

Macrostatistics and Fluctuating Hydrodynamics

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Abstract

We extend our earlier macrostatistical treatment of hydrodynamical fluctuations about nonequilibrium steady states to viscous fluids. Since the scale dependence of the Navier-Stokes equations precludes the applicability of any infinite scale (hydrodynamical) limit, this has to be based on the generic model of a large but finite system, rather than an infinite one. On this basis, together with the assumption of Onsager's regression hypothesis and conditions of local equilibrium and chaoticity, we show that the hydrodynamical fluctuations of a reservoir driven fluid about a nonequilibrium steady state execute a Gaussian Markov process that constitutes a mathematical structure for a generalised version of Landau's fluctuating hydrodynamics and generically carries long range spatial correlations.

Key Words. Macrostatistics, fluctuating hydrodynamics, nonequilibrium steady states, chaoticity conditions, long range correlations.

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1. Introduction

The theory of fluctuations is a key area of statistical physics, which is essential to both equilibrium [1] and nonequilibrium thermodynamics [2]. Further, Landau [3] introduced fluctuation theory into fluid dynamics by adding white noise terms to the Navier-Stokes equations and employing the fluctuation-dissipation theorem to relate their intensities to the viscosity and thermal conductivity coefficients.

In relatively recent works [4, 5], we have presented a macrostatistical approach to the theory of hydrodynamical fluctuations of reservoir driven quantum systems about nonequilibrium steady states. This was designed to relate the stochastic properties of the fluctuations to the phenomenological continuum mechanics of these systems on the basis of general arguments centred on the hydrodynamical observables, subject to assumptions of macroscopic classicality, local equilibrium, chaoticity and a generalised version of Onsager's regression hypothesis* [2], all pertaining to states that may be far from global thermal equilibrium. On this basis, it was shown that, in a certain large scale limit, the fluctuations of the hydrodynamical variables executed a classical macroscopic stochastic process, whose parameters were expressed in terms of purely phenomenological quantities, namely thermodynamical variables and transport coefficients, the underlying quantum mechanics of the system being buried in the forms of these variables. Among the results that ensued from a treatment of this stochastic process [4, 5] were a non-linear generalisation of Onsager's reciprocity relations and a proof that the spatial correlations of the hydrodynamical variables are generically of long range. However, as it stands, the theory of those works is limited to systems whose phenomenological evolutions are scale invariant** and is therefore not applicable to viscous fluids, as described by the Navier-Stokes (NS) equations, since these are scale dependent.

The object of the present article is to extend our macrostatistical treatment to viscous Navier-Stokes fluids and thereby to provide a mathematical structure for a generalised form of Landau's picture of hydrodynamical fluctuations and to establish that, generically, their spatial correlations in nonequilibrium steady states are of long range. However, the scale dependence of the NS equations precludes the applicability of a hydrodynamical limit***, which was basic to the methodology of the previous works. Consequently, the theory has to be based on the generic model of a large, but finite, system, rather than an infinite one.

Our treatment here is based on a classical macrostatistical model, whose presumed relationship to an underlying quantum mechanics is only briefly indicated in the concluding Section. Thus, the model comprises a continuous distribution of matter that is confined to

* In the present setting, this signifies essentially that small deviations of the hydrodynamical observables from their values in a nonequilibrium steady state evolve according to the same dynamical law whether this deviation arises from a spontaneous fluctuation or from a small external perturbation.

** Specifically, those evolutions had to be invariant under space- time scale transformations of the form $x \rightarrow \lambda x$, $t \rightarrow \lambda^2 t$.

*** This is a limit in which the space and time scales for the phenomenological description becomes infinite and the macroscopic equations of motion become exact.

a bounded spatial region Ω and coupled to an array of reservoirs at its boundary. We assume that the scales of mass, distance and time whose units represent characteristic values of these variables are macroscopic: thus, for example, the constants \hbar and k_B of Planck and Boltzmann are extremely small on these scales*. We take the hydrodynamical observables to be the position and time dependent densities of energy, mass and momentum, and we assume that their phenomenological evolution is given by the NS equations. We then construct the model of the hydrodynamical fluctuations about a nonequilibrium steady state on the basis of assumptions of the Onsager regression hypothesis, local equilibrium and chaoticity. Although no hydrodynamical limit is available here, we exploit the fact that Boltzmann's constant is extremely small on the employed macroscopic scaling: specifically, we pass to a limit in which $k_B \rightarrow 0$ in our formulation of local equilibrium conditions via Einstein's relation, $P = \text{const.} \exp(S/k_B)$, between the equilibrium probability distribution of the macroscopic variables and the entropy function S . On this basis we obtain a generalisation of Landau's picture, wherein the hydrodynamical fluctuations execute a Gaussian Markov process whose parameters are completely determined by macroscopic variables involved in the phenomenological thermodynamic and hydrodynamic pictures of the system: the underlying quantum mechanics is assumed to be buried in the forms of these variables as functions of the control parameters. Furthermore, we show that the spatial correlations of the hydrodynamical variables are generically of long range.

We present our treatment as follows. In Section 2 we formulate the thermodynamics and hydrodynamics of the model in purely phenomenological terms. In Section 3 we construct the stochastic process executed by the hydrodynamical fluctuations, subject to the assumptions specified in the previous paragraph. In Section 4 we prove that the spatial correlations of these fluctuations are generically of long range. We conclude in Section 5 with some brief comments on the basis of the model and on its presumed relationship to its underlying quantum mechanics. There are two Appendices: the first is devoted to a calculation leading to a key formula, the second to a proof of a lemma.

Note on distributions. As we shall represent the hydrodynamical fluctuations by distributions, in the sense of L. Schwartz [6], we now specify our notations for these. We denote by $\mathcal{D}(\Omega)$, $\mathcal{D}_V(\Omega)$ and $\mathcal{D}_T(\Omega)$ the Schwartz spaces of real valued, infinitely differentiable scalar, vector and second order tensor valued functions on the bounded open region Ω with support in that region. These spaces are reflexive and their duals are distributions, which we denote by $\mathcal{D}'(\Omega)$, $\mathcal{D}'_V(\Omega)$ and $\mathcal{D}'_T(\Omega)$, respectively. We define $\tilde{\mathcal{D}}(\Omega)$ and $\check{\mathcal{D}}(\Omega)$ to be the Cartesian products $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \times \mathcal{D}_V(\Omega)$ and $\mathcal{D}_V(\Omega) \times \mathcal{D}_V(\Omega) \times \mathcal{D}_T(\Omega)$, respectively, and we denote their duals by $\tilde{\mathcal{D}}'(\Omega)$ and $\check{\mathcal{D}}'(\Omega)$, respectively. Further, if α and ψ are elements of a \mathcal{D} -class space and its dual, respectively, then $\psi(\alpha)$ will sometimes be denoted by $\langle \psi, \alpha \rangle$. We also employ angular brackets to denote the L^2 inner product $\langle \alpha, \alpha' \rangle$ between pairs of elements α and α' of the same $\mathcal{D}(\Omega)$ space. Evidently, these two uses of the angular brackets are mutually consistent since any \mathcal{D} space is a subset of its dual. To avoid ambiguity in the definition of scalar products of tensor valued functions, we define

* For example, if SI units and degrees Kelvin are appropriate, \hbar and k_B are of the order of 10^{-34} and 10^{-23} , respectively.

that of elements α ($= \{\alpha_{ij}|i, j = (1, 2, \dots, d)\}$) and α' ($= \{\alpha'_{ij}|i, j = (1, 2, \dots, d)\}$) of $\mathcal{D}_T(\Omega)$ to be

$$\langle \alpha, \alpha' \rangle := \int_{\Omega} dx \alpha_{ij}(x) \alpha'_{ji}(x), \quad (1.1)$$

where, as elsewhere in this article, the repeated index summation convention has been employed.

2. The Phenomenological Picture

We assume that the fluid is a macroscopic system, Σ , that occupies a fixed bounded open connected region Ω of a d -dimensional Euclidean space X , which constitutes the laboratory reference frame. We assume that Σ is in contact at its boundary, $\partial\Omega$, with an array, \mathcal{R} , of reservoirs, and that these determine the boundary conditions for the flow of the system, in a way that we shall specify in Sec. 2.2. We further assume that its dynamics is Galilei covariant. We employ a continuum model for Σ , which we formulate on macroscopic scales wherein the magnitudes of its energy, mass and volume are of the order of unity: for simplicity, we take its volume to be unity.

2.1. The Thermodynamic Potentials

We assume that, at equilibrium, Σ is at rest, that its densities, e_0 and ρ_0 , of energy and mass are spatially uniform and that its entropy density is a function, s_0 , of these variables. Thus, as the volume of Ω is unity, e_0 , ρ_0 and $s_0(e_0, \rho_0)$ are also the total energy, mass and entropy, respectively, of Σ and satisfy the fundamental formula

$$ds_0 = \beta(de_0 - \mu d\rho_0), \quad (2.1)$$

where β^{-1} is the temperature, in degrees Kelvin, say, *not* in units of k_B , and μ is the chemical potential, as related to the mass*. Thus

$$\beta = \left(\frac{\partial s_0}{\partial e_0} \right)_{\rho_0} \quad (2.2)$$

and

$$\mu = -\beta^{-1} \left(\frac{\partial s_0}{\partial \rho_0} \right)_{e_0}. \quad (2.3)$$

The pressure is then

$$p = \beta^{-1} s_0(e_0, \rho_0) - e_0 + \mu \rho_0. \quad (2.4)$$

and the heat function (enthalpy density) is

$$\varepsilon = e_0 + p. \quad (2.5)$$

* We relate the chemical potential to mass, by Eq. (2.3), rather than particle number, since the continuum model does not involve any concept of the latter.

Turning now to the nonequilibrium situation, the densities of energy, mass and momentum are generally non-uniform. We assume that they are locally conserved and we denote their local densities by e , ρ and j , respectively. These are functions of position x ($\in\Omega$) and time t ($\in\mathbf{R}$). The local drift velocity is defined to be

$$u = j/\rho. \quad (2.6)$$

We denote the components of j and u , relative to some chosen coordinate system, by (j_1, \dots, j_d) and (u_1, \dots, u_d) , respectively.

We define a local rest frame for the point x to be one that moves with velocity $u(x)$ relative to the laboratory frame. It therefore follows from Galilei covariance that the local energy and mass densities relative to this frame are

$$e_0 = e - \frac{1}{2}\rho u^2 = e - \frac{j^2}{2\rho} \text{ and } \rho_0 = \rho. \quad (2.7)$$

We assume that, even in a nonequilibrium state, the system is in local equilibrium in the phenomenological sense* that its local enthalpy density ε is still given by Eq. (2.5) and its entropy density at a point x is the same function, s_0 , of the energy and mass densities relative to a local rest frame as that governing the equilibrium entropy density relative to the laboratory frame. Thus, by the Galilei invariance of entropy [7], its density at the point x , relative to the laboratory frame, is the function s of $e(x)$, $\rho(x)$ and $j(x)$ given by the formula

$$s(e, \rho, j) = s_0(e_0, \rho_0), \quad (2.8)$$

We now compactify the notation by denoting the triple (e, ρ, j) by ϕ . Thus, defining $\nu := (d+2)$, ϕ is the ν -component variable $(\phi_1, \dots, \phi_\nu) := (e, \rho, j_1, \dots, j_d)$ and s is a function of ϕ . Its derivative $s'(\phi) := (\partial s/\partial\phi_1, \dots, \partial s/\partial\phi_\nu)$ is then the conjugate, $\theta = (\theta_1, \dots, \theta_\nu)$ of ϕ and, by Eqs. (2.2), (2.3), (2.7) and (2.8), it is given explicitly by the formula

$$\theta = s'(\phi) = \beta(1, -\mu + u^2/2, -u_1, \dots, -u_d) \quad (2.9)$$

or, more compactly,

$$\theta = s'(\phi) = \beta(1, -\mu + u^2/2, -u). \quad (2.9a)$$

The thermodynamic conjugate, π , of s is the function of θ defined by the formula

$$\pi(\theta) = \sup_\phi (s(\phi) - \theta \cdot \phi). \quad (2.10)$$

We assume that all parts of this system are in a single thermodynamical phase at all times and consequently that the function s' is invertible and the supremum on the r.h.s. of Eq. (2.10) is attained when $\phi = [s']^{-1}(\theta)$. Hence

$$\pi(\theta) = s(\phi) - \theta \cdot \phi, \text{ with } s'(\phi) = \theta, \quad (2.11)$$

* A further, macrostatistical kind of local equilibrium will be assumed in Section 3.3

which, together with Eqs. (2.2)-(2.4), (2.8) and (2.9a), implies that

$$\pi(\theta) = \beta p. \quad (2.12)$$

Furthermore, it follows from Eq. (2.11) that

$$\pi'(\theta) = -\phi. \quad (2.13)$$

Consequently, the Hessians $s''(\phi)$ ($= ([\partial^2 s / \partial \phi_j \partial \phi_k])$) and $\pi''(\theta)$ are related by the equations

$$\pi''(\theta)s''(\phi) = s''(\phi)\pi''(\theta) = -I,$$

i.e.

$$\pi''(\theta) = -s''(\phi)^{-1}. \quad (2.14)$$

2.2. The Navier-Stokes Equations.

These hydrodynamical equations comprise local conservation laws for the energy, mass and momentum, together with constitutive equations for the associated fluxes. The local conservation laws are

$$\frac{\partial e}{\partial t} + \nabla \cdot q = 0; \quad \frac{\partial \rho}{\partial t} + \nabla \cdot j = 0; \quad \text{and} \quad \frac{\partial j}{\partial t} + \nabla \cdot \tau = 0, \quad (2.15)$$

where q is the energy current, τ is the stress tensor $[\tau_{kl}]$ and $(\nabla \cdot \tau)_k := \partial \tau_{kl} / \partial x_l$. The constitutive equations are then [3]

$$q = \left(\varepsilon + \frac{1}{2} \rho u^2 \right) u - \sigma \cdot u + \kappa \nabla \beta, \quad (2.16)$$

$$\tau = pI + \rho u u - \sigma \quad (2.17)$$

and

$$\sigma = \gamma_1 (Du - 2d^{-1}(\nabla \cdot u)I) + \gamma_2 (\nabla \cdot u)I, \quad (2.18)$$

where

$$(Du)_{kl} = \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}, \quad (2.19)$$

uu is the dyadic whose kl component is $u_k u_l$ and κ , γ_1 and γ_2 are positive, scalar valued functions of β and μ that represent the thermal conductivity, the bulk viscosity and the shear viscosity, respectively.

Thus, in view of the one-to-one correspondence between the variables (e, ρ, j) and (β, μ, u) , the combination of the conservation laws (2.15) and the constitutive equations (2.16)-(2.18) describe an autonomous evolution of the hydrodynamical variables. Further, as remarked at the beginning of Section 2, the boundary conditions are determined by the reservoirs with which the system is in contact. Specifically, we assume that the contact

is restricted to the surface, $\partial\Omega$, of Ω and the boundary conditions are taken to be the following ones.

(a) $u = 0$ on $\partial\Omega$.

(b) At any point of $\partial\Omega$ that is in contact with a reservoir, the values of the temperature and chemical potential of Σ are just those of the reservoir.

(c) Any part of $\partial\Omega$ not in contact with a reservoir is insulated and the normal part of $\nabla\theta$ is zero there.

We now compactify the hydrodynamical equations by expressing them in terms of the triples $\phi := (e, \rho, j)$ and $\chi := (q, j, \tau)$, the latter of which we shall term the *flux*. Inserting subscripts t to indicate time dependence of the hydrodynamical variables, the local conservation laws (2.15) may be expressed as the single equation

$$\frac{\partial\phi_t}{\partial t} + \nabla \cdot \chi_t = 0, \quad (2.20)$$

and Eqs. (2.16)-(2.18) comprise a constitutive equation for χ_t of the form

$$\chi_t(x) = \mathcal{G}(\phi_t; x). \quad (2.21)$$

By Eqs. (2.20) and (2.21), ϕ_t evolves according to the autonomous equation of motion

$$\frac{\partial\phi_t(x)}{\partial t} = \mathcal{F}(\phi_t; x), \quad (2.22)$$

where

$$\mathcal{F}(\phi_t; x) := -\nabla \cdot \mathcal{G}(\phi_t; x). \quad (2.23)$$

We assume that the equation of motion (2.22) has a unique stationary solution, subject to the prevailing boundary conditions, described above. We denote this solution by $\bar{\phi}(x) = (\bar{e}(x), \bar{\rho}(x), \bar{j}(x))$ and denote the corresponding steady state value of $\theta(x)$ by $\bar{\theta}(x)$.

The Perturbed flow. We assume that the stationary flow is stable under ‘small’ perturbations $\delta\phi_t(x) = (\delta e_t(x), \delta\rho_t(x), \delta j_t(x))$ of $\bar{\phi}(x)$, that preserve the boundary conditions. The resultant linearised equation of motion is then

$$\frac{\partial}{\partial t} \delta\phi_t(x) = (\mathcal{L}\delta\phi_t)(x) := \frac{\partial}{\partial\lambda} \mathcal{F}(\bar{\phi} + \lambda\delta\phi_t; x)|_{\lambda=0}. \quad (2.24)$$

Thus, by Eqs. (2.23) and (2.24),

$$\mathcal{L}\delta\phi_t = -\nabla \cdot \mathcal{K}\delta\phi_t, \quad (2.25)$$

where

$$(\mathcal{K}\delta\phi_t)(x) = \frac{\partial}{\partial\lambda} \mathcal{G}(\bar{\phi} + \lambda\delta\phi_t; x)|_{\lambda=0}. \quad (2.26)$$

Further, by Eqs. (2.21) and (2.26), the increment $\delta\chi_t$ in χ_t is given by the formula

$$\delta\chi_t(x) = \mathcal{K}\delta\phi_t(x). \quad (2.27)$$

In anticipation of the demands of the macrostatistical picture of the system, formulated in the next Section, we assume that $\delta\phi_t$ is a distribution, in the sense of L. Schwartz [6]. Thus, since $\delta\phi_t = (\delta e_t, \delta\rho_t, \delta j_t)$, where the first two components are scalar fields and the third is a vector field, we assume that $\delta\phi_t$ is an element of the space $\tilde{\mathcal{D}}'(\Omega)$, defined at the end of Section 1. Thus, employing the notations introduced there, it follows from the definition of $\delta\phi_t$ that, for $F = (f, g, h)$, with $f, g \in \mathcal{D}(\Omega)$ and $h \in \mathcal{D}_V(\Omega)$,

$$\delta\phi_t(F) = \delta e_t(f) + \delta\rho_t(g) + \delta j_t(h), \quad (2.28)$$

where the three terms on the r.h.s. are the integrals of δe_t , $\delta\rho_t$ and δj_t against f, g and h , respectively.

We now assume that the linear operator \mathcal{L} , defined in Eq. (2.24), is the generator of a one parameter semigroup $T := \{T_t | t \in \mathbf{R}_+\}$ of linear transformations of $\tilde{\mathcal{D}}'(\Omega)$. Hence, by Eq. (2.24),

$$\delta\phi_t = T_t \delta\phi_0 \quad \forall t \in \mathbf{R}_+, \quad (2.29)$$

from which it follows that

$$\delta\phi_t(F) = \delta\phi(T_t^* F) \quad \forall F \in \tilde{\mathcal{D}}(\Omega). \quad t \in \mathbf{R}_+, \quad (2.30)$$

where $T^* := \{T_t^* | t \in \mathbf{R}_+\}$ is the dual of the semigroup T : its generator, \mathcal{L}^* , is the dual of \mathcal{L} . We assume the dissipativity condition that, for all perturbations $\delta\phi$, $T_t \delta\phi$ tends to zero, in the $\tilde{\mathcal{D}}'(\Omega)$ topology, as $t \rightarrow \infty$. As the space $\tilde{\mathcal{D}}(\Omega)$ is reflexive, this is equivalent to the assumption that

$$\tilde{\mathcal{D}}(\Omega) : \lim_{t \rightarrow \infty} T_t^* F = 0 \quad \forall F \in \tilde{\mathcal{D}}(\Omega). \quad (2.31)$$

Re-expression of NS in terms of θ . Since the constitutive equations (2.16)-(2.19) relate the heat and mass currents, as well as the stress tensor, directly to θ , it is sometimes convenient to express the r.h.s.'s of Eqs. (2.22)-(2.24) in terms of this variable rather than ϕ . Thus we rewrite Eq. (2.22) as

$$\frac{\partial\phi_t}{\partial t} = -\nabla \cdot \Psi(\theta_t; x), \quad (2.32)$$

where $\Psi(\theta_t; x) = \mathcal{F}(\phi_t; x)$. Eq. (2.24) for the perturbed flow then becomes

$$\frac{\partial}{\partial t} \delta\phi_t = \Lambda \delta\theta_t := -\frac{\partial}{\partial \lambda} \nabla \cdot \Psi(\bar{\theta} + \lambda \delta\theta_t; x)|_{\lambda=0}. \quad (2.33)$$

Further, by Eq. (2.13) and the equivalence of Eqs. (2.24) and (2.33),

$$\Lambda \delta\theta = \mathcal{L} \delta\phi = -\mathcal{L} \pi''(\bar{\theta}) \delta\theta$$

for all perturbations $\delta\theta$ of $\bar{\theta}$ and hence

$$\Lambda = -\mathcal{L}\pi''(\bar{\theta}). \quad (2.34)$$

It follows immediately from this equation and the symmetry of $\pi(\theta)''$ that the dual, Λ^* , of Λ is related to that, \mathcal{L}^* , of \mathcal{L} by the formula

$$\Lambda^* = -\pi''(\bar{\theta})\mathcal{L}^*. \quad (2.35)$$

In particular, as we shall show in Appendix A, it follows from our definitions that the equilibrium form, Λ_{eq}^* , of Λ^* is given by the equation

$$\begin{aligned} \Lambda_{\text{eq}}^*(f, g, h) = & \\ & -\left(\kappa\Delta f - \beta^{-1}\varepsilon\nabla.h, -\beta^{-1}\rho\nabla.h, \right. \\ & \left. -\beta^{-1}\varepsilon\nabla f - \beta^{-1}\rho\nabla g + \beta^{-1}\gamma_1\nabla.(Dh - 2d^{-1}(\nabla.h)I) + \beta^{-1}\gamma_2\nabla(\nabla.h)\right) \\ & \forall (f, g, h) \in \tilde{D}\Omega). \end{aligned} \quad (2.36)$$

3. The Stochastic Fluctuation Process

According to elementary statistical mechanics, the thermodynamic and hydrodynamic variables undergo fluctuations, which are not taken into account in the phenomenological picture of the previous Section. We now seek to provide a general description of the hydrodynamical fluctuations about nonequilibrium steady states by treating the fields $\phi = (e, \rho, j)$ and $\chi = (q, j, \tau)$ as expectation values of random fields $\hat{\phi} = (\hat{e}, \hat{\rho}, \hat{j})$ and $\hat{\chi} = (\hat{q}, \hat{j}, \hat{\tau})$, respectively, where the tensor $\hat{\tau}$, like the phenomenological τ , is symmetric*. We assume that these fields satisfy the local conservation law given by the canonical analogue of Eq. (2.20), namely

$$\frac{\partial \hat{\phi}_t}{\partial t} + \nabla \cdot \hat{\chi}_t = 0. \quad (3.1)$$

The differences between the random fields $\hat{\phi}_t$ and $\hat{\chi}_t$ and their classical expectation values then represent the hydrodynamical fluctuations. In a standard way, we normalise them by a factor $\mathcal{N}^{1/2}$, where \mathcal{N} is chosen to be the ratio of characteristic values of corresponding macroscopic and microscopic quantities. In the present situation, where we are formulating the model on a macroscopic scale, a natural choice for \mathcal{N} is the reciprocal, k_B^{-1} , of Boltzmann's constant, which arises in Einstein's formula, $P = \text{const.}\exp(S/k_B)$, relating the equilibrium probability distribution of the macroscopic variables to the entropy S . Thus, we define the fields representing the fluctuations of ϕ_t and χ_t about their steady state values to be

$$\xi_t(x) = k_B^{-1/2}(\hat{\phi}_t(x) - \bar{\phi}) \quad (3.2)$$

* This symmetry property may be regarded as basic, as it prevails in standard microscopic pictures of stress tensors.

and

$$\eta_t(x) = k_B^{1/2}(\hat{\chi}_t(x) - \bar{\chi}), \quad (3.3)$$

respectively. Hence, defining

$$\zeta_{t,s} := \int_s^t du \eta_u, \quad (3.4)$$

it follows from Eqs. (3.1)-(3.4) that

$$\xi_t - \xi_s = -\nabla \cdot \zeta_{t,s} \quad \forall t, s \in \mathbf{R}. \quad (3.5)$$

In accordance with the general requirements of field theories [8], we assume that these fields are distributions, in the sense of L. Schwartz [6]. Specifically, in the notation of the last Note of Section 1, we assume that $\xi_t \in \tilde{\mathcal{D}}'(\Omega)$ and $\eta_t \in \tilde{\mathcal{D}}'(\Omega)$. We denote the smeared fields obtained by integrating ξ_t and $\zeta_{t,s}$ against test functions $F \in \tilde{\mathcal{D}}(\Omega)$ and $G \in \tilde{\mathcal{D}}(\Omega)$ by $\xi_t(F)$ and $\eta_t(G)$, respectively;

We aim to derive the stochastic process executed by the fluctuation fields ξ and ζ from the assumptions of

- (a) a generalised version of Onsager's regression hypothesis,
- (b) a macrostatistical local equilibrium hypothesis and
- (c) a chaoticity hypothesis for the random currents and stresses.

Further, as the continuum model harbours only macroscopic variables, we take any counterparts to microscopic correlation lengths and memory times of this model to be zero.

3.1. Regression Hypothesis.

The regression hypothesis signifies that the fluctuations of the hydrodynamical observables evolve, from a given starting point, according to the same dynamical law that governs the evolution of small perturbations of those variables from their steady state values. Thus, noting the formula (2.30) for the latter evolution and denoting conditional expectations of the stochastic variables with respect to the value of the field ξ at time t_0 by $E(\cdot | \xi_{t_0})$, we assume that

$$E(\xi_t(F) | \xi_{t_0}) = \xi_{t_0}((T^*(t - t_0)F)) \quad \forall F \in \tilde{\mathcal{D}}(\Omega), \quad t_0, t(\geq t_0) \in \mathbf{R}. \quad (3.6)$$

Since the process ξ_t is stationary, an immediate consequence of this formula is that

$$E(\xi_t(F)\xi_{t'}(F')) = E(\xi[T^*(t - t')F]\xi(F')) \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega), \quad t, t'(\leq t) \in \mathbf{R}. \quad (3.7)$$

We now note that, by Eq. (2.27), the increment in the integrated phenomenological flux, $\int_s^t du \chi_u$, due to a perturbation $\delta\phi$ of ϕ is $\int_s^t du \mathcal{K} \delta\phi_u$. Correspondingly, as the

phenomenological dynamics of the model is secular, we designate the secular part of the integrated fluctuation flux, $\zeta_{t,s}$, to be

$$\zeta_{t,s}^{\text{sec}} := \int_s^t du \mathcal{K} \xi_u. \quad (3.8)$$

We then define the remaining part, $\tilde{\zeta}_{t,s}$, of $\zeta_{t,s}$ to be the stochastic part of the integrated flux. Thus

$$\tilde{\zeta}_{t,s} := \zeta_{t,s} - \int_s^t du \mathcal{K} \xi_u. \quad (3.9)$$

We shall presently show, in Prop. 3.1, that this field does indeed enjoy strong, Wiener-like, stochastic properties.

By Eqs. (2.25), (3.5) and (3.9),

$$\xi_t(F) - \xi_s(F) = \int_s^t du \xi_u(\mathcal{L}^* F) + w_{t,s}(F) \quad \forall s, t \in \mathbf{R}, F \in \tilde{\mathcal{D}}(\Omega), \quad (3.10)$$

i.e.

$$\xi_t - \xi_s = \int_s^t du \mathcal{L} \xi_u + w_{t,s}, \quad (3.10a)$$

where

$$w_{t,s}(F) = \tilde{\zeta}_{t,s}(\nabla F). \quad (3.11)$$

The following Proposition, which was proved in [5, Prop. 5.1], establishes that w simulates a Wiener process, at least as far as its two-point function is concerned; and hence that Eq. (3.10a) is an integrated Langevin equation.

Proposition 3.1. *Under the assumption of the regression hypothesis, w has the following properties.*

(i)

$$E(w_{t,s}(F) \xi_u(F')) = 0 \quad \forall t \geq s \geq u, F, F' \in \tilde{\mathcal{D}}(\Omega). \quad (3.12)$$

(ii)

$$\begin{aligned} E(w_{t,s}(F) w_{t',s'}(F')) &= E(\xi(\mathcal{L}^* F) \xi(F') + \xi(F) \xi(\mathcal{L}^* F')) |[s, t] \cap [s', t']| \\ &\quad \forall s, t (\geq s), s', t' (\geq s) \in \mathbf{R}, F, F' \in \tilde{\mathcal{D}}(\Omega), \end{aligned} \quad (3.13)$$

where $|I|$ denotes the length of an interval I in \mathbf{R} .

3.2. The Chaoticity Hypothesis.

We now strengthen Prop. 3.1 by the assumption that the stochastic part, $\tilde{\zeta}$ of the integrated fluctuation flux is chaotic in the sense that its space-time correlations are of microscopic range, idealised here as zero range on our macroscopic scale. This assumption is designed to represent Boltzmann's molecular chaos hypothesis, as transferred to the

stochastic flux. Here we take it to signify that the field $\tilde{\zeta}$ is Gaussian, due to statistical independence of its values in disjoint space-time regions. Thus, our chaoticity hypothesis comprises the following conditions.

(C.1) The process $\tilde{\zeta}$ is Gaussian, and

(C.2) $E(\tilde{\zeta}_{s,t}(G)\tilde{\zeta}_{s',t'}(G'))$ vanishes if the intersection of either the intervals $[s, t]$ and $[s', t']$ or of the supports of G and G' is empty.

Further, we supplement these conditions by the following continuity assumption.

\mathcal{C} . The correlation function of (C.2) is continuous in its time variables.

The following Proposition was proved in [5, Prop. 5.2].

Proposition 3.2. *Under the assumption of (C.2) and \mathcal{C} , the two-point function of $\tilde{\zeta}$ takes the following form.*

$$E(\tilde{\zeta}_{t,s}(G)\tilde{\zeta}_{s',t'}(G')) = \Gamma(G, G')|[s, t] \cap [s', t']| \quad \forall G, G' \in \tilde{\mathcal{D}}'(\Omega), \quad s, t, s', t' \in \mathbf{R}, \quad (3.14)$$

where Γ is a continuous bilinear form on $\tilde{\mathcal{D}}(\Omega) \otimes \tilde{\mathcal{D}}(\Omega)$, whose support lies in the region $\{(x, x') \in \Omega^2 | x = x'\}$.

We shall derive the explicit form of Γ from the local equilibrium hypothesis in Section 3.3. The following Corollary to Prop. 3.2 is a simple consequence of Eq. (3.13), assumption (C.1) and Prop. (3.2).

Corollary 3.3. *Under the above assumptions, the process w is Gaussian with zero mean and its two-point function takes the form*

$$E(w_{t,s}(F)w_{t',s'}(F')) = \Gamma(\nabla F, \nabla F')|[s, t] \cap [s', t']| \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega), \quad t, s, t', s' \in \mathbf{R}. \quad (3.15)$$

The following Proposition was shown in [5, Prop. 5.5] to ensue from the Langevin equation (3.10a), Prop. 3.1 (i) and Cor. 3.3.

Proposition 3.4. *Under the above assumptions, ξ is a Gaussian Markov process, and the fields $w_{t,s}$ and ξ_u are statistically independent of one another if s and t are greater than or equal to u .*

3.3 Equilibrium and Local Equilibrium Conditions.

Equilibrium Statistics of ξ . We base our formulation of these statistics on the canonical version of Einstein's formula for the probability distribution, P of the hydrodynamical variables. This is given formally by the equation

$$P_{eq} = \text{const.} \exp(k_B^{-1} \int_{\Omega} dx [s(\hat{\phi}(x) - \bar{\theta} \cdot \hat{\phi}(x))]),$$

i.e., by Eq. (3,2),

$$P_{eq} = \text{const.} \exp(k_B^{-1} \int_{\Omega} dx [s(\bar{\phi} + k_B^{1/2} \xi(x)) - \bar{\theta} \cdot (\bar{\phi} + k_B^{1/2} \xi(x))]). \quad (3.16)$$

Since the integrand of this formula is maximised at $\xi = 0$, its ratio to k_B reduces to $(\xi(x) \cdot s''(\bar{\theta}) \xi(x))/2$ in the limit $k_B \rightarrow 0$, which we term the *Boltzmann* limit: here the dot denotes the \mathbf{R}^{ν} scalar product. Thus, operating henceforth in this limit, the equilibrium characteristic function for the fluctuation field ξ is given formally by

$$E_{eq}(\exp[i\xi(F)]) = \text{const.} \frac{\int \mathcal{D}\xi(x) \exp[\int_{\Omega} dx (\xi(x) \cdot s''(\bar{\phi}) \xi(x)/2 + i\xi(x) \cdot F(x))]}{\int \mathcal{D}\xi(x) \exp[\int_{\Omega} dx (\xi(x) \cdot s''(\bar{\phi}) \xi(x)/2)]} \quad \forall F \in \tilde{\mathcal{D}}(\Omega), \quad (3.17)$$

where $\mathcal{D}\xi$ denotes functional integration w.r.t. the field ξ . This formula may properly be defined by resolving Ω into a set of cells, ΔJ , denoting the values of ξ and F at the centre of ΔJ by ξ_J and F_J , respectively, and then expressing Eq. (3.17) as

$$E_{eq}(\exp[i\xi(F)]) = \lim_{\Delta} \Pi_J \left[\frac{\int_{\mathbf{R}^{\nu}} d\xi_J \exp(\xi_J \cdot s''(\bar{\theta}) \xi_J/2 + i\xi_J \cdot F_J)}{\int_{\mathbf{R}^{\nu}} d\xi_J \exp(\xi_J \cdot s''(\bar{\theta}) \xi_J/2)} \right], \quad (3.17a)$$

where \lim_{Δ} is the limit in which the cells shrink to points. It now follows easily from Eqs.(2.14) and (3.17a) that, in the notation specified at the end of Section 1,

$$E_{eq}(\exp[i\xi(F)]) = \exp(-\frac{1}{2} \langle F, \pi''(\bar{\theta}) F \rangle) \quad \forall F \in \tilde{\mathcal{D}}(\Omega). \quad (3.18)$$

Thus, at equilibrium, ξ is a Gaussian random field, with zero mean and two- point function given by the formula

$$E_{eq}(\xi(F))^2 = \frac{1}{2} \langle F, \pi''(\bar{\theta}) F \rangle \quad \forall F \in \tilde{\mathcal{D}}(\Omega),$$

or equivalently, by polarisation,

$$E_{eq}(\xi(F)\xi(F')) = \langle F, \pi''(\bar{\theta}) F' \rangle \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega). \quad (3.19)$$

Equilibrium two-point function for $\tilde{\zeta}$. By Eqs. (2.34), (3.11), (3.13) and (3.19),

$$\begin{aligned} E_{eq}(\tilde{\zeta}_{t,s}(\nabla F) \tilde{\zeta}_{t',s'}(F')) &= [\langle \Lambda_{eq}^* F, F' \rangle + \langle F, \Lambda_{eq}^* F' \rangle] |[s, t] \cap [s', t']| \\ &\quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega), \quad s, t, s', t' \in \mathbf{R}. \end{aligned} \quad (3.20)$$

Hence, for $F = (f, g, h)$ and $F' = (f', g', h')$, it follows from Eqs. (2.36) and (3.20) that

$$E_{eq}(\tilde{\zeta}_{t,s}(\nabla F) \tilde{\zeta}_{t',s'}(\nabla F')) =$$

$$[2\kappa\langle\nabla f, \nabla f'\rangle + \beta^{-1}\gamma_1(\langle Dh, Dh\rangle - 2d^{-1}\langle\nabla.h, \nabla.h'\rangle) + 2\beta^{-1}\gamma_2\langle\nabla.h, \nabla.h'\rangle] \times \\ |[s, t] \cap [s', t']|. \quad (3.21)$$

We now express $\tilde{\zeta}_{t,s}$ in the form

$$\tilde{\zeta}_{t,s} = (\tilde{\zeta}_{t,s}^{(1)}, \tilde{\zeta}_{t,s}^{(2)}, \tilde{\zeta}_{t,s}^{(3)}), \quad (3.22)$$

where the components are the time integrals of the stochastic fluctuations of q, j, τ and therefore lie in $\mathcal{D}'_V(\Omega)$, $\mathcal{D}'_V(\Omega)$ and $\mathcal{D}'_T(\Omega)$, respectively. Thus,

$$\tilde{\zeta}_{t,s}(\nabla F) = \tilde{\zeta}_{t,s}^{(1)}(\nabla f) + \tilde{\zeta}_{t,s}^{(2)}(\nabla g) + \tilde{\zeta}_{t,s}^{(3)}(\nabla h)$$

and similarly

$$\tilde{\zeta}_{t,s}(\nabla F') = \tilde{\zeta}_{t,s}^{(1)}(\nabla f') + \tilde{\zeta}_{t,s}^{(2)}(\nabla g') + \tilde{\zeta}_{t,s}^{(3)}(\nabla h').$$

The substitution of these last two equations in Eq. (3.21) yields the formulae

$$E_{\text{eq}}(\tilde{\zeta}_{t,s}^{(1)}(\nabla f)\tilde{\zeta}_{t',s'}^{(1)}(\nabla f')) = 2\kappa\langle\nabla f, \nabla f'\rangle|[s, t] \cap [s', t']|, \quad (3.23)$$

and

$$E_{\text{eq}}(\tilde{\zeta}_{t,s}^{(3)}(\nabla h)\tilde{\zeta}_{t',s'}^{(3)}(\nabla h')) = [2\beta^{-1}\gamma_1(\langle Dh, Dh'\rangle - 2d^{-1}\langle\nabla.h, \nabla.h'\rangle) + \beta^{-1}\gamma_2\langle\nabla.h, \nabla.h'\rangle] \\ |[s, t] \cap [s', t']|. \quad (3.24)$$

and implies that all other two-point functions for the components of $\tilde{\zeta}_{t,s}(\nabla F)$ are zero

The following lemma for the generalised functions $\tilde{\zeta}_{t,s}^{(1)}$ and $\tilde{\zeta}_{t,s}^{(3)}$ on Ω corresponding to the distributions denoted by the same symbols will be proved in Appendix B.

Lemma 3.5. *Under the above assumptions, supplemented by the condition that the equilibrium two-point function of $\tilde{\zeta}$ is locally translationally and rotationally invariant, and indicating the components of X -vectors by subscripts i, j, l, m*

$$E_{\text{eq}}(\tilde{\zeta}_{t,s;i}^{(1)}(x)\tilde{\zeta}_{t',s';j}^{(1)}(x')) = 2\kappa\delta(x - x')|[s, t] \cap [s', t']|, \quad (3.25)$$

$$E_{\text{eq}}(\tilde{\zeta}_{t,s;il}^{(3)}(x)\tilde{\zeta}_{t',s';jm}^{(3)}(x')) = \\ 2\beta^{-1}(\gamma_1(\delta_{ij}\delta_{lm} + \delta_{im}\delta_{jm} - 2d^{-1}\delta_{il}\delta_{jm}\delta_{jl}) + \gamma_2\delta_{il}\delta_{jm})\delta(x - x')|[s, t] \cap [s', t']| \quad (3.26)$$

and all other two-point functions of the components of $\tilde{\zeta}_{t,s}$ are zero. This result concurs with that of Landau [3, Eqs. (132.11-13)].

In order to re-express the two-point functions of $\tilde{\zeta}$ in terms of smeared fields, we denote elements G and G' of $\tilde{\mathcal{D}}(\Omega)$ by triples (a, b, c) and (a', b', c') , respectively, where the first

two components of both G and G' lie in $\mathcal{D}_V(\Omega)$ and the third lie in $\mathcal{D}_T(\Omega)$. We then define the elements $c^{(1)}$, $c'^{(1)}$ of $\mathcal{D}_T(\Omega)$ and $c^{(2)}$, $c'^{(2)}$ of $\mathcal{D}(\Omega)$ by the formulae

$$c_{ij}^{(1)} := c_{ij} + c_{ji}; \quad c'_{ij}{}^{(1)} := c'_{ij} + c'_{ji} \quad (3.27)$$

and

$$c^{(2)} := c_{jj}; \quad c'^{(2)} = c'_{jj}. \quad (3.28)$$

and infer from Eqs. (3.26)-(3.28) that

$$\begin{aligned} E_{\text{eq}}(\tilde{\zeta}_{t,s}(G)\tilde{\zeta}_{t',s'}(G')) &= [2\kappa\langle a, a' \rangle + \\ &\beta^{-1}\gamma_1\langle c^{(1)}, c'^{(1)} \rangle + 2\beta^{-1}(\gamma_2 - 2d^{-1}\gamma_1)\langle c^{(2)}, c'^{(2)} \rangle] |[s, t] \cap [s', t']|. \end{aligned} \quad (3.29)$$

This formula and Eq. (3.19) constitute our equilibrium conditions. In order to obtain their local properties, we consider their forms when their test functions are concentrated around points of Ω . Thus, for $x_0 \in \Omega$ and $\epsilon \in \mathbf{R}_+$, we define the transformations $F \rightarrow F_{x_0, \epsilon}$ and $G \rightarrow G_{x_0, \epsilon}$ of $\tilde{\mathcal{D}}(\Omega)$ and $\tilde{\mathcal{D}}(\Omega)$, respectively, by the equations

$$F_{x_0, \epsilon} = \epsilon^{-d/2} F(\epsilon^{-1}(x - x_0)) \quad (3.30)$$

and

$$G_{x_0, \epsilon} = \epsilon^{-d/2} G(\epsilon^{-1}(x - x_0)) \quad (3.31).$$

We then remark that Eqs. (3.19) and (3.29) are invariant under these transformations and therefore that they enjoy the local (punctual!) property that

$$\lim_{\epsilon \downarrow 0} E_{\text{eq}}(\xi(F_{x_0, \epsilon})\xi(F'_{x_0, \epsilon})) = \langle F, \pi''(\bar{\theta})F' \rangle \quad \forall x_0 \in \Omega, F, F' \in \tilde{\mathcal{D}}(\Omega). \quad (3.32)$$

and

$$\begin{aligned} \lim_{\epsilon \downarrow 0} E_{\text{eq}}(\tilde{\zeta}_{t,s}(G_{x_0, \epsilon})\tilde{\zeta}_{t',s'}(G'_{x_0, \epsilon})) &= 2\kappa\langle a, a' \rangle + \\ &\beta^{-1}\gamma_1\langle c^{(1)}, c'^{(1)} \rangle + 2\beta^{-1}(\gamma_2 - 2d^{-1}\gamma_1)\langle c^{(2)}, c'^{(2)} \rangle |[s, t] \cap [s', t']|. \end{aligned} \quad (3.33)$$

Local Equilibrium Conditions. We now assume that, even in a nonequilibrium steady state, the two point functions of ξ and $\tilde{\zeta}$ enjoy the same local properties as at equilibrium. Thus, bearing in mind that β , $\bar{\theta}$, κ , γ_1 and γ_2 , are generally position dependent in the nonequilibrium situation, we take the local equilibrium conditions to be the following ones.

$$\lim_{\epsilon \downarrow 0} E(\xi(F_{x_0, \epsilon})\xi(F'_{x_0, \epsilon})) = \langle F, \pi''(\bar{\theta}(x_0))F' \rangle \quad \forall x_0 \in \Omega, F, F' \in \tilde{\mathcal{D}}(\Omega). \quad (3.34)$$

and

$$\begin{aligned} \lim_{\epsilon \downarrow 0} E(\tilde{\zeta}_{t,s}(G_{x_0, \epsilon})\tilde{\zeta}_{t',s'}(G'_{x_0, \epsilon})) &= [2\kappa(x_0)\langle a, a' \rangle + \\ &\beta(x_0)^{-1}\gamma_1(x_0)\langle c^{(1)}, c'^{(1)} \rangle + 2\beta(x_0)^{-1}(\gamma_2(x_0) - 2d^{-1}\gamma_1(x_0))\langle c^{(2)}, c'^{(2)} \rangle] |[s, t] \cap [s', t']|. \end{aligned} \quad (3.35)$$

The following Proposition, whose proof ensues from a trivial modification of that of [5, Prop. 5.3], provides an explicit formula for the bilinear form Γ , which governs the form of the two-point function for $\check{\zeta}$ according to Eq. (3.14).

Proposition 3.6. *Under the assumptions of Prop. 3.2, together with the local equilibrium condition (3.35), Γ is given by the following formula.*

$$\Gamma(G, G') = 2\langle a, \kappa a' \rangle + \langle c^{(1)}, \beta^{-1} \gamma_1 c'^{(1)} \rangle + 2\langle c^{(2)}, \beta^{-1} (\gamma_2 - 2d^{-1} \gamma_1) c'^{(2)} \rangle, \quad (3.36)$$

where $a, a', c^{(1)}, c'^{(1)}, c^{(2)}, c'^{(2)}$ are related to G and G' according to the above specifications and now $\beta^1, \kappa, \gamma_1$ are functions of position, through their dependence on β and μ , that act multiplicatively on $\tilde{\mathcal{D}}(\Omega)$.

3.4. The Macrostatistical Model.

The stationary processes ξ and w , which are connected by the integrated Langevin equation (3.10a), comprise our macrostatistical model. In view of Cor. (3.3) and Prop. (3.4), these processes are both Gaussian, and the two-point function of w is given by Eqs. (3.15) and (3.36). To complete the formulation of the model, it remains for us to obtain the two-point function of the process ξ .

To this end, we infer from Eq. (3.10a) that, since \mathcal{L} is the generator of the semigroup T ,

$$\xi_t = T_{t-t_0} \xi_{t_0} + \int_{t_0}^t T_{t-s} dw_{s,t_0} \quad \forall t \geq t_0$$

and hence that

$$\xi_t(F) = \xi_{t_0}(T_{t-t_0}^* F) + \int_{t_0}^t dw_{s,t_0}(T_{t-s}^* F) \quad \forall F \in \tilde{\mathcal{D}}(\Omega), \quad t, t_0 (\leq t) \in \mathbf{R}. \quad (3.37)$$

In view of the stationarity of the ξ -process, it follows from Eq. (3.37) and Prop. 3.4 that the static two-point function for the field ξ is

$$\begin{aligned} W(F, F') &:= E(\xi(F)\xi(F')) = E(\xi(T_{t-t_0}^* F)\xi(T_{t-t_0}^* F')) + \\ &\int_{t_0}^t ds \Gamma(\nabla T_s^* F, \nabla T_s^* F') \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega), \quad t, t_0 (\leq t) \in \mathbf{R} \end{aligned} \quad (3.38)$$

On invoking the dissipative condition (2.31) and passing to the limiting form of Eq. (3.38) as $t_0 \rightarrow -\infty$, we obtain the formula

$$W(F, F') = \int_0^\infty ds \Gamma(\nabla T_s^* F, \nabla T_s^* F') \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega). \quad (3.39)$$

We note that, as \mathcal{L}^* is the generator of the semigroup T^* , it follows from this formula and Eq. (3.38) that

$$W(\mathcal{L}^* F, F') + W(F, \mathcal{L}^* F') + \Gamma(\nabla F, \nabla F') = 0. \quad (3.40)$$

Moreover, in view of the stationarity of the process ξ , it follows from Eqs. (3.38) and (3.39) that the two-point space time correlation function is given by the formula

$$E(\xi_t(F)\xi_{t'}(F')) = \int_0^\infty ds \Gamma(\nabla T_{t-t'+s}^* F, \nabla T_s^* F') \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega), \quad t, t' (\leq t) \in \mathbf{R}. \quad (3.41)$$

As the process ξ is Gaussian, this completes our formulation of the model.

4. Long Range Correlations.

We term the space correlations of ξ to be of short range, which we idealise as zero range in our macroscopic scaling, if the support of its static two-point function W lies in the region $\{(x, x') \in \Omega^2 | x = x'\}$. Then, by direct analogy with the proof of [5, Cor. 5.4], it follows from Schwartz's point and compact support theorems [6, Ths. 35 and 26] that if the correlations are of short range then W takes the form

$$W(F, F') = \langle F, \pi''(\theta) F' \rangle \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega). \quad (4.1)$$

On the other hand, we term the static space correlations of ξ to be of long range if this condition is violated. Thus, in the present context, 'long' is taken to mean non-zero on the employed macroscopic scale.

Our aim now is to establish that these correlations are generically of long range, i.e. that it is only in exceptional circumstances that the condition (4.1) is valid. To this end, we note that, in view of Eqs. (2.35) and (3.40), this condition may be expressed in the form

$$\langle F, \Lambda^* F' \rangle + \langle F', \Lambda^* F \rangle = \Gamma(\nabla F, \nabla F') \quad (4.2)$$

. Thus, the condition for long range correlations is that of the violation of Eq. (4.2) for some F and F' in $\tilde{\mathcal{D}}(\Omega)$.

Proposition 4.1. *A sufficient condition for the process ξ to have long range space correlations is that one of the following ones is violated.*

$$u = 0 \quad (4.3)$$

and

$$\nabla \cdot ((\beta^{-1} \mu \kappa_\mu - \kappa_\beta) \nabla \beta) = 0, \quad (4.4)$$

where κ_β and κ_μ are the derivatives of κ w.r.t. β and μ , respectively. Since these conditions can be satisfied only by certain particular forms of the space-dependent variable β , μ and u , this signifies that the space correlations of ξ are generically of long range.

Remark. It will be seen that the proof of this Proposition is based on the choice $F = F' = (f, 0, 0)$ for the forms of the test functions, with the result that long range correlations prevail if either the condition (4.3) for the drift velocity or (4.4) for the thermal conductivity is violated. We remark here that further sufficient conditions for long range

correlations, expressed in terms of the bulk and shear viscosities, may similarly be derived from other choices of the forms of F and F' .

Proof of Prop. 4.1. It suffices to show that the condition (4.2) for zero range spatial correlations implies Eqs. (4.3) and (4.4). To this end, we choose both F and F' to be $(f, 0, 0)$ and infer from Eq. (3.36) that

$$\Gamma(\nabla F, \nabla F) = 2 \int_{\Omega} dx \kappa (\nabla f)^2. \quad (4.5)$$

On the other hand, by Eqs. (2.32) and (2.33),

$$\langle F, \Lambda^* F' \rangle = \langle \Lambda F, F' \rangle = \frac{\partial}{\partial \lambda} \langle \Psi(\bar{\theta} + \lambda F), \nabla F' \rangle_{|\lambda=0} \quad \forall F, F' \in \tilde{\mathcal{D}}(\Omega). \quad (4.6)$$

In order to treat this formula for the case where $F = F' = (f, 0, 0)$, we define

$$\theta_{\lambda f}(x) := \bar{\theta} + \lambda(f(x), 0, 0) \quad (4.7)$$

On asserting the θ -dependence of the energy current q by referring to $q(x)$ as $q(\theta; x)$ —, defining

$$q_{\lambda f}(x) := q(\theta_{\lambda f}; x) \quad (4.8)$$

and noting that, by identification of Eq. (2.15) with Eq. (2.32), $\Psi := (q, j, \tau)$, we infer from Eq. (4.6) that

$$\langle F, \Lambda^* F \rangle = \left[\frac{\partial}{\partial \lambda} \int_{\Omega} dx \nabla f(x) \cdot q_{\lambda f}(x) \right]_{|\lambda=0}. \quad (4.9)$$

In order to determine the explicit form of $q_{\lambda f}$ in terms of the position dependent variables β , μ and u , we define their canonical counterparts $\beta_{\lambda f}$, $\mu_{\lambda f}$ and $u_{\lambda f}$, respectively, that correspond to $\theta_{\lambda f}$ by the version of Eq. (2.9) obtained by imposing the subscript λf to each of its terms. Thus

$$\theta_{\lambda f} = \beta_{\lambda f} \left(1, -\mu_{\lambda f} + \frac{1}{2} u_{\lambda f}^2, -u_{\lambda f} \right). \quad (4.10)$$

It then follows from Eqs. (2.9) and (4.10) that

$$\begin{aligned} \beta_{\lambda f} &= \beta + \lambda f, \quad \mu_{\lambda f} = (1 + \lambda \beta^{-1} f)^{-1} \mu - \lambda \beta f (1 + \lambda \beta^{-1} f)^{-2} u^2 \quad \text{and} \\ u_{\lambda f} &= (1 + \lambda \beta^{-1} f)^{-1} u. \end{aligned} \quad (4.11)$$

Further, since the energy current, q , defined by Eq. (2.16), is a functional of θ , we express $q_{\lambda f}$ as the corresponding functional of $\theta_{\lambda f}$, i.e. of $\beta_{\lambda f}$, $\mu_{\lambda f}$ and $u_{\lambda f}$. Thus,

$$\begin{aligned} q_{\lambda f} &= \varepsilon(\beta_{\lambda f}, \mu_{\lambda f}) u_{\lambda f} + \frac{1}{2} \rho(\beta_{\lambda f}, \mu_{\lambda f}) u_{\lambda f}^2 u_{\lambda} - \\ &\gamma_1(\beta_{\lambda f}, \mu_{\lambda f}) (D u_{\lambda f} - 2d^{-1} \nabla \cdot u_{\lambda f} I) \cdot u_{\lambda f} - \gamma_2(\beta_{\lambda f}, \mu_{\lambda f}) (\nabla \cdot u_{\lambda f}) u_{\lambda f} + \kappa(\beta_{\lambda f}, \mu_{\lambda f}) \nabla \beta_{\lambda f}. \end{aligned} \quad (4.12)$$

Since, by Eq. (4.11),

$$\left(\frac{\partial\beta_{\lambda f}}{\partial\lambda}\right)_{|\lambda=0} = f; \quad \left(\frac{\partial\mu_{\lambda f}}{\partial\lambda}\right)_{|\lambda=0} = -\beta^{-1}(\mu + u^2)f; \quad \text{and} \quad \left(\frac{\partial u_{\lambda f}}{\partial\lambda}\right)_{|\lambda=0} = -\beta^{-1}uf, \quad (4.13)$$

it follows from Eqs. (4.11)-(4.13) that

$$\begin{aligned} \left(\frac{\partial q_{\lambda f}}{\partial\lambda}\right)_{|\lambda=0} &= \kappa\nabla f + \beta^{-1}\gamma_1(u\nabla f + (\nabla f)u) - 2d^{-1}\nabla\cdot(fu)I\cdot u + (\gamma_2(u\cdot\nabla f)u + \\ &\quad (\kappa_\beta - \beta^{-1}\mu\kappa_\mu)(\nabla\beta)f + C(\beta, \mu; u)f, \end{aligned} \quad (4.14)$$

where κ_β and κ_μ are the derivatives of κ w.r.t. β and μ , respectively, and C is a vector valued functional of β , μ and u that vanishes when $u = 0$.

It follows now from Eqs. (4.5) and (4.14) that the condition (4.2) for short range correlations reduces to the formula

$$\begin{aligned} \int_{\Omega} dx \beta^{-1} [\gamma_1((\nabla f)^2 u^2 + (1 - 2d^{-1})(u\cdot\nabla f)^2) + \gamma_2(u\cdot\nabla f)^2] + \\ \int_{\Omega} dx \frac{1}{2} f^2 \nabla\cdot((\beta^{-1}\mu\kappa_\mu - \kappa_\beta)\nabla\beta - C) = 0 \quad \forall f \in \mathcal{D}(\Omega). \end{aligned} \quad (4.15)$$

In particular, if we replace f here by $f_{x_0, \epsilon}$ where $x_0 \in \Omega$, $\epsilon \in \mathbf{R}_+$ and

$$f_{x_0, \epsilon} = \epsilon^{(1-d)/2} f(\epsilon^{-1}(x - x^0)), \quad (4.16)$$

then the passage to the limit $\epsilon \rightarrow 0$ annihilates the second integral and yields the equation

$$\begin{aligned} \int_{\Omega} dx \beta(x_0)^{-1} [\gamma_1(x_0)((\nabla f(x))^2 u(x_0)^2 + (1 - 2d^{-1})(u(x_0)\cdot\nabla f(x))^2) + \\ \gamma_2(x_0)(u(x_0)\cdot\nabla f(x))^2] = 0 \quad \forall x_0 \in \Omega, \quad f \in \mathcal{D}(\Omega). \end{aligned} \quad (4.17)$$

For $d \geq 2$, this implies that the velocity field u vanishes and consequently that the condition (4.15) reduces to that of the vanishing of the second integral of that equation, with $C = 0$; and for $d = 1$, γ_1 may be equated to zero in the NS equation (2.18) and therefore the same conclusion is valid. Thus, the short range correlation condition implies Eqs. (4.3) and (4.4), and therefore the violation of either of those formulae implies that the ξ process has long range spatial correlations

5. Concluding Remarks

We have shown that the stochastic process executed by the hydrodynamical fluctuations of the continuum model about a nonequilibrium steady state is completely determined by the conditions of Onsagerian regression, local stability and chaoticity. This process, fully specified in Section 3.4, constitutes a mathematical generalisation of Landau's picture of

hydrodynamical fluctuations. In particular, by Prop. 4.1, the process generically carries long range spatial correlations. This result appears to be new within the framework of the Navier-Stokes equations, though it has been previously suggested on rather general heuristic grounds [9, 10] and proved for a certain classical stochastic (non-Hamiltonian) model [11]-[13]. Most notably, it marks an important difference between equilibrium and nonequilibrium properties of hydrodynamical fluctuations, as the spatial correlations of the former are of short (microscopic) range, except at critical points. We remark here that there is no corresponding qualitative difference between the time correlations of the hydrodynamical fluctuations about equilibrium and nonequilibrium, since the regression hypothesis implies that the time scales of both are the macroscopic ones of the Navier-Stokes flow.

The treatment of hydrodynamical fluctuations in this article has been based on a classical macroscopic continuum model, Σ , of a fluid. Presumably this should arise from an underlying quantum mechanics, at the microscopic level, in the following way. One assumes that the quantum system, Σ_{qu} , consists of N particles of one species that is confined to the region Ω and coupled at the boundary $\partial\Omega$ to an array, \mathcal{R} , of reservoirs whose temperatures and chemical potentials are just those of the macrostatistical model. Assuming that these control variables are not the same for all the reservoirs, the system ($\Sigma_{\text{qu}} + \mathcal{R}$) will evolve, under rather general conditions, to a nonequilibrium steady state ω [14]. The hydrodynamical observables of Σ_{qu} may then be formulated along the lines of [4, 5] in terms of the natural counterparts $\hat{\phi}_{\text{qu}}$ and $\hat{\chi}_{\text{qu}}$, of the random classical fields $\hat{\phi}$ and $\hat{\chi}$ of the model Σ , though those quantum fields are now operator valued functions of the positions and momenta of the particles of the system. We denote their evolutes at time t , as governed by the dynamics of the composite ($\Sigma_{\text{qu}} + \mathcal{R}$) by $\hat{\phi}_{\text{qu};t}$ and $\hat{\chi}_{\text{qu};t}$, and we assume that these are distributions of class $\tilde{\mathcal{D}}'(\Omega)$ and $\check{\mathcal{D}}'(\Omega)$ and that their expectation values for the state ω are the classical steady state fields ϕ and χ , respectively, of Section 2. We then define the quantum fluctuation fields $\xi_{\text{qu};t}$ and $\zeta_{\text{qu};t,s}$ by the canonical analogues of Eqs. (3.2) and (3.4) and denote the smeared fields obtained by integrating them against test functions F ($\in \tilde{\mathcal{D}}(\Omega)$) and G ($\in \check{\mathcal{D}}(\Omega)$) by $\xi_{\text{qu};t}(F)$ and $\zeta_{\text{qu};t,s}(G)$, respectively. The correlation functions given by the expectation values, for the state ω of the monomials in the $\xi_{\text{qu};t}(F)$'s and $\zeta_{\text{qu};t,s}(G)$'s then represent the quantum stochastic process [15] executed by the hydrodynamical fluctuations. Further, under a condition of *macroscopic classicality*, whereby the correlation functions are invariant under reordering of the constituent smeared fields, this process is classical. Thus, under the assumption that this condition is fulfilled, possibly up to corrections that are $o(1)$ w.r.t. \hbar , k_B and microscopic relaxation times and attenuation lengths, the fluctuation process simulates a classical one. The further assumptions of Onsagerian regression, local stability and chaoticity then lead precisely to the classical macrostatistical one presented here. The ultimate test of the physical validity of that model is that its correlation functions are those of the hydrodynamical observables of Σ_{qu} , up to the above microscopic corrections

Appendix A: Derivation of the Formula (2.36)

We assume that, at equilibrium, $u = 0$ and β and μ are spatially uniform. The same is therefore true of e , ρ , κ , γ_1 and γ_2 , since these are functions of the latter two variables.

Thus, by Eq. (2.9a),

$$\bar{\theta} = (\beta, -\beta\mu, 0) \quad (A.1)$$

and, as $\phi = (e, \rho, \rho u)$, it follows from Eq. (2.33), together with the identification of the NS equations (2.15)-(2.19) with Eq. (2.32), that the equilibrium form, Λ_{eq} , of Λ is given by the formula

$$\Lambda_{\text{eq}}\delta(\beta, -\beta\mu, -\beta u) = (-\varepsilon\nabla.\delta u - \kappa\Delta\delta\beta, -\rho\nabla.\delta u, -p_\beta\nabla\delta\beta - p_\mu\nabla\delta\mu + \gamma_1\nabla.(D\delta u - 2d^{-1}(\nabla.\delta u)I) + \gamma_2\nabla(\nabla.\delta u)), \quad (A.2)$$

where p_β and p_μ are the derivatives of p w.r.t. β and μ , respectively. Since $\phi = (e, \rho, \rho u)$, it follows from Eqs. (2.5) and (2.10)-(2.12) that

$$p_\beta = -\beta^{-1}(\varepsilon - \rho\mu) \text{ and } p_\mu = \rho. \quad (A.3)$$

Hence, by Eqs. (A.2) and (A.3),

$$\begin{aligned} \Lambda_{\text{eq}}\delta\theta \equiv \Lambda_{\text{eq}}(\delta\beta, -\beta\delta\mu - \mu\delta\beta, -\beta\delta u) = \\ (-\varepsilon\nabla.\delta u - \kappa\Delta\delta\beta, -\rho\nabla.\delta u, \beta^{-1}(\varepsilon - \mu\rho)\nabla\delta\beta - \rho\nabla\delta\mu + \\ \gamma_1\nabla.(D\delta u - 2d^{-1}(\nabla.\delta u)I) + \gamma_2\nabla(\nabla.\delta u)). \end{aligned} \quad (A.4)$$

Equivalently, defining $(\delta\theta^{(1)}, \delta\theta^{(2)}, \delta\theta^{(3)}) := \delta\theta = (\delta\beta, -\beta\delta\mu - \mu\delta\beta, \beta\delta u)$,

$$\begin{aligned} \Lambda_{\text{eq}}\delta\theta = \beta^{-1}(\varepsilon\nabla.\delta\theta^{(3)} - \beta\kappa\Delta\delta\theta^{(1)}, \rho\nabla.\delta\theta^{(2)}, \\ \varepsilon\nabla\delta\theta^{(1)} + \rho\nabla\delta\theta^{(2)} + \gamma_1\nabla.(D\delta\theta^{(3)} - 2d^{-1}\nabla.\delta\theta^{(3)}) + \gamma_2\nabla(\nabla.\delta\theta^{(3)}). \end{aligned} \quad (A.5)$$

Further, the dual, Λ_{eq}^* , of Λ_{eq} is defined by the identity

$$(\delta\theta, \Lambda_{\text{eq}}^*F) \equiv (\Lambda_{\text{eq}}\delta\theta, F) \quad \forall F \in \tilde{\mathcal{D}}(\Omega). \quad (A.6),$$

The formula (2.36) follows immediately from Eqs. (A.5) and (A.6).

Appendix B. Proof of Lemma 3.5.

By the standard relationship between distributions and the corresponding generalised functions,

$$\tilde{\zeta}_{t,s}^{(1)}(\nabla f) = - \int_{\Omega} dx f(x) \frac{\partial}{\partial x_i} \tilde{\zeta}_{t,s;i}^{(1)}(x) \quad \forall f \in \mathcal{D}(\Omega) \quad (B.1)$$

and

$$\tilde{\zeta}_{t,s}^{(3)}(\nabla h) = - \int_{\Omega} dx h_i(x) \frac{\partial}{\partial x_j} \tilde{\zeta}_{t,s;i,j}^{(3)}(x) \quad \forall h \in \mathcal{D}_T(\Omega). \quad (B.2)$$

On combining these formulae with Eqs. (2.19), (3.21) and (3.22), we see that

$$\int_{\Omega^2} dx dx' f(x) f'(x') \frac{\partial^2}{\partial x_i \partial x'_j} E_{\text{eq}}(\tilde{\zeta}_{t,s;i}^{(1)}(x) \tilde{\zeta}_{t',s';j}^{(1)}(x')) =$$

$$2\kappa \int_{\Omega^2} dx dx' f(x) f'(x') \frac{\partial^2}{\partial x_i \partial x'_j} \delta(x - x') \times \\ |[s, t] \cap [s', t']| \quad \forall f, f' \in \mathcal{D}(\Omega) \quad t, s, t', s' \in \mathbf{R}. \quad (B.3)$$

and

$$\int_{\Omega^2} dx dx' h_i(x) h_j(x') \frac{\partial^2}{\partial x_l \partial x'_m} E_{\text{eq}}(\tilde{\zeta}_{t,s;il}^{(3)}(x) \tilde{\zeta}_{t',s';jm}^{(3)}(x')) = \\ 2\beta^{-1} \int_{\Omega^2} dx dx' h_i(x) h_j(x') \frac{\partial^2}{\partial x_l \partial x'_m} (\gamma_1 (\delta_{ij} \delta_{lm} + \delta_{im} \delta_{jl} - 2d^{-1} \delta_{il} \delta_{jm}) + \gamma_2 \delta_{il} \delta_{jm}) \delta(x - x') \times \\ |[s, t] \cap [s', t']| \quad \forall h, h' \in \mathcal{D}_V(\Omega), \quad t, s, t', s' \in \mathbf{R}. \quad (B.4)$$

Eqs. (B.3) and (B.4) signify that

$$\frac{\partial^2}{\partial x_i \partial x'_j} [E_{\text{eq}}(\tilde{\zeta}_{t,s;i}^{(1)}(x) \tilde{\zeta}_{t',s';j}^{(1)}(x')) - 2\kappa \delta(x x') |[s, t] \cap [s', t']] = 0 \quad (B.5)$$

and

$$\frac{\partial^2}{\partial x_l \partial x'_m} [E_{\text{eq}}(\tilde{\zeta}_{t,s;il}^{(3)}(x) \tilde{\zeta}_{t',s';jm}^{(3)}(x')) - \\ 2\beta^{-1} (\gamma_1 (\delta_{ij} \delta_{lm} + \delta_{im} \delta_{jl} - 2d^{-1} \delta_{il} \delta_{jm}) + (\gamma_2 \delta_{il} \delta_{jm})) \delta(x - x') \times \\ |[s, t] \cap [s', t']] = 0. \quad (B.6)$$

We now invoke our assumptions that the tensor $\hat{\tau}$, and hence $\tilde{\zeta}^{(3)}$, is symmetric, that the equilibrium two-point functions of $\tilde{\zeta}^{(1)}$ and $\tilde{\zeta}^{(3)}$ are translationally and rotationally invariant and that they are of zero range. It then follows that these functions take the following forms.

$$E_{\text{eq}}(\tilde{\zeta}_{t,s;i}^{(1)}(x) \tilde{\zeta}_{t',s';j}^{(1)}(x')) = A(x - x') \delta_{ij} \quad (B.7)$$

and

$$E_{\text{eq}}(\tilde{\zeta}_{t,s;il}^{(3)}(x) \tilde{\zeta}_{t',s';jm}^{(3)}(x')) = B(x - x') (\delta_{ij} \delta_{lm} + \delta_{im} \delta_{lj}) + C(x - x') \delta_{il} \delta_{jm}, \quad (B.8)$$

where A, B and C are generalised functions on Ω that depend on t, s, t' and s' . It follows from Eqs. (B.5) and (B.7) that

$$\Delta [A(x - x') - 2\kappa \delta(x - x') |[s, t] \cap [s', t']] = 0 \quad (B.9)$$

and from Eqs. (B.6) and (B.8) that

$$\Delta [B(x - x') - 2\beta^{-1} \gamma_1 \delta(x - x') |[s, t] \cap [s', t']] = 0 \quad (B.10)$$

and

$$\Delta [C(x - x') - 2\beta^{-1} (\gamma_2 - 2d^{-1} \gamma_1) \delta(x - x') |[s, t] \cap [s', t']] = 0. \quad (B.11)$$

In view of Schwartz's point support theorem [6, Th. 35], these last three equations signify that

$$A(x - x') = 2\kappa\delta(x - x')|[s, t] \cap [s', t']|, \quad (B.12)$$

$$B(x - x') = 2\beta^{-1}\gamma_1\delta(x - x')|[s, t] \cap [s', t']| \quad (B.13)$$

and

$$C(x - x') = 2\beta^{-1}(\gamma_2 - 2d^{-1}\gamma_1)\delta(x - x')|[s, t] \cap [s', t']|. \quad (B.14)$$

Eq. (3.25) now follows from Eqs. (B.7) and (B.12); and Eq. (3.26) follows from Eqs. (B.8), (B.13) and (B.14). Finally, in view of the observation following Eq. (3.24), a parallel treatment of the other components of the two-point functions of $\tilde{\zeta}$ reveals that they all vanish.

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