# Conditions for certain ruin for the generalised Ornstein-Uhlenbeck process and the structure of the upper and lower bounds 

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#### Abstract

For a bivariate Lévy process $\left(\xi_{t}, \eta_{t}\right)_{t \geq 0}$ the generalised Ornstein-Uhlenbeck (GOU) process is defined as $$
V_{t}:=e^{\xi_{t}}\left(z+\int_{0}^{t} e^{-\xi_{s-}} \mathrm{d} \eta_{s}\right), \quad t \geq 0
$$ where $z \in \mathbb{R}$. We present conditions on the characteristic triplet of $(\xi, \eta)$ which ensure certain ruin for the GOU. We present a detailed analysis on the structure of the upper and lower bounds and the sets of values on which the GOU is almost surely increasing, or decreasing. This paper is the sequel to [2], which stated conditions for zero probability of ruin, and completes a significant aspect of the study of the GOU.


Key words: Lévy processes, Generalised Ornstein-Uhlenbeck process, Exponential functionals of Lévy processes, Ruin probability
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## 1 Notation and Theoretical Background

For a review of publications and applications for the GOU, see [2]. In Section 2 of this paper, we state results on certain ruin for the GOU. Theorem 3.1 of Paulsen [10] gives conditions for certain ruin for the GOU in the special case in which $\xi$ and $\eta$ are independent. In [2] it is shown that this theorem does not hold for the general case. Theorems 1 and 3 of Section 2 give the required
generalization, stated in terms of the characteristic triplet of $(\xi, \eta)$. Section 3 begins with results, in particular Proposition 6 and Theorem 9, which describe the structure of the upper and lower bounds and the sets of values on which the GOU is almost surely increasing, or decreasing. Section 3 then outlines the ruin probability implications of these structural results, in particular with Theorems 13 and 14, which state conditions for certain ruin in terms of upper and lower bound structure. Section 3 concludes with technical propositions used to prove the major theorems. Section 4 contains proofs of the results in Section 2 and 3, and concludes with a number of examples which illustrate and extend certain results. For the remainder of this section we set up some notation, which builds on that of [2], and outline some basic results which we will need.

Let $(\xi, \eta)$ be a bivariate Lévy process on a filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$ and define the GOU process $V$, and the associated stochastic integral process $Z$, as

$$
\begin{equation*}
V_{t}:=e^{\xi_{t}}\left(z+\int_{0}^{t} e^{-\xi_{s-}} \mathrm{d} \eta_{s}\right) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
Z_{t}:=\int_{0}^{t} e^{-\xi_{s-}} \mathrm{d} \eta_{s} \tag{2}
\end{equation*}
$$

To avoid trivialities, assume that neither $\xi$ nor $\eta$ are identically zero. It was shown in [2] that

$$
\begin{equation*}
\Delta V_{t}=e^{\Delta \xi_{t}}\left(\Delta \eta_{t}-V_{t-}\left(e^{-\Delta \xi_{t}}-1\right)\right) \tag{3}
\end{equation*}
$$

The characteristic triplet of $(\xi, \eta)$ will be written $\left(\left(\tilde{\gamma}_{\xi}, \tilde{\gamma}_{\eta}\right), \Sigma_{\xi, \eta}, \Pi_{\xi, \eta}\right)$. The characteristic triplet of $\xi$ as a one-dimensional Lévy process will be written $\left(\gamma_{\xi}, \sigma_{\xi}^{2}, \Pi_{\xi}\right)$, where

$$
\begin{equation*}
\gamma_{\xi}=\tilde{\gamma}_{\xi}+\int_{\{|x|<1\} \cap\left\{x^{2}+y^{2} \geq 1\right\}} x \Pi_{\xi, \eta}(\mathrm{d}(x, y)), \tag{4}
\end{equation*}
$$

and $\sigma_{\xi}^{2}$ is the upper left entry in the matrix $\Sigma_{\xi, \eta}$, and $\eta$ is symmetric. The random jump measure and Brownian motion components of $(\xi, \eta)$ will be denoted respectively by $N_{\xi, \eta, t}$ and $\left(B_{\xi}, B_{\eta}\right)$.

For a Lebesgue set $\Lambda$ define the hitting time of $\Lambda$ by $V$ to be $T_{z, \Lambda}:=\inf \{t>$ $\left.0: V_{t} \in \Lambda \mid V_{0}=z\right\}$, where $T_{z, \Lambda}:=\infty$ whenever $V_{t} \notin \Lambda$ for all $t>0$ and $V_{0}=z$. When the context makes it obvious we will simply write $T_{\Lambda}$. Define the infinite horizon ruin probability for the GOU by

$$
\psi(z):=P\left(\inf _{t>0} V_{t}<0 \mid V_{0}=z\right)=P\left(\inf _{t>0} Z_{t}<-z\right)=P\left(T_{z,(-\infty, 0)}<\infty\right) .
$$

Note that for all $t>0, V_{t}$ is increasing as a function of the initial value $z$ and hence, if $0 \leq z_{1} \leq z_{2}$, then $\psi\left(z_{1}\right) \geq \psi\left(z_{2}\right)$. For further explanation of the
above terms, as well as extra definitions and results for Lévy processes, see Section 1 of [2]. We now outline notation and theory needed for the present paper, which were not dealt with in Section 1 of [2].

The total variation of an $\mathbb{R}^{n}$-valued function over the interval $[a, b]$ is defined by

$$
V_{f}([a, b]):=\sup \sum_{i=1}^{n}\left|f\left(t_{i}\right)-f\left(t_{i-1}\right)\right|,
$$

where the supremum is taken over all finite partitions $a=t_{0}<t_{1}<\cdots<$ $t_{n}=b$. A Lévy process $X$ on $\mathbb{R}^{n}$, with characteristic triplet $\left(\gamma_{X}, \Sigma_{X}, \Pi_{X}\right)$ and random jump measure $N_{X, t}$, is said to be of finite variation if, with probability 1 , its sample paths $X_{t}(\omega)$ are of finite total variation on $[0, t]$ for every $t>0$. It is shown in [5], p.86, this occurs iff $\Sigma_{X}=0$ and $\int_{|z| \leq 1}|z| \Pi_{X}(\mathrm{~d} z)<\infty$. Further, if this occurs then we can write

$$
X_{t}=d_{X} t+\int_{\mathbb{R}^{n}} z N_{X, t}(\cdot, \mathrm{~d} z)=d t+\sum_{0<s \leq t} \Delta X_{s}
$$

where

$$
\begin{equation*}
d_{X}=\gamma_{X}-\int_{|z|<1} z \Pi_{X}(\mathrm{~d} z) \in \mathbb{R}^{n}=E\left(X_{1}-\int_{\mathbb{R}^{n}} z N_{X, 1}(\cdot, \mathrm{~d} z)\right) \tag{5}
\end{equation*}
$$

is called the drift vector of $X$. A 1-dimensional Lévy process $X$ is said to be a subordinator if $X_{t}(\omega)$ is an increasing function of $t$, a.s., and it is shown in [5], p.88, that the following conditions are equivalent:
(1) $X$ is a subordinator.
(2) $X_{t} \geq 0$ a.s for some $t>0$.
(3) $X_{t} \geq 0$ a.s for every $t>0$.
(4) The characteristic triplet satisfies

$$
\sigma_{X}^{2}=0, \quad \int_{(-\infty, 0]} \Pi_{X}(\mathrm{~d} x)=0, \quad \int_{(0,1)} x \Pi_{X}(\mathrm{~d} x)<\infty, \text { and } d_{X} \geq 0
$$

That is, there is no Brownian component, no negative jumps, the positive jumps are of finite variation and the drift is non-negative.

A 1-dimensional Lévy process $X$ will drift to $\infty$, drift to $-\infty$ or oscillate between $\infty$ and $-\infty$, namely, one of the following must hold:

$$
\begin{gather*}
\lim _{t \rightarrow \infty} X_{t}=\infty \text { a.s.; }  \tag{6}\\
\lim _{t \rightarrow \infty} X_{t}=-\infty \text { a.s.; }  \tag{7}\\
-\infty=\liminf _{t \rightarrow \infty} X_{t}<\limsup _{t \rightarrow \infty} X_{t}=\infty \text { a.s. } \tag{8}
\end{gather*}
$$

Necessary and sufficient conditions for these cases are given in [6]. Whenever the expected value of $X_{1}$ is a well-defined member of the extended real numbers, cases (6), (7), and (8) equate respectively to $E\left(X_{1}\right)>0, E\left(X_{1}\right)<0$, and
$E\left(X_{1}\right)=0$. For the case in which the expected value does not exist, we need more notation. For $x>0$, denote the tail functions of the Lévy measure by $\bar{\Pi}_{X}^{+}(x):=\Pi_{X}((x, \infty)), \quad \bar{\Pi}_{X}^{-}(x):=\Pi_{X}((-\infty,-x)), \quad \bar{\Pi}_{X}(x):=\bar{\Pi}_{X}^{+}(x)+\bar{\Pi}_{X}^{-}(x)$.

Define, for $x \geq 1$,

$$
A_{X}^{+}(x):=\max \left\{\bar{\Pi}_{X}^{+}(1), 1\right\}+\int_{1}^{x} \bar{\Pi}_{X}^{+}(u) \mathrm{d} u
$$

and

$$
A_{X}^{-}(x):=\max \left\{\bar{\Pi}_{X}^{-}(1), 1\right\}+\int_{1}^{x} \bar{\Pi}_{X}^{-}(u) \mathrm{d} u
$$

and define the integrals

$$
J_{X}^{+}:=\int_{1}^{\infty}\left(\frac{x}{A_{X}^{-}(x)}\right)\left|\bar{\Pi}_{X}^{+}(\mathrm{d} x)\right| \text { and } J_{X}^{-}:=\int_{1}^{\infty}\left(\frac{x}{A_{X}^{+}(x)}\right)\left|\bar{\Pi}_{X}^{-}(\mathrm{d} x)\right| .
$$

In [6] it is shown that if $E\left(X_{1}\right)$ is not well defined, that is, if

$$
\int_{1}^{\infty} x \Pi_{X}(\mathrm{~d} x)=\int_{-\infty}^{-1}|x| \Pi_{X}(\mathrm{~d} x)=\infty
$$

then (6) occurs iff $J_{X}^{-}<\infty$, (77) occurs iff $J_{X}^{+}<\infty$ and (8) occurs iff $J_{X}^{-}=$ $J_{X}^{+}=\infty$.

It is shown in [4] that the GOU is a time homogenous strong Markov process. In [7], necessary and sufficient conditions are stated for a.s. convergence of $Z_{t}$ to a finite random variable $Z_{\infty}$ as $t$ approaches $\infty$, whilst in [8], necessary and sufficient conditions are stated for stationarity of $V$. We will need to use these conditions, and to describe them we need some further notation.

For a bivariate Lévy process $(X, Y)$ define the integral

$$
I_{X, Y}:=\int_{(e, \infty)} \frac{\ln (y)}{A_{X}^{+}(\ln (y))}\left|\bar{\Pi}_{Y}(\mathrm{~d} y)\right|
$$

and the auxiliary Lévy process $K^{X, Y}$ by

$$
K_{t}^{X, Y}:=Y_{t}+\sum_{0<s \leq t}\left(e^{\Delta X_{s}}-1\right) \Delta Y_{s}-t \operatorname{Cov}\left(B_{X, 1}, B_{Y, 1}\right),
$$

where Cov denotes the covariance. Theorem 2 of [7] states that $Z_{t}$ converges a.s. to a finite random variable $Z_{\infty}$ as $t \rightarrow \infty$ iff $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a.s. and $I_{\xi, \eta}<\infty$. There is a special case in which, for some $c \in \mathbb{R}$,

$$
\begin{equation*}
Z_{t}=c\left(e^{-\xi_{t}}-1\right) \quad \text { and } \quad V_{t}=e^{\xi_{t}}(z-c)+c \tag{9}
\end{equation*}
$$

a.s. for all $t \geq 0$. Exact conditions for this degenerate situation, given in terms of the characteristic triplet of $(\xi, \eta)$, will be stated in Proposition 8. In this
situation， $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a．s．implies that $Z_{t}$ converges a．s．to the constant random variable $Z_{\infty}=-c$ as $t \rightarrow \infty$ ，and in［3］it is shown that this is the only case in which $Z_{\infty}$ is not a continuous random variable．Note that，regardless of the asymptotic behaviour of $\xi$ ，if（9）holds then $V$ is strictly stationary iff $V_{0}=c$ ．If（9）does not hold for any $c \in \mathbb{R}$ ，then Theorem 2.1 of［8］states that $V$ is strictly stationary iff the stochastic integral $\int_{0}^{\infty} e^{\xi_{s-}} \mathrm{d} K_{s}^{\xi, \eta}$ converges a．s． or，equivalently，iff $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$ a．s．and $I_{-\xi, K^{\xi}, \eta}<\infty$ ．In this case the stationary random variable $V_{\infty}$ satisfies $V_{\infty}={ }_{D} \int_{0}^{\infty} e^{\xi_{s}-} \mathrm{d} K_{s}^{\xi, \eta}$ ．

## 2 Conditions for Certain Ruin

In Theorem 1 of［2］，exact conditions were given on the characteristic triplet of $(\xi, \eta)$ for the existence of $u \geq 0$ such that $\psi(u)=0$ ，and a precise value was given for the value $\inf \{u \geq 0: \psi(u)=0\}$ ，where we use the convention that $\inf \{\emptyset \cap[0, \infty)\}=\infty$ ．It is a consequence of Theorem $⿴ 囗 十$ below，that when the relevant assumptions are satisfied，there exists $z \geq 0$ such that $\psi(z)<1$ iff there exists $u \geq 0$ such that $\psi(u)=0$ ．Thus，even though they are not stated explictly，Theorem 1 implies exact conditions on the characteristic triplet of $(\xi, \eta)$ for certain ruin．

Statements（1）and（2）of Theorem 1 are generalizations to the dependent case of Paulsen＇s Theorem 3．1，parts（a）and（b），respectively．Statement（1）of Theorem 1 also removes Paulsen＇s assumption of finite mean for $\xi$ ，and replaces his moment conditions with the precise necessary and sufficient conditions for stationarity of $V$ ．For statement（2）of Theorem［1，a finite mean assumption and moment conditions remain necessary．

Theorem 1 Let $m:=\inf \{u \geq 0: \psi(u)=0\}$ ．
（1）Suppose $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$ a．s．and $I_{-\xi, K}{ }^{\xi, \eta}<\infty$ ．Then $0<\psi(z)<1$ iff $0 \leq z<m<\infty$ ．
（2）Suppose $E\left(\xi_{1}\right)=0, E\left(e^{\left|\xi_{1}\right|}\right)<\infty$ and there exist $p, q>1$ with $1 / p+1 / q=$ 1 such that $E\left(e^{-p \xi_{1}}\right)<\infty$ and $E\left(\left|\eta_{1}\right|^{q}\right)<\infty$ ．If，for all $c \in \mathbb{R}$ ，the degenerate case（9）does not hold，then $0<\psi(z)<1$ iff $0 \leq z<m<\infty$ ． If there exists $c \in \mathbb{R}$ such that equation（9）holds，then $\psi(z)<1$ iff $\psi(z)=0$ ，which occurs iff $0 \leq c \leq z$ ．

Remark 2 （1）In proving［10］Theorem 3.1 （b），Paulsen discretizes the GOU at integer time points and then uses a recurrence result from［1］．His ar－ gument uses the inequality $P\left(V_{1}<0 \mid V_{0}=z\right)>0$ for all $z \geq 0$ ，which is true in the independent case if either $\xi$ or $\eta$ has a Brownian com－ ponent，or can have negative jumps．However，even in the independent case，this inequality can fail to hold when $V_{t}$ decreases due to a determin－
istic drift. For example, let $N$ and $M$ be independent Poisson processes with parameter 1 and define $\xi_{t}:=-t+N_{t}$ and $\eta_{t}:=-t+M_{t}$. Let $T_{z}:=\inf \left\{t>0: V_{t}<0 \mid V_{0}=z\right\}$. Then $V_{t} \geq(z+1) e^{-\xi_{t}}-1:=V_{t}^{\prime}$ on $t \leq T_{z}$ and $P\left(V_{1}^{\prime}<0 \mid V_{0}^{\prime}=z\right)=0$ whenever $z>e^{1}-1$. In proving statement (2) of Theorem 1 we get around this difficulty by discretizing the GOU at random times $T_{i}$ and then showing that the stated conditions result in $P\left(V_{T_{1}}<0 \mid V_{0}=z\right)>0$ for all $z \geq 0$ in the general case.
(2) Assume $\xi$ and $\eta$ are independent and $\eta$ is not a subordinator. In this case, whenever $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates between $\infty$ and $-\infty$ a.s., it is a consequence of Theorem 1 in [2], that $\psi(u)>0$ for all $u \geq 0$, and hence $m=\infty$. Thus, by statement (1) of Theorem 1, if $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$ a.s. and $I_{-\xi, K^{\xi}, \eta}<\infty$, then $\psi(z)=1$ for all $z \geq 0$. This result is a slight strengthening of Paulsen's Theorem 3.1 (a). Further, statement (2) simplifies exactly to Paulsen's Theorem 3.1 (b). Since $\xi$ and $\eta$ are independent the conditions in statement (2) simplify to $E\left(\xi_{1}\right)=0, E\left(e^{\left|\xi_{1}\right|}\right)<\infty$ and $E\left(\eta_{1}\right)<\infty$. Since $m=\infty, \psi(z)=1$ for all $z \geq 0$ whenever these conditions hold. The simplification of conditions occurs because Hölder's inequality is not needed in the proof, and a simpler argument using independence suffices. When transferred onto the Lévy measure, these conditions are equivalent to those in Paulsen's Theorem 3.1 (b).

We now present Theorem 3, which is the generalization to the dependent case of Paulsen's Theorem 3.1, part (c). In addition, Paulsen's assumption of finite mean for $\xi$ is removed, and his moment conditions are replaced with the precise necessary and sufficient conditions for a.s. convergence of $Z_{t}$ to a finite random variable $Z_{\infty}$, as $t \rightarrow \infty$. A formula for the ruin probability in this situation was given in Theorem 4 of [2], however no conditions for certain ruin were found. Theorem 3 gives exact conditions on the characteristic triplet of $(\xi, \eta)$ for certain ruin. To state these conditions, we need the following definitions.

Let $A_{1}:=\left\{(x, y) \in \mathbb{R}^{2}: x \geq 0, y \geq 0\right\}$, and similarly, let $A_{2}, A_{3}$ and $A_{4}$ be the quadrants in which $\{x \geq 0, y \leq 0\},\{x \leq 0, y \leq 0\}$ and $\{x \leq 0, y \geq 0\}$ respectively. For each $i=1,2,3,4$ and $u \in \mathbb{R}$ let

$$
B_{i}^{u}:=\left\{(x, y) \in A_{i}: y-u\left(e^{-x}-1\right)>0\right\}
$$

and define

$$
\begin{aligned}
& \theta_{1}^{\prime}:=\left\{\begin{array}{c}
\inf \left\{u \leq 0: \Pi_{\xi, \eta}\left(B_{1}^{u}\right)>0\right\} \\
0 \\
\text { if } \Pi_{\xi, \eta}\left(A_{1} \backslash A_{2}\right)=0,
\end{array} \quad \theta_{3}^{\prime}:=\left\{\begin{array}{l}
\sup \left\{u \leq 0: \Pi_{\xi, \eta}\left(B_{3}^{u}\right)>0\right\} \\
-\infty \\
\text { if } \Pi_{\xi, \eta}\left(A_{3} \backslash A_{2}\right)=0,
\end{array}\right.\right. \\
& \theta_{2}^{\prime}:=\left\{\begin{array}{c}
\inf \left\{u \geq 0: \Pi_{\xi, \eta}\left(B_{2}^{u}\right)>0\right\} \\
\infty \\
\text { if } \Pi_{\xi, \eta}\left(A_{2} \backslash A_{3}\right)=0,
\end{array} \quad \theta_{4}^{\prime}:=\left\{\begin{array}{c}
\sup \left\{u \geq 0: \Pi_{\xi, \eta}\left(B_{4}^{u}\right)>0\right\} \\
0 \\
\text { if } \Pi_{\xi, \eta}\left(A_{4} \backslash A_{3}\right)=0 .
\end{array}\right.\right.
\end{aligned}
$$

Theorem 3 Suppose $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a.s. and $I_{\xi, \eta}<\infty$. Then $\psi(0)=1$ if and only iff $-\eta$ is a subordinator, or there exists $z>0$ such that $\psi(z)=1$. The latter occurs if and only if $\Pi_{\xi, \eta}\left(A_{1}\right)=0, \theta_{4}^{\prime} \leq \theta_{2}^{\prime}$, and there exists $u \in\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]$ such that

$$
\Sigma_{\xi, \eta}=\left[\begin{array}{rr}
1 & -u  \tag{10}\\
-u & u^{2}
\end{array}\right] \sigma_{\xi}^{2}
$$

and

$$
\begin{equation*}
g(u):=\tilde{\gamma}_{\eta}+u \tilde{\gamma}_{\xi}-\frac{1}{2} u \sigma_{\xi}^{2}-\int_{\left\{x^{2}+y^{2}<1\right\}}(u x+y) \Pi_{\xi, \eta}(\mathrm{d}(x, y)) \leq 0 \tag{11}
\end{equation*}
$$

If there exists $z \geq 0$ such that $\psi(z)=1$ and, for all $c \in \mathbb{R}$, the equation (9) does not hold, then the following hold:
(1) If $\sigma_{\xi}^{2}=0$ then $\psi(z)=1$ for all $z \leq m:=\sup \left\{u \in\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]: g(u) \leq 0\right\}$, and $0 \leq \psi(z)<1$ for all $z>m$;
(2) If $\sigma_{\xi}^{2} \neq 0$ then $\psi(z)=1$ for all $z \leq m:=-\frac{\sigma_{\xi, n}}{\sigma_{\xi}}$, and $0<\psi(z)<1$ for all $z>m$.

If there exists $z \geq 0$ such that $\psi(z)=1$ and there exists $c \in \mathbb{R}$ such that (9) holds, then $0<c=\theta_{4}^{\prime}=\theta_{2}^{\prime}, \psi(z)=1$ for all $z<c$, and $\psi(z)=0$ for all $z \geq c$.

Remark 4 (1) When $\Pi_{\xi, \eta}\left(A_{1}\right)=0, \theta_{4}^{\prime} \leq \theta_{2}^{\prime}$ and $u \in\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]$ the function $g(u)$ is a well-defined member of the extended reals. The existence and finiteness of $g$ is fully analysed in point (1) of Remark 19 ,
(2) Assume $\xi$ and $\eta$ are independent. Then all jumps occur at the axes of the sets $A_{i}$, and $\sigma_{\xi, \eta}=0$. With a little work, Theorem 3 simplifies to the following statement: Suppose $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a.s. and $I_{\xi, \eta}<\infty$. Then $\psi(0)=1$ iff $-\eta$ is a subordinator, or $\psi(z)=1$ for some $z>0$. The latter occurs iff $\xi$ and $\eta$ are each of finite variation and have no positive jumps, and $g(z) \leq 0$. Note that when $(\xi, \eta)$ is finite variation, $g$ simplifies to $g(u)=d_{\eta}+u d_{\xi}$, as explained in equation (131). Since $\xi$ drifts to $\infty$ a.s., it must be that $d_{\xi}>0$. Thus, $g(z) \leq 0$ for some $z>0$ iff $d_{\eta}<0$. In particular, $-\eta$ is a subordinator.
(3) In Paulsen [10], Theorem 3.1 (c), it is stated that when $\xi$ and $\eta$ are independent, $E\left(\xi_{1}\right)>0$, and a set of moment conditions hold, then $\psi(z)=1$ iff $\xi_{t}=\alpha t, \eta_{t}=\beta t$ and $\beta<-\alpha z$ for real constants $\alpha$ and $\beta$. This statement contradicts the independence version of Theorem 3 stated above, and is false. A simple counterexample is $(\xi, \eta)_{t}:=\left(t,-t-N_{t}\right)$ where $N$ is a Poisson process. Paulsen's moment conditions are satisfied trivially. However, Theorem 3 implies that $\psi(z)=1$ for all $z \leq 1$, and this is confirmed by elementary calculations. If we denote the jump times of $N_{t}$
by $0=T_{0}<T_{1}<T_{2}<\cdots$ then

$$
V_{t}=1+e^{t}\left(z-1-\sum_{i=1}^{N_{t}} e^{-T_{i}}\right) .
$$

Thus, if $z=1$, then $V_{T_{2}}=-e^{T_{2}-T_{1}}<0$ a.s. and so $\psi(1)=1$.
The following proposition fully explains the ruin probability function for the degenerate situation (9). It will be used to prove that Theorems 1 and 3 correctly allow for this case.

Proposition 5 Suppose that there exists $c \in \mathbb{R}$ such that $V_{t}=e^{\xi_{t}}(z-c)+c$. If $c \geq 0$ then $\psi(z)=0$ for all $z \geq c$, and the following statements hold for all $0 \leq z<c$ :
(1) If $\xi$ drifts to $-\infty$ a.s. then $0<\psi(z)<1$;
(2) If $\xi$ oscillates between $\infty$ and $-\infty$ a.s. then $\psi(z)=1$;
(3) If $\xi$ drifts to $\infty$ a.s. then $\psi(z)=1$.

If $c<0$ then the following statements hold for all $z \geq 0$ :
(4) If $\xi$ drifts to $-\infty$ a.s. then $\psi(z)=1$;
(5) If $\xi$ oscillates between $\infty$ and $-\infty$ a.s. then $\psi(z)=1$;
(6) If $\xi$ drifts to $\infty$ a.s. then $0<\psi(z)<1$.

## 3 Structure of the upper and lower bounds, and relationship with certain ruin

Define the lower bound function $\delta$ and the upper bound function $\Upsilon$ by

$$
\delta(z):=\inf \left\{u \in \mathbb{R}: P\left(\inf _{t \geq 0} V_{t} \leq u \mid V_{0}=z\right)>0\right\}
$$

and

$$
\Upsilon(z):=\sup \left\{u \in \mathbb{R}: P\left(\sup _{t \geq 0} V_{t} \geq u \mid V_{0}=z\right)>0\right\}
$$

where we use the convention that $\inf \{\emptyset \cap \mathbb{R}\}=\infty$ and $\sup \{\emptyset \cap \mathbb{R}\}=-\infty$. When $V_{0}=z$, the probability that the sample paths $V_{t}$ will ever rise above $\Upsilon(z)$, or below $\delta(z)$, is zero. In particular, the ruin probability function $\psi$ satisfies $\psi(z)=0$ iff $\delta(z) \geq 0$. Define the sets $L$ and $U$ by

$$
L:=\{u \in \mathbb{R}: \delta(u)=u\} \text { and } U:=\{u \in \mathbb{R}: \Upsilon(u)=u\}
$$

It will be a consequence of Proposition 17 that $L$ and $U$ must each be of the form

$$
\begin{equation*}
\emptyset,\{a\},[a, b],[a, \infty), \text { or }(-\infty, b] \tag{12}
\end{equation*}
$$

for some $a, b \in \mathbb{R}$. The fact that $L$ and $U$ are both connected sets is of great importance.

This section contains a detailed analysis of $\delta, \Upsilon, U$ and $L$ and their relationship with the ruin function. In particular, we are interested in which combinations of $L$ and $U$ can exist. For each combination we are also interested in the possible asymptotic behaviour of $\xi$, namely, whether $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates between $\infty$ and $-\infty$ a.s. We are interested in this asymptotic behaviour because of its link with the conditions for convergence of $Z_{t}$ and stationarity of $V$, as discussed in Section 1. As well as being of independent interest, the results contained in this section are essential for the proofs of Theorems 1 and 3 .

We begin with comments on $\delta$, and $L$. The analogues for $\Upsilon$ and $U$ are obvious through symmetry. Firstly, note that $\delta(z) \leq z$ for all $z \in \mathbb{R}$, whilst the fact that $V_{t}$ is increasing in $z$ for all $t \geq 0$ implies that $\delta\left(z_{1}\right) \leq \delta\left(z_{2}\right)$ whenever $z_{1}<z_{2}$. The following proposition explains the behaviour of the $\delta$ outside the set $L$, and states that $L$ is precisely the set of starting parts $V_{0}=z$ for which almost all sample paths $V_{t}$ are increasing for some time period. Recall that $T_{z, \Lambda}:=\inf \left\{t>0: V_{t} \in \Lambda\right\}$, and define $L^{c}:=\mathbb{R} \backslash L$.

Proposition 6 The following statements hold for $L$ and $\delta$, and the symmetric statements hold for $U$ and $\Upsilon$ :
(1) If $z \geq \sup L$ then $\delta(z)=\sup L$;
(2) If $z<\inf L$ then $\delta(z)=-\infty$;
(3) For $z \in L, P\left(V_{t}\right.$ is increasing on $\left.0<t \leq T_{z, L^{c}} \mid V_{0}=z\right)=1$;
(4) For $z \in L^{c}, P\left(V_{t}\right.$ is increasing on $\left.0<t \leq T_{z, L} \mid V_{0}=z\right)<1$.

In Section 1 we assumed that neither $\xi$ nor $\eta$ are identically zero in order to avoid trivialities. The following proposition explains these trivialities.

Proposition 7 (1) $L=\mathbb{R}$ iff $\xi_{t}=0$ a.s. for all $t>0$ and $\eta$ is a subordinator.
(2) $U=\mathbb{R}$ iff $\xi_{t}=0$ a.s. for all $t>0$ and $-\eta$ is a subordinator.
(3) $L=U=\mathbb{R}$ iff $\xi_{t}=\eta_{t}=0$ a.s. for all $t>0$.

For the rest of this paper we again assume that neither $\xi$ nor $\eta$ are identically zero. The following proposition explains the degenerate situation described in equation (9). Note that the deterministic case $(\xi, \eta)_{t}:=(\alpha, \beta) t$ for non-zero constants $\alpha$ and $\beta$ satisfies the conditions of this proposition for $c=-\beta / \alpha$. Recall that a Borel set $\Lambda \subsetneq \mathbb{R}$ is an absorbing set for $V$, if for all $0 \leq s \leq t$, $P\left(V_{t} \in \Lambda \mid V_{s}=x\right)=1$ for all $x \in \Lambda$. That is, whenever a sample path $V_{t}$ hits $\Lambda$, it never leaves. The stochastic exponential will be denoted by $\epsilon$.

Proposition 8 The following are equivalent for $c \neq 0$ :
(1) $L \cap U \neq \emptyset$;
(2) $L \cap U=\{c\}$;
(3) $V_{t}=e^{\xi_{t}}(z-c)+c$ and $Z_{t}=c\left(e^{-\xi_{t}}-1\right)$;
(4) $\{c\}$ is an absorbing set;
(5) $\Sigma_{\xi, \eta}$ satisfies (10) for $u=c, \Pi_{\xi, \eta}=0$ or is supported on the curve $\left\{(x, y): y-c\left(e^{-x}-1\right)=0\right\}$, and $g(c)=0$;
(6) $e^{-\xi_{t}}=\epsilon(\eta / c)_{t}$.

If the above conditions hold and $\Sigma_{\xi, \eta} \neq 0$ then $L=U=\{c\}$ and there exist Lévy processes $(\xi, \eta)$ for this situation such that $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates a.s. If the above conditions hold and $\Sigma_{\xi, \eta}=0$ then:
(a) $U=(-\infty, c]$ and $L=[c, \infty)$ iff $\xi$ is a subordinator;
(b) $L=(-\infty, c]$ and $U=[c, \infty)$ iff $-\xi$ is a subordinator;
(c) $L=U=\{c\}$ iff neither $\xi$ or $-\xi$ is a subordinator. There exist Lévy processes $(\xi, \eta)$ for this situation such that $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates a.s.

We present a theorem which describes all possible combinations of $L$ and $U$ and the associated asymptotic behaviour of $\xi$, for the case in which $L \cap U=\emptyset$.

Theorem 9 Suppose that $L \cap U=\emptyset$. If $\Sigma_{\xi, \eta} \neq 0$ then only the following cases can exist:
(1) $L=U=\emptyset$;
(2) $L=\{a\}$ for some $a \in \mathbb{R}$ and $U=\emptyset$;
(3) $U=\{a\}$ for some $a \in \mathbb{R}$ and $L=\emptyset$.

If $\Sigma_{\xi, \eta}=0$ then only the following cases can exist:
(a) If $L=\emptyset$ then $U$ is of the form $\emptyset,\{a\},[a, b],[a, \infty)$, or $(-\infty, b]$ for some $a, b \in \mathbb{R}$;
(b) If $U=\emptyset$ then $L$ is of the form $\emptyset,\{a\},[a, b],[a, \infty)$, or $(-\infty, b]$ for some $a, b \in \mathbb{R}$;
(c) If $L \neq \emptyset$ and $U \neq \emptyset$ then there exist $a<b$ such that $L=(-\infty, a]$ and $U=[b, \infty)$, or $U=(-\infty, a]$ and $L=[b, \infty)$.

If $U=(-\infty, a]$ or $L=[b, \infty)$ (or both) then $\xi$ is a subordinator. If $L=$ $(-\infty, a]$ or $U=[b, \infty)$ (or both) then $-\xi$ is a subordinator. For all of the other combinations of $L$ and $U$ above, there exist Lévy processes $(\xi, \eta)$ such that $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates a.s.

An absorbent set $\Lambda \subsetneq \mathbb{R}$ is a maximal absorbing set if it is not properly contained in any other absorbing set. Note that if $\Lambda$ is a maximal absorbing
set, then $\mathbb{R} \backslash \Lambda$ contains no absorbing sets otherwise we could take the union of $\Lambda$ with the absorbing set, and this would be an absorbing set properly containing $\Lambda$. The following corollary is immediate. For each statement (1)(4), the claim that the sets $\Lambda$ are maximal absorbing follows from Proposition 6. The remaining statements follow immediately from Theorem 9 .

Corollary 10 There exist Lévy processes $(\xi, \eta)$ with $L \cap U=\emptyset$ such that the associated GOU has the following maximal absorbing sets $\Lambda$ :
(1) $\Lambda=U \cup L$, where $U=(-\infty, a]$ and $L=[b, \infty)$;
(2) $\Lambda=U$, where $U=(-\infty, a]$ and $L=\emptyset$;
(3) $\Lambda=L$, where $L=[b, \infty)$ and $U=\emptyset$;
(4) $\Lambda=(a, b)$ where $L=(-\infty, a]$ and $U=[b, \infty)$.

If $(\xi, \eta)$ has $L \cap U=\emptyset$ and does not have $U$ and $L$ satisfying one of (1)-(4), then no absorbing sets exist.

We examine two striking cases of $L$ and $U$ structure, and state exact conditions on the characteristic triplet of $(\xi, \eta)$ for such behaviour. Note that similar conditions can be found for each of the other $L$ and $U$ structures stated in Theorem 9, however, the statements are longer and unwieldy.

Proposition 11 Suppose $L \cap U=\emptyset$. Then $U=(-\infty, a]$ and $L=[b, \infty)$ for $-\infty<a<b<\infty$ iff $(\xi, \eta)$ is of finite variation and the following hold:

- There is no Brownian component $\left(\Sigma_{\xi, \eta}=0\right)$;
- The drift of $\xi$ is non-negative $\left(d_{\xi} \geq 0\right)$;
- The Lévy measure satisfies $\Pi_{\xi, \eta}\left(A_{3}\right)=\Pi_{\xi, \eta}\left(A_{4}\right)=0, \theta_{1}^{\prime}>-\infty$, and $\theta_{2}<\infty$.

If these conditions hold then $\xi$ is a subordinator and, for any $V_{0}=z \in \mathbb{R}$, $\lim _{t \rightarrow \infty}\left|V_{t}\right|=\infty$ a.s.

Similarly $L=(-\infty, a]$ and $U=[b, \infty)$ for $-\infty<a<b<\infty$ iff $(\xi, \eta)$ is of finite variation and the following hold:

- There is no Brownian component $\left(\Sigma_{\xi, \eta}=0\right)$;
- The drift of $\xi$ is non-positive $\left(d_{\xi} \leq 0\right)$;
- The Lévy measure satisfies $\Pi_{\xi, \eta}\left(A_{1}\right)=\Pi_{\xi, \eta}\left(A_{2}\right)=0, \theta_{4}^{\prime}<\infty$ and $\theta_{3}>-\infty$.

If these conditions hold then $-\xi$ is a subordinator, and $V$ is strictly stationary and converges in distribution as $t \rightarrow \infty$ to a random variable $V_{\infty}$ supported on $(a, b)$.

We now present a theorem describing the relationship between the sets $L$ and $U$, and the upper and lower bounds of the limit random variable $Z_{\infty}$ of $Z_{t}$ as $t \rightarrow \infty$.

Theorem 12 Let $a, b \in \mathbb{R}$ and suppose $Z_{t} \rightarrow Z_{\infty}$ a.s. as $t \rightarrow \infty$, where $Z_{\infty}$ is a finite random variable. If, for all $c \in \mathbb{R}$, the degenerate case (9) does not hold, then $a \leq \sup U$ iff $Z_{\infty}<-a$ a.s., whilst $b \geq \inf L$ iff $Z_{\infty}>-b$ a.s. Further, $-\sup U=\inf \left\{u \in \mathbb{R} \mid Z_{\infty}<u\right.$ a.s. $\}$ and $-\inf L=\sup \left\{u \in \mathbb{R} \mid Z_{\infty}>u\right.$ a.s. $\}$. Alternatively, if there exists $c \in \mathbb{R}$ such that equation (9) holds, then $Z_{\infty}=-c$ a.s. and $\inf L=\sup U=c$.

The next theorem presents results on certain ruin which occur when $L$ and $U$ are of a particular structure.

Theorem 13 Suppose that $L \cap U=\emptyset$. Then the following statements hold:
(1) If $\sup U \geq 0$ and $L \cap[0, \sup U]=\emptyset$, then $\psi(z)=1$ for all $z \leq \sup U$;
(2) If $\sup L \geq 0$ and $U \cap[0, \sup L]=\emptyset$, then $0<\psi(z)<1$ for all $0 \leq z<$ $\inf L$. If $\sup L \geq 0$ and $U \cap[0, \sup L] \neq \emptyset$, then $\psi(z)<1$ for all $z>\sup U$.

Note that in statement (2) above, when $\sup L \geq 0$ and $L \cap U \neq \emptyset$, Theorem 9 ensures that $\sup U<\inf L$, and statement (1) above ensures that $\psi(z)=1$ for all $z \leq \sup U$. Also, by definition of $L, \psi(z)=0$ whenever $z \geq \inf L$.

We now present a major theorem which utilises Theorems 9,12 and 13, and is the major tool in proving Theorems 1 and 3 . For the non-degenerate case, and for $(\xi, \eta)$ which satisfies various asymptotic and stability criteria, this theorem presents iff conditions for certain ruin, stated in terms of $L$ and $U$ structure. In particular, it completely describes the $L$ and $U$ structures for which certain ruin occurs.

Theorem 14 Suppose $L \cap U=\emptyset$.
(1) Suppose $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$ a.s. and $I_{-\xi, K^{\xi}, \eta}<\infty$. There exists $z \geq 0$ such that $\psi(z)<1$ iff $L \cap[0, \infty) \neq \emptyset$. If this occurs then $0<\psi(z)<1$ for all $0 \leq z<\inf L, \psi(z)=0$ for all $z \geq \inf L$, and one of the following must hold:
(a) $L=[a, b]$ and $U=\emptyset$, where $-\infty \leq a \leq b<\infty$, and $b \geq 0$;
(b) $L=(-\infty, a]$ and $U=[b, \infty)$ where $0 \leq a<b<\infty$.
(2) Suppose $E\left(\xi_{1}\right)=0, E\left(e^{\left|\xi_{1}\right|}\right)<\infty$ and there exist $p, q>1$ with $1 / p+1 / q=$ 1 such that $E\left(e^{-p \xi_{1}}\right)<\infty$ and $E\left(\left|\eta_{1}\right|^{q}\right)<\infty$. There exists $z \geq 0$ such that $\psi(z)<1$ iff $L \cap[0, \infty) \neq \emptyset$. If this occurs then $L=[a, b]$ and $U=\emptyset$, where $-\infty<a \leq b<\infty$ and $b \geq 0$, in which case $0<\psi(z)<1$ for all $0 \leq z<a$ and $\psi(z)=0$ for all $z \geq a$;
(3) Suppose $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a.s. and $I_{\xi, \eta}<\infty$. There exists $z \geq 0$ such that $\psi(z)=1$ iff $U \cap[0, \infty) \neq \emptyset$. If this occurs then one of the following must hold:
(c) $U=[a, b]$ and $L=\emptyset$, where $-\infty \leq a \leq b<\infty$ and $b \geq 0$, in which case $\psi(z)=1$ for all $z \leq b$ and $0<\psi(z)<1$ for all $z>b$;
(d) $U=(-\infty, a]$ and $L=[b, \infty)$ where $0 \leq a<b<\infty$, in which case $\psi(z)=1$ for all $z \leq a, 0<\psi(z)<1$ for all $a<z<b$ and $\psi(z)=0$ for all $z \geq b$.

Remark 15 The characteristic triplet conditions which equate to the iff result in statement (3) above, are given in Theorem 3, and are obtained using the forthcoming Proposition 20, Further, exact characteristic triplet conditions for the structure $U=(-\infty, a]$ and $L=[b, \infty)$ in case (d) above, are given in Proposition 11 .

### 3.1 Technical results on the upper and lower bounds

We present a series of important technical propositions on $\delta, L, \Upsilon$ and $U$. As well as being of independent interest, they are essential in proving the previously stated theorems. The first proposition is obtained by combining and restating parts of Proposition 6, Theorem 7 and Theorem 9 of [2], and no proof is given. When put into this form the proposition completely describes the relationship between the Lévy measure of $(\xi, \eta)$ and the lower bound function $\delta$. We recall some notation from [2]. For $A_{i}$ as in Section 2] define $A_{i}^{u}:=$ $\left\{(x, y) \in A_{i}: y-u\left(e^{-x}-1\right)<0\right\}$. For $u \leq 0$ define

$$
\theta_{1}:=\left\{\begin{array}{l}
\sup \left\{u \leq 0: \Pi_{\xi, \eta}\left(A_{1}^{u}\right)>0\right\} \\
-\infty \quad \text { if } \Pi_{\xi, \eta}\left(A_{1} \backslash A_{4}\right)=0,
\end{array} \quad \theta_{3}:=\left\{\begin{array}{l}
\inf \left\{u \leq 0: \Pi_{\xi, \eta}\left(A_{3}^{u}\right)>0\right\} \\
0 \quad \text { if } \Pi_{\xi, \eta}\left(A_{3} \backslash A_{4}\right)=0
\end{array}\right.\right.
$$

and for $u \geq 0$ define

$$
\theta_{2}:=\left\{\begin{array}{l}
\sup \left\{u \geq 0: \Pi_{\xi, \eta}\left(A_{2}^{u}\right)>0\right\} \\
0 \quad \text { if } \Pi_{\xi, \eta}\left(A_{2} \backslash A_{1}\right)=0,
\end{array} \quad \theta_{4}:=\left\{\begin{array}{l}
\inf \left\{u \geq 0: \Pi_{\xi, \eta}\left(A_{4}^{u}\right)>0\right\} \\
\infty \quad \text { if } \Pi_{\xi, \eta}\left(A_{4} \backslash A_{1}\right)=0
\end{array}\right.\right.
$$

Throughout, let $W$ be the Lévy process such that $e^{-\xi_{t}}=\epsilon(W)_{t}$.
Proposition 16 (lower bound) The following statements are equivalent:
(1) The lower bound $\delta(z)>-\infty$ for some $z \in \mathbb{R}$;
(2) There exists $u \in \mathbb{R}$ such that $\delta(u)=u$;
(3) There exists $u \in \mathbb{R}$ such that the Lévy process $\eta-u W$ is a subordinator.

Statements (2) and (3) hold for a particular value $u \neq 0$ iff the following three conditions are satisfied: (i) the Gaussian covariance matrix satisfies equation (10); (ii) one of the following is true:
(a) $\Pi_{\xi, \eta}\left(A_{3}\right)=0, \Pi_{\xi, \eta}\left(A_{2}\right) \neq 0, \theta_{2} \leq \theta_{4}$ and $u \in\left[\theta_{2}, \theta_{4}\right]$;
(b) $\Pi_{\xi, \eta}\left(A_{2}\right)=0, \Pi_{\xi, \eta}\left(A_{3}\right) \neq 0, \theta_{1} \leq \theta_{3}$ and $u \in\left[\theta_{1}, \theta_{3}\right]$;
(c) $\Pi_{\xi, \eta}\left(A_{3}\right)=\Pi_{\xi, \eta}\left(A_{2}\right)=0$ and $u \in\left[\theta_{1}, \theta_{4}\right]$;
and, (iii), in addition, $u$ satisfies $g(u) \geq 0$ for the function $g$ in equation (11).
From the definition of $L$ it is an immediate corollary, firstly, that $L=\emptyset$ iff none of conditions (1)-(3) of Proposition 16 hold, and secondly, that $\eta$ is a subordinator iff $0 \in L$. The next proposition adds further information concerning $L$. Most importantly, it shows that the set $L$ is always connected, and gives concrete values for the endpoints.

Proposition 17 If $\sigma_{\xi}^{2} \neq 0$ and any of conditions (1)-(3) of Proposition 16 hold, then $L=\left\{-\frac{\sigma_{\xi, \eta}}{\sigma_{\xi}^{2}}\right\}$. If $\sigma_{\xi}^{2}=0$ and any of (1)-(3) hold, then $\sigma_{\eta}^{2}=0$ and one of the following holds:

- $\eta$ is a subordinator and condition (ii) of Proposition 16 does not hold for any $u \neq 0$, in which case $L=\{0\}$;
- Condition (ii) is satisfied for some $u \neq 0$, in which case there exists $-\infty \leq$ $a \leq b \leq \infty$ such that $L=[a, b]$.

In the latter case, if condition (a) of Proposition 16 holds then $0 \leq a=$ $\max \left\{\theta_{2}, m_{1}\right\}$ and $b=\min \left\{\theta_{4}, m_{2}\right\}$ for $m_{1}:=\inf \{u \in \mathbb{R}: g(u) \geq 0\}$ and $m_{2}:=\sup \{u \in \mathbb{R}: g(u) \geq 0\}$. If (b) holds then $a=\max \left\{\theta_{1}, m_{1}\right\}$ and $b=\min \left\{\theta_{3}, m_{2}\right\} \leq 0$. If (c) holds then $a=\max \left\{\theta_{1}, m_{1}\right\}$ and $b=\min \left\{\theta_{4}, m_{2}\right\}$.

Define $L^{*}$ to be the set of starting values on which the GOU has no negative jumps, namely

$$
L^{*}:=\left\{u \in \mathbb{R}: \forall t>0 P\left(\Delta V_{t}<0 \mid V_{t-}=u\right)=0\right\} .
$$

It is a consequence of Proposition 6 that $L \subseteq L^{*}$. The next proposition describes $L^{*}$. In particular, it shows that the set $L^{*}$ is always connected, and gives concrete values for the endpoints. It also shows that whenever $V_{t-}>\sup L^{*}$ and a negative jump $\Delta V_{t}$ occurs, then the jump cannot be so negative as to cause $V_{t} \leq \sup L^{*}$. Thus, $L^{*}$ acts as a barrier for negative jumps of $V$.

Proposition 18 (1) If $L^{*} \neq \emptyset$ then, for any $t \geq 0, V_{t-}>\sup L^{*}$ implies $V_{t}>\sup L^{*}$ a.s.;
(2) $L^{*}=\{u \in \mathbb{R}: \eta-u W$ has no negative jumps $\}$;
(3) $L^{*} \neq \emptyset$ iff condition (ii) of Proposition 16 is satisfied for some $u \neq 0$, or $\eta$ has no negative jumps;
(4) $L^{*}=\{0\}$ iff $\eta$ has no negative jumps and condition (ii) does not hold for any $u \neq 0$;
(5) If condition (ii) of Proposition 16 holds for some $u \neq 0$ then $L^{*}=\left[\theta_{2}, \theta_{4}\right]$, $\left[\theta_{1}, \theta_{3}\right]$ or $\left[\theta_{1}, \theta_{4}\right]$, corresponding to conditions (a), (b) or (c) of Proposition 16.

Remark 19 (1) If $(\xi, \eta)$ is an infinite variation Lévy process then, as noted in Section 1. $\int_{\left\{x^{2}+y^{2}<1\right\}}|(x, y)| \Pi_{\xi, \eta}(\mathrm{d}(x, y))=\infty$. Thus, it may be the case that for a particular $u \in \mathbb{R}$ the integral $\int_{\left\{x^{2}+y^{2}<1\right\}}(u x+y) \Pi_{\xi, \eta}(\mathrm{d}(x, y))$, and hence the function $g(u)$ in (11), may not exist as a well-defined member of the extended real numbers. However, it is a consequence of the proof of Theorem 9 in [2], that if $u \in L^{*}$ then $g(u)$ is a well defined member of the extended reals, and $g(u) \in[-\infty, \infty)$. Under such conditions, it is also shown that

$$
\Pi_{\xi, \eta}\left(\left\{y-u\left(e^{-x}-1\right)<0\right\}\right)=0
$$

and so the domain of integration for the integral component of $g$ can be decreased to $\left\{x^{2}+y^{2}<1\right\} \cap\left\{y-u\left(e^{-x}-1\right) \geq 0\right\}$.
(2) Note that $g$ is a linear function on $\mathbb{R}$ iff the Lévy measure of $(\xi, \eta)$ is of finite variation, namely

$$
\int_{\left\{x^{2}+y^{2}<1\right\}}|(x, y)| \Pi_{\xi, \eta}(\mathrm{d}(x, y))<\infty
$$

In this case the drift vector $\left(d_{\xi}, d_{\eta}\right)$ is finite, and we can write

$$
\begin{align*}
g(u) & =\gamma_{\eta}-\int_{(-1,1)} y \Pi_{\eta}(\mathrm{d} y)+u\left(\gamma_{\xi}-\frac{1}{2} \sigma_{\xi}^{2}-\int_{(-1,1)} x \Pi_{\xi}(\mathrm{d} x)\right) \\
& =d_{\eta}+u\left(d_{\xi}-\frac{1}{2} \sigma_{\xi}^{2}\right) \tag{13}
\end{align*}
$$

where the first equality follows by converting $\left(\tilde{\gamma}_{\xi}, \tilde{\gamma}_{\eta}\right)$ to $\left(\gamma_{\xi}, \gamma_{\eta}\right)$ using equation (4) and the symmetric version for $\eta$, and the second equality follows by converting $\left(\gamma_{\xi}, \gamma_{\eta}\right)$ to $\left(d_{\xi}, d_{\eta}\right)$ using equation (5). It will be a consequence of the proof of Proposition 17, that if $a, b \in L$ and $a \neq b$ then $g$ is a linear function on $\mathbb{R}$.
(3) In Section 1 we stated exact conditions for a Lévy process to be a subordinator. When $u \neq 0$ the Lévy measure conditions in Proposition 16 are exactly the requirements for $\eta-u W$ to be a subordinator. Equation (10) is equivalent to the condition $\sigma_{\eta-u W}=0$. The requirement that one of the conditions (a), (b) and (c) holds is equivalent to the requirement that there exists $u \neq 0$ such that $\Pi_{\eta-u W}((-\infty, 0))=0$. Note that this implies that $L^{*} \backslash\{0\}$ is precisely the set of all $u \neq 0$ such $\eta-u W$ has no negative jumps. Finally, if $u \in L^{*}$ then $g(u)=d_{\eta-u W}$, and hence condition (11) is equivalent to the requirement that $\eta-u W$ has positive drift. The fact that $\eta-u W$ is of finite variation actually follows from the two conditions $\Pi_{\eta-u W}((-\infty, 0))=0$ and $d_{\eta-u W} \geq 0$. To see this, note that when $\Pi_{\eta-u W}((-\infty, 0))=0$, the equation (5) simplifies to

$$
d_{\eta-u W}=\gamma_{\eta-u W}-\int_{(0,1)} x \Pi_{\eta-u W}(\mathrm{~d} x)
$$

and hence $d_{\eta-u W}$ is a member of the extended reals regardless of whether $\eta-u W$ is finite variation. In particular, $d_{\eta-u W} \in[-\infty, \infty)$, and $d_{\eta-u W}=$ $-\infty$ iff $\int_{(0,1)} x \Pi_{\eta-u W}(\mathrm{~d} x)=\infty$ which occurs iff $\eta-u W$ is infinite variation.

Although the situation is symmetric, we explicitly state the parallel version for $U$ and $\Upsilon$, to Proposition 16. No proof is given. We state the parallel result explicitly because some of the statements are not obvious, and we need to use them for Theorem 3. Also, we will need to combine them with the statements for $L$ and $\delta$ in order to prove Theorem 9, 13 and 14. If we define

$$
U^{*}:=\left\{u \in \mathbb{R}: \forall t>0 P\left(\Delta V_{t}>0 \mid V_{t-}=u\right)=0\right\},
$$

then the symmetric versions of Proposition 17, Proposition 18 and Remark 19 also hold. We will need to use these results, however the parallels are obvious in this case, so we do not state them explicitly.

Proposition 20 (upper bound) The following are equivalent:
(1) The upper bound $\Upsilon(z)<\infty$ for some $z \in \mathbb{R}$;
(2) There exists $u \in \mathbb{R}$ such that $\Upsilon(u)=u$;
(3) There exists $u \in \mathbb{R}$ such that the Lévy process $-(\eta-u W)$ is a subordinator.

Statements (2) and (3) hold for a particular value $u \neq 0$ iff the following three conditions are satisfied: (i) the Gaussian covariance matrix satisfies equation (10); (ii) one of the following is true:
(a) $\Pi_{\xi, \eta}\left(A_{1}\right)=0, \Pi_{\xi, \eta}\left(A_{4}\right) \neq 0, \theta_{4}^{\prime} \leq \theta_{2}^{\prime}$ and $u \in\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]$;
(b) $\Pi_{\xi, \eta}\left(A_{4}\right)=0, \Pi_{\xi, \eta}\left(A_{1}\right) \neq 0, \theta_{3}^{\prime} \leq \theta_{1}^{\prime}$ and $u \in\left[\theta_{3}^{\prime}, \theta_{1}^{\prime}\right]$;
(c) $\Pi_{\xi, \eta}\left(A_{1}\right)=\Pi_{\xi, \eta}\left(A_{4}\right)=0$ and $u \in\left[\theta_{3}^{\prime}, \theta_{2}^{\prime}\right]$;
and,(iii), in addition, $u$ satisfies $g(u) \leq 0$ for the function $g$ in equation (11).
Remark 21 Symmetric statements to those for $L$ and $L^{*}$ in Remark 19, hold for $U$ and $U^{*}$. The following remarks relate to the combination of $L$ and $U$, and $L^{*}$ and $U^{*}$.
(1) Parallel to 1 and 2 of Remark 19, whenever $u \in U^{*}, g(u)$ from (11) is a well-defined member of the extended reals, $g(u) \in(-\infty, \infty]$, and $-g(u)=$ $d_{-(\eta-u W)}$. Since $d_{-(\eta-u W)}=-d_{\eta-u W}$, we know that if $u \in U^{*} \cup L^{*}$ then $g(u)$ is a well-defined member of the extended reals and $g(u)=d_{\eta-u W}$.
(2) If $a \in L, b \in U$ and $a \neq b$ then $g$ is linear and $(\xi, \eta)$ is finite variation. This statement is proved easily using similar arguments to those in the proof of Proposition 17 .

We state a proposition, describing the possible combinations of $L^{*}$ and $U^{*}$, which will be essential for proving Theorem 9 .

Proposition 22 The following statements hold for $L^{*}$, and the symmetric statements hold for $U^{*}$ :
(1) $L^{*}=\mathbb{R}$ then $U^{*}=\emptyset$ or $U^{*}=\mathbb{R}$;
(2) If $L^{*}=[a, b]$ for some $-\infty<a \leq b<\infty$, then $U^{*}=\emptyset$ or $U^{*}=L^{*}=$ $\{a\}=\{b\} ;$
(3) If $L^{*}=[b, \infty)$ for some $b \in \mathbb{R}$, then $U^{*}=\emptyset$ or $U^{*}=(-\infty, a]$ for some $-\infty<a \leq b<\infty ;$
(4) If $L^{*}=(-\infty, a]$ for some $a \in \mathbb{R}$, then $U^{*}=\emptyset$ or $U^{*}=[b, \infty)$ for some $-\infty<a \leq b<\infty$.

We end the section with two lemmas. No proof will be given. The first follows by considering the definitions of $\theta_{i}$ and $\theta_{i}^{\prime}$. It will be used several times as a calculation tool. The second gives conditions on the Lévy measure of $\xi$ and $\eta$ which ensure that $\sup _{0 \leq t \leq 1}\left|Z_{t}\right|$ has finite mean. It will be needed to prove statement (2) of Theorem 1. The proof is similar to that of Lemma 11 in [2] and uses the Burkholder-Davis-Gundy inequalities, and various Doob's inequalities.

Lemma 23 (1) If $\Pi_{\xi, \eta}\left(A_{1}\right) \neq 0$ then $\theta_{1}^{\prime} \leq \theta_{1} \leq 0$;
(2) If $\Pi_{\xi, \eta}\left(A_{2}\right) \neq 0$ then $0 \leq \theta_{2}^{\prime} \leq \theta_{2}$;
(3) If $\Pi_{\xi, \eta}\left(A_{3}\right) \neq 0$ then $\theta_{3} \leq \theta_{3}^{\prime} \leq 0$;
(4) If $\Pi_{\xi, \eta}\left(A_{4}\right) \neq 0$ then $0 \leq \theta_{4} \leq \theta_{4}^{\prime}$.

## Further:

(a) $\Pi_{\xi, \eta}\left(A_{1}\right)=0$ iff $\theta_{1}=-\infty$ and $\theta_{1}^{\prime}=0$;
(b) $\Pi_{\xi, \eta}\left(A_{2}\right)=0$ iff $\theta_{2}=0$ and $\theta_{2}^{\prime}=\infty$;
(c) $\Pi_{\xi, \eta}\left(A_{3}\right)=0$ iff $\theta_{3}=0$ and $\theta_{3}^{\prime}=-\infty$;
(d) $\Pi_{\xi, \eta}\left(A_{4}\right)=0$ iff $\theta_{4}=\infty$ and $\theta_{4}^{\prime}=0$.

Lemma 24 Suppose there exist $r>0$ and $p, q>1$ with $1 / p+1 / q=1$ such that $E\left(e^{-\max \{1, r\} p \xi_{1}}\right)<\infty$ and $E\left(\left|\eta_{1}\right|^{\max \{1, r\} q}\right)<\infty$. Then

$$
\begin{equation*}
E\left(\sup _{0 \leq t \leq 1}\left|\int_{0}^{t} e^{-\xi_{s}-} \mathrm{d} \eta_{s}\right|^{\max \{1, r\}}\right)<\infty . \tag{14}
\end{equation*}
$$

## 4 Proofs and Examples

The proofs are presented in mathematically chronological order rather than the order in which the statements of the results are presented. For all proofs, except the proof of Proposition 7, we assume that neither $\xi$ nor $\eta$ are zero.

PROOF. [Proposition 18] We prove statements (1), (2) and (3). The proof of statements (4) and (5) follows trivially from the proof of statements (2) and (3).
(1) Suppose $L^{*}=\emptyset$. Assume that condition (a) of Proposition 16 holds and $L^{*}=\left[\theta_{2}, \theta_{4}\right]$. If condition (b) or (c) of Proposition 16 holds then the proof is similar. We use the following reformulation of equation (3):

$$
\begin{equation*}
\Delta V_{t}=\left(e^{\Delta \xi_{t}}-1\right) V_{t-}+e^{\Delta \xi_{t}} \Delta \eta_{t} \tag{15}
\end{equation*}
$$

Suppose $V_{t-}>\theta_{4}$. It follows immediately from the definitions of $\theta_{4}$ and $A_{4}^{u}$, and from equation (15), that there exists $(x, y) \in A_{4}^{V_{t-}}$ such that $\left(e^{x}-1\right) \theta_{4}+e^{x} y \geq 0$ and $\left(e^{x}-1\right) V_{t-}+e^{x} y<0$. Thus,

$$
\begin{aligned}
V_{t} & =V_{t-}+\left(e^{x}-1\right) V_{t-}+e^{x} y \\
& =V_{t-}+\left(e^{x}-1\right)\left(V_{t-}-\theta_{4}\right)+\left(e^{x}-1\right) \theta_{4}+e^{x} y \\
& \geq V_{t-}+\left(e^{x}-1\right)\left(V_{t-}-\theta_{4}\right)>\theta_{4} .
\end{aligned}
$$

(2) It is a consequence of Proposition 6 in [2] that

$$
\Delta\left(\eta_{t}-u W_{t}\right)=\Delta \eta_{t}-u\left(e^{-\Delta \xi_{t}}-1\right)
$$

Thus, equation (3) implies that whenever $V_{t-}=u$, a jump $\left(\Delta \xi_{t}, \Delta \eta_{t}\right)$ causes a negative jump $\Delta V_{t}$ iff $\Delta\left(\eta_{t}-u W_{t}\right)$ is negative. Hence $L^{*}$ is precisely the set of all $u$ such that $\eta_{t}-u W_{t}$ has no negative jumps.
(3) By (1) above, $L^{*} \neq \emptyset$ iff $\eta-u W$ has no negative jumps. If $u=0$, this occurs iff $\eta$ has no negative jumps. If $u \neq 0$, it is noted in point (3) of Remark 19. that this occurs iff $u \neq 0$ satisfies condition (ii) of Proposition 16.

PROOF. [Proposition 17 Assume that $\sigma_{\xi}^{2} \neq 0$ and statements (1)-(3) of Proposition 16 hold for some $u \neq 0$. Then equation (10) must hold for $u$, which implies that $u=-\frac{\sigma_{\xi, n},}{\sigma_{\xi}}$, and hence is the unique non-zero number satisfying statements (1)-(3) of Proposition 16. Since $-\frac{\sigma_{\xi, \eta}}{\sigma_{\xi}^{2}}$ satisfies condition (2), $L=$ $\left\{-\frac{\sigma_{\xi, n}}{\sigma_{\xi}^{\epsilon}}\right\}$ by definition.

Now assume that $\sigma_{\xi}^{2} \neq 0$ and statements (1)-(3) of Proposition 16 hold for $u=0$. By statement (2), $0 \in L$. By statement (3), $\eta$ is a subordinator, and hence $\sigma_{\eta}^{2}=\sigma_{\xi, \eta}=0$. Thus, by the above, no non-zero number can satisfy statements (1)-(3), and so $L=\{0\}=\left\{-\frac{\sigma_{\xi, \eta}}{\sigma_{\xi}^{2}}\right\}$.

Now assume that $\sigma_{\xi}^{2}=0$. If statements (1)-(3) of Proposition 16 hold for $u=0$ then $\eta$ is a subordinator by statement (3) and hence $\sigma_{\eta}^{2}=0$. Alternatively, If
statements (1)-(3) of Proposition 16 hold for some $u \neq 0$ then equation (10) must hold for $u$, which implies that $\sigma_{\eta}^{2}=u^{2} \sigma_{\xi}^{2}$, and so $\sigma_{\eta}^{2}=0$.

Now assume that $\sigma_{\xi}^{2}=0$ and condition (ii) of Proposition 16 does not hold for any $u \neq 0$. This immediately implies that $L \cap(\mathbb{R} \backslash\{0\})=\emptyset$. If, further, $\eta$ is a subordinator, then $0 \in L$, and hence $L=\{0\}$.

Now assume that $\sigma_{\xi}^{2}=0$ and condition (ii) of Proposition 16 holds for some $u \neq 0$. This occurs precisely when one of conditions (a), (b) or (c) of Proposition 16 holds, and equation (11) holds. Thus, $\inf L=a$ and $\sup L=b$ for the values of $a$ and $b$ given in the proposition statement. It remains to prove that the set $L$ is connected. Since $L^{*}$ is connected, this occurs iff $\{u \in \mathbb{R}: g(u) \geq 0\}$ is connected, which follows from the analysis below.

As noted in point (1) of Remark [19, whenever $u \in L^{*}$ we know $g(u) \in$ $[-\infty, \infty)$. There are three possibilities for behaviour of $g$ on $L^{*}$. Firstly, it may be that $g(u)=-\infty$ for all $u \in L^{*}$. Secondly there may exist $v \in L^{*}$ such that $g(v)$ is finite and $g(u)=-\infty$ for all $u \in L^{*}$ with $u \neq v$. We show that the only other possibility is that $g$ is linear on $\mathbb{R}$. Suppose there exists $u_{1}, u_{2} \in L^{*}$ with $u_{1} \neq u_{2}$, such that $g\left(u_{1}\right)$ and $g\left(u_{2}\right)$ are both finite. Then

$$
g\left(u_{1}\right)-g\left(u_{2}\right)=\left(\tilde{\gamma}_{\xi}-\frac{1}{2} \sigma_{\xi}^{2}-\int_{\left\{x^{2}+y^{2}<1\right\}} x \Pi_{\xi, \eta}(\mathrm{d}(x, y))\right)\left(u_{1}-u_{2}\right)
$$

is finite, which implies that $\int_{\left\{x^{2}+y^{2}<1\right\}} x \Pi_{\xi, \eta}(\mathrm{d}(x, y))$ exists, and is finite. Since $g\left(u_{1}\right)$ is finite, this implies that $\int_{\left\{x^{2}+y^{2}<1\right\}} y \Pi_{\xi, \eta}(\mathrm{d}(x, y))$ exists and is finite. Thus, $g$ is a linear function on $\mathbb{R}$.

PROOF. [Proposition 6] It is an immediate consequence of Proposition 16 that $\delta(\delta(z))=\delta(z)$ and

$$
\begin{equation*}
\delta(z)=\sup \{u \leq z: \delta(u)=u\} \tag{16}
\end{equation*}
$$

Now the first statement of Proposition 6 follows immediately from (16). To prove the second statement, assume $z<\inf L$. Suppose $-\infty<m:=\delta(z)$. Since $\delta(z) \leq z$, we have $-\infty<m \leq z<\inf L$. However, equation (16) implies that $m \in L$, which gives a contradiction. Hence $\delta(z)=-\infty$. The third and fourth statements follow immediately from the definitions of $\delta$ and $L$.

PROOF. [Proposition 7 Assume $L=\mathbb{R}$. This implies, using Proposition 16 and point (2) of Remark 19, that $\Sigma_{\xi, \eta}=0$ and $g$ is linear. Further, it must be the case that $\Pi_{\xi, \eta}\left(A_{3}\right)=\Pi_{\xi, \eta}\left(A_{2}\right)=0$ and $L^{*}=\left[\theta_{1}, \theta_{4}\right]=(-\infty, \infty)$. Now $\theta_{1}=$ $-\infty$ iff $\Pi_{\xi, \eta}((0, \infty) \times[0, \infty))$, whilst $\theta_{4}=-\infty$ iff $\Pi_{\xi, \eta}((-\infty, 0) \times[0, \infty))=0$. Hence $\xi$ can have no jumps and $\eta$ can only have positive jumps. By Proposition
16. $g(u) \geq 0$ on $\mathbb{R}$. Since $g(u)=d_{\eta}+u d_{\xi}$, this implies that $d_{\xi}=0$ and $d_{\eta} \geq 0$, thus proving one direction of the first claim. The converse is trivial since $V$ simplifies to $V_{t}=z+\eta_{t}$. The proof of the second claim is similar. The third claim follows immediately from the first two.

PROOF. [Proposition 22] We prove statements (1), (2) and (3). The proof of statement (4) is symmetrical to the proof of statement (3).
(1) Assume $L^{*}=\mathbb{R}$. Then condition (c) of Proposition 16 must hold, and so $\Pi_{\xi, \eta}\left(A_{2}\right)=\Pi_{\xi, \eta}\left(A_{3}\right)=0$, and $L^{*}=\left[\theta_{1}, \theta_{4}\right]$. Since $\theta_{1}=-\infty$ and $\theta_{4}=\infty$, it must be that $\Pi_{\xi, \eta}\left(A_{1} \backslash A_{4}\right)=0$ and $\Pi_{\xi, \eta}\left(A_{4} \backslash A_{1}\right)=0$, respectively. Thus, if $\Pi_{\xi, \eta}\left(A_{1} \cap A_{4}\right)=0$ then $\Pi_{\xi, \eta}\left(\mathbb{R}^{2}\right)=0$, in which case condition (c) of Proposition 20 holds, and $U^{*}=\mathbb{R}$. Alternatively, if $\Pi_{\xi, \eta}\left(A_{1} \cap A_{4}\right) \neq 0$ then $\eta$ has positive jumps and so $0 \notin U^{*}$, and (ii) of Proposition 20 cannot hold. Hence $U^{*}=\emptyset$.
(2) Assume $L^{*}=[a, b]$ for some $-\infty<a \leq b<\infty$. There are four ways in which this is possible, namely, when (a), (b) or (c) of Proposition 16 hold, or when $L^{*}=\{0\}$. For each case we show $U^{*}=\emptyset$ or $U^{*}=L^{*}=\{a\}=\{b\}$.

Suppose first that condition (a) of Proposition 16 holds, and $U^{*} \neq \emptyset$. The case in which condition (b) holds and $U^{*} \neq \emptyset$, is symmetric. Propositions 16 and 18 imply that $\Pi_{\xi, \eta}\left(A_{3}\right)=0, \Pi_{\xi, \eta}\left(A_{2}\right) \neq 0, \theta_{2} \leq \theta_{4}$ and $L^{*}=\left[\theta_{2}, \theta_{4}\right]$. Since $\theta_{4}<\infty$, it must be that $\Pi_{\xi, \eta}\left(A_{4} \backslash A_{1}\right) \neq 0$. Since $\Pi_{\xi, \eta}\left(A_{3}\right)=0$, this implies that $-\eta$ is not a subordinator, and so $0 \notin U^{*}$. Thus, since we have assumed that $U^{*} \neq \emptyset$, it must be that condition (a) of Proposition 20 holds, and so $\Pi_{\xi, \eta}\left(A_{1}\right)=0, \theta_{4}^{\prime} \leq \theta_{2}^{\prime}$, and $U^{*}=\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]$. However, statements (2) and (4) of Lemma 23 state that $\theta_{2}^{\prime} \leq \theta_{2}$ and $\theta_{4} \leq \theta_{4}^{\prime}$. Hence $\theta_{2}^{\prime}=\theta_{2}=\theta_{4}=\theta_{4}^{\prime}$.

Now suppose that condition (c) of Proposition 16 holds. Then $\Pi_{\xi, \eta}\left(A_{2}\right)=$ $\Pi_{\xi, \eta}\left(A_{3}\right)=0$, and $L^{*}=\left[\theta_{1}, \theta_{4}\right]$. Since $\theta_{4}<\infty$ and $\theta_{1}>-\infty$ it must be that $\Pi_{\xi, \eta}\left(A_{4} \backslash A_{1}\right) \neq 0$ and $\Pi_{\xi, \eta}\left(A_{1} \backslash A_{4}\right) \neq 0$, respectively. Hence condition (ii) of Proposition 20 cannot hold, and so $U^{*} \backslash\{0\}=\emptyset$. Further, $-\eta$ is not a subordinator, and so $U^{*}=\emptyset$.

Now suppose $L^{*}=\{0\}$, and $U^{*} \neq \emptyset$. By statement (4) of Proposition 18, $L^{*}=\{0\}$ iff $\eta$ has no negative jumps and at the same time $\Pi_{\xi, \eta}\left(A_{3} \cap A_{4}\right) \neq 0$ and $\Pi_{\xi, \eta}\left(A_{2} \cap A_{1}\right) \neq 0$. Hence, condition (ii) of Proposition 20 fails to hold, which implies $U^{*} \backslash\{0\}=\emptyset$. Thus, since $U^{*} \neq \emptyset$, it must be that $U^{*}=L^{*}=\{0\}$.
(3) Assume that $L^{*}=[b, \infty)$ for some $b \in \mathbb{R}$ and $U^{*}=\emptyset$. We show that $U^{*}=(-\infty, a]$ for some $-\infty<a \leq b<\infty$. By the symmetric version of statement (2) of Proposition [22, it is immediate that $U^{*} \neq\{0\}$.

Since $L^{*}=[b, \infty$ ), condition (a) or (c) of Proposition 16 must hold, with $\theta_{4}=\infty$. Thus, $\Pi_{\xi, \eta}\left(A_{3}\right)=0$, which implies that $\theta_{3}^{\prime}=-\infty$. Also, since $\theta_{4}=\infty$,
it must be that $\Pi_{\xi, \eta}\left(A_{4} \backslash A_{1}\right)=0$. Since $U^{*} \neq \emptyset$, it must be that $\Pi_{\xi, \eta}\left(A_{1} \cap\right.$ $\left.A_{4}\right)=0$, and so $\Pi_{\xi, \eta}\left(A_{4}\right)=0$. This implies that one of conditions (b) or (c) of Proposition 20 must hold, and so $U^{*}=\left(-\infty, \theta_{1}^{\prime}\right]$ or $U^{*}=\left(-\infty, \theta_{2}^{\prime}\right]$ respectively. Now, if condition (a) of Proposition 16 holds, then $L^{*}=\left[\theta_{2}, \infty\right)$. Note that Lemma 23 states that $\theta_{1}^{\prime} \leq 0 \leq \theta_{2}^{\prime} \leq \theta_{2}$, and hence the result is proved for either form of $U^{*}$.

Alternatively, if condition (c) of Proposition 16 holds, then $L^{*}=\left[\theta_{1}, \infty\right)$ where $\theta_{1}>-\infty$, which implies that $\Pi_{\xi, \eta}\left(A_{1} \backslash A_{4}\right) \neq 0$. Hence, condition (b) of Proposition 20 must hold and $U^{*}=\left(-\infty, \theta_{1}^{\prime}\right.$ ]. Lemma 23 states that $\theta_{1}^{\prime} \leq \theta_{1}$, and so we are done.

## PROOF. [Proposition [8]

$(1) \Leftrightarrow(2)$ Assume $L \cap U \neq \emptyset$ and let $z_{1}, z_{2} \in L \cap U$. We show $z_{1}=z_{2} \neq 0$. By Proposition 16, $z \in L$ iff $\eta-z W$ is increasing and by Proposition 20, $z \in U$ iff $\eta-z W$ is decreasing. Thus, $\eta-z_{1} W=\eta-z_{2} W=0$, which implies $z_{1} W=z_{2} W$. Since $\xi$ is not zero, $W$ is not zero, and thus $z_{1}=z_{2}$. Further, if $z_{1}=z_{2}=0$, then $\eta$ must be both increasing and decreasing, which requires that $\eta$ be identically zero. Since we have rejected this case, it must be that $z_{1}=z_{2} \neq 0$.
$(2) \Leftrightarrow(3)$ Suppose $L \cap U=\{c\}$. Then $V_{t}=c$ for all $t \geq 0$ whenever $V_{0}=c$, which implies $e^{\xi_{t}}\left(c+Z_{t}\right)=c$, which implies $V_{t}=e^{\xi_{t}}(z-c)+c$, as required. Conversely, suppose $V_{t}=e^{\xi_{t}}(z-c)+c$. Clearly, $c \in L \cap U$ and so $L \cap U \neq \emptyset$, which implies $L \cap U=\{c\}$ by the above.
$(2) \Leftrightarrow(4)$ By the definitions of $\delta$ and $\Upsilon$, it is clear that $c$ is an absorbing point iff $\delta(c)=\Upsilon(c)=c$, and the definitions of $L$ and $U$ imply that this occurs iff $c \in L \cap U$.
$(2) \Rightarrow(5)$ Assume $L \cap U=\{c\}$ where $c \neq 0$. Propositions 16 and Proposition 20 immediately imply that equation (10) is satisfied for $u=c$, and imply respectively that $g(c) \geq 0$ and $g(c) \leq 0$, thus giving $g(c)=0$. Finally, since $(2) \Rightarrow(3)$, the equation $Z_{t}:=\int_{0}^{t} e^{-\xi_{s-}} \mathrm{d} \eta_{s}=c\left(e^{-\xi_{t}}-1\right)$ holds, which implies that $e^{-\xi_{t-}} \Delta \eta_{t}=c\left(e^{-\xi_{t}}-1\right)-c\left(e^{-\xi_{t-}}-1\right)$ and so $\Delta \eta_{t}=$ $c\left(e^{-\Delta \xi_{t}}-1\right)$.
$(5) \Rightarrow(2)$ Assume that the conditions of statement (5) hold for $c \neq 0$. We prove $c \in L$. Since (10) is satisfied for $u=c$, and $g(c)=0$ holds, we know that conditions (i) and (iii) of Proposition 16 are respectively satisfied for $u=c$. Thus it suffices to prove condition (ii) of Proposition 16 is satisfied for $u=c$, or equivalently, show $c \in L^{*}$. If $\Pi_{\xi, \eta}=0$ then this is trivial since $L^{*}=\mathbb{R}$. Now suppose that $\Pi_{\xi, \eta}$ is supported on the curve $\left\{(x, y): y-c\left(e^{-x}-1\right)=0\right\}$ for $c \in \mathbb{R}$. If $c>0, \Pi_{\xi, \eta}\left(A_{2}\right) \neq 0$ and $\Pi_{\xi, \eta}\left(A_{4}\right) \neq 0$, then $\theta_{2}=\theta_{4}=c$ and so $L^{*}=\{c\}$. If $c \geq 0, \Pi_{\xi, \eta}\left(A_{2}\right)=0$ and $\Pi_{\xi, \eta}\left(A_{4}\right) \neq 0$, then $\theta_{2}=0$ and $\theta_{4}=c$, and so $L^{*}=[0, c]$. If $c \geq 0$,
$\Pi_{\xi, \eta}\left(A_{2}\right) \neq 0$ and $\Pi_{\xi, \eta}\left(A_{4}\right)=0$, then $\theta_{2}=c$ and $\theta_{4}=\infty$, and so $L^{*}=$ $[c, \infty)$. In each of these three cases, $c \in L^{*}$. The proof for $c<0$ is similar and we omit.

A symmetric argument proves that $c \in U$. Hence, $c \in L \cap U$ which, by the equivalence of statements (1) and (2), implies that $L \cap U=\{c\}$, as required.
$(2) \Leftrightarrow(6) L \cap U=\{c\}$ iff $\eta-c W=0$ where $e^{-\xi_{t}}=\epsilon(W)_{t}$ which occurs iff $e^{-\xi_{t}}=$ $\epsilon(\eta / c)_{t}$.

Now assume that the above statements (1)-(6) hold. If $\Sigma_{\xi, \eta} \neq 0$ and both $L$ and $U$ are non-empty, then Propositions 16 and 20 immediately imply that $L=U=\{c\}$ where $c=-\frac{\sigma_{\xi, \eta}}{\sigma_{\xi}}$. For examples of Lévy processes $(\xi, \eta)$ satisfying statements (1)-(6) and such that $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates a.s., see Example 26 .

If $\Sigma_{\xi, \eta}=0$ then the statements (a), (b) and (c) follow immediately by examining the equation for $V$ in statement (3) above. For examples of Lévy processes $(\xi, \eta)$ satisfying statement (c) and such that $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates a.s., see Example 27 .

PROOF. [Theorem 9 Assume that $L \cap U=\emptyset$. Suppose, firstly, that $\Sigma_{\xi, \eta} \neq 0$. We must show that $(\xi, \eta)$ exists such that (1), (2) or (3) occurs, and for each of these cases, we must show that $\xi$ can satisfy each of the three asymptotic behaviours. For case (1), this is obvious. Choosing $(\xi, \eta)$ such that $\Sigma_{\xi, \eta}$ does not satisfy equation (10) implies that $(\xi, \eta)$ fails both propositions, and so $L=U=\emptyset$, regardless of the choice of $\left(\tilde{\gamma}_{\xi}, \tilde{\gamma}_{\eta}\right)$ and $\Pi_{\xi, \eta}$. Clearly, we can make suitable choices for these objects to obtain the desired asymptotic behaviour of $\xi$. For case (2), our existence claims are proven by Example 25, and case (3) is symmetric. It follows from Proposition [17, and the symmetric version for $U$, that whenever $L$ and $U$ are non-zero, they are each equal to $\left\{-\sigma_{\xi, \eta} / \sigma_{\xi}^{2}\right\}$. Hence, no cases, other than (1), (2) and (3) of Theorem 9, can exist.

Now suppose that $\Sigma_{\xi, \eta}=0$. We must show that $(\xi, \eta)$ exists such that (a), (b) or (c) occurs, and for each of these cases, we must show that $\xi$ can satisfy the specified asymptotic behaviours. Examples 28 and 29 present $(\xi, \eta)$ such that $L=\emptyset$, whilst $U$ may be of form $\emptyset,\{a\}$ or $[a, b]$ for $-\infty<a<b<\infty$, and for each of these combinations, it is shown that $\xi$ can satisfy the three asymptotic behaviours. In Example 30, $L=\emptyset, U$ is of form $[b, \infty)$ for $b \in \mathbb{R}$, and $\xi$ drifts to $-\infty$ a.s. In Example 32, $L=\emptyset, U$ is of form $(-\infty, a]$ for $a \in \mathbb{R}$, and $\xi$ drifts to $\infty$ a.s. These four examples prove the existence claims for (a), and the case (b) is symmetric. In Example 31, $L=(-\infty, a], U=[b, \infty)$ for $-\infty<a<b<\infty$ and $\xi$ drifts to $-\infty$ a.s. In Example 33, $U=(-\infty, a]$, $L=[b, \infty)$ for $-\infty<a<b<\infty$, and $\xi$ drifts to $\infty$ a.s. These two examples prove the existence claims for (c).

We now assume that $\Sigma_{\xi, \eta}=0, L \neq \emptyset, U \neq \emptyset$ and $L \cap U=\emptyset$. We prove that no cases, other than those listed in (c), can exist. As noted in point (2) of Remark 21, it follows from our assumptions that $(\xi, \eta)$ is finite variation and $g$ is linear.

Suppose that $L=[a, b]$ for some $-\infty<a \leq b<\infty$. We show that this causes a contradiction with our assumptions. If $L^{*}=[c, d]$ for some $-\infty<$ $c \leq a \leq b \leq d<\infty$, then point (2) of Proposition 22 states that $U^{*}=\emptyset$ or $U^{*}=L^{*}=\{c\}=\{d\}$. Thus, $U=\emptyset$ or $U=L=\{a\}=\{b\}$, both of which contradict our assumptions. Hence, it must be the case that $L^{*}=[c, \infty)$ for some $-\infty<c \leq a$, or $L^{*}=(-\infty, d]$ for some $b \leq d<\infty$.

Thus, we suppose that $L=[a, b]$ and $L^{*}=[c, \infty)$ for some $-\infty<c \leq a \leq$ $b<\infty$. The case in which $L^{*}=(-\infty, d]$ for some $b \leq d<\infty$ is symmetric. We know $g(u)=d_{\eta}+u d_{\xi}$. If $d_{\xi} \geq 0$ then it must be that $b=\infty$, which we have rejected. Hence $d_{\xi}<0$, and we must have $b=-\frac{d_{\eta}}{d_{\xi}} \geq a$. Thus, since $U$ is non-empty, $L \cap U=\emptyset$, and $g(u) \leq 0$ on $U$, it must be that $U \subset[b, \infty)$. However, point (3) of Proposition 22 implies that $U^{*} \cap[b, \infty)=\emptyset$. Hence $U$ is empty, and we have a contradiction. This completes the proof that $L \neq[a, b]$ for some $-\infty<a \leq b<\infty$.

We now assume that $L=[b, \infty)$ for $b \in \mathbb{R}$. We first prove that $\xi$ is a subordinator, which is another of the statements of Proposition 17 and point (2) of Remark 19, imply respectively, that $(\xi, \eta)$ has no Brownian component, and $(\xi, \eta)$ is of finite variation. Thus, we can write $g(u)=d_{\eta}+u d_{\xi}$. Proposition 16 implies that $g(u) \geq 0$ on $[b, \infty)$ and hence $d_{\xi} \geq 0$. Finally, it must be that $L^{*}=[c, \infty)$ for some $-\infty \leq c \leq b$. It is a consequence of the proofs of statements (1) and (3) of Proposition [22, that $\xi$ has no negative jumps. Thus $\xi$ is a subordinator.

Now, we assume that $L=[b, \infty)$ for $b \in \mathbb{R}$ and $U=\emptyset$. We prove that $U=(-\infty, a]$ for some $-\infty<a<b<\infty$. Note that $L^{*}=[c, \infty)$ for some $-\infty \leq c \leq b$, so statement (3) of Proposition 22 implies that $U^{*}=(-\infty, d]$ for some $-\infty<d \leq c$. Since $g(u)=d_{\eta}+u d_{\xi}$ and $d_{\xi} \geq 0, U=(-\infty, a]$ for some $-\infty<a \leq d$. Since we have assumed $L \cap U=\emptyset, a<b$ as required.

If we assume that $U=(-\infty, a]$ for $a \in \mathbb{R}$, it can be shown, using a method of proof similar to the one above, that $\xi$ is a subordinator, and $L=\emptyset$ or $L=[b, \infty)$ for some $-\infty<a<b<\infty$. We omit the details.

Now, if we assume $L=(-\infty, a]$ for $a \in \mathbb{R}$, then symmetric proofs to the ones above, show that $-\xi$ is a subordinator, and $U=\emptyset$ or $U=[b, \infty)$ for $-\infty<a<b<\infty$. Similarly, if we assume $U=[b, \infty)$ for $b \in \mathbb{R}$, then symmetric proofs show that $-\xi$ is a subordinator, and $L=\emptyset$ or $L=(-\infty, a]$ for $-\infty<a<b<\infty$.

PROOF. [Proposition 11] Assume $L \cap U=\emptyset$. In the above proof of Theorem 9, it was shown that if $L=[b, \infty)$ for $b \in \mathbb{R}$ then $(\xi, \eta)$ is of finite variation, $\Sigma_{\xi, \eta}=0, d_{\xi} \geq 0, \Pi_{\xi, \eta}\left(A_{3}\right)=0, \Pi_{\xi, \eta}\left(A_{4} \backslash A_{1}\right)=0$, and $\theta_{2}<\infty$. It is clear from Propositions [16] and 17 that the converse also holds. A similar proof shows that $U=(-\infty, a]$ for $a \in \mathbb{R}$ iff $(\xi, \eta)$ is of finite variation, $\Sigma_{\xi, \eta}=0, d_{\xi} \geq 0$, $\Pi_{\xi, \eta}\left(A_{4}\right)=0, \Pi_{\xi, \eta}\left(A_{3} \backslash A_{2}\right)=0$, and $\theta_{1}^{\prime}>-\infty$. Combining these two sets of iff conditions immediately gives iff conditions for the case in which $U=(-\infty, a]$ and $L=[b, \infty)$ with $-\infty<a<b<\infty$. Since $V$ is increasing on $L$ and decreasing on $U$, and $V$ is a strong Markov process, it is clear that in this situation $\lim _{t \rightarrow \infty}\left|V_{t}\right|=\infty$ a.s. for any $V_{0}=z \in \mathbb{R}$.

It follows by symmetric methods that $L=(-\infty, a]$ and $U=[b, \infty)$ for $-\infty<$ $a<b<\infty$ iff the stated conditions in Proposition 11 hold. The only extra proof needed is to show that in this situation, $V$ is strictly stationary. In [8] it is shown that

$$
V_{t}={ }_{D} e^{\xi_{t}} z+\int_{0}^{t} e^{\xi_{s-}} \mathrm{d} K_{s}^{\xi, \eta}
$$

By Theorem 2 in [7] it is shown that if $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$ and and the integral condition $I_{-\xi, K}{ }^{\xi, \eta}=\infty$ holds, then $\left|\int_{0}^{t} e^{\xi_{s}-} \mathrm{d} K_{s}^{\xi, \eta}\right| \rightarrow_{P} \infty$ as $t \rightarrow \infty$.

As noted, if $L=(-\infty, a]$ and $U=[b, \infty)$ with $-\infty<a<b<\infty$ then $-\xi$ is a subordinator and so $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$ a.s. Now if $I_{-\xi, K \xi, \eta}=\infty$ then by the above, and since $\lim _{t \rightarrow \infty} e^{\xi_{t}}=-\infty$ a.s, it must be that $\left|V_{t}\right| \rightarrow_{D} \infty$. However this is impossible since $V$ is increasing on $L$ and decreasing on $U$. Thus, we must have $I_{-\xi, K}{ }^{\xi, \eta}<\infty$. Hence, by Theorem 2.1 in [8], $V$ is strictly stationary and converges in distribution to $\int_{0}^{\infty} e^{\xi_{s}-} \mathrm{d} K_{s}^{\xi, \eta}:=V_{\infty}$. Since $V$ is increasing on $L$ and decreasing on $U$, and $V$ is a strong Markov process, it is clear that $V_{\infty}$ has support $(a, b)$.

PROOF. [Theorem 12] Assume $Z_{t} \rightarrow Z_{\infty}$ a.s. as $t \rightarrow \infty$, where $Z_{\infty}$ is a finite random variable. Suppose that for all $c \in \mathbb{R}$, equation (9) does not hold. This implies that $Z_{\infty}$ is continuous. As noted in Section 1, a necessary condition for the convergence of $Z_{t}$, is $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a.s., which implies that $e^{\xi_{t}} \rightarrow \infty$ a.s. Since $Z_{\infty}$ is finite a.s., and $e^{\xi_{t}} \rightarrow \infty$ a.s., it is clear from the definition $V_{t}:=e^{\xi_{t}}\left(z+Z_{t}\right)$, that

$$
\begin{equation*}
P\left(\lim _{t \rightarrow \infty} V_{t}=\infty \mid V_{0}=z\right)=P\left(Z_{\infty}>-z\right) . \tag{17}
\end{equation*}
$$

Now let $a \leq \sup U$. By definition of $U, P\left(\lim _{t \rightarrow \infty} V_{t}=\infty \mid V_{0}=a\right)=0$ which implies, by equation (17), that $Z_{\infty}<-a$ a.s., as required.

Conversely, let $a>\sup U$. We prove $P\left(Z_{\infty}>-a\right)>0$. Since we have assumed that $\left|Z_{\infty}\right|<\infty$ a.s., we can choose $x>a$ such that $P\left(Z_{\infty}>-x\right)>0$. Note that $\Upsilon(a)=\infty$ and so there exists a fixed time $T>0$ such that $P\left(V_{T} \geq\right.$ $\left.x \mid V_{0}=a\right)>0$.

Hence, using (17), the law of conditional probability and the Markov property,

$$
\begin{aligned}
P\left(Z_{\infty}>-a\right) & =P\left(\lim _{t \rightarrow \infty} V_{t}=\infty \mid V_{0}=a\right) \\
& \geq P\left(\lim _{t \rightarrow \infty} V_{t}=\infty \mid V_{T} \geq x\right) P\left(V_{T} \geq x \mid V_{0}=a\right) \\
& \geq P\left(\lim _{t \rightarrow \infty} V_{t}=\infty \mid V_{0}=x\right) P\left(V_{T} \geq x \mid V_{0}=a\right)
\end{aligned}
$$

which is greater than zero by (17) and the choice of $x$ and $T$. Thus,

$$
\begin{equation*}
a \leq \sup U \text { iff } Z_{\infty}<-a \text { a.s. } \tag{18}
\end{equation*}
$$

Now we prove $-\sup U=m$ where $m:=\inf \left\{u \in \mathbb{R} \mid Z_{\infty}<u\right.$ a.s. $\}$. By equation (18), $Z_{\infty}<-\sup U$ and thus $-\sup U \geq m$. By assumption, $Z_{\infty}$ has no atoms and so $Z_{\infty}<m$ a.s. Thus, equation (18) implies that $-m \leq \sup U$. The proofs of the statements for $L$ are symmetric.

Now assume that there exists $c \in \mathbb{R}$ such that equation (9) holds, and assume that $Z_{t} \rightarrow Z_{\infty}$ a.s. as $t \rightarrow \infty$. By equation (9) it is immediate that $Z_{\infty}=-c$ a.s. Further, since $\xi$ drifts to $\infty$ a.s., Proposition 8 implies that $L=U=\{c\}$, or $U=(-\infty, c]$ and $L=[c, \infty)$. In both of these cases, $\inf L=\sup U=c$.

PROOF. [Theorem 13] (1) Assume $L \cap U=\emptyset, \sup U \geq 0$ and $L \cap[0, \sup U]=$ $\emptyset$, and let $0 \leq u \leq \sup U$. We want to prove that $\psi(u)=1$. Note that there exists $z \geq u$ such that $z \in U$, and so $\Upsilon(z)=z$. Since $\psi(u) \geq \psi(z)$, it suffices to prove that $\psi(z)=1$.

Since $L \cap[0, \sup U]=\emptyset$, we know $\delta(z)<0$, which implies that $P_{z}\left(\inf _{t>0} V_{t}<\right.$ $0)>0$. Thus, there exists a fixed time $T \in \mathbb{R}$ such that $P_{z}\left(\inf _{0<t \leq T} V_{t}<0\right):=$ $m>0$. Let $n \in \mathbb{N}$ and let $A$ be the distribution of $V_{n T}$ conditional on both $V_{0}=z$ and $\inf _{0<t \leq n T} V_{t} \geq 0$. Since $\Upsilon(z)=z$ we know $A \leq z$ a.s. Now

$$
P_{z}\left(\inf _{n T<t \leq(n+1) T} V_{t}<0 \mid \inf _{0<t \leq n T} V_{t} \geq 0\right)=P_{A}\left(\inf _{0<t \leq T} V_{t}<0\right) \geq m
$$

where the equality follows from the Markov property and the inequality follows from the fact that $A \leq z$ and $V_{t}$ is increasing in $z$. Define $P^{n}:=$ $P_{z}\left(\inf _{0<t \leq n T} V_{t}<0\right)$ for all $n \in \mathbb{N}$. By the law of total probability

$$
P^{n+1}=P^{n}+P_{z}\left(\inf _{n T<t \leq(n+1) T} V_{t}<0 \mid \inf _{0<t \leq n T} V_{t} \geq 0\right)\left(1-P^{n}\right)
$$

and so $P^{n+1} \geq P^{n}+\left(1-P^{n}\right) m$ where $P^{1}=m \in(0,1)$. This implies that $P^{n} \geq$ $1-(1-m)^{n}$ which implies that $\lim _{n \rightarrow \infty} P^{n}=1$, and hence $P_{z}\left(\inf _{0<t} V_{t}<0\right)=$ 1 by the continuity property of measures.
(2) Assume $L \cap U=\emptyset, \sup L \geq 0$, and $U \cap[0, \sup L]=\emptyset$. We let $z \geq 0$ and prove that $\psi(z)<1$. If $z \geq \inf L$ then $\psi(z)=0$ by definition. Thus, it suffices to assume $0 \leq z<\inf L$.

Suppose $\psi(z)=1$. By assumption, $\Upsilon(z)>\inf L$ and so, by definition, $P(C)>$ 0 where $C:=\left\{\sup _{t \geq 0} V_{t} \geq \inf L\right\}$. By definition of $L, \lim _{t \rightarrow \infty} V_{t} \geq \inf L$ a.s. for all $\omega \in C$. Let $T_{1}:=\inf \left\{t>0 \mid V_{t}<0\right\}$ and $T_{n}:=\inf \left\{t>T_{n-1} \mid V_{t}<V_{T_{n-1}}\right\}$ for integers $n>1$. By assumption, $\psi(z)=1$ and so $T_{1}$ is finite a.s. Further, the strong Markov property of $V$ implies that $\left\{T_{n}\right\}$ is a sequence of stopping times increasing towards infinity as $n \rightarrow \infty$, and each $T_{i}$ is a.s. finite. In particular, each $T_{i}$ is a.s. finite on $C$. However $V_{T_{n}}<0$ a.s. which contradicts the fact that $\lim _{t \rightarrow \infty} V_{t}>\inf L$ a.s. on $C$. Hence $\psi(z)<1$. The proof of the case in which $U \cap[0, \sup L] \neq \emptyset$ is almost identical, and we omit.

PROOF. [Theorem [14(1): Assume $L \cap U=\emptyset, \lim _{t \rightarrow \infty} \xi_{t}=-\infty$ a.s. and $I_{-\xi, K}{ }^{\xi, \eta}<\infty$. Suppose that $L \cap[0, \infty) \neq \emptyset$. Since $\xi$ drifts to $-\infty$ a.s., Propositions 8 and 9 imply that one of conditions (a) or (b) must hold. Further, it follows from statement (2) of Proposition 13 and the definition of $L$, that $0<\psi(z)<1$ for all $0 \leq z<\inf L$, and $\psi(z)=0$ for all $z \geq \inf L$.

Now suppose that $L \cap[0, \infty)=\emptyset$. We let $z \geq 0$ and prove that $\psi(z)=1$. Let $N$ be a Poisson process with parameter $\lambda$, let $D_{i}$ be an iid sequence of 1-dimensional exponential random variables and let $C_{i}=1$ for all $i$. Suppose that $N, D_{i}$ and $(\xi, \eta)$ are mutually independent and define the compound Poisson process $W_{t}:=\sum_{i=1}^{N_{t}}\left(C_{i}, D_{i}\right)$. Now define a new Lévy process $\left(\xi_{t}^{\diamond}, \eta_{t}^{\diamond}\right):=$ $\left(\xi_{t}, \eta_{t}\right)+W_{t}$, and denote the associated GOU by $V^{\diamond}$. For $V^{\diamond}$, denote the upper and lower bound functions, the sets of upper and lower bounds, and the ruin probability function by $\Upsilon^{\diamond}, \delta^{\diamond}, U^{\diamond}, L^{\diamond}$ and $\psi^{\diamond}$ respectively.

Define $T_{z}:=\inf \left\{t>0: V_{t}<0 \mid V_{0}=z\right\}$. Since $\sup L<0$, we know $\delta(z)<0$ and hence $T_{z}$ is finite a.s. Note that $V_{0}=V_{0}^{\triangleright}=z$. Also, whenever $V_{t-} \geq 0$, every jump $\Delta W_{t}$ causes a non-negative jump $\Delta V_{t}$. Hence $V_{t} \leq V_{t}^{\diamond}$ a.s. on $t \leq T_{z}$. This implies that $\psi(z) \geq \psi^{\diamond}(z)$. Thus it suffices to show that $\psi^{\diamond}(z)=1$. To do this, we first need to prove that $V^{\diamond}$ is strictly stationary.

We show that $\lambda>0$ can be chosen small enough such that $\lim _{t \rightarrow \infty} \xi_{t}^{\diamond}=-\infty$. Since $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$, either $E\left(\xi_{1}\right) \in[-\infty, 0)$ or $E\left(\xi_{1}\right)$ does not exist. If $E\left(\xi_{1}\right) \in[-\infty, 0)$ then $E\left(\xi_{1}^{\diamond}\right)=E\left(\xi_{1}\right)+\lambda$ and so we can choose $\lambda$ small enough such that $E\left(\xi_{1}^{\diamond}\right)<0$, which implies that $\lim _{t \rightarrow \infty} \xi_{t}^{\diamond}=-\infty$. If $E\left(\xi_{1}\right)$ does not exist then we know $E\left(\xi_{1}^{\diamond}\right)$ does not exist. We show that $\lim _{t \rightarrow \infty} \xi_{t}^{\diamond}=-\infty$ holds for any $\lambda>0$. Note that $\xi^{\diamond}=\xi+N$ and, as noted in Section 11, $J_{\xi}^{+}<\infty$ since $E\left(\xi_{1}\right)$ does not exist and $\lim _{t \rightarrow \infty} \xi_{t}=-\infty$. Also note that $\bar{\Pi}_{\xi^{\circ}}^{-}=\bar{\Pi}_{\xi}^{-}$and so $A_{\xi^{\circ}}^{-}=A_{\xi}^{-}$. Since $\xi$ and $N$ are independent we have $\bar{\Pi}_{\xi^{\circ}}^{+}=\bar{\Pi}_{\xi}^{+}+\bar{\Pi}_{N}^{+}$. Further $\bar{\Pi}_{N}^{+}(x)=0$ for all $x \geq 1$. Hence $J_{\xi^{\circ}}^{+}=J_{\xi}^{+}$and so is finite. As noted in Section

1. this implies that $\lim _{t \rightarrow \infty} \xi_{t}^{\diamond}=-\infty$.

We now show that $\left(\xi^{\diamond}, \eta^{\diamond}\right)$ satisfies $I_{-\xi^{\diamond}, K^{\xi^{\diamond}, \eta^{\triangleright}}}<\infty$. Since $(\xi, \eta)$ and $W$ are independent, it is clear from the definitions in Section 1 that $K_{t}^{\xi^{\circ}, \eta^{\circ}}=K_{t}^{\xi, \eta}+$ $K_{t}^{W}$ and $\bar{\Pi}_{K^{\xi}, \eta^{\circ}}(y)=\bar{\Pi}_{K^{\xi, \eta}}(y)+\bar{\Pi}_{K^{W}}(y)$. And, as above, $A_{-\xi^{\diamond}}^{+}=A_{-\xi}^{+}$. Hence

$$
I_{-\xi^{\prime}, K^{\circ}, \eta^{\triangleright}}=I_{-\xi, K^{\xi}, \eta}+\int_{(e, \infty)}\left(\frac{\ln (y)}{A_{-\xi}^{+}(\ln (y))}\right)\left|\bar{\Pi}_{K^{W}}(\mathrm{~d} y)\right| .
$$

By the choice of $W$ it is clear that $K_{1}^{W}$ has a finite expected value which implies that $\int_{(e, \infty)} y\left|\bar{\Pi}_{K^{W}}(\mathrm{~d} y)\right|<\infty$. Hence $I_{-\xi^{\prime}, K^{\xi^{\diamond}, \eta^{\triangleright}}}<\infty$. Thus $V^{\diamond}$ is strictly stationary.

For a Lebesgue set $\Lambda$ define $T_{\Lambda}^{\diamond}:=\inf \left\{t>0: V_{t}^{\diamond} \in \Lambda\right\}$. Note that $\theta_{1}^{\prime \diamond}=$ $-\infty$ and hence Proposition 20 implies that $\Upsilon^{\diamond}(u)=\infty$ for all $u \in \mathbb{R}$, or equivalently, $U^{\diamond}=\emptyset$. Also, $\theta_{1}^{\diamond}=0$, and so Proposition 16 implies that $L^{\diamond} \cap$ $(-\infty, 0)=\emptyset$, whilst the fact that $L \cap(0, \infty)=\emptyset$ clearly implies that $L^{\prime} \cap$ $(0, \infty)=\emptyset$.

These facts imply that, for all $a$ and $u$ in $\mathbb{R}, P\left(T_{(-\infty, a]}^{\diamond}<\infty \mid V_{0}^{\diamond}=u\right)>0$ and $P\left(T_{[a, \infty]}^{\diamond}<\infty \mid V_{0}^{\diamond}=u\right)>0$. Since $D$ is an exponential random variable, it is clear that $V_{t}^{\diamond}$ has a continuous density with respect to Lebesgue measure. Hence $P\left(T_{\Lambda}^{\diamond}<\infty\right)>0$ for any set $\Lambda$ with positive Lebesgue measure. This result, and the fact that $V^{\diamond}$ is strictly stationary, allows us to mimic the argument of Theorem 3.1 (a) in Paulsen [10]. Let $S$ be an independent standard exponential variable and define the resolvent kernel

$$
K(z, \Lambda):=\int_{0}^{\infty} P_{z}\left(V_{t}^{\diamond} \in \Lambda\right) e^{-t} \mathrm{~d} t=P_{z}\left(V_{S}^{\diamond} \in \Lambda\right)
$$

Proposition 2.1 of 9$]$ implies that $V^{\diamond}$ is $\phi$-irreducible for the measure $\phi=\lambda K$. Using the language of [9] p. 495 and 496, it is clear that $K$ has a continuous nontrivial component for all $z$ and hence is a T-process. Since $V^{\diamond}$ is strictly stationary it is clear that $V^{\diamond}$ is non-evanescent, as defined in [9] p.494. Thus Theorem 3.2 of [9] p. 494 implies that $V^{\diamond}$ is Harris recurrent, as defined in [9] p490, which clearly implies that $\psi^{\diamond}(z)=1$ as required.
(2) Assume that $L \cap U=\emptyset, E\left(\xi_{1}\right)=0, E\left(e^{\left|\xi_{1}\right|}\right)<\infty$ and there exist $p, q>1$ with $1 / p+1 / q=1$ such that $E\left(e^{-p \xi_{1}}\right)<\infty$ and $E\left(\left|\eta_{1}\right|^{q}\right)<\infty$.

Suppose that $L \cap[0, \infty) \neq \emptyset$. Since $\xi$ oscillates a.s., Proposition 9 implies that $L=[a, b]$ and $U=\emptyset$ where $-\infty<a \leq b<\infty$ and $b \geq 0$. Hence, it follows from statement (2) of Proposition 13 and the definition of $L$, that $0<\psi(z)<1$ for all $0<z<a$ and $\psi(z)=0$ for all $z \geq a$.

Now suppose that $L \cap[0, \infty)=\emptyset$. We let $z \geq 0$ and prove that $\psi(z)=1$. We
know that $P\left(\inf _{t>0} V_{t}<0 \mid V_{0}=z\right)>0$. However, it is possible that for some $z>0, P\left(V_{1}<0 \mid V_{0}=z\right)=0$. For example, this would happen if $(\xi, \eta)$ has no Brownian component and $\sup L^{*}>0$. Let $0=T_{0}<T_{1}<T_{2}<\ldots$ be random times such that $T_{i}-T_{i-1}$ are iid with exponential distribution and parameter $\lambda$. Since $T_{1}$ has infinite support it is clear that $\sup L<0$ implies $P\left(V_{T_{1}}<0 \mid V_{0}=z\right)>0$ for all $z \geq 0$. Equation (1) implies that a.s.

$$
V_{T_{n}}=e^{\xi_{T_{n}}-\xi_{T_{n-1}}}\left(e^{\xi_{T_{n-1}}}\left(z+\int_{0}^{T_{n-1}} e^{-\xi_{s-}} \mathrm{d} \eta_{s}\right)\right)+e^{\xi_{T_{n}}} \int_{T_{n-1}}^{T_{n}} e^{-\xi_{s-}} \mathrm{d} \eta_{s}
$$

Thus, if we define $A_{n}:=e^{\xi_{T_{n}}-\xi_{T_{n-1}}}, B_{n}:=e^{\xi_{T_{n}}} \int_{T_{n-1}}^{T_{n}} e^{-\xi_{s-}} \mathrm{d} \eta_{s}$ and the stochastic difference equation $W_{n}:=A_{n} W_{n-1}+B_{n}$ with $W_{0}:=V_{0}=z$ then $W_{n}=V_{T_{n}}$ a.s. for all $n \in \mathbb{N}$. Note that the term $e^{\xi_{T_{n}}}$ in $B_{n}$ cannot be brought under the integral sign because it is not predictable. Since a Lévy process has independent increments it is clear that $\left(A_{n}, B_{n}\right)$ is an independent sequence. Now,

$$
\begin{aligned}
\left(A_{2}, B_{2}\right) & =\left(e^{\xi_{T_{2}}-\xi_{T_{1}}}, e^{\xi_{T_{2}}-\xi_{T_{1}}} e^{\xi_{T_{1}}} \int_{T_{1}}^{T_{2}} e^{-\xi_{s-}} \mathrm{d} \eta_{s}\right) \\
& \left.=\left(e^{\xi_{T_{2}}-\xi_{T_{1}}}, e^{\xi_{T_{2}}-\xi_{T_{1}}} \int_{T_{1}}^{T_{2}} e^{-\left(\xi_{s-}-\xi_{T_{1}}\right.}\right) \mathrm{d} \eta_{s}\right) \\
& \left.=\left(e^{\xi_{T_{2}}-\xi_{T_{1}}}, e^{\xi_{T_{2}}-\xi_{T_{1}}} \int_{T_{1}}^{T_{2}} e^{-\left(\xi_{s-}-\xi_{T_{1}}\right.}\right) \mathrm{d}\left(\eta_{s}-\eta_{T_{1}}\right)\right) \\
& ={ }_{D}\left(e^{\xi_{T_{1}}}, e^{\xi_{T_{1}}} \int_{T_{1}}^{T_{2}} e^{-\xi_{s-T_{1}}} \mathrm{~d} \eta_{s-T_{1}}\right) \\
& =\left(e^{\xi_{T_{1}}}, e^{\xi_{T_{1}}} \int_{0}^{T_{1}} e^{-\xi_{s-}} \mathrm{d} \eta_{s}\right)=\left(A_{1}, B_{1}\right),
\end{aligned}
$$

where the second equality holds because $e^{\xi_{T_{1}}}$ is predictable with respect to the integral, the third equality holds because a Lévy process has identically distributed increments and the final equality is obtained using a change of variables. The argument for general $n$ is identical, and thus $\left(A_{n}, B_{n}\right)$ is an iid sequence.

Now Proposition 1.1 and Corollary 4.2 of [1] state that if $P\left(A_{1} z+B_{1}=z\right)<1$ for all $z \in \mathbb{R}, E\left(\ln A_{1}\right)=0, A_{1} \not \equiv 1$ and there exists $\alpha>0$ such that

$$
\begin{equation*}
E\left(\left(\left|\ln A_{1}\right|+\ln ^{+}\left|B_{1}\right|\right)^{2+\alpha}\right)<\infty \tag{19}
\end{equation*}
$$

then the discrete stochastic process $W$ has an invariant unbounded Radon measure $\mu$ unique up to a constant factor such that the sample paths $W_{n}$, with $W_{0}=z$, visit every open set of positive $\mu$-measure infinitely often with probability 1 , for every $z \in \mathbb{R}$. The first of these conditions follows from our assumption that $L \cap U=\emptyset$, using Proposition 8. The second and third
conditions follow respectively from our assumptions that $E\left(\xi_{1}\right)=0$, and $\xi_{1}$ is not identically zero. We will show later that our moment conditions on $\xi$ and $\eta$ ensure equation (19) holds. Note that the Babillot result implies that $\psi(z)=1$ if we can show $\mu((-\infty, 0))>0$. However by the definition of an invariant measure,

$$
\mu((-\infty, 0))=\int_{z \in \mathbb{R}} P\left(A_{1} z+B_{1}<0\right) \mu(\mathrm{d} z) \geq \int_{z \in \mathbb{R}} P\left(V_{T_{1}}<0 \mid V_{0}=z\right) \mu(\mathrm{d} z)
$$

Thus if $\mu([0, \infty))>0$ then $\mu((-\infty, 0))>0$ since $P\left(V_{T_{1}}<0 \mid V_{0}=z\right)>0$ for all $z \geq 0$. And if $\mu([0, \infty))=0$ then $\mu((-\infty, 0))>0$ since $\mu(\mathbb{R})>0$. Thus we are done if we can prove equation (19).

To do this, it suffices to assume $T_{1}=1$ and $\left(A_{1}, B_{1}\right):=\left(e^{\xi_{1}}, e^{\xi_{1}} \int_{0}^{1} e^{-\xi_{s}-} \mathrm{d} \eta_{s}\right)$ since we can choose the parameter $\lambda$ of the increments to be arbitrarily small. Note that if $x, y>0$ and $\alpha>0$ then there exists $c_{1}>0$ such that $(x+y)^{\alpha} \leq c_{1}\left(x^{\alpha}+y^{\alpha}\right)$. Also $\ln ^{+}(x+y) \leq \ln ^{+}(x)+\ln ^{+}(y)$ and $\ln ^{+}(x y) \leq$ $\ln ^{+}(x)+\ln ^{+}(y)$. Finally note that whenever $0<\alpha \leq 1$ there exists $c_{2}>0$ such that $\ln ^{+}(x)^{2+\alpha} \leq c_{2} x^{\alpha}$. Using these four inequalities it is clear that equation (19) is satisfied whenever there exists $0<\alpha \leq 1$ such that $E\left(e^{\alpha \xi_{1}}\right)<\infty$, $E\left(\left|\xi_{1}\right|^{2+\alpha}\right)<\infty$ and $E\left(\left|\int_{0}^{1} e^{-\xi_{s-}} \mathrm{d} \eta_{s}\right|^{\alpha}\right)<\infty$. By Proposition 24, and the fact that the existence of an absolute exponential moment implies the existence of absolute moments of all orders, the assumed moment conditions imply that these conditions are satisfied for $\alpha=1$.
(3) Assume that $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a.s. and $I_{\xi, \eta}<\infty$. Suppose that $-\infty \leq$ $\sup U<z$. Assume, for the sake of contradiction, that $\psi(z)=1$. Theorem 12 implies that $P(C)>0$ where $C:=\left\{Z_{\infty}>-z\right\}$. Since $\lim _{t \rightarrow \infty} \xi_{t}=\infty$, we know that $\lim _{t \rightarrow \infty} V_{t}=\infty$ a.s. on $C$. Now, the same strong Markov property argument used in the proof of statement (2) of Theorem 13, gives a contradiction. Hence $\psi(z)<1$.

Now suppose $U \cap[0, \infty) \neq \emptyset$. Since $\xi$ drifts to $\infty$ a.s., Theorem 9 implies that either $U=[a, b]$ and $L=\emptyset$ where $-\infty \leq z \leq b<\infty$ and $b \geq 0$, or $U=(-\infty, a]$ and $L=[b, \infty)$ for some $0 \leq a<b<\infty$. In both of these cases, statement (1) of Theorem 13 implies that $\psi(z)=1$ for all $z \leq \sup U$. Using the definition of $L$, and the above result, it is clear that $0<\psi(z)<1$ for all $\sup U<z<\inf L$ and $\psi(z)=0$ for all $z \geq \sup L$.

PROOF. [Proposition 5 Assume that $V_{t}=e^{\xi_{t}}(z-c)+c$. By definition of $L$, if $c \geq 0$ then $\psi(z)=0$ for all $z \geq c$.

Let $0 \leq z<c$. If $\xi$ drifts to $-\infty$ a.s. then $\lim _{t \rightarrow \infty} V_{t}=c$ a.s. Thus, the strong Markov property of $V$ implies that $\psi(z)<1$, using a proof similar to that used for statement (2) of Theorem 13. If $\xi$ oscillates a.s. then $-\infty=$
$\liminf _{t \rightarrow \infty} V_{t}<\limsup _{t \rightarrow \infty} V_{t}=c$, and so $\psi(z)=1$. If $\xi$ drifts to $\infty$ a.s. then $\lim _{t \rightarrow \infty} V_{t}=-\infty$ a.s. which implies $\psi(z)=1$.

Let $c<0 \leq z$. If $\xi$ drifts to $-\infty$ a.s. then $\lim _{t \rightarrow \infty} V_{t}=c$ a.s. and so $\psi(z)=1$. If $\xi$ oscillates a.s. then $c=\liminf _{t \rightarrow \infty} V_{t}<\limsup _{t \rightarrow \infty} V_{t}=\infty$, and so $\psi(z)=1$. If $\xi$ drifts to $\infty$ a.s. then $\lim _{t \rightarrow \infty} V_{t}=\infty$ a.s. which implies $\psi(z)<1$, using a strong Markov property argument.

PROOF. [Theorem 1] Suppose that for all $c \in \mathbb{R}$ the degenerate case (9) does not hold. Then, by Proposition 8, $L \cap U=\emptyset$. It follows immediately from Theorem 14 that $0<\psi(z)<1$ iff $0 \leq z<m<\infty$ whenever the assumptions for statement (1), or statement (2), of Theorem 1 are satisfied. Now suppose that there exists $c \in \mathbb{R}$ such that equation (9) holds. Then it follows immediately from Proposition 5 that $0<\psi(z)<1$ iff $0 \leq z<m<\infty$ whenever the assumptions for statement (1), or statement (2), of Theorem 1 are satisfied. In both these situations, $m=c$.

PROOF. [Theorem 3] Assume $\lim _{t \rightarrow \infty} \xi_{t}=\infty$ a.s. and $I_{\xi, \eta}<\infty$. Assume that for all $c \in \mathbb{R}$ equation (9) does not hold, or equivalently, $L \cap U=\emptyset$. Theorem 3 claims that $\psi(0)=1$ iff $-\eta$ is a subordinator, or there exists $z>0$ such that $\psi(z)=1$. This claim follows by combining two known results: $\psi(z)=1$ iff $\sup U \geq 0$ and $z<\sup U$, which is implied by statement (3) of Theorem 13: secondly, $0 \in U$ iff $-\eta$ is a subordinator, which is stated in Proposition 20 ,

Theorem 3 also states conditions on the characteristic triplet of $(\xi, \eta)$ and claims these are equivalent to the fact that there exists $z>0$ such that $\psi(z)=1$. However, using statement (3) of Theorem 13, we know there exists $z>0$ such that $\psi(z)=1$ iff $\sup U>0$. And Proposition 20 gives iff conditions on the characteristic triplet of $(\xi, \eta)$ for the case $\sup U>0$ to occur. These conditions are precisely the conditions stated in Theorem 3,

Finally, statements (1) and (2) of Theorem 33 contain values for $\sup \{z \geq$ $0: \psi(z)=1\}$. However, these follow from the unstated parallel version of Proposition 17 which gives exact values for the endpoints of $U$.

Now, assume that there exists $c \in \mathbb{R}$ such that the degenerate equation (9) holds, and $L=U=\{c\}$. Since $\xi$ drifts to $\infty$ a.s., Proposition 8 implies that $\sup U=c$. Thus, Proposition 5 implies that $\psi(z)=1$ iff $\sup U \geq 0$ and $z<\sup U$. Theorem 3 is proved for the degenerate case by combining this statement with Proposition 20 Proposition 20 and the parallel version of Proposition 17, in an identical manner to the above. The only difference is that the set $\{z \geq 0: \psi(z)=1\}$ does not contain its supremum in the degenerate case, since $\sup \{z \geq 0: \psi(z)=1\}=U=L$, and is an absorbing point.

### 4.1 Examples

Propositions 8, 9 and 11 claim that Lévy processes $(\xi, \eta)$ exist which satisfy particular combinations of $L$ and $U$, and particular asymptotic behaviour for $\xi$. We now present examples which prove these claims. We use the simplest Lévy processes possible. The Lévy measures will always be finite activity, namely $\Pi_{\xi, \eta}\left(\mathbb{R}^{2}\right)<\infty$. Hence, we can write $(\xi, \eta)$ in the form $(\xi, \eta)_{t}=$ $\left(d_{\xi}, d_{\eta}\right) t+\left(B_{\xi, t}, B_{\eta, t}\right)+\sum_{i=1}^{N_{t}} Y_{i}$ where $\left(B_{\xi, t}, B_{\eta, t}\right)$ is Brownian motion with covariance matrix $\Sigma_{\xi, \eta}, N$ is a Poisson process with parameter $\Lambda$ and $\left\{Y_{i}\right\}_{i=1}^{\infty}$ is an iid sequence of two dimensional random variables with distribution $Y$.

Examples with Brownian component The first example is of a Lévy process $(\xi, \eta)$ for which $L=\{a\}, U=\emptyset$. The second example is of a Lévy process for which $L=U=\{a\}$. For both examples we show how variables can be chosen so that $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. or $\xi$ oscillates a.s.

Example $25 \operatorname{Let}(\xi, \eta)_{t}:=\left(d_{\xi}, 2\right) t+\left(B_{t}, B_{t}\right)+\sum_{i=1}^{N_{t}} Y_{i}$ where $B$ is a onedimensional Brownian motion with variance 1, and $P(Y=(10,10))=1 / 2$ and $P(Y=(-10,10))=1 / 2$. The covariance matrix equation (10) is satisfied for $u=-1$. Condition (ii) of Proposition [16] is satisfied for $u=-1$, whilst condition (ii) of Proposition 20 is not satisfied. By equation (13), $g(-1)=$ $3 / 2-d_{\xi}$, and so choosing $d_{\xi} \leq 3 / 2$ implies that $L=-1$ and $U=\emptyset$. However $E\left(\xi_{1}\right)=d_{\xi}$ so if $0<d_{\xi}<3 / 2$ then $\xi$ drifts to $\infty$ a.s., if $d_{\xi}<0$ then $\xi$ drifts to $-\infty$ a.s., and if $d_{\xi}=0$ then $\xi$ oscillates a.s.

Example $26 \operatorname{Let}(\xi, \eta)_{t}:=\left(d_{\xi}, d_{\eta}\right) t+\left(B_{t},-B_{t}\right)$. Equation (10) is satisfied for $u=1$, whilst condition (ii) of Proposition 16 and condition (ii) of Proposition 20 are satisfied trivially. Equation (13) implies $g(1)=d_{\eta}+d_{\xi}-1 / 2$. Thus, choosing $d_{\xi}=1 / 2-d_{\eta}$ implies that $L=U=1$. Note $E\left(\xi_{1}\right)=d_{\xi}$, so if $d_{\eta}<1 / 2$ then $\xi$ drifts to $\infty$ a.s., if $d_{\eta}>1 / 2$ then $\xi$ drifts to $-\infty$ a.s., and if $d_{\eta}=1 / 2$ then $\xi$ oscillates a.s.

Examples with no Brownian component We now present seven examples of Lévy processes $(\xi, \eta)$ with no Brownian component. In Example 27, $L=$ $U=\{a\}$ and we indicate how the parameters can be changed in order to obtain each of the three asymptotic behaviours for $\xi$. In Examples 28 and 29, $L=\emptyset$, whilst $U$ may be of form $\emptyset,\{a\}$ or $[a, b]$ for $-\infty<a<b<\infty$. We indicate how parameters can be changed in order to obtain these different sets, and for each set, to obtain the three possible asymptotic behaviours for $\xi$. In Example 30, $L=\emptyset$ whilst $U$ is of form $[b, \infty)$ for $b \in \mathbb{R}$. In Example [31, $L=(-\infty, a]$ and $U=[b, \infty)$ for $-\infty<a<b<\infty$. For both these examples we show that $\xi$ drifts to $-\infty$ a.s. In Example 32, $L=\emptyset$ whilst $U$ is of form $(-\infty, a]$ for $a \in \mathbb{R}$. In Example 33, $U=(-\infty, a]$ and $L=[b, \infty)$ for $-\infty<a<b<\infty$. For both these examples we show that $\xi$ drifts to $\infty$ a.s.

Example 27 Let $(\xi, \eta)_{t}:=\left(d_{\xi}, d_{\eta}\right) t+\sum_{i=1}^{N_{t}} Y_{i}$ where $P\left(Y=\left(3,2 e^{-3}-2\right)\right)=$ $1 / 2$ and $P\left(Y=\left(-3,2 e^{3}-2\right)\right)=1 / 2$. Then $\theta_{2}=\theta_{2}^{\prime}=\theta_{4}=\theta_{4}^{\prime}=2$ and $L^{*}=U^{*}=\{2\}$. Note that $g(u)=d_{\eta}+u d_{\xi}$, so choosing $d_{\eta}=-2 d_{\xi}$ implies that $g(2)=0$ and hence $L=U=\{2\}$. Since $E\left(\xi_{1}\right)=d_{\xi}$, choosing $d_{\xi}>0$, $d_{\xi}<0$, and $d_{\xi}=0$, implies that $\xi$ drifts to $\infty$ a.s., $\xi$ drifts to $-\infty$ a.s. and $\xi$ oscillates a.s., respectively.

Example $28 \operatorname{Let}(\xi, \eta)_{t}:=\left(d_{\xi}, d_{\eta}\right) t+\sum_{i=1}^{N_{t}} Y_{i}$ where $P(Y=(4,-2))=1 / 3$ and $P(Y=(-2,-3))=1 / 3$ and $P(Y=(-2,1))=1 / 3$. Then $L=\emptyset$ since $\Pi_{\xi, \eta}\left(A_{2}\right)$ and $\Pi_{\xi, \eta}\left(A_{3}\right)$ are both non-zero, whilst $U^{*}=\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]=\left[\frac{-2}{e^{-4}-1}, \frac{1}{e^{2}-1}\right] \cong$ $[0.2,2]$. Now $U=\left\{u \in U^{*}: g(u) \leq 0\right\}$ and $g$ simplifies to $g(u)=d_{\eta}+u d_{\xi}$. Note that $E\left(\xi_{1}\right)=d_{\xi}$.

Choosing $d_{\xi}=0$ and $d_{\eta}>0$ implies that $U=\emptyset$ and $\xi$ oscillates a.s. Choosing $d_{\xi}>0$ and $d_{\eta}>-\theta_{4}^{\prime} d_{\xi}$ implies that $U=\emptyset$ and $\xi$ drifts to $\infty$ a.s. Choosing $d_{\xi}<0$ and $d_{\eta}>-\theta_{2}^{\prime} d_{\xi}$ implies that $U=\emptyset$ and $\xi$ drifts to $-\infty$ a.s.

Choosing $d_{\xi}=0$ and $d_{\eta}<0$ implies that $U=U^{*} \cong[0.2,2]$ and $\xi$ oscillates a.s. Choosing $d_{\xi}>0$ and $d_{\eta}<-\theta_{2}^{\prime} d_{\xi}$ implies that $U=U^{*} \cong[0.2,2]$ and $\xi$ drifts to $\infty$ a.s. Choosing $d_{\xi}<0$ and $d_{\eta}<-\theta_{4}^{\prime} d_{\xi}$ implies that $U=U^{*} \cong[0.2,2]$ and $\xi$ drifts to $-\infty$ a.s.

Choosing $d_{\xi}>0$ and $d_{\eta}=-\theta_{4}^{\prime} d_{\xi}$ implies that $U=\left\{\theta_{4}^{\prime}\right\} \cong\{0.2\}$ and $\xi$ drifts to $\infty$ a.s. Choosing $d_{\xi}<0$ and $d_{\eta}=-\theta_{2}^{\prime} d_{\xi}$ implies that $U=\left\{\theta_{2}^{\prime}\right\} \cong\{2\}$ and $\xi$ drifts to $-\infty$ a.s.

Note that for Example 32, no adjustment of $d_{\xi}$ and $d_{\eta}$ can result in $U=\{a\}$ with $\xi$ oscillating a.s. We now present a different example with this behaviour.

Example 29 Let $(\xi, \eta)_{t}:=(0,-2) t+\sum_{i=1}^{N_{t}} Y_{i}$ where $P\left(Y=\left(2, e^{-2}-1\right)\right)=$ $1 / 3$ and $P(Y=(-1, e-1))=1 / 3$ and $P(Y=(-1,-2))=1 / 3$. Then $L=\emptyset$, $\theta_{2}=\theta_{2}^{\prime}=\theta_{4}=\theta_{4}^{\prime}=1$, and $U^{*}=\{1\}$. Since $g$ simplifies to $g(u)=-2$ for all $u \in \mathbb{R}$ we obtain $U=\{1\}$. Since $E\left(\xi_{1}\right)=0$, $\xi$ oscillates a.s.

Example $30 \operatorname{Let}(\xi, \eta)_{t}:=(0,-2) t+\sum_{i=1}^{N_{t}} Y_{i}$ where $P(Y=(-1,2))=1 / 3$ and $P(Y=(-2,-3))=1 / 3$ and $P(Y=(0,-5))=1 / 3$. Then $L^{*}=\emptyset$ whilst $U^{*}=\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]=\left[\frac{2}{e-1}, \infty\right) \cong[1.2, \infty)$. Since $g(u)=-2$ for all $u \in \mathbb{R}$ we obtain $L=\emptyset$ and $U=U^{*}$ Since $E\left(\xi_{1}\right)=-1.5, \xi$ drifts to $-\infty$ a.s.

Example $31 \operatorname{Let}(\xi, \eta)_{t}:=\left(d_{\xi}, d_{\eta}\right) t+\sum_{i=1}^{N_{t}} Y_{i}$ where $P(Y=(-1,2))=1 / 2$ and $P(Y=(-2,-3))=1 / 2$. Then $L^{*}=\left[\theta_{1}, \theta_{3}\right]=\left(-\infty, \frac{-3}{e^{2}-1}\right] \cong(-\infty,-0.5]$ and $U^{*}=\left[\theta_{4}^{\prime}, \theta_{2}^{\prime}\right]=\left[\frac{2}{e-1}, \infty\right) \cong[1.2, \infty)$. Note that $g$ simplifies to $g(u)=$ $d_{\eta}+u d_{\xi}$ and hence choosing $d_{\xi} \leq 0$ and $d_{\eta}=0$ gives $L=L^{*}$ and $U=U^{*}$. Since $E\left(\xi_{1}\right)=-1.5+d_{\xi}$, $\xi$ drifts to $-\infty$ a.s.

Example 32 Let $(\xi, \eta)_{t}:=\sum_{i=1}^{N_{t}} Y_{i}$ where $P(Y=(1,2))=1 / 3$ and $P(Y=$
$(1,8))=1 / 3$ and $P(Y=(0,-5))=1 / 3$. Then $L^{*}=\emptyset$ whilst $U^{*}=\left[\theta_{3}^{\prime}, \theta_{1}^{\prime}\right]=$ $\left(-\infty, \frac{8}{e^{-1}-1}\right] \cong(-\infty,-12.6]$. Note that $g(u)=0$ for all $u \in \mathbb{R}$ so $L=L^{*}$ and $U=U^{*}$. Since $E\left(\xi_{1}\right)=1, \xi$ drifts to $\infty$ a.s.

Example 33 Let $(\xi, \eta)_{t}:=\sum_{i=1}^{N_{t}} Y_{i}$ where $P(Y=(1,2))=1 / 2$ and $P(Y=$ $(1,8))=1 / 2$. Then $L^{*}=\left[\theta_{1}, \theta_{4}\right]=\left[\frac{2}{e^{-1}-1}, \infty\right) \cong[-3.2, \infty)$ and $U^{*}=\left[\theta_{3}^{\prime}, \theta_{1}^{\prime}\right]=$ $\left(-\infty, \frac{8}{e^{-1}-1}\right] \cong(-\infty,-12.6]$. Note that $g(u)=0$ for all $u \in \mathbb{R}$ so $L=L^{*}$ and $U=U^{*}$. Since $E\left(\xi_{1}\right)=1, \xi$ drifts to $\infty$ a.s.

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