TOWARDS A RELATIVISTIC KMS-CONDITION

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ABSTRACT

It is shown that, under quite general conditions, thermal correlation functions in relativistic quantum field theory have stronger analyticity properties in configuration space than those imposed by the KMS-condition. These analyticity properties may be understood as a remnant of the relativistic spectrum condition in the vacuum sector and lead to a Lorentz-covariant formulation of the KMS-condition involving all space-time variables.

1. Introduction

In relativistic quantum field theory, one characterizes thermal equilibrium states in the same way as their counterparts in non-relativistic quantum statistical mechanics by means of the KMS-condition [BR,H]. If one denotes by $\mathcal{A}$ the algebra of observables and by α_t the automorphism of $\mathcal{A}$ inducing the time translations in the rest frame of the considered heat bath, this condition reads as follows :

KMS-Condition : The state ω_β on $\mathcal{A}$ satisfies the KMS-condition at inverse temperature $\beta > 0$ iff for every pair of operators $A, B \in \mathcal{A}$ there exists an analytic function F in the strip $S_\beta = \{z \in \mathbb{C} : 0 < \text{Im } z < \beta\}$ with continuous boundary values at $\text{Im } z = 0$ *and* $\text{Im } z = i\beta$, *given respectively (for $t \in \mathbb{R}$) by:*

$$F(t) = \omega_\beta \left(A\alpha_t(B)\right), \qquad F(t + i\beta) = \omega_\beta \left(\alpha_t(B)A\right) \ .$$

The notions used in this quite general formulation of the condition are related to the conventional field-theoretical setting as follows. The algebra $\mathcal{A}$ may be thought of as being generated by bounded functions of the underlying observable fields, currents, etc. If $\phi(x)$ is any such field and if $f(x)$ is any real test function with support in a bounded region of spacetime, then the corresponding (unitary) operator $A = e^{i \int dx\, f(x)\phi(x)}$ would be a typical element of $\mathcal{A}$. Conversely, one can recover the fields from such operators in $\mathcal{A}$ by taking (functional) derivatives. We assume here that the algebra $\mathcal{A}$ is defined on the vacuum Hilbert space $\mathcal{H}$ of the theory. The vacuum state ω_∞ can then be represented in the form $\omega_\infty(A) = (\Omega, A\Omega)$, $A \in \mathcal{A}$, where $\Omega \in \mathcal{H}$ is the vacuum vector. Similarly, the thermal states ω_β are positive, linear and normalized functionals on $\mathcal{A}$ which, according to the reconstruction theorem, can be represented by vectors Ω_β in the thermal Hilbert spaces $\mathcal{H}_\beta$. Thus the functions $F(t)$ in the formulation of the KMS-condition are the correlation functions of an arbitrary pair of observables in these states. The automorphism α_t inducing the time translations can be represented on the vacuum Hilbert space $\mathcal{H}$ according to $\alpha_t(A) = e^{itH} A e^{-itH}$, $A \in \mathcal{A}$, where H is the Hamiltonian. A similar representation holds on the thermal Hilbert spaces $\mathcal{H}_\beta$. But, whereas the action of α_t on $\mathcal{A}$ does not depend on the state ω_β which one considers, the respective unitaries e^{itH_β} implementing this action on the spaces $\mathcal{H}_\beta$ do. The present somewhat abstract approach is therefore appropriate if one wants to consider all thermal states at the same time.

The characterization of equilibrium states by the KMS-condition recalled above is well justified [HHW, HKTP, PW], and the breaking of Lorentz invariance in this condition through the choice of a distinguished time-axis is fully understood [O,N]. Nevertheless it would seem natural in a relativistic setting to incorporate in the KMS-condition the properties of equilibrium states with respect to observations *in arbitrary Lorentz frames*. It is the aim of the present article to shed some light on this problem.

Let us begin with some considerations based on a property of thermal equilibrium states about which all uniformly moving observers ought to agree : it is impossible to extract from such a state an arbitrarily large amount of energy by local operations, namely the energy content of equilibrium states is locally finite. We first advocate that, for an observer in the rest frame of the state, this property is essentially encoded in the analyticity requirement of the KMS-condition.

In fact, let us make the assumption considered as physically well-motivated in the general algebraic setting of quantum field theory, that the KMS-state ω_β is locally normal with respect to the vacuum sector [H]. By applying standard arguments in the theory of operator algebras [D], one deduces from this assumption that for each bounded space-time region $\mathcal{O}$ there exists some vector $\Omega_{\beta,\mathcal{O}} \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space describing the vacuum sector, such that $\omega_\beta(A) = (\Omega_{\beta,\mathcal{O}}, A\, \Omega_{\beta,\mathcal{O}})$ for all observables A which are localized in $\mathcal{O}$. The KMS-condition for ω_β then entails that for observables A which are localized in the interior of $\mathcal{O}$ and for sufficiently small $|t|$ the vector valued function $t \longrightarrow \alpha_t(A)\Omega_{\beta,\mathcal{O}}$ has an analytic continuation into the strip $S_{\beta/2}$. Hence $\Omega_{\beta,\mathcal{O}}$ behaves "locally" in the same way as an analytic vector for the energy H (lying in the domain of $\mathrm{e}^{\frac{\beta}{2}H}$). This indicates that large local energy contributions, though present due to energy fluctuations, are exponentially suppressed in the state ω_β.

Since in the vacuum sector of a relativistic theory the notion of "analytic vector for the energy" is a Lorentz invariant concept (as a consequence of the relativistic spectrum condition), the above argument suggests that, with respect to all time-like directions $f \in V_+$, the function $t \longrightarrow \alpha_t^{(f)}(A)\Omega_{\beta,\mathcal{O}}$ should have, for sufficiently small $|t|$, an analytic continuation into some domain of the upper half plane. Because of local normality this would in turn imply that also the correlation functions $t \longrightarrow \omega_\beta\left(A\ \alpha_t^{(f)}(B)\right)$ can be continued analytically into that domain. This condition could be regarded as a *stability requirement*, giving a formal expression to the idea that the local energy content of equilibrium states is finite in all Lorentz frames.

Unfortunately we have not been able to cast these heuristic considerations into a rigorous argument without further input, the subtle point being the question in which precise sense the vectors $\Omega_{\beta,\mathcal{O}}$ may be regarded as being locally analytic for the energy operator in the various Lorentz frames.

We will therefore adopt a different approach which relies in its spirit on the original introduction of KMS states (due to [HHW]) as thermodynamic limits of appropriate local Gibbs states. The method which we will apply for carrying out this approach in a rigorous way is the one introduced in [BJ] whose main result is the following: if the underlying theory satisfies an appropriate "nuclearity condition" proposed in [BW] which restricts the number of local degrees of freedom in a physically sensible manner, then, in a generic way, KMS states ω_β can be approximated by states representable by a rigorous (local) version of the Gibbs formula $Z^{-1}\mathrm{e}^{-\beta H}$. For the correlation functions of these approximations, the relativistic spectrum condition can be applied and it has implications in terms of space-time analyticity properties which are of the type indicated above. Then, if the thermodynamic limit can be controlled sufficiently well, namely if long range boundary effects are negligible in a way which will be made precise below, we can establish similar analyticity properties of the correlation functions.

These analyticity properties are a remnant of the spectral properties of energy and momentum in the vacuum sector and may be regarded as an appropriate substitute to the relativistic spectrum condition for the case of thermal equilibrium states. Moreover, they suggest a specific Lorentz-covariant formulation of the KMS-condition. In the simplest case of equilibrium states in a minkowskian space $\mathbb{R}^d$ which are invariant under space-time translations and unsensitive to boundary effects, this condition reads as follows (V_+ and V_- denoting respectively the cones of future and past events in $\mathbb{R}^d$) :

Relativistic KMS-condition : *The state ω_β on $\mathcal{A}$ satisfies the relativistic KMS-condition at inverse temperature $\beta > 0$ iff there exists some positive timelike vector $e \in V_+$, $e^2 = 1$, such that for every pair of local operators $A, B \in \mathcal{A}$ there is a function F which is analytic in the tube $\mathcal{T}_{\beta e} = \left\{z \in \mathbb{C}^d : \mathrm{Im}\ z \in V_+ \cap (\beta e + V_-)\right\}$ and continuous at the boundary sets*[1] $\mathrm{Im}\ z = 0$, $\mathrm{Im}\ z = \beta e$ *with boundary values given by*

$$F(x) = \omega_\beta\left(A\alpha_x(B)\right), \qquad F(x + i\beta e) = \omega_\beta\left(\alpha_x(B)A\right)$$

for $x \in \mathbb{R}^d$.

In this condition, α_x denotes the automorphism of $\mathcal{A}$ which induces the space-time translations in $\mathbb{R}^d$, all space-time variables $x \in \mathbb{R}^d$ being treated on an equal footing. The condition thereby applies to all observers moving with constant velocity and allows them to identify ω_β as an equilibrium state which fixes a distinguished rest frame with time direction along e.

It may be instructive to illustrate this condition on the case where A, B are replaced respectively by field operators $\phi(f) = \int dx\, f(x)\phi(x)$ and $\phi(g) = \int dx\, g(x)\phi(x)$ so that $\omega_\beta(\phi(f)\phi(g)) = \int dx \int dy f(x)g(y)\mathcal{W}_\beta(x - y)$, where $\mathcal{W}_\beta(x-y)$ is the thermal two-point (Wightman) function of the field ϕ. The relativistic KMS-condition then amounts to the following analyticity property of $\mathcal{W}_\beta$: There exists a function W_β, analytic in the tube $\mathcal{T}_{\beta e}$, with boundary values given by $W_\beta(x) = \mathcal{W}_\beta(-x)$ and $W_\beta(x + i\beta e) = \mathcal{W}_\beta(x)$. Analogous analyticity properties can be stated for the n-point functions.

We believe that the relativistic KMS-condition covers a large area of equilibrium situations with the possible exception of phase-transition points, where boundary effects matter. Some variants of the condition, dealing with equilibrium states which are more sensitive to boundary effects or not invariant under (space) translations will be derived in the main text.

In the subsequent Sec. 2 we state our assumptions and recall the construction of thermal equilibrium states given in [BJ], which is based on an approximating family of local equilibrium states in the vacuum sector. We will then exhibit (in Sec. 3) analyticity properties of these approximations with respect to space-time translations which follow from the relativistic spectrum condition. Sec. 4 is devoted to a study of the

[1] The precise definition of continuity at the edges of tubes with conical bases is given in Appendix A.

influence of boundary effects on the analyticity properties of the limit states. These results are taken as an input in Sec. 5 for determining the primitive domains of analyticity of thermal correlation functions in various cases ; the pertinent mathematical facts used in this analysis are gathered in two appendices. It turns out that the correlation functions exhibit analyticity properties with respect to the space-time variables which corroborate in a rigorous way the conclusions of the previous heuristic discussion. The article concludes with a remark on a possible alternative approach towards the justification of a relativistic KMS-condition.

2. Local approximations of KMS-states

For analysing the consequences of the relativistic spectrum condition for the structure of KMS-states, it is necessary to know how these states are related to the vacuum sector. Such a link has been established in [BJ] for theories with a reasonable number of local degrees of freedom. Since our argument is based on that approach, we recall here the relevant assumptions and results of [BJ].

The setting used in this investigation is algebraic quantum field theory, where basic physical principles such as locality, relativistic covariance and stability of matter are expressed in terms of a family of algebras $\mathcal{A}(\mathcal{O})$ representing the observables of the underlying theory which are localized in the space-time region $\mathcal{O}$, (cf.[H]).

1. *(Locality)* There is a family of (concrete) C^*-algebras $\mathcal{A}(\mathcal{O})$ which are labelled by the open bounded space-time regions $\mathcal{O} \subset \mathbb{R}^d$ and act on a Hilbert space $\mathcal{H}$ representing the states in the vacuum sector. These algebras are subject to the condition of isotony,

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2 \ ; \tag{2.1}$$

so the assignment $\mathcal{O} \longrightarrow \mathcal{A}(\mathcal{O})$ defines a local net over $\mathbb{R}^d$. The C^*-inductive limit of this net is denoted by $\mathcal{A}$ and assumed to act irreducibly on $\mathcal{H}$. (We note that in this investigation we do not rely on any form of spacelike commutation relations.)

2. *(Covariance)* The space-time translations $x \in \mathbb{R}^d$ act on $\mathcal{A}$ by automorphisms α_x which are unitarily implemented on $\mathcal{H}$ by

$$\alpha_x(A) = \mathrm{e}^{iPx} A \, \mathrm{e}^{-iPx}, \qquad A \in \mathcal{A} \tag{2.2}$$

where P are the energy-momentum operators. The action of α_x on $\mathcal{A}$ is strongly continuous[2)] and covariant, i.e.

$$\alpha_x(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + x) \ , \tag{2.3}$$

where $\mathcal{O} + x$ is the region $\mathcal{O}$ shifted by x.

3. *(Stability)* The energy-momentum operators P satisfy the relativistic spectrum condition, sp $P \subset \bar{V}_+$, and there is a unique ground state $\Omega \in \mathcal{H}$, such that $P\Omega = 0$, which represents the vacuum. The vector Ω satisfies the Reeh-Schlieder property, i.e. the sets of vectors $\mathcal{A}(\mathcal{O})\Omega$ and[3)] $\mathcal{A}(\mathcal{O})'\Omega$ are dense in $\mathcal{H}$ for every $\mathcal{O}$.

4. *(Nuclearity)* Let $H = P^\mu \cdot e_\mu$ be the Hamiltonian in a given Lorentz system with time-direction fixed by the positive timelike vector $e \in V_+$, $e^2 = 1$; let $\beta > 0$ and let $\theta_{\beta,\mathcal{O}} : \mathcal{A}(\mathcal{O}) \longrightarrow \mathcal{H}$ be the linear mapping

$$\theta_{\beta,\mathcal{O}}(A) = \mathrm{e}^{-\beta H} A \, \Omega, \quad A \in \mathcal{A}(\mathcal{O}) \ . \tag{2.4}$$

This mapping is nuclear for every $\mathcal{O}$ and $\beta > 0$. This means that for fixed β and $\mathcal{O}$ there is a sequence of vectors $\Phi_i \in \mathcal{H}$ and of bounded linear functionals φ_i on $\mathcal{A}(\mathcal{O})$ such that $\sum_i \|\varphi_i\| \, \|\Phi_i\| < \infty$ and

$$\theta_{\beta,\mathcal{O}}(A) = \sum_i \varphi_i(A) \cdot \Phi_i, \qquad A \in \mathcal{A}(\mathcal{O}) \ . \tag{2.5}$$

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2) More specifically, there holds $\lim_{x \longrightarrow 0} \|\alpha_x(A) - A\| = 0$ for all $A \in \mathcal{A}$, where $\| \cdot \|$ is the operator norm.

3) Given a set $\mathcal{C}$ of bounded operators on $\mathcal{H}$, the symbol $\mathcal{C}'$ denotes the set of bounded operators commuting with all elements of $\mathcal{C}$.

The nuclear norm of $\theta_{\beta,\mathcal{O}}$ is defined by

$$\|\theta_{\beta,\mathcal{O}}\|_1 = \inf \sum_i \|\varphi_i\| \, \|\Phi_i\| \ , \tag{2.6}$$

where the infimum is to be taken with respect to all decompositions of $\theta_{\beta,\mathcal{O}}$ of the form (2.5). One then postulates that, for sufficiently small $\beta > 0$ and large balls $\mathcal{O}$ with radius r, the following bound holds:

$$\|\theta_{\beta,\mathcal{O}}\|_1 \leq \mathrm{e}^{cr^m \beta^{-n}} \ , \tag{2.7}$$

where c, m, n are positive numbers which do not depend on r and β.

As it was discussed in [BW], the nuclear norm $\|\theta_{\beta,\mathcal{O}}\|_1$ may be regarded as a substitute to the partition function of the theory in finite volume at temperature β^{-1}. The bound (2.7) may thus be understood as the requirement that the free energy of the system exhibits a regular behaviour in the thermodynamic limit and at large temperatures. In fact, one expects that in physically reasonable theories one can put m, n equal to the dimension of space (Stefan-Boltzmann law). It is note-worthy that the nuclearity condition holds in all Lorentz systems if it holds in some.

Having listed the relevant properties of the underlying theory in terms of the vacuum sector, let us now recall from [BJ] how one one can proceed from this sector to thermal equilibrium states at temperature $\beta^{-1} > 0$. Let $\mathcal{O}_r$ be the ball of radius r centered at the origin of $\mathbb{R}^d$ and let

$$\Lambda = (\mathcal{O}_r, \mathcal{O}_R) \, , \qquad r < R \tag{2.8}$$

be any pair of such balls. (In order not to overburden the notation we do not indicate the dependence of Λ on r and R). It follows from the above assumptions that, for every Λ, there exists a product vector $\eta_\Lambda \in \mathcal{H}$ with the factorization property

$$(\eta_\Lambda, AB'\eta_\Lambda) = (\Omega, A\Omega)\,(\Omega, B'\Omega) \quad \text{for} \quad A \in \mathcal{A}(\mathcal{O}_r) \ , \quad B' \in \mathcal{A}(\mathcal{O}_R)' \ . \tag{2.9}$$

The vector η_Λ is not completely fixed by this equation and it can be modified by applying any isometry in $\mathcal{A}(\mathcal{O}_r)' \wedge \mathcal{A}(\mathcal{O}_R)''$. Yet there is a canonical choice for it in the so-called natural cone $P^\natural$ affiliated with Ω and with the von Neumann algebra $\mathcal{A}(\mathcal{O}_r)'' \vee \mathcal{A}(\mathcal{O}_R)'$. The latter fact was used in some arguments in [BJ], but it is of no relevance in the sequel. From the physical viewpoint, the vector η_Λ describes a state in which no correlations exist between the finite region $\mathcal{O}_r$ and the causal complement of $\mathcal{O}_R$. The ambiguities on its determination correspond to the variety of possible boundary conditions for systems in a finite volume, but the general arguments developed below are in fact independent of this variety of situations. It is only in the passage to the thermodynamic limit (done in Sec. 4) that we shall be led to exhibit (in a certain sense) the role of the boundary conditions in the derivation of the final result.

With the help of the vectors η_Λ one constructs subspaces $\mathcal{H}(\Lambda) \doteq \overline{\mathcal{A}(\mathcal{O}_r)\,\eta_\Lambda} \subset \mathcal{H}$, where the bar denotes closure. The vectors in $\mathcal{H}(\Lambda)$ describe excitations of the vacuum which are strictly localized in $\mathcal{O}_R$ and exhaust all partial states on $\mathcal{A}(\mathcal{O}_r)$. Thus the spaces $\mathcal{H}(\Lambda)$ replace in the present setting the state spaces which arise in finite volume theories. Let us denote by $E(\Lambda)$ the orthogonal projection onto $\mathcal{H}(\Lambda)$; it has been shown in [BJ] that $E(\Lambda)$ converges to 1 if the radii r, R of the underlying balls approach infinity in an appropriate manner. Moreover, there holds the following crucial lemma [BJ].

<u>Lemma 2.1</u> : *The operators* $E(\Lambda)\mathrm{e}^{-\beta H}$ *are of trace class for every* $\Lambda = (\mathcal{O}_r, \mathcal{O}_R)$, $\beta > 0$ *and there holds*

$$\mathrm{Tr}\left|E(\Lambda)\mathrm{e}^{-\beta H}\right| \leq \mathrm{e}^{cR^m\beta^{-n}} \ . \tag{2.10}$$

Here c, m, n *are the constants appearing in the nuclearity condition and* R, β^{-1} *have to be sufficiently large.*

This result allows one to define on $\mathcal{H}$ the density matrices

$$\rho_{\beta,\Lambda} \doteq \frac{1}{Z_{\beta,\Lambda}} \cdot E(\Lambda)\mathrm{e}^{-\beta H} E(\Lambda) \ , \tag{2.11}$$

where $Z_{\beta,\Lambda}$ is a normalization factor entailing that Tr $\rho_{\beta,\Lambda} = 1$. The corresponding states $\omega_{\beta,\Lambda}$, given by

$$\omega_{\beta,\Lambda}(A) = \mathrm{Tr}\ \rho_{\beta,\Lambda} A \ , \qquad A \in \mathcal{A} \ , \tag{2.12}$$

satisfy a local version of the KMS-condition [BJ] which indicates that these states are close to thermal equilibrium in the region $\mathcal{O}_r$. Moreover, by making use of the fact that $E(\Lambda)$ tends to 1 in the limit of large r, R it has been shown in [BJ] that the states $\omega_{\beta,\Lambda}$ have limit points

$$\omega_\beta = \lim_i \omega_{\beta,\Lambda_i} \tag{2.13}$$

which satisfy the KMS-condition at temperature β^{-1} with respect to the time translations in the given Lorentz frame. The states ω_β thus describe systems at thermal equilibrium [HKTP], [PW].

In the subsequent section we will analyse the properties of the approximating states $\omega_{\beta,\Lambda}$ more closely in order to gain further information on their limits ω_β.

3. Implications of the relativistic spectrum condition

We will now study the consequences of the relativistic spectrum condition for the properties of the approximating states $\omega_{\beta,\Lambda}$. In a first step we will show that the correlation functions of these states satisfy a local version of the relativistic KMS-condition, as outlined in the Introduction.

Lemma 3.1. : *Let $\Lambda = (\mathcal{O}_r, \mathcal{O}_R)$, $\beta > 0$ and let $\omega_{\beta,\Lambda}$ be the corresponding state defined in relation (2.12). For any $\rho < r$ and any pair of operators $A, B \in \mathcal{A}(\mathcal{O}_\rho)$ there exists a function $F(z)$ which is analytic in the tube $\mathcal{T}_{\beta e} = \{z \in \mathbb{C}^d : \mathrm{Im}\ z \in V_+ \cap (\beta e + V_-)\}$ and continuous at the boundaries* Im $z = 0$, Im $z = \beta e$ *with boundary values given by*

$$F(x) = \omega_{\beta,\Lambda}(A\ \alpha_x(B)) \qquad \text{and} \qquad F(x + i\beta e) = \omega_{\beta,\Lambda}(\alpha_x(B)A)$$

for $|x| < r - \rho$.

Proof : The relativistic spectrum condition implies that the operator functions $z \longrightarrow \mathrm{e}^{izP}$ and $z \longrightarrow \mathrm{e}^{-izP-\beta H}$ are analytic on $\mathcal{T}_{\beta\cdot e}$ and continuous on $\overline{\mathcal{T}_{\beta\cdot e}}$ in the strong operator topology. Moreover, since[4] $\left|\mathrm{e}^{iwP}\right| \leq \mathrm{e}^{-(\mathrm{Im} w_0\ -\ |\mathrm{Im}\mathbf{w}|)H}$ it follows from Lemma 2.1 that the function

$$z \longrightarrow F(z) = \frac{1}{Z_{\beta,\Lambda}} \mathrm{Tr}\ E(\Lambda) \mathrm{e}^{izP} B\ \mathrm{e}^{-izP-\beta H} E(\Lambda) A$$

is well-defined on $\overline{\mathcal{T}_{\beta\cdot e}}$. If we denote by X the operator under the trace, there holds for $z \in \overline{\mathcal{T}_{\alpha\cdot e}}$, $0 < \alpha < \beta$, the following operator inequality:

$$(X^*X)^{1/2} \leq \|B\| \cdot \left|\mathrm{e}^{-(\beta-\alpha)H} E(\Lambda) A\right| \ ,$$

and similarly for $z \in \overline{\mathcal{T}_{\alpha\cdot e}} + i(\beta - \alpha)e$, $0 < \alpha < \beta$,

$$(XX^*)^{1/2} \leq \|A\| \|B\| \cdot \left|\mathrm{e}^{-(\beta-\alpha)H} E(\Lambda)\right| \ .$$

Thus, because of the uniformity of these bounds in z and of Lemma 2.1 one may conclude [K, Chap.7] that the function F is analytic in $\mathcal{T}_{\beta e}$ and continuous at all boundary points Im $z = 0$ and Im $z = i\beta e$.

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[4] In the following, we introduce proper coordinates and denote the time and space components of (complex) d-vectors w by $(w_0, \mathbf{w})$.

The calculation of the boundary values of F is accomplished by making use of the cyclicity of the trace and of the fact that the projection $E(\Lambda)$ commutes with all operators in $\mathcal{A}(\mathcal{O}_r)$. In fact, for $B \in \mathcal{A}(\mathcal{O}_\rho)$ and $|x| < r - \rho$ there holds

$$\begin{aligned} F(x) &= \frac{1}{Z_{\beta,\Lambda}} \text{Tr } E(\Lambda)\alpha_x(B)\mathrm{e}^{-\beta H} E(\Lambda) A \\ &= \frac{1}{Z_{\beta,\Lambda}} \text{Tr } E(\Lambda)\mathrm{e}^{-\beta H} E(\Lambda) A \; \alpha_x(B) = \omega_{\beta,\Lambda}(A \; \alpha_x(B)) \; , \end{aligned}$$

and similarly

$$\begin{aligned} F(x + i\beta e) &= \frac{1}{Z_{\beta,\Lambda}} \text{Tr } E(\Lambda)\mathrm{e}^{-\beta H} \alpha_x(B) E(\Lambda) A \\ &= \frac{1}{Z_{\beta,\Lambda}} \text{Tr } E(\Lambda)\mathrm{e}^{-\beta H} E(\Lambda) \alpha_x(B) A = \omega_{\beta,\Lambda}(\alpha_x(B) A) \; . \end{aligned}$$

This completes the proof of the statement.

This result is a first indication that the thermal equilibrium states ω_β arising as limit points of the family of states $\omega_{\beta,\Lambda}$ for large r, R satisfy the relativistic KMS-condition, since the restrictions on the localization properties of the operators A, B and the size of $|x|$ disappear in this limit. We will discuss in the next section conditions on the approximating states $\omega_{\beta,\Lambda}$ which allow one to establish this fact rigorously.

Before we enter into that discussion, let us complement the preceding lemma by a result of a similar nature. We shall make use of the fact that any positive linear functional on $\mathcal{A}$ induces, by the GNS-construction, a representation of $\mathcal{A}$ on some Hilbert space. The functionals of interest here are $\omega_{\beta,\Lambda}$; so there exists a Hilbert space $\mathcal{H}_{\beta,\Lambda}$, a distinguished unit vector $\Omega_{\beta,\Lambda} \in \mathcal{H}_{\beta,\Lambda}$, and a homomorphism $\pi_{\beta,\Lambda}$ of $\mathcal{A}$ into the algebra of bounded operators on $\mathcal{H}_{\beta,\Lambda}$ such that

$$\omega_{\beta,\Lambda}(A) = (\Omega_{\beta,\Lambda}, \pi_{\beta,\Lambda}(A)\Omega_{\beta,\Lambda}) \quad \text{for} \quad A \in \mathcal{A} \; . \tag{3.1}$$

In the subsequent lemma we make use of a more concrete realization of this representation.

<u>Lemma 3.2</u> : *Let* $\Lambda = (\mathcal{O}_r, \mathcal{O}_R)$, $\beta > 0$ *and* $\rho < r$. *Then the vector valued functions*

$$x \longrightarrow \pi_{\beta,\Lambda}(\alpha_x(A))\,\Omega_{\beta,\Lambda}, \qquad A \in \mathcal{A}(\mathcal{O}_\rho)$$

can be continued analytically from the real ball $|x| < r - \rho$ *into the domain* $\mathcal{T}_{\frac{\beta}{2}e}$, *and they are weakly continuous at the boundary sets* $\text{Im } z = 0$, $|\text{Re } z| < r - \rho$, *and* $\text{Im } z = \beta/2 \cdot e$, $\text{Re } z \in \mathbb{R}^d$.

<u>Remark</u> : These functions are also strongly continuous at the boundary points if $x \longrightarrow \alpha_x(A)$ is differentiable in norm.

<u>Proof</u> : We begin by recalling the standard representation of $\mathcal{A}$ induced by density matrices in the vacuum sector $\mathcal{H}$. Let $\mathcal{K}$ be the space of Hilbert-Schmidt operators on $\mathcal{H}$, equipped with the scalar product

$$\langle K_1 \mid K_2 \rangle = \text{Tr } K_1^* K_2 \quad \text{for} \quad K_1, K_2 \in \mathcal{K} \; ,$$

and let π be the representation of $\mathcal{A}$ which acts on $\mathcal{K}$ by left multiplication,

$$\pi(A)K = AK \quad \text{for} \quad A \in \mathcal{A}, \quad K \in \mathcal{K} \; .$$

Then there holds

$$\langle K_1 \mid \pi(A) K_2 \rangle = \text{Tr } K_1^* A \; K_2 = \text{Tr } K_2 K_1^* A \; .$$

Within this setting one can identify the vector $\Omega_{\beta,\Lambda}$ with the trace class (hence Hilbert-Schmidt class) operator $Z_{\beta,\Lambda}^{-1/2} E(\Lambda)\mathrm{e}^{-\beta/2 \; H}$, the Hilbert space $\mathcal{H}_{\beta,\Lambda}$ with the subspace $\overline{\mathcal{A} \; E(\Lambda)\mathrm{e}^{-\beta/2 \; H}}$ of $\mathcal{K}$, and the representation $\pi_{\beta,\Lambda}$ with the restriction of π to this subspace. So, for the proof of the statement, we have to study the properties of the function (taking its values in $\mathcal{K}$)

$$x \longrightarrow \alpha_x(A)E(\Lambda)\mathrm{e}^{-\beta/2\ H}, \qquad A \in \mathcal{A}(\mathcal{O}_\rho)$$

which coincides for $|x| < r - \rho$ with

$$x \longrightarrow E(\Lambda)\alpha_x(A)\mathrm{e}^{-\beta/2\ H}\ .$$

The latter function can be extended (as a function with values in $\mathcal{K}$) to the domain $\mathcal{T}_{\beta/2\cdot e}$, by setting

$$z \longrightarrow K(z) = E(\Lambda)\mathrm{e}^{izP}A\ \mathrm{e}^{-izP-\beta/2\ H}\ .$$

In view of the arguments in the preceding lemma, it is apparent that $K(z)$ is analytic on $\mathcal{T}_{\frac{\beta}{2}e}$ and continuous at the boundary set Im $z = i\beta/2\cdot e$ of this domain. It requires more work to prove that it is also continuous at the real boundary points Im $z = 0$, $|\text{Re } z| < r - \rho$.

The spectrum condition implies that $z \longrightarrow K(z)$, regarded as an element of the space of bounded operator-valued functions, is continuous at the real boundary points in the strong operator topology. Hence if $K_0 \in \mathcal{K}$ is any operator of *finite rank*, the function

$$z \longrightarrow \langle K_0 \mid K(z)\rangle = \text{Tr } K_0^* E(\Lambda)\mathrm{e}^{izP}A\ \mathrm{e}^{-izP-\beta/2\ H}$$

is continuous at the real boundary set. But the finite rank operators are dense in $\mathcal{K}$; so the proof is complete if we can show that $K(z)$ is uniformly bounded in $\mathcal{K}$ in a neighbourhood of the boundary points. To verify this, we pick any real x, $|x| < r - \rho$, and any $y \in V_+$ with $y_0 + |\underline{y}| = \varepsilon(x) = \min(\beta/2, r - \rho - |x|)$. Keeping these data fixed for a moment, we consider for $w \in \mathbb{C}$ with $0 \le \text{Re } w \le 1$, $|\text{Im } w| \le 1$ the function

$$w \longrightarrow K_{x,y}(w) = K(x + iwy)\ .$$

This function is analytic as a function with values in $\mathcal{K}$ in the interior of its definition domain, as a consequence of the analyticity properties of $z \longrightarrow K(z)$. By making use of the restrictions on x, y, of the spectrum condition and of Lemma 2.1, we get for $0 < \text{Re } w \le 1$ the estimate

$$\begin{aligned}\text{Tr}\,|K_{x,y}(w)| &\le \|A\| \cdot \text{Tr}\left|E(\Lambda)\mathrm{e}^{-\text{Re } w\left(y_0 - |\underline{y}|\right)H}\right| \\ &\le \|A\| \cdot \mathrm{e}^{cR^m\left(\text{Re } w\left(y_0 - |\underline{y}|\right)\right)^{-n}}\ .\end{aligned}$$

On the other hand, if Re $w = 0$, $|\text{Im } w| \le 1$ we get, by using again the fact that $E(\Lambda)$ commutes with the elements of $\mathcal{A}(\mathcal{O}_r)$,

$$\text{Tr}\,|K_{x,y}(w)| \le \|A\| \cdot \text{Tr}\left|E(\Lambda)\mathrm{e}^{-\beta/2\ H}\right|\ .$$

These crude a priori bounds suffice to complete the proof by an argument of Phragmén-Lindelöf type. Consider the region

$$D = \left\{w \in \mathbb{C}\ : 0 \le \text{Re } w \le (1 - |\text{Im } w|)^2\right\}$$

and the auxiliary function on D given by

$$a(w) = \exp\left(-\mathrm{e}^{1/(1+iw)} - \mathrm{e}^{1/(1-iw)} + 2e\right)\ .$$

This function is analytic on the interior of D, continuous at the boundary, and $a(\pm i) = 0$. Moreover, there holds for real w

$$a(w) \ge 1, \qquad 0 \le w \le 1\ .$$

We multiply $K_{x,y}$ with this function in order to suppress a possibly singular behaviour of $K_{x,y}$ in the neighbourhood of the imaginary axis. By using the preceding bounds on $\text{Tr}\,|K_{x,y}(w)|$, we obtain for all x, y as specified above the following estimate on the boundary of D

$$\sup_{w\in\partial D} |a(w)|\text{Tr}\,|K_{x,y}(w)| \le M\left(y_0 - |\underline{y}|\right)\ ,$$

where $u \longrightarrow M(u)$, $u > 0$ is continuous and monotonically decreasing. We pick now any $K_0 \in \mathcal{K}$ of finite rank and consider the function

$$w \longrightarrow \langle K_0 \,|a(w)K_{x,y}(w)\,\rangle$$

which is continuous in D and analytic in the interior. By setting $\|K\|_2 = \langle K \mid K\rangle^{1/2}$, $K \in \mathcal{K}$, and taking into account that $\|K\|_2 \leq \mathrm{Tr}|K|$, it follows from the maximum modulus principle and from the preceding estimate that, for $w \in D$

$$|\langle K_0 \,|a(w)K_{x,y}(w)\,\rangle| \leq M\left(y_0 - |\underline{y}|\right) \cdot \|K_0\|_2 \ .$$

Since the operators K_0 of finite rank are dense in $\mathcal{K}$, this implies in particular that for real w, $0 \leq w \leq 1$,

$$\|K(x+iwy)\|_2 = \|K_{x,y}(w)\|_2 \leq \frac{1}{a(w)} M\left(y_0 - |\underline{y}|\right) \leq M\left(y_0 - |\underline{y}|\right) \ .$$

By bearing in mind the special choice of y, we conclude from this estimate that

$$\|K(x+iy)\|_2 \leq M\left(\varepsilon(x)\frac{y_0 - |\underline{y}|}{y_0 + |\underline{y}|}\right)$$

for all x, $|x| < r - \rho$, and $y \in V_+$, $y_0 + |\underline{y}| \leq \varepsilon(x) = \min\left(\beta/2,\ r - \rho - |x|\right)$. This completes the proof of the weak continuity of $z \longrightarrow K(z)$ at the real boundary points.

We conclude this section with the remark that, for all $\rho < r$, the vector $\Omega_{\beta,\Lambda}$ is separating for the algebra $\pi_{\beta,\Lambda}\left(\mathcal{A}\left(\mathcal{O}_\rho\right)\right)^-$. This can most easily be seen in the standard representation used in the proof of the preceding lemma: if $Z \cdot \Omega_{\beta,\Lambda} = 0$ for some $Z \in \pi_{\beta,\Lambda}\left(\mathcal{A}\left(\mathcal{O}_\rho\right)\right)^-$, then $Z\ E(\Lambda)\mathrm{e}^{-\beta/2\ H} = 0$ and consequently $Z\ E(\Lambda) = 0$ since $\mathrm{e}^{-\beta/2\ H}$ is invertible. Moreover, $[\alpha_x(Z), E(\Lambda)] = 0$ if $|x| < r - \rho$, hence $Z = 0$ by a theorem of Schlieder [Sch].

4. The role of boundary effects and the thermodynamic limit

After having seen how analyticity properties of the correlation functions affiliated with the approximating states $\omega_{\beta,\Lambda}$ result from the relativistic spectrum condition, we now turn to the formulation of conditions which imply that these properties persist in the limit states ω_β.

Our starting point is the heuristic idea that the restrictions of ω_β and $\omega_{\beta,\Lambda}$ to any given local algebra $\mathcal{A}(\mathcal{O})$ should practically look alike if the regions $\mathcal{O}_r$, $\mathcal{O}_R$ are sufficiently large compared to $\mathcal{O}$. Any differences between the restricted (partial) states should be due to boundary effects in the state $\omega_{\beta,\Lambda}$ which are localized in a layer $\partial\Lambda$ in the spacelike (causal) complement of $\mathcal{O}_r$, where the equilibrium situation in $\mathcal{O}_r$ is decoupled from the exterior vacuum (cf. the definition of $\mathcal{H}(\Lambda)$ and relation (2.9)).

One may expect that these boundary effects are removable in generic cases[5)] by operations in the layer $\partial\Lambda$ and hence *a fortiori* by operations in the causal complement of $\mathcal{O}$. The latter operations can be described in a relativistic theory by operators T which commute with all observables in $\mathcal{O}$.

To substantiate this idea, let us assume that $\omega_\beta \restriction \mathcal{A}(\mathcal{O})$ (namely, the restriction of ω_β to $\mathcal{A}(\mathcal{O})$) is normal with respect to $\omega_{\beta,\Lambda} \restriction \mathcal{A}(\mathcal{O})$. By taking into account the remark at the end of the preceding section, it follows [D] that $\omega_\beta \restriction \mathcal{A}(\mathcal{O})$ is a vector state in the GNS-representation $(\pi_{\beta,\Lambda},\ \mathcal{H}_{\beta,\Lambda},\ \Omega_{\beta,\Lambda})$ induced by $\omega_{\beta,\Lambda} \restriction \mathcal{A}(\mathcal{O})$, i.e. there is a vector $\Omega_\beta \in \mathcal{H}_{\beta,\Lambda}$ such that $\omega_\beta(A) = (\Omega_\beta, \pi_{\beta,\Lambda}(A)\Omega_\beta)$ for $A \in \mathcal{A}(\mathcal{O})$. It is therefore possible [S, Chap.2.7] to introduce a linear operator T in $\mathcal{H}_{\beta,\Lambda}$, by setting

$$T \cdot \pi_{\beta,\Lambda}(A)\Omega_{\beta,\Lambda} = \pi_{\beta,\Lambda}(A)\Omega_\beta \qquad \text{for} \qquad A \in \mathcal{A}(\mathcal{O}). \tag{4.1}$$

This operator is well defined on the dense domain $\pi_{\beta,\Lambda}(\mathcal{A}(\mathcal{O}))\Omega_{\beta,\Lambda}$, as it follows likewise from the remark at the end of the preceding section, and it is also apparent from (4.1) that this operator commutes on its

5) Boundary effects play a prominent role at phase transition points, where our arguments are less conclusive.

domain with the elements of the algebra $\pi_{\beta,\Lambda}(\mathcal{A}(\mathcal{O}))$. Hence T has the heuristically expected properties of an operator which removes boundary effects.

We propose to classify the strength of boundary effects by the continuity properties of T. Depending on the nature of these effects, T may be a bounded operator, a closable unbounded operator or even, in some instances, a non-closable operator. In the following criterion we distinguish two generic cases.

Criterion : *The state ω_β is said to be strongly resistant to boundary effects if for each bounded region $\mathcal{O}$ there is a $\Lambda = (\mathcal{O}_r, \mathcal{O}_R)$ and a bounded operator $T \in \pi_{\beta,\Lambda}(\mathcal{A}(\mathcal{O}))'$ such that*

$$\omega_\beta(A) = (T\Omega_{\beta,\Lambda}, \pi_{\beta,\Lambda}(A)T\Omega_{\beta,\Lambda}) \qquad \text{for} \qquad A \in \mathcal{A}(\mathcal{O}) \ .$$

It is said to be resistant if T is a closable unbounded operator which commutes on its domain $\pi_{\beta,\Lambda}(\mathcal{A}(\mathcal{O}))\Omega_{\beta,\Lambda}$ with all the elements of $\pi_{\beta,\Lambda}(\mathcal{A}(\mathcal{O}))$ and satisfies $\|T^*T\Omega_{\beta,\Lambda}\| < \infty$.

It is of interest here that these conditions can be reformulated in terms of the underlying functionals $\omega_{\beta,\Lambda}$ and ω_β, thereby permitting a different physical interpretation.

Lemma 4.1 : *The state ω_β is strongly resistant to boundary effects iff for each bounded region $\mathcal{O}$ there is a $\Lambda = (\mathcal{O}_r, \mathcal{O}_R)$ and a positive constant c such that*

$$\omega_\beta\,(A^*A) \leq c \cdot \omega_{\beta,\Lambda}\,(A^*A) \ , \qquad A \in \mathcal{A}(\mathcal{O}) \ . \tag{4.2}$$

It is resistant iff

$$|\omega_\beta(A)|^2 \leq c \cdot \omega_{\beta,\Lambda}\,(A^*A)\,, \qquad A \in \mathcal{A}(\mathcal{O}) \ . \tag{4.3}$$

Proof : We begin by proving the second part of the statement: it is a direct consequence of condition (4.3) that the state $\omega_\beta \restriction \mathcal{A}(\mathcal{O})$ is normal with respect to $\omega_{\beta,\Lambda} \restriction \mathcal{A}(\mathcal{O})$. Hence, as explained before, there is a vector $\Omega_\beta \in \mathcal{H}_{\beta,\Lambda}$ inducing the state $\omega_\beta \restriction \mathcal{A}(\mathcal{O})$ in the representation $(\pi_{\beta,\Lambda}, \mathcal{H}_{\beta,\Lambda}, \Omega_{\beta,\Lambda})$, and a linear operator T such that relation (4.1) holds. It also follows from (4.3) that for any $A, B \in \mathcal{A}(\mathcal{O})$

$$\begin{gathered}|(\pi_{\beta,\Lambda}(B)\Omega_\beta, \pi_{\beta,\Lambda}(A)\Omega_\beta)|^2 = |\omega_\beta\,(B^*A)|^2 \leq c\ \omega_{\beta,\Lambda}\,(A^*BB^*A) \\ \leq c\,\|BB^*\|\,\omega_{\beta,\Lambda}\,(A^*A) = c\|B\|^2\,\|\pi_{\beta,\Lambda}(A)\Omega_{\beta,\Lambda}\|^2 \ ,\end{gathered}$$

and consequently

$$|(\pi_{\beta,\Lambda}(B)\Omega_\beta, T\ \pi_{\beta,\Lambda}(A)\Omega_{\beta,\Lambda})| \leq c^{1/2}\|B\|\,\|\pi_{\beta,\Lambda}(A)\Omega_{\beta,\Lambda}\| \ .$$

This inequality shows that the adjoint T^* is defined on the range of T, hence T is closable. By setting $B = 1$ in the latter inequality, it is also clear that $\|T^*T\ \Omega_{\beta,\Lambda}\| = \|T^*\Omega_\beta\| \leq c^{1/2}$, which proves that T has all the properties required in our criterion for resistance. Conversely, if T is such an operator, there holds

$$\begin{gathered}|\omega_\beta(A)|^2 = |(T\Omega_{\beta,\Lambda}, T\ \pi_{\beta,\Lambda}(A)\Omega_{\beta,\Lambda})|^2 \\ \leq \|T^*T\ \Omega_{\beta,\Lambda}\|^2\,\|\pi_{\beta,\Lambda}(A)\Omega_{\beta,\Lambda}\|^2 = \|T^*T\ \Omega_{\beta,\Lambda}\|^2\,\omega_{\beta,\Lambda}\,(A^*A) \ ,\end{gathered}$$

which completes our proof of the second part of the lemma. The first part concerning strong resistance is an immediate consequence of this result.

The first condition in the preceding lemma says that $\omega_\beta \restriction \mathcal{A}(\mathcal{O})$ describes a subensemble of $\omega_{\beta,\Lambda} \restriction \mathcal{A}(\mathcal{O})$. Note that the latter state is faithful and, as such, would dominate any other state in the sense of relation (4.2) in locally finite theories (spin systems). Since the nuclearity condition characterizes theories which are in some specific sense close to being locally finite, we believe that condition (4.2) is also meaningful in the present field-theoretical setting. The second less stringent condition amounts to the requirement that the difference between the expectation values of any observable $A \in \mathcal{A}(\mathcal{O})$ in the states ω_β and $\omega_{\beta,\Lambda}$, respectively, is dominated by the fluctuations of A in the state $\omega_{\beta,\Lambda}$. By bearing in mind the physical situation described by ω_β and $\omega_{\beta,\Lambda}$, this seems to be a rather mild requirement.

Let us now turn to the discussion of the consequences of our criterion for the analyticity properties of the correlation functions induced by ω_β. These functions are, according to relations (4.2) and (4.3), respectively, and by Lemma 3.1, dominated on the *reals* by boundary values of analytic functions. Because of the underlying Hilbert space structure, it is possible to carry over the analyticity properties of these upper bounds to the correlation functions.

Proposition 4.2 : *Let ω_β be a KMS-state on $\mathcal{A}$ and let $A, B \in \mathcal{A}$ be local operators with corresponding correlation function*

$$(x_1, x_2) \longrightarrow \omega_\beta \left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) \ .$$

If ω_β is strongly resistant to boundary effects, this correlation function has an analytic continuation into the domain $-\mathcal{T}_{\frac{\beta}{2}e} \times \mathcal{T}_{\frac{\beta}{2}e}$, and its boundary values on $\mathbb{R}^d \times \mathbb{R}^d$ and $\left(\mathbb{R}^d - i\frac{\beta}{2}e\right) \times \left(\mathbb{R}^d + i\frac{\beta}{2}e\right)$ are continuous. If ω_β is (merely) resistant to boundary effects, the correlation function has an analytic continuation with respect to the variable x_1 (resp. x_2) into the domain $-\mathcal{T}_{\frac{\beta}{2}e} \times \mathbb{R}^d$ (resp. $\mathbb{R}^d \times \mathcal{T}_{\frac{\beta}{2}e}$) with continuous boundary values on $\mathbb{R}^d \times \mathbb{R}^d$ and $\left(\mathbb{R}^d - i\frac{\beta}{2}e\right) \times \mathbb{R}^d$ (resp. $\mathbb{R}^d \times \mathbb{R}^d$ and $\mathbb{R}^d \times \left(\mathbb{R}^d + i\frac{\beta}{2}e\right)$).

Proof : For fixed local operators $A, B \in \mathcal{A}$ and translations x_1, x_2 varying within any given bounded subset $\mathcal{R} \subset \mathbb{R}^d$, there exists a bounded region $\mathcal{O} \subset \mathbb{R}^d$ such that $\alpha_{x_1}(A), \alpha_{x_2}(B) \in \mathcal{A}(\mathcal{O})$. By assumption, there is then a $\Lambda = (\mathcal{O}_r, \mathcal{O}_R)$ and a corresponding operator T such that

$$\begin{aligned}\omega_\beta \left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) &= \left(T\ \Omega_{\beta,\Lambda}, \pi_{\beta,\Lambda}\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) T\ \Omega_{\beta,\Lambda}\right) \\ &= \left(T\ \pi_{\beta,\Lambda}\left(\alpha_{x_1}(A^*)\right)\Omega_{\beta,\Lambda},\ T\ \pi_{\beta,\Lambda}\left(\alpha_{x_2}(B)\right)\Omega_{\beta,\Lambda}\right) \ ,\end{aligned}$$

where we made use of the fact that T commutes with the elements of $\pi_{\beta,\Lambda}(\mathcal{A}(\mathcal{O}))$. If ω_β is strongly resistant to boundary effects, i.e. if T is bounded, the stated analyticity and continuity properties of the correlation function follow from Lemma 3.2. (We note in this context that the *joint* continuity at the boundaries in all variables is a consequence of the continuity of the correlation functions and of the maximum modulus principle, cf. Appendix A.) If ω_β is only resistant to boundary effects, the preceding representation of the correlation function is not adequate since the analytic continuation of the underlying vector-valued functions might not remain in the domain of the unbounded operator T. But we have

$$\begin{aligned}\omega_\beta \left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) &= \left(T\ \Omega_{\beta,\Lambda},\ T\ \pi_{\beta,\Lambda}\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right)\Omega_{\beta,\Lambda}\right) \\ &= \left(\pi_{\beta,\Lambda}\left(\alpha_{x_1}(A^*)\right) \cdot T^*T\ \Omega_{\beta,\Lambda},\ \pi_{\beta,\Lambda}\left(\alpha_{x_2}(B)\right)\Omega_{\beta,\Lambda}\right) \ ,\end{aligned}$$

and similarly

$$\omega_\beta \left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) = \left(\pi_{\beta,\Lambda}\left(\alpha_{x_1}(A^*)\right)\Omega_{\beta,\Lambda},\ \pi_{\beta,\Lambda}\left(\alpha_{x_2}(B)\right) T^*T\ \Omega_{\beta,\Lambda}\right) \ .$$

Lemma 3.2, when applied to the appropriate expression, now yields the stated analyticity and continuity properties in x_1, resp. x_2.

It is apparent from this argument that the conclusions hold under much weaker conditions. For example, the operator T in the preceding proof could in principle depend on the choice of the operators $A, B \in \mathcal{A}$. One may therefore expect that the correlation functions exhibit analyticity properties of the type established in this proposition also under more general circumstances.

5. Analyticity domains of correlation functions and the relativistic KMS-condition

In the preceding section, we have exhibited certain specific analyticity properties of thermal correlation functions in relativistic quantum field theory. By using this initial information, we will now determine the full primitive domains of analyticity of these functions by applying geometrical techniques of analytic completion, as described e.g. in [BEGS]. Since these techniques may be not so well-known, we have added an appendix where the relevant notions and results are summarized.

The most general (or weakest) formulation of the expected regularity properties of correlation functions affiliated with thermal equilibrium states ω_β in relativistic quantum field theory is encoded in the following analyticity properties.

a) *(KMS-condition)* Let $e \in V_+$, $e^2 = 1$ be the time direction of the privileged Lorentz frame fixed by ω_β and let $A, B \in \mathcal{A}$ be any pair of local operators. Then there exists an analytic function F_0 in the flat tube $\mathcal{T}_{\mathcal{B}_0} \subset \mathbb{C}^d \times \mathbb{C}^d$ with basis[6)]

$$\mathcal{B}_0 = \{(y_1, y_2) : y_1 = t_1 e,\ y_2 = t_2 e,\ 0 < t_2 - t_1 < \beta\}\ , \tag{5.1}$$

which is continuous on the closure of $\mathcal{T}_{\mathcal{B}_0}$ and has, for $x_1, x_2 \in \mathbb{R}^d$, the boundary values

$$\begin{aligned} F_0\left(x_1, x_2\right) &= \omega_\beta\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) \\ F_0\left(x_1 - i\frac{\beta}{2}e,\ x_2 + i\frac{\beta}{2}e\right) &= \omega_\beta\left(\alpha_{x_2}(B)\alpha_{x_1}(A)\right)\ . \end{aligned} \tag{5.2}$$

(We note that F_0 is invariant under time-translations, $F_0\left(x_1 + te,\ x_2 + te\right) = F_0\left(x_1, x_2\right)$, as a consequence of this assumption, [BR].)

b) *(Stability)* For any positive timelike unit vector $f \in V_+$, there exists a function F_f which is analytic in the flat tube $\mathcal{T}_{\mathcal{B}_f}$ with basis

$$\mathcal{B}_f = \{(y_1, y_2) : y_1 = 0,\ y_2 = tf\ ,\quad 0 < t < t_f\}\ , \tag{5.3}$$

(t_f being a positive number depending of f), and has continuous boundary values on the reals $\mathbb{R}^d \times \mathbb{R}^d$ given by

$$F_f\left(x_1, x_2\right) = \omega_\beta\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right)\ . \tag{5.4}$$

(Note that by replacing A, B by B^*, A^* and taking complex conjugates one obtains another function $F_f^\dagger$ which is analytic in the flat tube

$$\mathcal{T}_{\mathcal{B}_f^\dagger} = \{(z_1, z_2) : y_1 = -tf,\ 0 < t < t_f,\ y_2 = 0\} \tag{5.5}$$

and coincides with F_f on the reals $\mathbb{R}^d \times \mathbb{R}^d$.)

Condition b) expresses the type of stability requirement presented in the Introduction (finiteness of local energy in all Lorentz frames). It is of course fulfilled when the state ω_β satisfies the criteria of the preceding section (cf. Proposition 4.2). We shall now prove:

Proposition 5.1 : *Under the previous conditions a) and b) there exists an analytic function F in a convex open tube $\mathcal{T}_\mathcal{B} \subset \mathbb{C}^d \times \mathbb{C}^d$ with basis $\mathcal{B}$ which extends the correlation function $x_1, x_2 \longrightarrow \omega_\beta\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right)$ for any given local $A, B \in \mathcal{A}$. More specifically:*

i) The basis $\mathcal{B}$ is a neighbourhood in $\mathbb{R}^d \times \mathbb{R}^d$ of the basis $\mathcal{B}_0$ of condition a) and is invariant under time translations $(y_1, y_2) \longrightarrow (y_1 + te,\ y_2 + te)\,,\ t \in \mathbb{R}$. Moreover, F satisfies in its domain the invariance condition

$$F\left(z_1 + ue,\ z_2 + ue\right) = F\left(z_1, z_2\right), \qquad u \in \mathbb{C}.$$

ii) At every real boundary point (x_1, x_2) of $\mathcal{T}_\mathcal{B}$, the profile $\Lambda_{(0,0)}$ of $\mathcal{T}_\mathcal{B}$ (corresponding to the conical boundary point $(0,0)$ of $\mathcal{B}$) is the cone

$$\Lambda_{(0,0)} = \{(y_1, y_2) : y_1 \in V_- + te,\ y_2 \in V_+ + te,\ t \in \mathbb{R}\}\ ,$$

and one has

$$\lim_{\Lambda_{(0,0)} \ni (\eta_1, \eta_2) \longrightarrow (0,0)} F\left(x_1 + i\eta_1,\ x_2 + i\eta_2\right) = \omega_\beta\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right)\ .$$

6) In the following we reserve the letters x, y for the real and imaginary parts of $z \in \mathbb{C}^d$, respectively.

iii) Similarly, at every boundary point $\left(x_1 - i\frac{\beta}{2}e,\ x_2 + i\frac{\beta}{2}e\right)$ *of* $\mathcal{T}_{\mathcal{B}}$*, the profile* $\Lambda_{\left(-\frac{\beta}{2}e,\frac{\beta}{2}e\right)}$ *of* $\mathcal{T}_{\mathcal{B}}$ *(corresponding to the conical boundary point* $\left(-\frac{\beta}{2}e, \frac{\beta}{2}e\right)$ *of* $\mathcal{B}$*) is the cone*

$$\Lambda_{\left(-\frac{\beta}{2}e,\frac{\beta}{2}e\right)} = \left(-\frac{\beta}{2}e, \frac{\beta}{2}e\right) - \Lambda_{(0,0)} \ ,$$

and one has

$$\lim_{\Lambda_{\left(-\frac{\beta}{2}e,\frac{\beta}{2}e\right)} \ni (\eta_1,\eta_2) \longrightarrow \left(-\frac{\beta}{2}e,\frac{\beta}{2}e\right)} F\left(x_1 + i\eta_1,\ x_2 + i\eta_2\right) = \omega_\beta\left(\alpha_{x_2}(B)\alpha_{x_1}(A)\right) \ .$$

In ii) and iii) the boundary values taken from the tube $\mathcal{T}_{\mathcal{B}}$ *exist in the sense of continuous functions near boundary points, as specified in Appendix A.*

Proof : The desired analytic continuation is accomplished in five steps by repeated applications of the (flat) tube theorem (cf. Appendix A). We note that the successive continuations will be labelled by the number of the steps where they appear.

1. We first notice that the functions $F_f,\ f \in V_+$ have a common analytic continuation F_1 in the flat tube $\mathcal{T}_{\mathcal{B}_1}$ whose basis $\mathcal{B}_1$ is the convex hull of the set

$$\{(y_1, y_2) : y_1 = 0,\ y_2 = tf,\ f \in V_+,\ 0 < t < t_f\} \ .$$

This follows from an iterated application of the flat tube theorem to the whole set of flat tubes $\mathcal{T}_{\mathcal{B}_f}$, with f sweeping the set of all directions in V_+, and a subsequent application of the tube theorem with respect to the variable z_2. Similarly, the functions $F_f^\dagger,\ f \in V_+$, have a common analytic continuation $F_1^\dagger$ in the flat tube $\mathcal{T}_{\mathcal{B}_1^\dagger}$ whose basis $\mathcal{B}_1^\dagger$ is the convex hull of the set

$$\{(y_1, y_2) : y_1 = -tf,\ f \in V_+,\ 0 < t < t_f,\ y_2 = 0\} \ .$$

At all real points, $\mathcal{T}_{\mathcal{B}_1}$ and $\mathcal{T}_{\mathcal{B}_1^\dagger}$ have the respective profiles

$$\begin{aligned} \Lambda_1 &= \{(y_1, y_2) : y_1 = 0,\ y_2 \in V_+\} \\ \Lambda_1^\dagger &= \{(y_1, y_2) : y_1 \in V_-,\ y_2 = 0\} \ . \end{aligned}$$

2. Since the functions F_1 and $F_1^\dagger$ coincide on the reals $\mathbb{R}^d \times \mathbb{R}^d$, they satisfy all the conditions of Lemma A.2 in the Appendix. As a result they have a common analytic continuation F_2 in the tube whose basis is the convex hull of $\mathcal{B}_1 \cup \mathcal{B}_1^\dagger$. The function F_2 is analytic in particular in the tube $\mathcal{T}_{\mathcal{B}_2}$ with basis

$$\mathcal{B}_2 = \left\{(y_1, y_2) : (2y_1, 0) \in \mathcal{B}_1^\dagger,\ (0, 2y_2) \in \mathcal{B}_1\right\}$$

(corresponding to $\mathcal{B}_\lambda$ with $\lambda = 1/2$ in Lemma A.2). It is clear that at each real point (x_1, x_2) the profile of $\mathcal{T}_{\mathcal{B}_2}$ is the cone

$$\Lambda_2 = \{(y_1, y_2) : y_1 \in V_-,\ y_2 \in V_+\} \ ,$$

and there holds the boundary relation

$$\lim_{\Lambda_2 \ni (\eta_1,\eta_2) \longrightarrow (0,0)} F_2\left(x_1 + i\eta_1,\ x_2 + i\eta_2\right) = \omega_\beta\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) \ .$$

3. Next we consider the function $x_1, x_2 \longrightarrow \omega_\beta\left(\alpha_{x_2}(B)\alpha_{x_1}(A)\right)$. An argument similar to the one given in the preceding steps establishes the existence of an analytic continuation G of this function into the tube

$\mathcal{T}_{-\mathcal{B}_2}$ with profile $-\Lambda_2$ at the real boundary points (x_1, x_2). For later convenience we proceed from G to the function F_3, given by

$$F_3(z_1, z_2) \doteq G\left(z_1 + i\frac{\beta}{2}e,\ z_2 - i\frac{\beta}{2}e\right) \ ,$$

which is analytic in the tube $\mathcal{T}_{\mathcal{B}_3}$ with basis

$$\mathcal{B}_3 = \left\{(y_1, y_2) : (y_1, y_2) \in \left(-\frac{\beta}{2}e, \frac{\beta}{2}e\right) - \mathcal{B}_2\right\}$$

and profile $\Lambda_3 = \left(-\frac{\beta}{2}e, \frac{\beta}{2}e\right) - \Lambda_2$ at each boundary point $\left(x_1 - i\frac{\beta}{2}e,\ x_2 + i\frac{\beta}{2}e\right)$. The following boundary relation then holds:

$$\lim_{\Lambda_3 \ni (\eta_1,\eta_2) \longrightarrow \left(-\frac{\beta}{2}e, \frac{\beta}{2}e\right)} F_3(x_1 + i\eta_1,\ x_2 + i\eta_2) = \omega_\beta\left(\alpha_{x_2}(B)\alpha_{x_1}(A)\right) \ .$$

4. So far we have only exploited the stability condition b). Now the KMS-condition a) tells us (by an application of Lemma A.1) that, on the one hand, the functions F_2 and F_0 coincide on the flat tube $\mathcal{T}_{\mathcal{B}_2} \cap \mathcal{T}_{\mathcal{B}_0}$ since they have the same boundary value $\omega_\beta\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right)$ on $\mathbb{R}^d \times \mathbb{R}^d$. On the other hand, the functions F_3 and F_0 coincide on the flat tube $\mathcal{T}_{\mathcal{B}_3} \cap \mathcal{T}_{\mathcal{B}_0}$ since they have the same boundary value $\omega_\beta\left(\alpha_{x_2}(B)\alpha_{x_1}(A)\right)$ on $\left(\mathbb{R}^d - i\frac{\beta}{2}e\right) \times \left(\mathbb{R}^d + i\frac{\beta}{2}e\right)$. It follows that the functions F_0, F_2 and F_3 define a unique function F_4, analytic in the dumbbell-shaped tube $\mathcal{T}_{\mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{L}}$, where $\mathcal{L}$ is the segment $\left\{(y_1, y_2) : y_2 = -te,\ y_2 = te,\ 0 < t < \frac{\beta}{2}\right\}$ in $\mathcal{B}_0$. As a result of Lemma A.3, F_4 can therefore be analytically continued in the convex open tube $\mathcal{T}_{\mathcal{B}_4}$ whose basis $\mathcal{B}_4$ is the convex hull of $\mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{L}$. Hence $\mathcal{B}_4$ contains in particular a full neighbourhood of $\mathcal{L}$ in $\mathbb{R}^d \times \mathbb{R}^d$.

5. In order to obtain the complete results stated in the proposition, it remains to establish the property of translation invariance of the function F_4 under the transformations $(z_1, z_2) \longrightarrow (z_1 + te,\ z_2 + te)$ and to derive its geometrical consequences. Let

$$N(z_1, z_2) = \frac{\mathrm{d}}{\mathrm{d}t} F_4\left(z_1 + te,\ z_2 + te\right)\Big|_{t=0} \ .$$

This function is analytic in the same domain $\mathcal{T}_{\mathcal{B}_4}$ as F_4 itself and, in view of the remark in condition a), its boundary value on $\mathbb{R}^d \times \mathbb{R}^d$ vanishes (in the sense of distributions). Therefore, in view of Lemma A.1, N is equal to 0 in $\mathcal{T}_{\mathcal{B}_4}$, which in turn implies that $F_4(z_1 + te,\ z_2 + te) = F_4(z_1, z_2)$ in the whole domain $\mathcal{T}_{\mathcal{B}_4}$ (since, in view of the convexity of $\mathcal{T}_{\mathcal{B}_4}$, each orbit $(z_1 + ue, z_2 + ue)$, $u \in \mathbb{C}$ has a connected intersection with $\mathcal{T}_{\mathcal{B}_4}$).

From this invariance property of F_4, it follows that F_4 can be analytically continued to an analytic function F in the tube $\mathcal{T}_{\mathcal{B}}$ with basis

$$\mathcal{B} = \{(y_1, y_2) : y_1 = \eta_1 + te,\ y_2 = \eta_2 + te,\ (\eta_1, \eta_2) \in \mathcal{B}_4,\ t \in \mathbb{R}\} \ .$$

Since $\mathcal{B}_4$ contains a neigbhourhood of $\mathcal{L}$ (as shown in step 4), one easily checks that the region $\mathcal{B}$ contains a neighbourhood of $\mathcal{B}_0$ in $\mathbb{R}^d \times \mathbb{R}^d$; moreover, the profile of $\mathcal{T}_{\mathcal{B}}$ at any boundary point of $\mathbb{R}^d \times \mathbb{R}^d$, (resp. $\left(\mathbb{R}^d - i\frac{\beta}{2}e\right) \times \left(\mathbb{R}^d + i\frac{\beta}{2}e\right)$), is the cone

$$\Lambda_{(0,0)} = \{(y_1, y_2) : y_1 = \eta_1 + te,\ y_2 = \eta_2 + te,\ (\eta_1, \eta_2) \in \Lambda_2,\ t \in \mathbb{R}\}$$

(resp. $\Lambda_{\left(-\frac{\beta}{2}e, \frac{\beta}{2}e\right)} = \left(-\frac{\beta}{2}e, \frac{\beta}{2}e\right) - \Lambda_{(0,0)}$). Finally, from the boundary relations in steps 2 and 3 it follows that the function F has the boundary values stated in the proposition.

Let us now specialize this proposition to the cases when the correlation functions have the stronger analyticity properties established in the previous section under more restrictive assumptions. In the case when ω_β is resistant to boundary effects (in the sense of our criterion) we have given a direct proof of the existence of analytic functions F_1 and $F_1^\dagger$ in respective flat tubes $\mathcal{T}_{\mathcal{B}_1}$ and $\mathcal{T}_{\mathcal{B}_1^\dagger}$ with (convex) bases

$$\mathcal{B}_1 = \left\{ (y_1, y_2) : y_1 = 0,\ y_2 \in V_+ \cap \left(\frac{\beta}{2} e + V_- \right) \right\} \tag{5.6}$$

$$\mathcal{B}_1^\dagger = \left\{ (y_1, y_2) : y_1 \in V_- \cap \left(-\frac{\beta}{2} e + V_+ \right),\ y_2 = 0 \right\} \tag{5.7}$$

which extend the correlation functions $x_1, x_2 \longrightarrow \omega_\beta \left(\alpha_{x_1}(A) \alpha_{x_2}(B) \right)$. As a matter of fact, the complete holomorphy envelope can be computed in this case according to the following statement.

<u>Proposition 5.2</u> : *If the state ω_β is resistant to boundary effects, then the functions F of Proposition 5.1 which extend the correlation functions, are analytic in the (flatly bordered) convex tube $\mathcal{T}_\mathcal{B}$ whose basis $\mathcal{B}$ is defined as follows:*

$$\mathcal{B} = \{ (y_1, y_2) : y_1 = \eta_1 + te,\ y_2 = \eta_2 + te,\ (\eta_1, \eta_2) \in \mathcal{B}_U,\ t \in \mathbb{R} \} \tag{5.8}$$

where $\mathcal{B}_U = \bigcup_{0 \le \lambda \le 1} \mathcal{C}_\lambda^- \times \mathcal{C}_{1-\lambda}^+$ and

$$\mathcal{C}_\mu^+ = -\mathcal{C}_\mu^- = \left\{ y : y \in V_+ \cap \left(\frac{\beta}{2} e + V_- \right),\ \ \left| y^2 - (e \cdot y)^2 \right|^{1/2} \le \mu \frac{\beta}{4} \right\} \tag{5.9}$$

for $0 \le \mu \le 1$.

The proof of this proposition is given in Appendix B. The result can be compared with the case when ω_β is supposed to be strongly resistant to boundary effects. There Proposition 4.2 has yielded the tube $\mathcal{T}_{\mathcal{B}_M} = -\mathcal{T}_{\frac{\beta}{2}e} \times \mathcal{T}_{\frac{\beta}{2}e}$ with basis $\mathcal{B}_M = \mathcal{C}_1^- \times \mathcal{C}_1^+ \supset \mathcal{B}_U$ as a domain of analyticity of F. (A pictorial comparison of the bases $\mathcal{B}_U$ and $\mathcal{B}_M$ is given in Fig.1.) Hence in that case the complete answer (obtained by time-translation invariance of F) is:

<u>Proposition 5.3</u> : *If the state ω_β is strongly resistant to boundary effects, then the functions F of Proposition 5.1, extending the correlation functions, are analytic in the tube $\mathcal{T}_\mathcal{B}$ with basis*

$$\mathcal{B} = \{ (y_1, y_2) : y_1 = \eta_1 + te,\ y_2 = \eta_2 + te,\ (\eta_1, \eta_2) \in \mathcal{B}_M,\ t \in \mathbb{R} \}\ . \tag{5.10}$$

This is the maximal domain of analyticity of thermal correlation functions which one may expect in our general setting.

Fig.1 : The bases $\mathcal{B}_U = \bigcup_{0 \le \lambda \le 1} \mathcal{C}_\lambda^- \times \mathcal{C}_{1-\lambda}^+$ and $\mathcal{B}_M = \mathcal{C}_1^- \times \mathcal{C}_1^+$.

We now restrict our attention to the physically interesting cases when the state ω_β is invariant under the full translation group $\mathbb{R}^d$, i.e.

$$\omega_\beta\left(\alpha_{x_1}(A)\alpha_{x_2}(B)\right) = \omega_\beta\left(\alpha_{x_1+x}(A)\alpha_{x_2+x}(B)\right) \tag{5.11}$$

for all $x \in \mathbb{R}^d$. By an argument (based on Lemma A.1) completely analogous to the one previously used for the time-invariance property of F, one shows that the function F then satisfies the invariance relation

$$F\left(z_1, z_2\right) = F\left(z_1 + z,\ z_2 + z\right)\ , \quad z \in \mathbb{C}^d \tag{5.12}$$

in the convex open tube $\mathcal{T}_\mathcal{B}$ where it is defined. It follows that the resulting domain of the function $\underline{F}$, defined by $\underline{F}\left(z_2 - z_1\right) = F\left(z_1, z_2\right)$, is obtained by taking the projection of the open tube $\mathcal{T}_\mathcal{B}$ in the space $\mathbb{C}^d$ of the vector variable $z_2 - z_1$. One can then state:

Corollary 5.4 : *Let ω_β be a KMS-state which is invariant under space-time translations and satisfies conditions a) and b). There exists a function $\underline{F}$, analytic in a convex open tube $\mathcal{T}_\mathcal{C} \subset \mathbb{C}^d$ with basis $\mathcal{C} \subset \mathbb{R}^d$, which extends the correlation function $x \longrightarrow \omega_\beta\left(A\alpha_x(B)\right)$ for given local $A, B \in \mathcal{A}$. More specifically:*

i) The basis $\mathcal{C}$ is a neighbourhood in $\mathbb{R}^d$ of the linear segment $\{y : y = \lambda e,\ 0 < \lambda < \beta\}$, and the profile of $\mathcal{T}_\mathcal{B}$ at all boundary points in $\mathbb{R}^d$ (resp. in $\mathbb{R}^d + i\beta e$) is the light cone V_+ (resp. $\beta e + V_-$). Moreover, one has

$$\lim_{V_+ \ni \eta \longrightarrow 0} \underline{F}(x + i\eta) = \omega_\beta\left(A\alpha_x(B)\right)$$
$$\lim_{V_- \ni \eta \longrightarrow 0} \underline{F}(x + i\beta e + i\eta) = \omega_\beta\left(\alpha_x(B)A\right)\ .$$

A precise shape can be given for the tube $\mathcal{T}_\mathcal{C}$ in the following cases.

ii) If ω_β satisfies the condition of resistance to boundary effects, then

$$\mathcal{C} = \left\{ y : y \in V_+ \cap \left(\beta e + V_-\right),\ \ \left|y^2 - (e\cdot y)^2\right|^{1/2} < \frac{\beta}{4} \right\}\ .$$

iii) If ω_β satisfies the condition of strong resistance to boundary effects, then

$$\mathcal{C} = \left\{ y : y \in V_+ \cap \left(\beta e + V_-\right) \right\}\ ,$$

(i.e., $\mathcal{C}$ is the basis of the tube $\mathcal{T}_{\beta e}$ introduced in the Introduction).

The three typical situations considered in this corollary are depicted in Fig.2.

Fig.2 - The basis $\mathcal{C}$ in the three statements of Corollary 5.4.

The results of this section provide various formulations of the KMS-condition which take into account the expected features of thermal correlation functions in a relativistic theory. Even in their weakest form, namely Proposition 5.1, they involve all space-time variables and reveal the existence of a maximal propagation speed through the special role of the light cone V_+. Moreover, they apply in a canonical way to observers in any Lorentz frame. We therefore regard these results as proper versions of a relativistic KMS-condition.

6. Concluding remarks

In the present investigation we have given arguments which suggest a relativistic formulation of the KMS-condition for thermal equilibrium states. Although we have used the framework of algebraic quantum field theory for mathematical convenience, it is apparent that the conclusions of our analysis are of a quite general nature and can be applied to unbounded field operators (Wightman fields) as well. The resulting analyticity domains of the Wightman n-point functions are of the type described (for $n = 2$) at the end of the Introduction. These domains are the analogues of the primitive tube domains implied by the relativistic spectrum condition in the vacuum case.

As in the latter, these primitive analyticity properties combined with the condition of locality should define, through the edge-of-the-wedge technique, the full analytic structure of correlation functions in complex spacetime. First results in this direction have been obtained in [BB1] and [BB2].

In view of its expected significance for the structural analysis, it would be desirable to provide further arguments in favour of a relativistic version of the KMS-condition. On the one hand, it would be important to check the latter in models. We note in this context that the relativistic KMS-condition in its most restrictive form (case iii) of Corollary 5.4) is satisfied by the thermal equilibrium states of local non-interacting fields and has also been verified in perturbation theory [St]. On the other hand, it may well be possible to derive this condition from more fundamental principles, such as the second law of thermodynamics.

To motivate the latter statement, we recall that the standard KMS-condition can be established for states ω which are passive [PW,BR], i.e. states from which one cannot extract energy by a cyclic process. The condition of passivity is expressed in the mathematical setting by the requirement that

$$\Delta E = i\frac{\mathrm{d}}{\mathrm{d}t}\omega\left(U^*\alpha_{t\cdot e}(U)\right)\big|_{t=0} \leq 0 \tag{6.1}$$

for all (differentiable) unitary operators $U \in \mathcal{A}$. The quantity ΔE can be interpreted as the energy gained in a cyclic process between the initial state $\omega(\cdot)$ and final state $\omega\left(U^* \cdot U\right)$ [PW].

Whereas condition (6.1) is an appropriate expression of the second law for an observer in the rest frame of the state ω, it is not adequate for an observer who is moving. In fact, the quantity ΔE does not take into account the energy which is necessary to maintain the motion of such an observer in the presence of dissipative forces; as a result ΔE can become strictly positive.

It is clear, however, that the energy ΔE has to be smaller than the energy fed into the system by the moving observer. This fact suggests to amend the passivity condition (6.1) by the following assumption: for any positive timelike vector $f \in V_+$, $f^2 = 1$, and any bounded spacetime region $\mathcal{O}$ there exists a constant $E_{f,\mathcal{O}}$ such that

$$i\frac{\mathrm{d}}{\mathrm{d}t}\omega\left(U^*\alpha_{t\cdot f}(U)\right)\big|_{t=0} \leq E_{f,\mathcal{O}} \tag{6.2}$$

for all (differentiable) unitary operators $U \in \mathcal{A}(\mathcal{O})$. The essence of this condition is the assumption that the energy $E_{f,\mathcal{O}}$ which is necessary to proceed from the rest system to the moving system, characterized by f, is locally finite. The dependence of $E_{f,\mathcal{O}}$ on f will be submitted to the specific properties of the state ω, but one may expect that, quite generally, $E_{f,\mathcal{O}}$ is proportional to the size of $\mathcal{O}$ for large spacetime regions $\mathcal{O}$.

It seems worthwhile to explore the consequences of condition (6.2) for the structure of the correlation functions. We hope that by applying the powerful methods developped in [PW] it will be possible to establish analyticity properties of these functions, as anticipated in our stability condition of Sec. 5. This would then provide an alternative and quite fundamental justification of our relativistic version of the KMS-condition.

Appendix A

We collect here some definitions and results of the theory of analytic functions of several complex variables which are used throughout Sec. 5.

The functions F which we encounter are typically defined in subsets of $\mathbb{C}^N$ which are tubes, namely sets of the form $\mathcal{T}_{\mathcal{B}} = \mathbb{R}^N + i\mathcal{B}$, the set $\mathcal{B} \subset \mathbb{R}^N$ being called the basis of the tube $\mathcal{T}_{\mathcal{B}}$. If $\mathcal{B}$ is an open set in $\mathbb{R}^N$, $\mathcal{T}_{\mathcal{B}}$ is an open tube in $\mathbb{C}^N$; the analyticity of a function F in $\mathcal{T}_{\mathcal{B}}$ then means analyticity of $F(z)$ with respect to all complex variables $z = (z_1, ..., z_N)$ varying in $\mathcal{T}_{\mathcal{B}}$.

If $\mathcal{B}$ is a linear submanifold of $\mathbb{R}^N$ of dimension $n < N$, $\mathcal{T}_{\mathcal{B}}$ will be called a *flat tube*; by analyticity of a function F in the flat tube $\mathcal{T}_{\mathcal{B}}$, we shall always mean: joint continuity with respect to all variables varying in $\mathcal{T}_{\mathcal{B}}$ and analyticity of F with respect to a (maximal) set of n complex variables in all the complex n-dimensional (linear) submanifolds which generate $\mathcal{T}_{\mathcal{B}}$. A simple but important example of a flat tube in $\mathbb{C}^2$ is $\mathcal{T}_{\mathcal{B}} = \mathbb{R}^2 + i\mathcal{B}$, where $\mathcal{B} = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < b,\ y_2 = 0\}$.

We shall also be led to consider tubes $\mathcal{T}_{\mathcal{B}}$ with a non-empty interior which are obtained by adjoining one or several flat tubes to their interior (in other words, the basis $\mathcal{B}$ of such a tube is the union of an open connected set and of one or several linear manifolds belonging to the boundary of the latter): such tubes may be produced by taking the convex hull of the union of two or several flat tubes (see Lemma A.2 below and in particular the first case presented in the proof of the latter). In such a case, we shall say that $\mathcal{T}_{\mathcal{B}}$ is a "flatly-bordered tube" and that a function F is analytic in $\mathcal{T}_{\mathcal{B}}$ if the following conditions are fulfilled: i) F is continuous in $\mathcal{T}_{\mathcal{B}}$, ii) the restriction of F to the interior of $\mathcal{T}_{\mathcal{B}}$ is analytic; this implies that the restrictions of F to the various bordering flat tubes in $\mathcal{T}_{\mathcal{B}}$ are also analytic in the sense of the previous definition.

If a point b of $\mathbb{R}^N$ belongs to the boundary of $\mathcal{B}$, we call *profile* of the tube $\mathcal{T}_{\mathcal{B}}$ at any point $c = a + ib$ of its boundary the cone Λ_b with apex b in $\mathbb{R}^N$ which is the union of all closed half-lines starting from b and intersecting $\mathcal{B}$. We will say that a function F, analytic in $\mathcal{T}_{\mathcal{B}}$, admits a *continuous boundary value* near a boundary point $c = a + ib$ of $\mathcal{T}_{\mathcal{B}}$ if it can be extended as a continuous function on some neighbourhood of c in $\mathcal{T}_{\mathcal{B}\cup\{b\}}$ in the following sense: for each closed subcone $\bar{\Lambda}$ of Λ_b there exists a real neighbourhood $\mathcal{N}_a$ of a such that the extension of F is continuous in the region $\mathcal{N}_a + i\bar{\Lambda}$. Similarly, F is said to be continuous on a given open set of boundary points if it is continuous in the above sense near each of these points.

The following result can be seen as an extension of the principle of uniqueness of the analytic continuation (see e.g. [SW] Theorem 2.17).

Lemma A.1 : *Let $\mathcal{T}_{\mathcal{B}}$ be a given (open or flat) tube and $c = a + ib$ a boundary point of $\mathcal{T}_{\mathcal{B}}$. If two functions F and G, analytic in $\mathcal{T}_{\mathcal{B}}$, admit coinciding (continuous) boundary values near the point c, then $F = G$.*

The basic result concerning analytic functions in tube-shaped domains is the following *tube theorem* (see e.g. [W] and references therein): if a function F is analytic in a tube $\mathcal{T}_{\mathcal{B}}$ with open connected basis $\mathcal{B}$, then it can be analytically continued in the tube $\mathcal{T}_{\widehat{\mathcal{B}}}$ whose basis $\widehat{\mathcal{B}}$ is the convex hull of $\mathcal{B}$. In other words: $\mathcal{T}_{\widehat{\mathcal{B}}}$ is the holomorphy envelope of $\mathcal{T}_{\mathcal{B}}$.

A non-trivial refinement of the tube theorem is the fact that it can be extended to the case of flat tubes (see e.g. [BEGS], the first result of this type being due to Malgrange and Zerner). We shall need the following version of this *flat tube theorem*.

Lemma A.2 : *Let $\mathcal{T}_{\mathcal{B}_0}$ and $\mathcal{T}_{\mathcal{B}_1}$ be two flat tubes in $\mathbb{C}^N$ whose bases $\mathcal{B}_i$, $i = 0, 1$ are convex and have closures $\bar{\mathcal{B}}_i$ which contain the origin 0 and are star-shaped with respect to 0. Let F_0 and F_1 be any pair of functions which are analytic in $\mathcal{T}_{\mathcal{B}_0}$ and $\mathcal{T}_{\mathcal{B}_1}$ respectively, and have continuous boundary values on $\mathbb{R}^N$ which coincide, i.e. $F_0 \restriction \mathbb{R}^N = F_1 \restriction \mathbb{R}^N$. Then there exists a unique function F which is analytic in the tube $\mathcal{T}_{\widehat{\mathcal{B}_0\cup\mathcal{B}_1}}$, has continuous boundary values on $\mathbb{R}^N$, and extends the given functions:*

$$F \restriction \mathcal{T}_{\mathcal{B}_i} = F_i, \qquad i = 0, 1 \ .$$

Moreover, the convex tube $\mathcal{T}_{\widehat{\mathcal{B}_0\cup\mathcal{B}_1}}$ can be described as follows:

$$\mathcal{T}_{\widehat{\mathcal{B}_0\cup\mathcal{B}_1}} = \bigcup_{0\le\lambda\le1} \mathcal{T}_{\mathcal{B}_\lambda} \ ,$$

where

$$\mathcal{B}_\lambda = (1-\lambda)\mathcal{B}_0 + \lambda\mathcal{B}_1 = \left\{ y \in \mathbb{R}^N : y = (1-\lambda)y^{(0)} + \lambda y^{(1)},\ y^{(0)} \in \mathcal{B}_0,\ y^{(1)} \in \mathcal{B}_1 \right\} .$$

We notice that if $\mathcal{B}_0$ and $\mathcal{B}_1$ are linearly independent and of respective dimensions n_0 and n_1, $\widehat{\mathcal{B}_0 \cup \mathcal{B}_1} = \bigcup_{0\le\lambda\le1} \mathcal{B}_\lambda$ is of dimension $n_0 + n_1$. Thus the non-trivial aspect of this result is that analyticity with respect to $n_0 + n_1$ variables is obtained from two assumptions of analyticity with respect to n_0 and n_1 variables in the distinct sets $\mathcal{T}_{\mathcal{B}_0}$ and $\mathcal{T}_{\mathcal{B}_1}$. Of course, the coincidence condition on the reals is crucial and plays the role of the connectedness of the tube $\mathcal{T}_\mathcal{B}$ in the standard tube theorem. We also notice that the uniqueness of the function F is a direct consequence of the analytic continuation principle (respectively of Lemma A.1).

<u>Proof of Lemma A.2</u> : Let us first consider a simple case covered by the Lemma which is in fact basic for the proof of the general case, as indicated below. This is the two-dimensional situation where $\mathcal{T}_{\mathcal{B}_i} \subset \mathbb{C}^2$, $i = 0, 1$ are tubes with bases $\mathcal{B}_0 = \{(y_1, y_2) : \ 0 < y_1 < a_1,\ y_2 = 0\}$ and $\mathcal{B}_1 = \{(y_1, y_2) : \ y_1 = 0,\ 0 < y_2 < a_2\}$. The corresponding convex hull is then

$$\widehat{\mathcal{B}_0 \cup \mathcal{B}_1} = \left\{ (y_1, y_2) : 0 \le y_1,\ 0 \le y_2,\ 0 < \frac{y_1}{a_1} + \frac{y_2}{a_2} < 1 \right\} ,$$

equivalently described as

$$\widehat{\mathcal{B}_0 \cup \mathcal{B}_1} = \left(\bigcup_{0<\lambda<1} \{(y_1, y_2) : 0 < y_1 < (1-\lambda)a_1,\ 0 < y_2 < \lambda a_2\} \right) \bigcup (\mathcal{B}_0 \cup \mathcal{B}_1)$$

The proof of Lemma A.2 in this case is essentially given in [BEGS]. In the version presented there, F_0 and F_1 are assumed to be C^∞ functions[7] .

However, the present version (in which F_0 and F_1 are only assumed to be continuous) can be easily traced back to the situation considered in [BEGS] by approximating the given functions F_0, F_1 by sequences of C^∞-functions $F_i^{(n)} = F_i * \delta_n$, $i = 0, 1$, where $\{\delta_n,\ n \in \mathbb{N}\}$ is a regularizing sequence of test functions in $\mathcal{D}\left(\mathbb{R}^2\right)$ which tends to the Dirac measure in the limit of large n. In view of the results of [BEGS] (and of the Cauchy integral method used therein[7]), the smooth functions $F_i^{(n)}$, $i = 0, 1$, can be continued to analytic functions $F^{(n)}$ in $\mathcal{T}_{\widehat{\mathcal{B}_0 \cup \mathcal{B}_1}}$ which (by virtue of the maximum modulus principle) are uniformly bounded on all compact subsets of $\mathcal{T}_{\widehat{\mathcal{B}_0 \cup \mathcal{B}_1}} \cup \mathbb{R}^2$. It follows that the functions $F^{(n)}$ form a normal family of analytic functions, and that the desired analytic continuation F of the functions F_0, F_1 is then defined as a limit point of this normal family.

The proof of the multi-dimensional version of Lemma A.2 relies on the two-dimensional result thanks to the following simple geometrical argument. One sweeps each of the (star-shaped) bases $\mathcal{B}_0$, $\mathcal{B}_1$ of the tubes $\mathcal{T}_{\mathcal{B}_0}$, $\mathcal{T}_{\mathcal{B}_1}$ by linear segments $\mathcal{L}_0 = \{y : y = \lambda \cdot b_0,\ \ 0 < \lambda < 1\}$ for any $b_0 \in \mathcal{B}_0$, and $\mathcal{L}_1 = \{y : y = \lambda \cdot b_1,\ 0 < \lambda < 1\}$ for any $b_1 \in \mathcal{B}_1$. By applying the two-dimensional flat tube theorem, quoted above, to the couple of functions $F_0 \restriction \mathcal{T}_{\mathcal{L}_0}$ and $F_1 \restriction \mathcal{T}_{\mathcal{L}_1}$ one obtains a corresponding analytic continuation $F_{b_0 b_1}$ in the flat convex tube $\mathcal{T}_{\widehat{\mathcal{L}_0 \cup \mathcal{L}_1}} = \bigcup_{0\le\lambda\le1} \mathcal{T}_{(1-\lambda)\mathcal{L}_0 + \lambda\mathcal{L}_1}$. As a matter of fact, the germs of functions $F_{b_0 b_1}$ obtained in each of these flat tubes are analytic not only with respect to the two complex variables associated with the complex two-plane fixed by b_0, b_1, but with respect to all those variables which vary in the full tube $\mathcal{T}_{\widehat{\mathcal{B}_0 \cup \mathcal{B}_1}}$. This can be seen directly by inspection of the proof of Lemma 1 in [BEGS], where the Cauchy integrals used for defining $F_{b_0 b_1}$ now exhibit (through F_0, F_1) an analytic dependence with respect to all variables involved in $\mathcal{T}_{\mathcal{B}_0}$ and $\mathcal{T}_{\mathcal{B}_1}$. Finally, patching together all these germs of analytic functions $F_{b_0 b_1}$ for $b_0 \in \mathcal{B}_0$

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[7] In [BEGS], a direct proof of the flat tube property, based on a Cauchy integral method, is first given for the case when F_0 and F_1 are C^∞ and sufficiently decreasing at infinity (Lemma 1); a *localized* version of the flat tube property is then derived from the latter by an appropriate use of conformal mappings (Lemma 2); finally, as indicated in a subsequent remark, a by-product of this localized version is that it allows one to get rid of any restriction on the behaviour at infinity of F_0, F_1 in the first result; it actually displays the purely local character of the analytic completion procedure which is at work in the (flat) tube property.

and $b_1 \in \mathcal{B}_1$ yields a univalent analytic function F in the (simply connected) domain $\bigcup_{0\le\lambda\le 1} \mathcal{T}_{\mathcal{B}_\lambda}$, since $\mathcal{B}_\lambda = \bigcup_{b_0\in\mathcal{B}_0, b_1\in\mathcal{B}_1} ((1-\lambda)\mathcal{L}_0 + \lambda\mathcal{L}_1)$. The fact that $\bigcup_{0\le\lambda\le 1}\mathcal{B}_\lambda$ is convex and therefore equal to the convex hull of $\mathcal{B}_0 \cup \mathcal{B}_1$ can be checked directly.

We shall also make use of a variant of the tube and flat tube theorems, in which the basis $\mathcal{B}$ of the tube $\mathcal{T}_\mathcal{B}$ is "*dumb-bell shaped*", i.e. of the form $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{L}$, where $\mathcal{B}_0$ and $\mathcal{B}_1$ are two disjoint convex open sets of $\mathbb{R}^N$ and $\mathcal{L}$ is a linear segment whose end-points b_0, b_1 belong respectively to $\mathcal{B}_0$ and $\mathcal{B}_1$. A function F is said to be analytic in the tube $\mathcal{T}_\mathcal{B}$ with dumb-bell shaped basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{L}$ if it is continuous on $\mathcal{T}_\mathcal{B}$, analytic in $\mathcal{T}_{\mathcal{B}_0}$ and $\mathcal{T}_{\mathcal{B}_1}$ (as a function of N complex variables) and analytic in $\mathcal{T}_\mathcal{L}$ (as a function of one complex variable).

Lemma A.3 : *Any function F which is analytic in a tube $\mathcal{T}_\mathcal{B}$ with dumb-bell shaped basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{L}$ can be analytically continued (as a function of N complex variables) in the tube $\mathcal{T}_{\widehat{\mathcal{B}}}$ whose basis $\widehat{\mathcal{B}}$ is the convex hull of $\mathcal{B}_0 \cup \mathcal{B}_1$.*

Proof : Without restriction of generality we may assume that the endpoint b_0 of $\mathcal{L}$ is the origin 0 of $\mathbb{R}^N$, i.e. that $0 \in \mathcal{B}_0$. Let then $a_1, ..., a_N$ be N points in $\mathcal{B}_0$ whose convex hull $\mathcal{H}$ is an $(N-1)$-dimensional simplex containing 0 as an interior point, and let us consider the two flat tubes $\mathcal{T}_\mathcal{H}$ and $\mathcal{T}_\mathcal{L}$. It is clear that any function F, analytic in $\mathcal{T}_\mathcal{B}$, defines a pair of functions $F_0 = F \restriction \mathcal{T}_\mathcal{H}$ and $F_1 = F \restriction \mathcal{T}_\mathcal{L}$ which satisfy all the conditions of Lemma A.2. As a result, there exists an analytic function F_{01} of N variables which is the common analytic continuation of F_0 and F_1 in the convex tube $\mathcal{T}_{\widehat{\mathcal{H}\cup\mathcal{L}}} = \bigcup_{0\le\lambda\le 1}\mathcal{T}_{(1-\lambda)\mathcal{H}+\lambda\mathcal{L}}$. Since $F_{01} \restriction \mathbb{R}^N = F \restriction \mathbb{R}^N$, it follows from Lemma A.1 that the restrictions of F_{01} and F to the domain $\mathcal{T}_{\mathcal{B}_0} \cap \mathcal{T}_{\widehat{\mathcal{H}\cup\mathcal{L}}}$ coincide; similarly, since $F_{01} \restriction \left(\mathbb{R}^N + ib_1\right) = F \restriction \left(\mathbb{R}^N + ib_1\right)$, it follows that the restrictions of F_{01} and F to the domain $\mathcal{T}_{\mathcal{B}_1} \cap \mathcal{T}_{\widehat{\mathcal{H}\cup\mathcal{L}}}$ coincide. The function F_{01} therefore provides an analytic continuation of F in the connected open tube $\mathcal{T}_{\mathcal{B}_0} \cup \mathcal{T}_{\mathcal{B}_1} \cup \mathcal{T}_{\widehat{\mathcal{H}\cup\mathcal{L}}}$. Hence, by applying the standard tube theorem, we conclude that F can be analytically continued in the tube $\mathcal{T}_{\widehat{\mathcal{B}}}$, whose basis $\widehat{\mathcal{B}}$ is the convex hull of $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \left(\widehat{\mathcal{H}\cup\mathcal{L}}\right)$ and therefore coincides with the convex hull of $\mathcal{B}_0 \cup \mathcal{B}_1$.

Appendix B

We give here the proof of Proposition 5.2. The argument is similar to the one given in Proposition 5.1, but the more detailed statement about the shape of the tube $\mathcal{T}_\mathcal{B}$ requires some extra calculations.

From the KMS-condition a) and the assumption that ω_β is resistant to boundary effects, we obtain analyticity of F (together with continuity at the edges) in the four flat tubes $\mathcal{T}_{\mathcal{B}_1} = \mathbb{R}^d \times \mathcal{T}_{\frac{\beta}{2}e}$, $\mathcal{T}_{\mathcal{B}_1^\dagger} = \left(-\mathcal{T}_{\frac{\beta}{2}e}\right) \times \mathbb{R}^d$, $\mathcal{T}_{\mathcal{B}_2} = \left(\mathbb{R}^d - i\frac{\beta}{2}e\right) \times \mathcal{T}_{\frac{\beta}{2}e}$ and $\mathcal{T}_{\mathcal{B}_2^\dagger} = \left(-\mathcal{T}_{\frac{\beta}{2}e}\right) \times \left(\mathbb{R}^d + i\frac{\beta}{2}e\right)$. Lemma A.2 can now be applied to the four pairs $\left(\mathcal{T}_{\mathcal{B}_1}, \mathcal{T}_{\mathcal{B}_1^\dagger}\right)$, $\left(\mathcal{T}_{\mathcal{B}_2}, \mathcal{T}_{\mathcal{B}_2^\dagger}\right)$, $\left(\mathcal{T}_{\mathcal{B}_1}, \mathcal{T}_{\mathcal{B}_2^\dagger}\right)$ and $\left(\mathcal{T}_{\mathcal{B}_2}, \mathcal{T}_{\mathcal{B}_1^\dagger}\right)$. Since the four resulting convex tubes are flatly bordered tubes in $\mathbb{C}^d \times \mathbb{C}^d$, a final application of the standard tube theorem to the union of the interiors of the latter yields analyticity of F in the convex tube $\mathcal{T}_\mathcal{B}$, whose basis $\mathcal{B}$ is the convex hull of $\mathcal{B}_1 \cup \mathcal{B}_1^\dagger \cup \mathcal{B}_2 \cup \mathcal{B}_2^\dagger$. (The uniqueness of F, continued into the various common domains, is ensured by Lemma A.1.)

We are now just led to the technical problem of computing the convex hull $\mathcal{B}$, which we treat as follows: let $\mathcal{L}_1^\dagger = \{(y_1, y_2) : y_1 = \lambda e,\ -\beta/2 < \lambda < 0,\ y_2 = 0\}$ be the "diagonal" of the base $\mathcal{B}_1^\dagger$. The convex hull of $\mathcal{B}_1 \cup \mathcal{L}_1^\dagger \cup \mathcal{B}_2$ is the ($d+1$ -dimensional) set $\mathcal{C}_0^- \times \mathcal{C}_1^+$, where (consistently with the notations used in the statement of the proposition)

$$\mathcal{C}_0^- = \left\{y : y = \lambda e,\ -\frac{\beta}{2} < \lambda < 0\right\} \quad \text{and} \quad \mathcal{C}_1^+ = \left\{y : y \in V^+ \cap \left(\frac{\beta}{2}e + V^-\right)\right\} .$$

Similarly, if $\mathcal{L}_1 = \{(y_1, y_2) : y_1 = 0,\ y_2 = \lambda{\cdot}e,\ 0 < \lambda < \frac{\beta}{2}\}$ is the diagonal of $\mathcal{B}_1$, the convex hull of $\mathcal{B}_1^\dagger \cup \mathcal{L}_1 \cup \mathcal{B}_2^\dagger$

is the set $\mathcal{C}_1^- \times \mathcal{C}_0^+$, where

$$\mathcal{C}_1^- = \left\{ y : y \in V^- \cap \left(-\frac{\beta}{2} e + V^+ \right) \right\} \quad \text{and} \quad \mathcal{C}_0^+ = \left\{ y : y = \lambda e,\ 0 < \lambda < \frac{\beta}{2} \right\} .$$

It is now straightforward to compute $\mathcal{B}$ by noting that it is the convex hull of the region $\left(\mathcal{C}_0^- \times \mathcal{C}_1^+\right) \cup \left(\mathcal{C}_1^- \times \mathcal{C}_0^+\right)$. Alternatively, $\mathcal{B}$ can be characterized as being the union of all interpolating products $\mathcal{C}_\lambda^- \times \mathcal{C}_{1-\lambda}^+$ for $0 \leq \lambda \leq 1$. The latter fact can be seen by taking arbitrary two-dimensional meridian sections of the double cones $\mathcal{C}_1^-$ and $\mathcal{C}_1^+$. One thereby obtains products of interpolating trapezia for the corresponding convex completions in the chosen products of meridian sections, as indicated in Fig.3.

Fig.3 : Interpolating trapezia

a) a meridian section of $\mathcal{C}_1^-$ (represented with the origin at $-\frac{\beta}{2}e$) with hatchings inside $\mathcal{C}_\lambda^-$ b) a meridian section of $\mathcal{C}_1^+$ with hatchings inside $\mathcal{C}_{1-\lambda}^+$.

By taking finally into account the time invariance of F, as in the proof of Proposition 5.1, the statement then follows.

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