde{\psi}}(x_2,t_2))\eta_m^{C;F}(\underrightarrow{\widetilde{\psi}}(x_3,t_3))\eta_n^{D;G}(\underrightarrow{\widetilde{\psi}}(x_4,t_4))}
$$

$$
\times\overline{\Gamma_k^H(t_{1+})\Gamma_l^E(t_{2+})\Gamma_m^F(t_{3+})\Gamma_n^G(t_{4+})}
$$

$$
=\sum_{Hk}\sum_{El}\sum_{Fm}\sum_{Gn}\overline{\eta_k^{A;H}(\underrightarrow{\widetilde{\psi}}(x_1,t_1))\eta_l^{B;E}(\underrightarrow{\widetilde{\psi}}(x_2,t_2))\eta_m^{C;F}(\underrightarrow{\widetilde{\psi}}(x_3,t_3))\eta_n^{D;G}(\underrightarrow{\widetilde{\psi}}(x_4,t_4))}
$$

$$
\times\left\{\begin{array}{c}(\delta_{kl}\delta_{HE}\delta(t_1-t_2))(\delta_{mn}\delta_{FG}\delta(t_3-t_4))+(\delta_{km}\delta_{HF}\delta(t_1-t_3))(\delta_{nl}\delta_{EG}\delta(t_2-t_4))\\+(\delta_{kn}\delta_{HG}\delta(t_1-t_4))(\delta_{ml}\delta_{EF}\delta(t_2-t_3))\end{array}\right\}
$$

$$
\begin{aligned}
=\;&\overline{\left[\sum_{Hk}\eta_k^{A;H}(\underrightarrow{\widetilde{\psi}}(x_1,t_1))\sum_{El}\eta_l^{B;E}(\underrightarrow{\widetilde{\psi}}(x_2,t_2))\delta_{kl}\delta_{HE}\delta(t_1-t_2)\right]}\\
&\overline{\times\left[\sum_{Fm}\eta_m^{C;F}(\underrightarrow{\widetilde{\psi}}(x_3,t_3))\sum_{Gn}\eta_n^{D;G}(\underrightarrow{\widetilde{\psi}}(x_4,t_4))\delta_{mn}\delta_{FG}\delta(t_3-t_4)\right]}
\end{aligned}
$$

$$
\begin{aligned}
+\;&\overline{\left[\sum_{Hk}\eta_k^{A;H}(\underrightarrow{\widetilde{\psi}}(x_1,t_1))\sum_{Fm}\eta_m^{C;F}(\underrightarrow{\widetilde{\psi}}(x_3,t_3))\delta_{km}\delta_{HF}\delta(t_1-t_3)\right]}\\
&\overline{\times\left[\sum_{El}\eta_l^{B;E}(\underrightarrow{\widetilde{\psi}}(x_2,t_2))\sum_{Gn}\eta_n^{D;G}(\underrightarrow{\widetilde{\psi}}(x_4,t_4))\delta_{nl}\delta_{EG}\delta(t_2-t_4)\right]}
\end{aligned}
$$

$$
\begin{aligned}
+\;&\overline{\left[\sum_{Hk}\eta_k^{A;H}(\underrightarrow{\widetilde{\psi}}(x_1,t_1))\sum_{Gn}\eta_n^{D;G}(\underrightarrow{\widetilde{\psi}}(x_4,t_4))\delta_{kn}\delta_{HG}\delta(t_1-t_4)\right]}\\
&\overline{\times\left[\sum_{El}\eta_l^{B;E}(\underrightarrow{\widetilde{\psi}}(x_2,t_2))\sum_{Fm}\eta_m^{C;F}(\underrightarrow{\widetilde{\psi}}(x_3,t_3))\delta_{lm}\delta_{EF}\delta(t_2-t_3)\right]}
\end{aligned}
$$

$$
\begin{aligned}
=\;&\overline{\left[D_{AB}(\underrightarrow{\widetilde{\psi}}(x_{1,2},t_{1,2}),x_{1,2})\right]}\overline{\left[D_{CD}(\underrightarrow{\widetilde{\psi}}(x_{3,4},t_{3,4}),x_{3,4})\right]}\\
&\times\delta(x_1-x_2)\delta(x_3-x_4)\delta(t_1-t_2)\delta(t_3-t_4)\\
+&\overline{\left[D_{AC}(\underrightarrow{\widetilde{\psi}}(x_{1,3},t_{1,3}),x_{1,3})\right]}\overline{\left[D_{BD}(\underrightarrow{\widetilde{\psi}}(x_{2,4},t_{2,4}),x_{2,4})\right]}\\
&\times\delta(x_1-x_3)\delta(x_2-x_4)\delta(t_1-t_3)\delta(t_2-t_4)\\
+&\overline{\left[D_{AD}(\underrightarrow{\widetilde{\psi}}(x_{1,4},t_{1,4}),x_{1,4})\right]}\overline{\left[D_{BC}(\underrightarrow{\widetilde{\psi}}(x_{2,3},t_{2,3}),x_{2,3})\right]}\\
&\times\delta(x_1-x_4)\delta(x_2-x_3)\delta(t_1-t_4)\delta(t_2-t_3)
\end{aligned}
$$

$$\text{(F.24)}$$

using the results in (F.29). This is not quite the same as

$$\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\}}$$

$$\times\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4),\underset{\rightarrow}{\Gamma}(t_{4+}))\right)\}}$$

$$+\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\}}$$

$$\times\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4),\underset{\rightarrow}{\Gamma}(t_{4+}))\right)\}}$$

$$+\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_A(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t_1),\underset{\rightarrow}{\Gamma}(t_{1+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_D(\underset{\rightarrow}{\widetilde{\psi}}(x_4,t_4),\underset{\rightarrow}{\Gamma}(t_{4+}))\right)\}}$$

$$\times\overline{\{\left(\frac{\partial}{\partial t}\widetilde{G}_B(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t_2),\underset{\rightarrow}{\Gamma}(t_{2+}))\right)\left(\frac{\partial}{\partial t}\widetilde{G}_C(\underset{\rightarrow}{\widetilde{\psi}}(x_3,t_3),\underset{\rightarrow}{\Gamma}(t_{3+}))\right)\}}$$

$$\text{(F.25)}$$

because in general

$$\overline{\left[D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t_{1,2}),x_{1,2})\right]}\;\overline{\left[D_{CD}(\underset{\rightarrow}{\widetilde{\psi}}(x_{3,4},t_{3,4}),x_{3,4})\right]}$$

$$\neq\;\;\overline{D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t_{1,2}),x_{1,2})}\times\overline{D_{CD}(\underset{\rightarrow}{\widetilde{\psi}}(x_{3,4},t_{3,4}),x_{3,4})}$$

$$\text{(F.26)}$$

etc., so the noise terms are not themselves Gaussian-Markov processes, though there is some similarity.

Appendix F.5. Supplementary Equations

Functional Fokker-Planck equation for two mode case

$$\frac{\partial P}{\partial t}\;=\;\sum_A\int dx\,\frac{\delta}{\delta\psi_A(x)}A_A(\underset{\rightarrow}{\psi}(x),x)\,P$$

$$+\sum_{A\le B}\int\int dx\,dy\,\frac{\delta}{\delta\psi_A(x)}\frac{\delta}{\delta\psi_B(y)}H_{AB}(\underset{\rightarrow}{\psi}(x),x,\underset{\rightarrow}{\psi}(y),y)\,P\quad\text{(F.27)}$$

Summation results

$$\sum_{Dk}\eta_k^{A;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t))\eta_k^{B;D}(\underset{\rightarrow}{\widetilde{\psi}}(x_2,t))$$

$$=\;D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_1,t),x_1,\underset{\rightarrow}{\widetilde{\psi}}(x_2,t),x_2)\qquad Two\ Mode$$

$$\text{(F.28)}$$

$$=\;D_{AB}(\underset{\rightarrow}{\widetilde{\psi}}(x_{1,2},t),x_{1,2})\,\delta(x_1-x_2)\qquad One\ Mode$$

$$\text{(F.29)}$$

Gaussian-Markoff rules

$$
\begin{aligned}
\overline{\Gamma_k^D(t_1)} &= 0 \\
\overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\}} &= \delta_{DE}\delta_{kl}\delta(t_1 - t_2) \\
\overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\Gamma_m^F(t_3)\}} &= 0 \\
\overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\Gamma_m^F(t_3)\Gamma_n^G(t_4)\}} &= \overline{\{\Gamma_k^D(t_1)\Gamma_l^E(t_2)\}}\,\overline{\{\Gamma_m^F(t_3)\Gamma_n^G(t_4)\}} \\
&\quad +\overline{\{\Gamma_k^E(t_1)\Gamma_m^F(t_3)\}}\,\overline{\{\Gamma_l^E(t_2)\Gamma_n^G(t_4)\}} \\
&\quad +\overline{\{\Gamma_k^D(t_1)\Gamma_n^G(t_4)\}}\,\overline{\{\Gamma_l^E(t_2)\Gamma_m^F(t_3)\}} \\
&\quad ... \tag{F.30}
\end{aligned}
$$

Decorrelation Rule

$$
\begin{aligned}
&\overline{F(\underrightarrow{\widetilde{\alpha}}(t_1))\{\Gamma_k^D(t_2)\Gamma_l^E(t_3)\Gamma_m^F(t_4)...\Gamma_a^X(t_l)\}} \\
&= \overline{F(\underrightarrow{\widetilde{\alpha}}(t_1))}\,\overline{\{\Gamma_k^D(t_2)\Gamma_l^E(t_3)\Gamma_m^F(t_4)...\Gamma_a^X(t_l)\}} \qquad t_1 < t_2, t_3, .., t_l
\end{aligned} \tag{F.31}
$$