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LARGE DEFLECTIONS OF CIRCULAR ISOTROPIC MEMBRANES SUBJECTED TO ARBITRARY AXISYMMETRIC LOADING

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SUMMARY

Circular membranes with fixed peripheral edges, subjected to arbitrary axisymmetric loading were analyzed. A single governing differential equation in terms of radial stress was used. This nonlinear governing equation was solved using the finite difference method in conjunction with Newton-Raphson method. Three loading cases, namely (a) uniformly loaded membrane, (b) a membrane with uniform load over an inner portion, and (c) a membrane with ring load, were analyzed. Calculated central displacement and the central and edge radial stresses for uniformly loaded membrane, agreed extremely well with the classical solution.

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INTRODUCTION

Composite laminates are being extensively used in aerospace applications because of their high strength to weight ratios. Because of their widespread use, these laminates in addition to several types of loads are subjected to impact loads. Due to these impact loads composite laminates suffer visible and invisible damage. Damage characterization and residual strength evaluation of these impact-damaged composite laminates is of utmost importance.

During low-velocity impact on thin circular 8-ply quasi-isotropic laminates, the laminates exhibit large deflection behavior. The displacement in the laminates can exceed up to one laminate thickness [1]. Because of these large deflections, the laminates exhibit mid-plane stretching. The mid-plane stretching contributes to the membrane action. Therefore, a damaged composite laminate can be analyzed by decomposing the problem into two parts. The first part consists of a thin plate with shear and flexural stiffness but no mid-plane extensional stiffness. The second part consists of a membrane with mid-plane extensional stiffness but no shear and flexural stiffness. The results of these two types of analyses will yield complete knowledge of displacements and stresses in a quasi-isotropic laminate due to low-velocity impact. This knowledge is needed for the damage characterization and residual strength evaluations of these laminates.

The first part of the analysis mentioned above is the well known small deflection analysis of isotropic plates [2]. The second part, which is the analysis of isotropic membranes with arbitrary axisymmetric loading, however, needs further attention. The problem of an isotropic membrane subjected to surface and edge loads has been studied by many investigators [3-10]. Hencky [3], Dickey [4], and Shaw and Perrone [5] determined the deflection of a uniformly loaded membrane. Kao and Perrone [6] used a nonlinear relaxation method to obtain deflections in a uniformly loaded membrane. Goldberg and

Pifko [7] and Weinitschke [8,9] employed power series approaches to obtain the solutions for annular membranes. In addition to the power series method Weinitschke [8,9] presented an integral equation approach to the solution of annular membranes subjected to surface and edge loads. Callegari and Reiss [10] studied the axisymmetric deformations of a circular membrane subjected to arbitrary normal pressure by using the shooting method. The reason for the limited numerical solutions for arbitrary loaded membranes may be due to difficulties in using the conventional approaches used by earlier investigators. One of the difficulties may be due to the large differences in magnitudes of the radial and transverse displacements and the satisfaction of the two governing equations involving these displacements.

The purpose of this paper, therefore, is to present an alternate formulation for the analysis of circular isotropic membranes subjected to arbitrary axisymmetric loading. In contrast to the earlier governing equations involving radial and transverse displacements, in this formulation a single nonlinear governing differential equation is used in terms of the radial stress. The nonlinear differential equation was then replaced by a set of nonlinear algebraic equations using difference quotients. Then by using the Newton-Raphson method, these nonlinear algebraic equations were numerically solved to obtain stresses and displacements. This type of formulation involving a single governing equation avoids difficulties encountered with the differing magnitudes of radial and transverse displacements.

This approach is used to analyze (1) a uniformly loaded circular membrane, (2) a membrane with uniformly distributed load over the inner portion, and (3) a ring loaded circular membrane. The results for the radial and tangential stresses as well as the radial and transverse displacements are presented for each case. The results for the uniformly loaded circular membrane are compared with those from the literature.

LIST OF SYMBOLS

a	radius of membrane
E	Young's modulus
h	thickness of membrane
m	number of regions
n	node number
N_r	radial tension force
N_θ	tangential tension force
$p(r)$	pressure at any radius r
r, θ, z	cylindrical coordinate system
Δr	elemental length
u	radial displacement
$\bar{u}$	normalized radial displacement
w	transverse displacement
$\bar{w}$	normalized transverse displacement
$\varepsilon_r, \varepsilon_\theta$	radial and tangential strain
σ_r, σ_θ	radial and tangential stress
$\bar{\sigma}_r, \bar{\sigma}_\theta$	normalized radial and tangential stress
ν	Poisson's ratio

ANALYSIS

Membrane Configuration and Strain-Displacement Relations

Figure 1 shows an axisymmetrically loaded circular membrane with fixed peripheral edge, with thickness h and radius a. Since the deflection surface is axisymmetrical, the displacement can be resolved into two components: (1) a component u in the radial direction, and (2) a component w

perpendicular to the plane of the membrane. From large deflection theory [2], strain-displacement relations for an isotropic membrane are

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

$$\varepsilon_\theta = \frac{u}{r} \tag{1a}$$

and the corresponding stresses are,

$$\sigma_r = \frac{E}{(1 - \nu^2)}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right]$$

$$\sigma_\theta = \frac{E}{(1 - \nu^2)}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] \tag{1b}$$

Note that the assumption of large deflection but with small strains is made in this formulation.

Equilibrium Equations

The equation of equilibrium in the radial direction is

$$N_r - N_\theta + r\frac{dN_r}{dr} = 0 \tag{2}$$

If N_r and N_θ in Eq. (2) are replaced by $\sigma_r h$ and $\sigma_\theta h$, respectively, and the thickness of the membrane is assumed constant, the equation of equilibrium in radial direction is,

$$\sigma_r - \sigma_\theta + r\frac{d\sigma_r}{dr} = 0 \tag{3}$$

The equation of equilibrium in the direction perpendicular to the plane of the membrane, with general axisymmetric loading $p(r)$ can be written as

$$2\pi r h\sigma_r \frac{dw}{dr} + \int_0^r p(\xi)\ 2\pi\xi\ d\xi = 0 \tag{4}$$

The stress and strain displacement relations (Eq. (1)) and equilibrium equations (Eqs. (3) and (4)) when combined form four nonlinear partial differential equations with four unknowns σ_r, σ_θ, u, and w.

Derivation of a Governing Equation

Using the stress and strain displacement relations (Eq. (1)), the radial displacement u was expressed as

$$u = \frac{r}{E}\left(\sigma_\theta - \nu\sigma_r\right) \tag{5}$$

The term $\frac{du}{dr}$ was obtained by differentiating the radial displacement u, with respect to r.

$$\frac{du}{dr} = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) + \frac{r}{E}\left(\frac{d\sigma_\theta}{dr} - \nu\,\frac{d\sigma_r}{dr}\right) \tag{6}$$

The strain in the radial direction ε_r was expressed in terms of the radial and tangential stresses and by using the strain-displacement relations (Eq. (1)),

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) = \left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2\right] \tag{7}$$

By using $\frac{du}{dr}$ from Eq. (6) in Eq. (7) the following relationship was obtained:

$$\left(\sigma_\theta - \sigma_r\right)\left(\frac{1+\nu}{E}\right) + \frac{r}{E}\left[\frac{d\sigma_\theta}{dr} - \nu\,\frac{d\sigma_r}{dr}\right] + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = 0 \tag{8}$$

From the equation of equilibrium in radial direction (Eq. (3)), $\frac{d\sigma_r}{dr}$ was written as

$$\frac{d\sigma_r}{dr} = \frac{1}{r}\left(\sigma_r - \sigma_\theta\right) \tag{9}$$

Substituting Eq. (9) in Eq. (8) and rearranging the terms the following relationship was obtained:

$$\frac{r}{E}\left(\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr}\right) + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = 0 \tag{10}$$

Further, $\frac{d\sigma_\theta}{dr}$ was obtained by differentiating the equation of equilibrium (Eq. (3)) in radial direction with respect to r, as

$$\frac{d\sigma_\theta}{dr} = 2\frac{d\sigma_r}{dr} + r\frac{d^2\sigma_r}{dr^2} \tag{11}$$

Substituting this value of $\frac{d\sigma_\theta}{dr}$ in the governing equation (Eq. (10)), the following relationship was obtained:

$$\frac{r}{E}\left(3\frac{d\sigma_r}{dr} + r\frac{d^2\sigma_r}{dr^2}\right) + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = 0 \tag{12}$$

By using the equation of equilibrium (Eq. (4)), in the direction perpendicular to the plane of membrane, $\left(\frac{dw}{dr}\right)^2$ was written as

$$\left(\frac{dw}{dr}\right)^2 = \left[\frac{\int_0^r p(\xi)2\pi\xi\,d\xi}{2\pi r h\sigma_r}\right]^2 \tag{13}$$

Substituting this value of $\left(\frac{dw}{dr}\right)^2$ in the governing equation (Eq. (12)) gives

$$\frac{r}{E}\left(3\frac{d\sigma_r}{dr} + r\frac{d^2\sigma_r}{dr^2}\right) + \frac{1}{2}\left[\frac{\int_0^r p(\xi)\xi\,d\xi}{rh\sigma_r}\right]^2 = 0 \tag{14}$$

Equation (14) is a nonlinear differential equation in terms of the radial stress that governs the large deflection response of the membrane with arbitrary axisymmetric loading. Several investigators obtained the governing equation in terms of the radial stress σ_r, similar to Eq. (14). Dickey [4] and Weinitschke [8,9] presented the differential equation for uniformly distributed load. Callegari and Reiss [10] obtained the differential equation for a membrane with an arbitrary axisymmetric loading.

SOLUTION METHOD

To solve the governing equation (Eq. (14)) for the radial stress, Dickey [4] used integral equation method, Weinitschke [8,9] used integral equation and power series approaches and Callegari and Reiss [10] used the shooting method. In contrast, here a numerical method of solution is proposed. The nonlinear differential equation was replaced by a set of nonlinear algebraic equations using difference quotients. Then using the Newton-Raphson method [11], these nonlinear algebraic equations were solved numerically to obtain the stresses and displacements. The details of the solution method are as follows.

The solution domain was discretized into m regions and $(m + 1)$ nodes. Denoting $\sigma_r(n)$ as the radial stress at the nth node, the governing equation (Eq. (14)) was rewritten as, at any node n:

$$\frac{r_n}{E}\left[3\,\frac{d\sigma_r(n)}{dr} + r_n\,\frac{d^2\sigma_r(n)}{dr^2}\right] + \frac{1}{2}\left[\frac{\int_0^{r_n} p(\xi)\xi\,d\xi}{r_n h\sigma_r(n)}\right]^2 = 0 \tag{15}$$

To simplify the evaluation of the integral in the above equation, the following assumption will be made. Consider an ith region with $r_{i-1} < r < r_i$. Although the applied pressure varies within the region $r_{i-1} < r < r_i$, the pressure will be assumed to be uniform in this region with a value p_i. The magnitude of p_i is assumed equal to the value of the pressure at the mid-point of this region, i.e., at $r = (r_i + r_{i-1})/2$. As the number of regions in the model become large, the size of each region reduces and hence the variation of the pressure within each region also becomes insignificant. With this assumption the governing equation (Eq. (15)), reduces to

$$\frac{r_n}{E}\left[3\,\frac{d\sigma_r(n)}{dr} + r_n\,\frac{d^2\sigma_r(n)}{dr^2}\right]$$

$$+\frac{1}{8}\left[\frac{1}{r_n h\sigma_r(n)}\sum_{i=1}^{n} p_i\left(r_i^2 - r_{i-1}^2\right)\right]^2 = 0 \tag{16}$$

First and second derivatives of the radial stress from the governing equation (Eq. (16)) were replaced by

$$\left.\frac{d\sigma_r}{dr}\right|_{\text{nth node}} = \frac{\sigma_r(n+1) - \sigma_r(n-1)}{2\ \Delta r}$$

and

$$\left.\frac{d^2\sigma_r}{dr^2}\right|_{\text{nth node}} = \frac{\sigma_r(n+1) - 2\sigma_r(n) + \sigma_r(n-1)}{(\Delta r)^2}$$

For the circular isotropic membrane with axisymmetric loading the boundary conditions were:

(1) Both the radial displacement u and the transverse displacement w equal zero at the fixed edge ($r = a$).

(2) The radial displacement u and the slope $\frac{dw}{dr}$ equal zero at the center ($r = 0$).

Since the governing equation (Eq. (14)) was derived in terms of the radial stress, the transformation of boundary conditions was done by using stress and strain-displacement relations (Eq. (1)), and equilibrium equations (Eqs. (3) and (4)). When $u = 0$ and $r = a$ are substituted into Eq. (5) for radial displacement, the boundary condition (1) above can be transformed to

$$\sigma_\theta = \nu\sigma_r, \quad \text{at} \quad r = a \tag{17}$$

Using this boundary condition in the equilibrium equation in radial direction (Eq. (3)), (Eq. (17)), was further transformed to

$$\sigma_r(1 - \nu) + r\frac{d\sigma_r}{dr} = 0, \quad \text{at } r = a \tag{18}$$

The boundary condition (2) above is a statement of symmetry about $r = 0$, the center of the membrane. This symmetry condition can be expressed in terms of the radial stresses,

$$\frac{d\sigma_r}{dr} = 0, \quad \text{at} \quad r = 0 \tag{19}$$

The derivative $\frac{d\sigma_r}{dr}$ was replaced by the finite difference quotient as

$$4\sigma_r(2) - 3\sigma_r(1) - \sigma_r(3) = 0, \quad \text{at} \quad r = 0.$$

Using the governing equation (Eq. (16)),

$$\frac{r_n}{E}\left[3\,\frac{d\sigma_r(n)}{dr} + r_n\,\frac{d^2\sigma_r(n)}{dr^2}\right] + \frac{1}{8}\left\{\sum_{i=1}^{n}\frac{p_i\left(r_i^2 - r_{i-1}^2\right)}{r_n h\sigma_r(n)}\right\}^2 = 0$$

and boundary conditions (Eqs. (18) and (19)) in the form of finite difference quotients at each node, $m + 1$, nonlinear algebraic equations were obtained. These algebraic equations contained $m + 1$ unknowns, viz., $\sigma_r(1) \ldots \sigma_r(m + 1)$. By using the Newton-Raphson technique [11], these $(m + 1)$ equations were solved to obtain the radial stress at each node.

The equation of equilibrium in the radial direction (Eq. (3)), was used to obtain the tangential stress, σ_θ, at each node. The radial displacement, u, and the slope, $\frac{dw}{dr}$, at any node were obtained by using Eqs. (5) and (12), respectively, and the σ_r and σ_θ values at that node. Then using the boundary conditions,

$$w = 0 \quad \text{at} \quad r = a,$$

$$\frac{dw}{dr} = 0 \quad \text{at} \quad r = 0$$

and the finite difference representation of the slope, $\frac{dw}{dr}$, at all the $(m + 1)$ nodes, the transverse displacements, w, were determined.

RESULTS AND DISCUSSION

In this section, first a convergence study for the above outlined method is presented. Then, the present method is illustrated for circular membranes with different loadings, shown in figure 2:

(1) Uniformly loaded circular membrane.

(2) A membrane with uniformly distributed load over the inner portion.

(3) A membrane with a ring load.

For all the cases analyzed a Poisson's ratio of 0.3 was assumed. The stresses and displacements were expressed in the dimensionless forms, using Hencky's normalized formulation as,

$$\bar{u} = \frac{u}{a\left(\frac{pa}{Eh}\right)^{2/3}}$$

$$\bar{w} = \frac{w}{a\left(\frac{pa}{Eh}\right)^{1/3}}$$

$$\bar{\sigma}_r = \frac{\sigma_r}{\left(\frac{p^2 a^2 E}{h^2}\right)^{1/3}}$$

$$\bar{\sigma}_\theta = \frac{\sigma_\theta}{\left(\frac{p^2 a^2 E}{h^2}\right)^{1/3}}$$

Convergence Study

To study the convergence of the present method, the circular membrane was idealized into m number of regions with (m + 1) nodes, (where nodes are numbered from the center to the outside). The number of regions, m, used in this convergence study were 8, 16, 32, 60, and 64. Figures 3(a) and 3(b) present the relative errors in the normalized maximum deflections and stresses, for a uniformly loaded membrane and for a membrane loaded uniformly over the region $0 < \frac{r}{a} < 0.5$. The solution shows rapid convergence and about 60 regions were found to be necessary for a membrane loaded uniformly over the region $0 < \frac{r}{a} < 0.5$, to yield a solution which is within 0.001 percent of the converged solution. In contrast much fewer than 60 regions were necessary for the uniformly loaded membrane. However, a 60 region idealization is used in the analysis and all the results are presented for this idealization.

The Newton-Raphson method [11] used here needs initial $\bar{\sigma}_r$ values for the iterative process. The sensitivity of the present method to these initial values was studied. Table 1 presents the number of iterations needed to achieve convergence for a range of initial $\bar{\sigma}_r$ values for the three configurations studied. The initial $\bar{\sigma}_r$ values were as low as three orders of magnitude smaller than the maximum converged value of $\bar{\sigma}_r$ and as high as three orders of magnitude higher. For the uniformly loaded membrane with each of these initial values the present method converged to the same solution. The other two configurations showed similar convergence, thus showing insensitivity to the initial $\bar{\sigma}_r$ values. When the initial $\bar{\sigma}_r$ values were farther away from the converged values the number of iterations needed were around 40 compared to about 8 when the initial $\bar{\sigma}_r$ values were closer to the converged $\bar{\sigma}_r$ values.

Uniformly Loaded Circular Membrane

The first problem analyzed was that of a uniformly loaded circular membrane, for which classical solution [ref. 3] exists. The uniform loading was represented by setting the magnitude of loading terms p_1 through p_{61} equal to unit values. Using these values in Eq. (16), the governing equation for the membrane was solved by using the procedure outline earlier. The values of $\bar{u}$ and $\bar{w}$ displacements and stresses $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ are presented in figures 4 and 5, respectively. A comparison of the present solution with Hencky's classical solution [3] and Kao and Perrone [6] nonlinear relaxation method solution is shown in Table 2. The central displacement as well as central and edge radial stresses obtained by present method are in excellent agreement with the earlier reported results as shown in Table 2.

A Membrane With Uniformly Distributed Load Over the Inner Portion

The second example is that of a membrane with uniformly distributed load over an inner portion. A particular case of loading over the region $0 < \frac{r}{a} < 0.1$ was analyzed. In the corresponding solution, this loading was represented by setting the magnitude of loading terms p_1 through p_6 equal to unit values and terms p_7 through p_{61} equal to zero. Again using the governing equation (Eq. (16)) with the loading terms discussed before, the solution was obtained for stresses and displacements at each node. The corresponding values of $\bar{u}$ and $\bar{w}$ displacements and $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ stresses are shown in figures 6 and 7, respectively.

A Membrane With a Ring Load:

The last problem considered here is one where the membrane carried a uniformly distributed ring load. As a specific case the ring load was assumed to be spread over the region $0.5 < \frac{r}{a} < 0.6$. Since loading was considered over the region $0.5 < \frac{r}{a} < 0.6$, in the corresponding solution, this loading was represented by setting the magnitude of loading terms p_1 through p_{31} equal to zero, p_{32} through p_{37} equal to unit values; and p_{38} through p_{61} equal to zero. Using these values in Eq. (16), the governing equation for the membrane with the ring load was solved by using the procedure outlined earlier.

The corresponding values of $\bar{u}$ and $\bar{w}$ displacements are presented in figure 8 and stresses $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ are presented in figure 9. Figure 8 shows that the transverse displacement $\bar{w}$ is constant up to the ring load and then starts decreasing and becomes zero at the clamped edge. In contrast, the radial displacement $\bar{u}$ is zero at the center, increases gradually until it reaches the peak value in the loading region and then starts decreasing and becomes zero at the fixed edge. On the other hand, the normalized stresses $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$ are of the identical magnitude in the unloaded region $0 < \frac{r}{a} < 0.5$ and then decrease for larger values of r.

As pointed out in the introduction, when $\bar{u}$ and $\bar{w}$ differ by large amounts a simultaneous method of solution presents difficulties. As the present method does not use $\bar{u}$ and $\bar{w}$ as parameters, rather uses a single parameter, $\bar{\sigma}_r$ these difficulties are avoided. Therefore, it is interesting to compare the differences in magnitudes of $\bar{u}$ and $\bar{w}$ values for various problems analyzed. For a membrane with uniformly distributed load $\bar{u}$ and $\bar{w}$ are of the same magnitude, whereas in the case of membrane loaded over an inner portion and for the ring loaded membrane, the displacements differ by two and one order of magnitude, respectively.

Because $\bar{u}$ and $\bar{w}$ do not differ by large amounts in the case of uniformly loaded membrane, one would expect the simultaneous solution to be efficient and feasible. Indeed, it is so as demonstrated by Kao and Perrone [6]. Because $\bar{u}$ and $\bar{w}$ differ by large amounts for the arbitrarily loaded membranes, one would expect difficulties with simultaneous solution method. This may be the reason for the limited numerical of the solutions for these two loading cases. The present method on the other hand avoided these problems by using a governing equation in a single parameter, the radial stress. The present method is demonstrated to have good convergence characteristics even when the initial $\bar{\sigma}_r$ values are unrealistically large or small. Therefore, the nonlinear governing equation in terms of the radial stress and the Newton-Raphson technique appear to be the ideal choice for large deflection problems of arbitrarily loaded membranes.

CONCLUSIONS

Circular membranes with fixed peripheral edges, subjected to arbitrary axisymmetric loading, were analyzed. Earlier finite difference methods used transverse and radial displacements as parameters and developed two nonlinear simultaneous equations. These differential equations were solved simultaneously, to obtain solutions for transverse and radial displacements. However because of solution difficulties, only limited results are available. In the present formulation a different apprach was taken. Instead of two nonlinear simultaneous differential equations, a single nonlinear differential equation in terms of radial stress was developed. This nonlinear equation was solved by using the finite difference method in conjunction with Newton-Raphson method. Three different loading cases, namely, (a) uniformly loaded membrane, (b) a membrane with uniform load over an inner portion, and (c) a membrane with ring load, were analyzed.

The results from the present method show that for arbitrary loaded membrane, transverse and radial displacements differ by large amounts, which probably explains why the simultaneous solution method fails to yield a satisfactory solution.

The present method shows good convergence characteristics for various types of loadings. Also central displacement and the central and edge radial stresses for uniformly loaded membrane by the present method agreed very well with the classical solution.

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Table 1. Sensitivity of the present method of the initial values of $\bar{\sigma}_r$[a]

$$\bar{\sigma}_r = \frac{\sigma_r}{\left(\frac{p^2 a^2 E}{h^2}\right)^{1/3}}$$

Initial $\bar{\sigma}_r$ values[b]	Number of iterations required for convergence		
	Uniform loading over the region		
	$0 < \frac{r}{a} < 1$	$0 < \frac{r}{a} < 0.1$	$0.5 < \frac{r}{a} < 0.6$
0.0001	25	22	23
0.001	20	16	17
0.01	14	10	11
0.1	8	8	5
1.0	9	19	15
10.0	20	27	26
100.0	31	41	37
Converged $\bar{\sigma}_r$ (r = 0)	0.4310	0.1104	0.1237
Converged $\bar{\sigma}_r$ (r = a)	0.3329	0.0235	0.0971

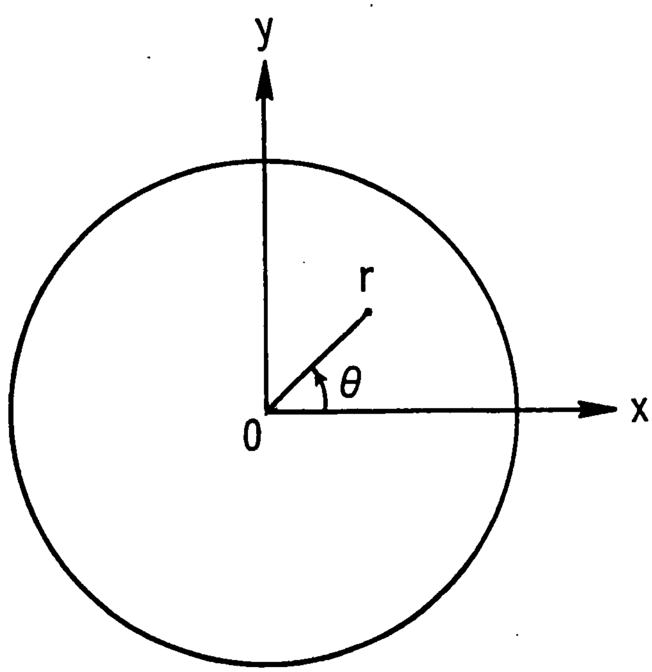
[a]Membrane idealized with 60 regions.
[b]Constant radial stress $\bar{\sigma}_r$ was assumed at all 61 nodes.

TABLE 2. Comparison of Normalized Displacements and Stresses for Uniformly Loaded Membrane

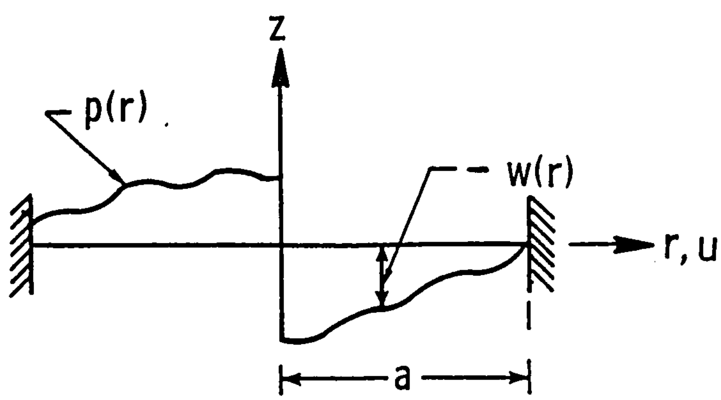
$$\bar{w} = \frac{w}{a\left(\frac{pa}{Eh}\right)^{1/3}}, \quad \bar{\sigma}_r = \frac{\sigma_r}{\left(\frac{p^2a^2E}{h^2}\right)^{1/3}}$$

	Central transverse deflection $\bar{w}$ at $r = 0$	Central radial stresss $\bar{\sigma}_r$ at $r = 0$	Edge radial stress $\bar{\sigma}_r$ at $r = a$
Hencky* (ref. 3)	0.6536	0.4310	0.3280
Kao and Perrone (ref. 6)	0.6541	0.4289	0.3306
Present results	0.6534	0.4310	0.3329

*Values taken from reference 6.

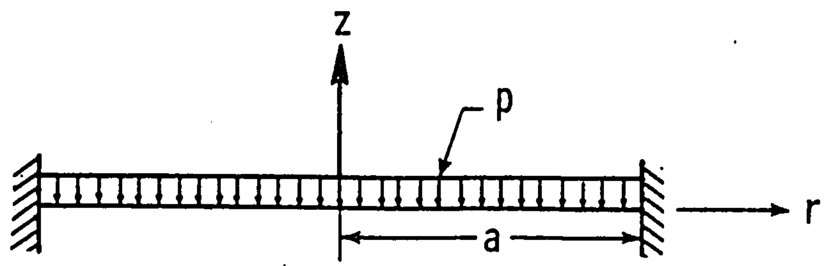


(a) Circular Membrane

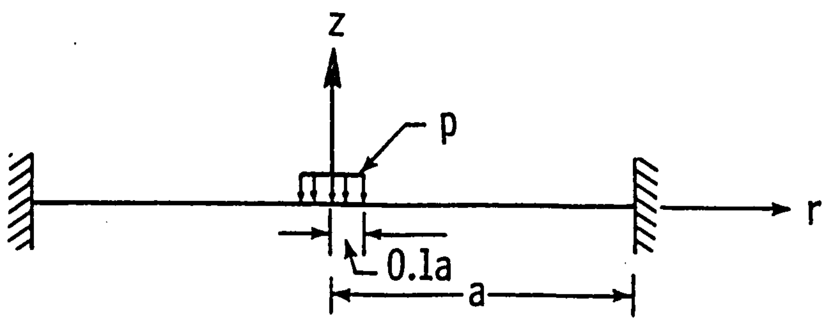


(b) Loading and deformations

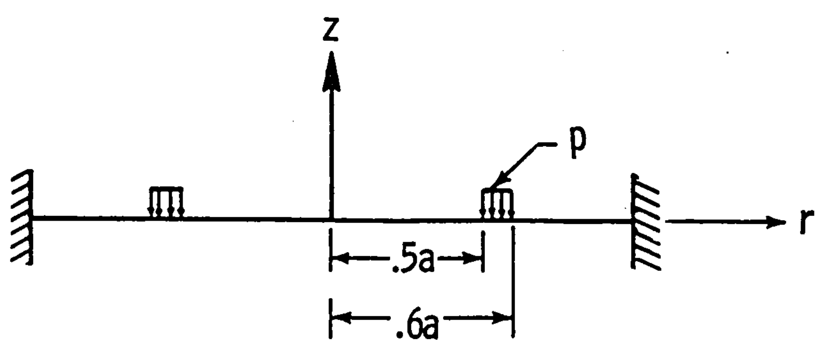
Figure 1.- Membrane configuration.



(a) Uniform loading

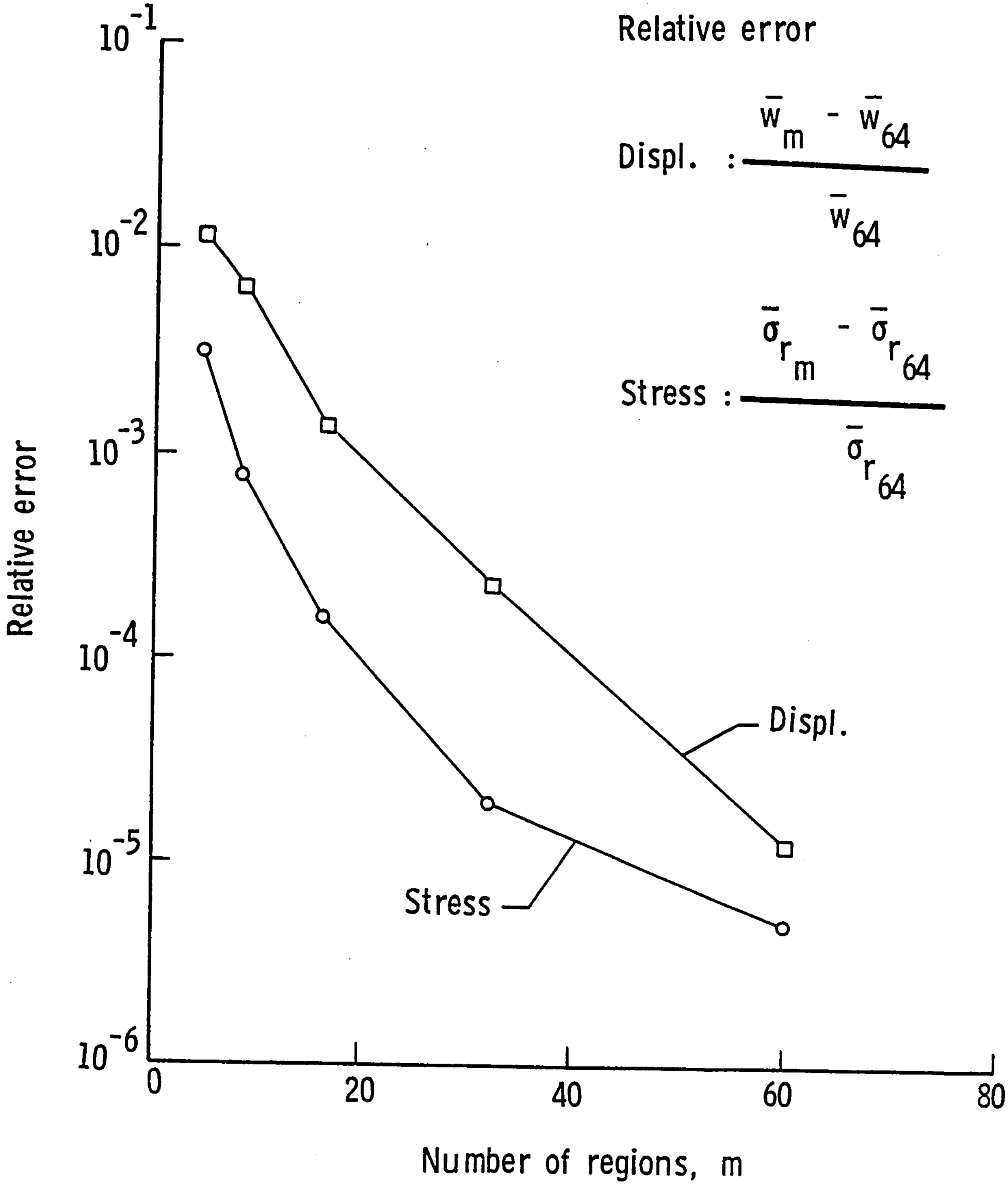


(b) Uniform load over inner portion



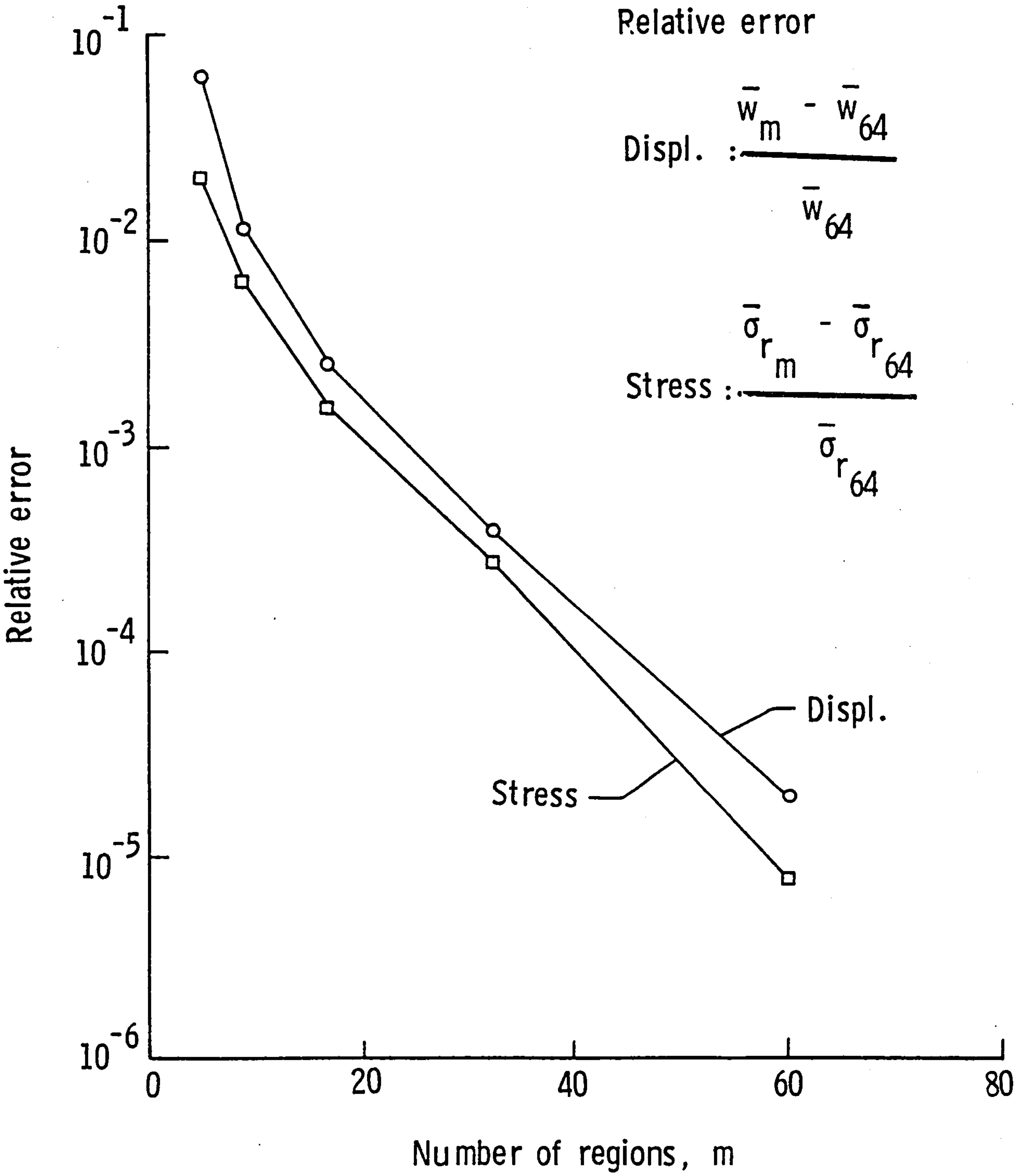
(c) Ring loading

Figure 2.- Types of loading on the membrane.



(a) Uniform loading.

Figure 3.- Convergence study for the membrane.



(b) Uniform loading over the region $0 \leq r/a \leq 0.5$.

Figure 3 - Convergence study for the membrane.

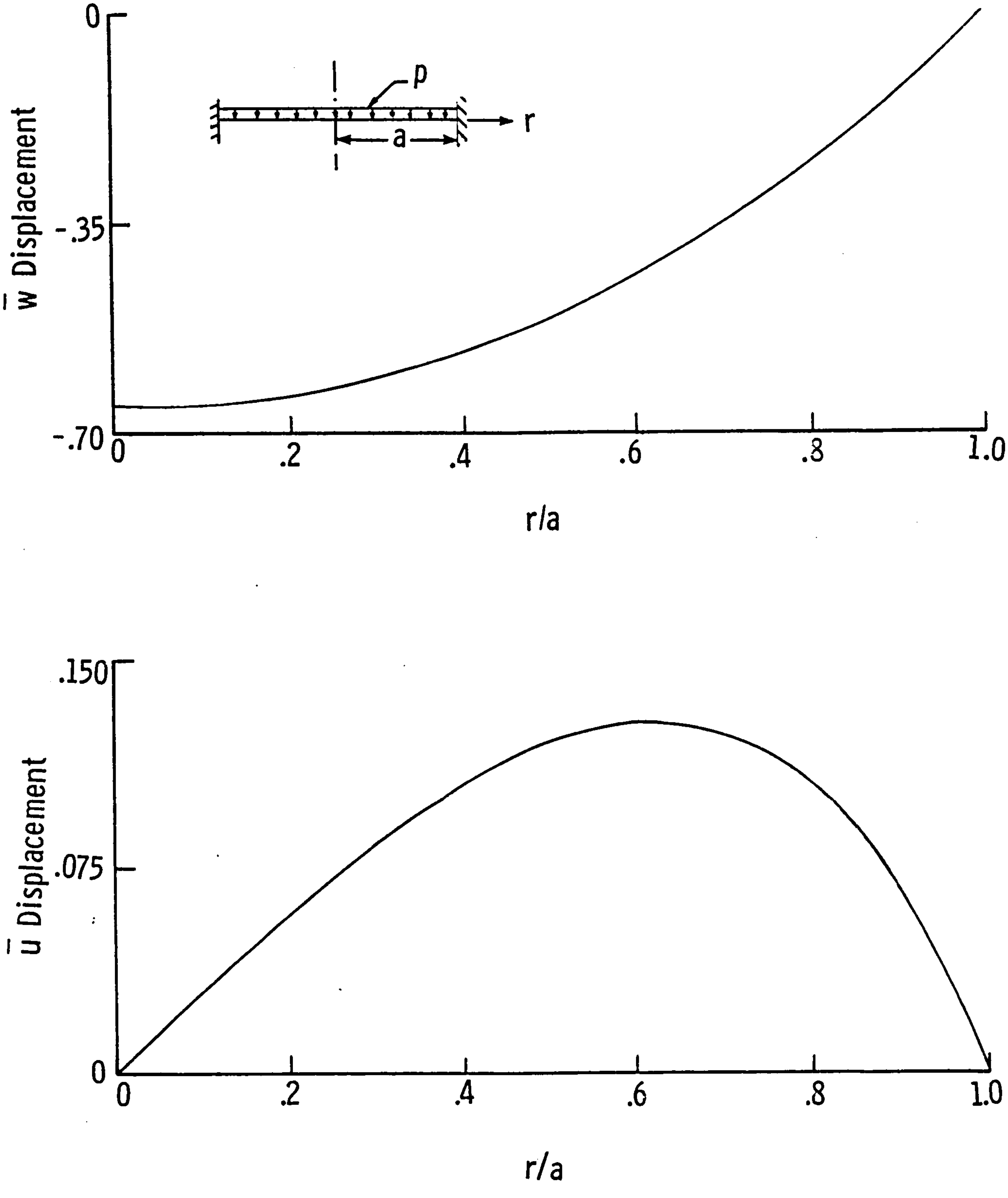


Figure 4.- Normalized displacements for uniformly loaded membrane.

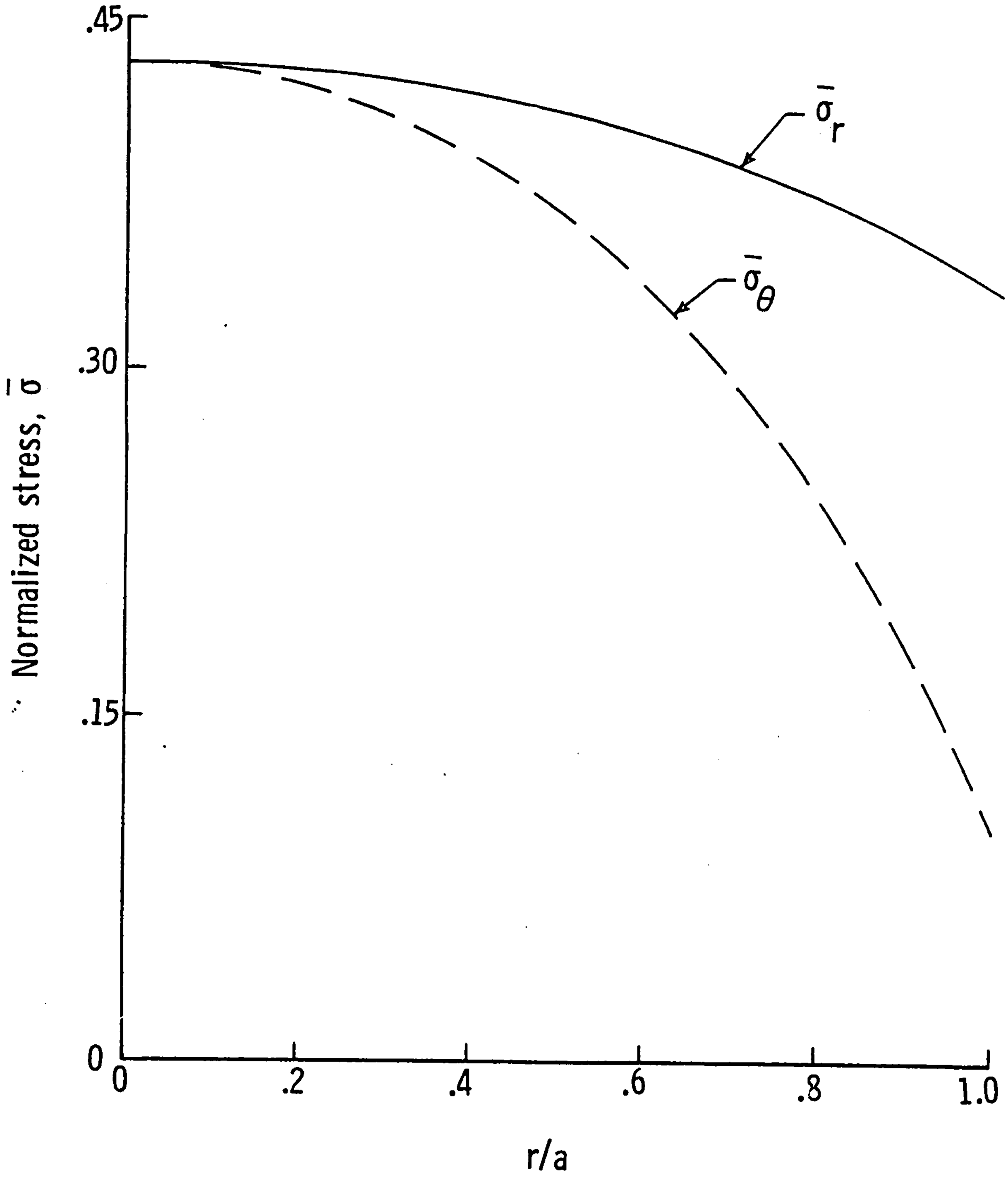


Figure 5.- Normalized stresses for uniformly loaded membrane.

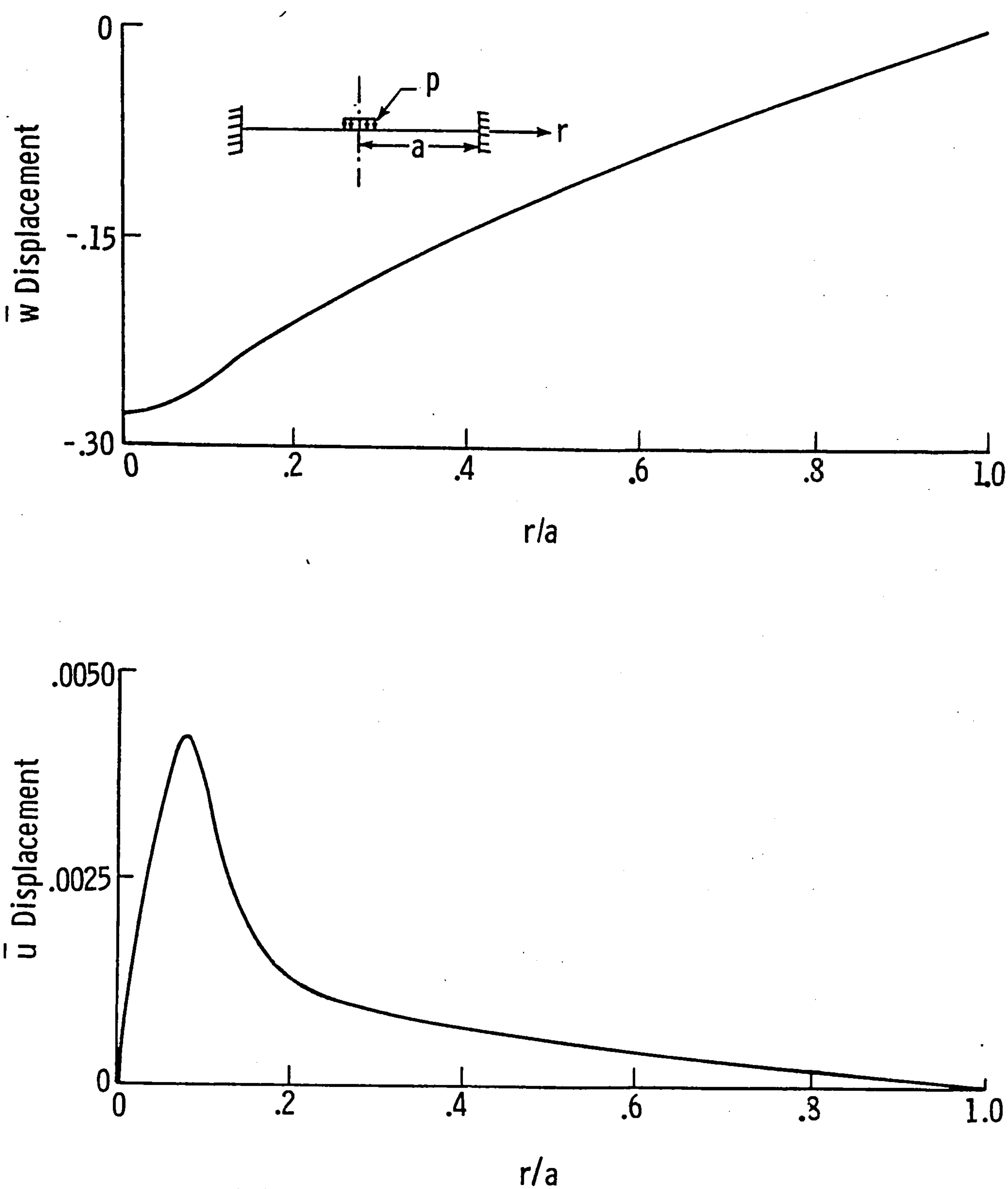


Figure 6.- Normalized displacements in a membrane uniformly loaded over the inner portion $0 \leq r/a \leq 0.1$.

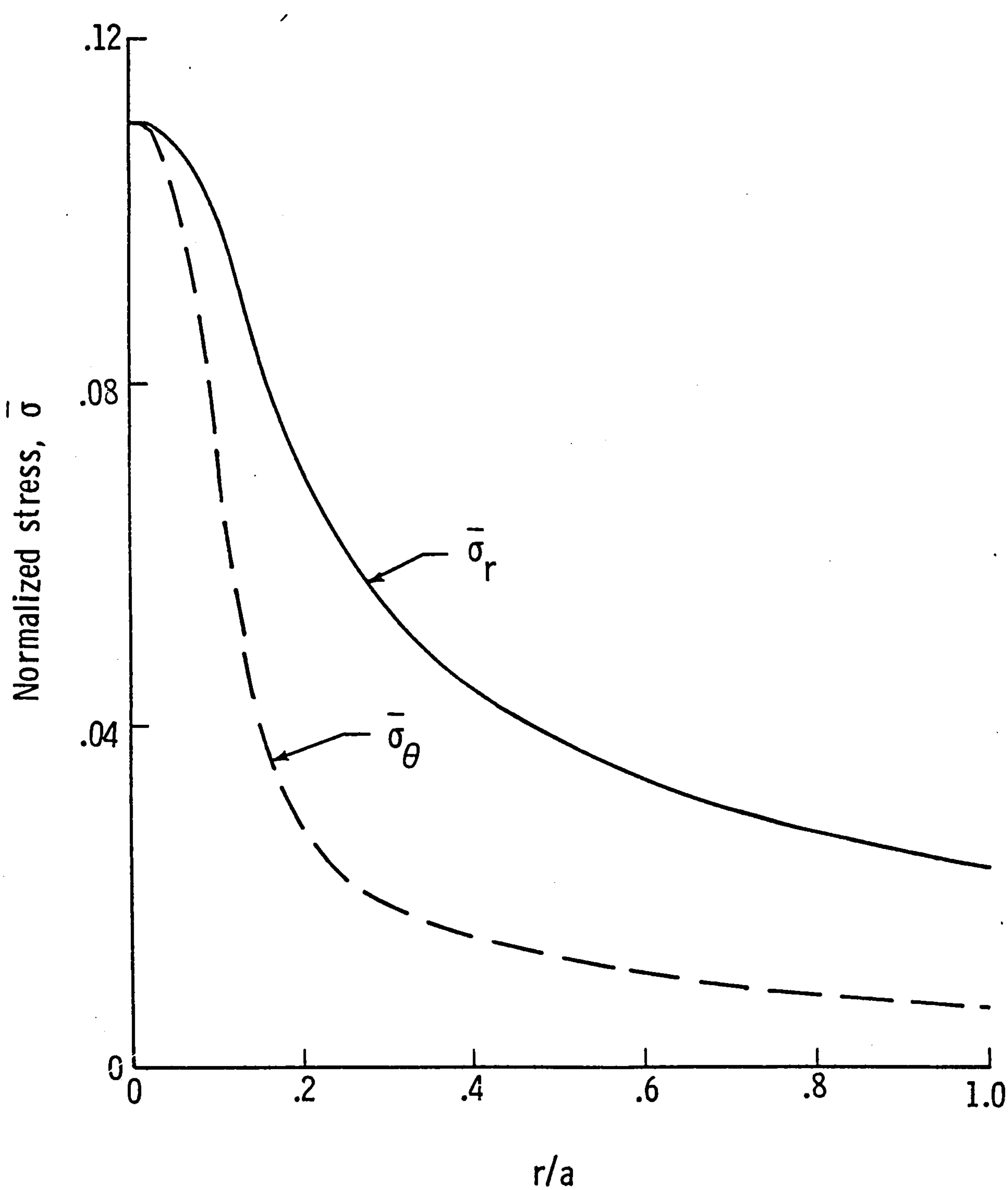


Figure 7.- Normalized stresses in a membrane uniformly loaded over the inner portion $0 \leq r/a \leq 0.1$.

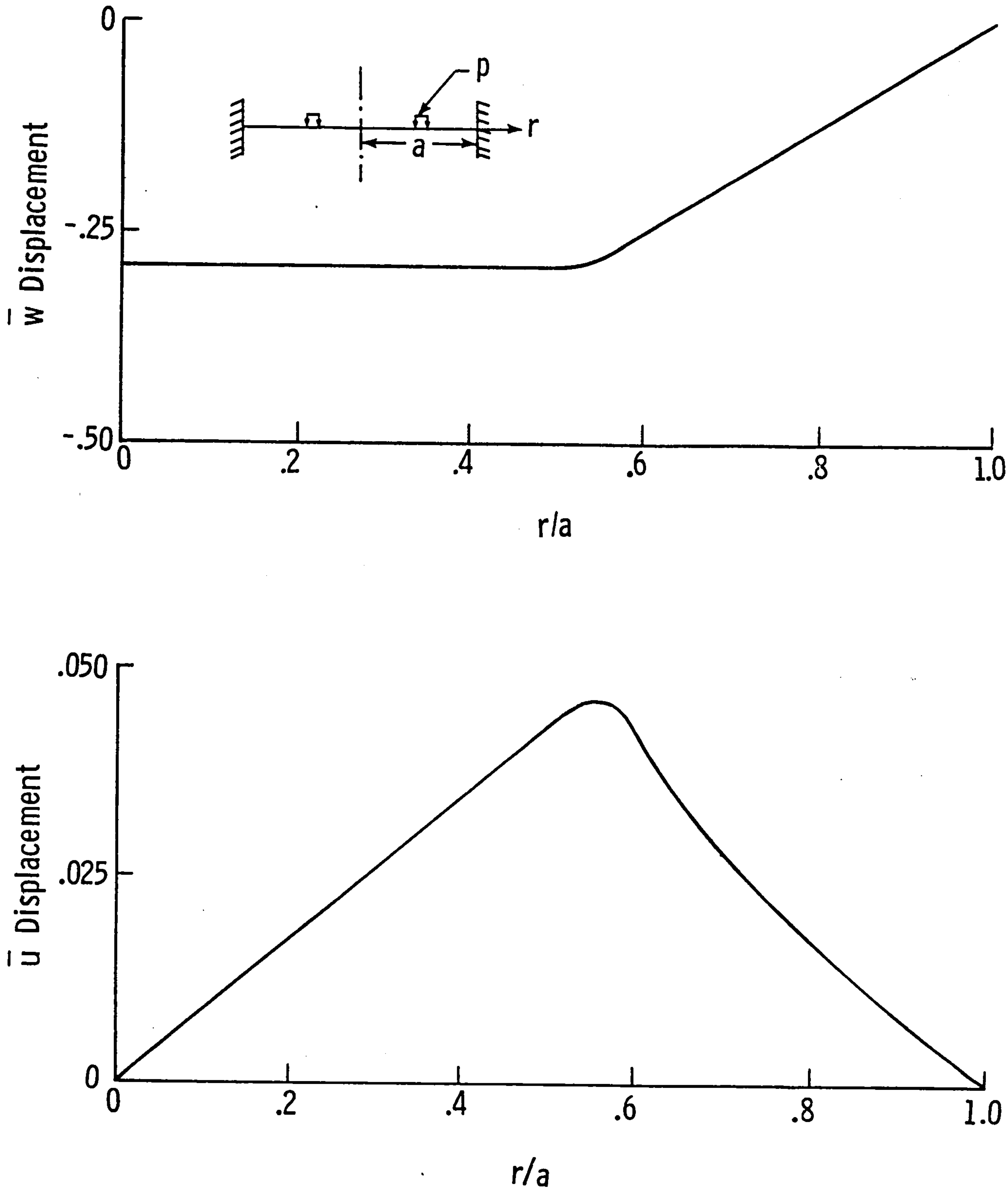


Figure 8.- Normalized displacements in a membrane with ring loading over the region $0.5 \leq r/a \leq 0.6$.

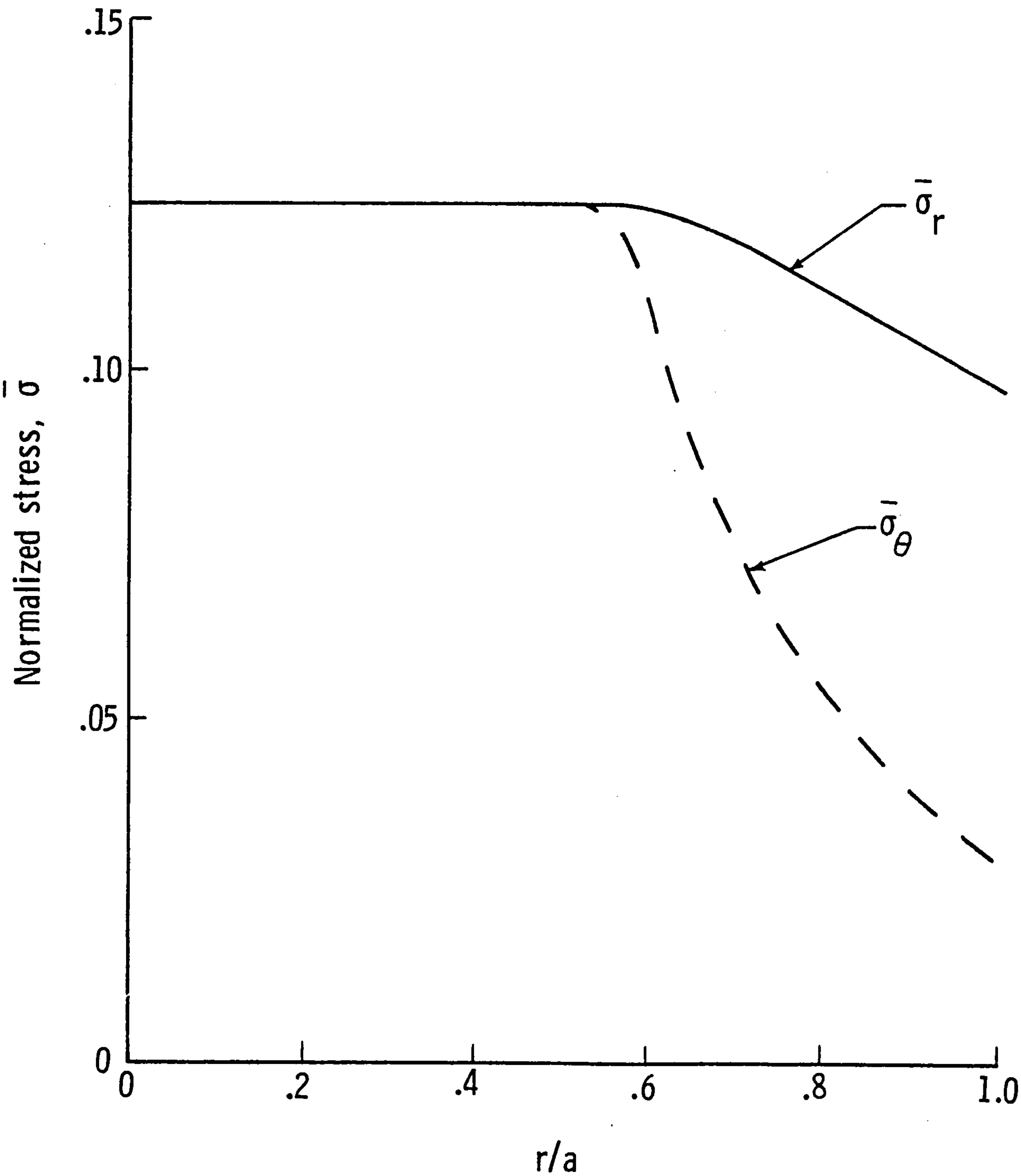


Figure 9.- Normalized stresses in a membrane with ring loading over the region $0.5 \leq r/a \leq 0.6$.

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15. Supplementary Notes

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16. Abstract

Circular membranes with fixed peripheral edges, subjected to arbitrary axisymmetric loading were analyzed. A single governing differential equation in terms of radial stress was used. This nonlinear governing equation was solved using the finite difference method in conjunction with Newton-Raphson method. Three loading cases, namely (a) uniformly loaded membrane, (b) a membrane with uniform load over an inner portion, and (c) a membrane with ring load, were analyzed. Calculated central displacement and the central and edge radial stresses for uniformly loaded membrane, agreed extremely well with the classical solution.

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