

ON THE ROOTS OF THE EQUATION $\zeta(s) = a$

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ABSTRACT. Given any complex number a , we prove that there are infinitely many simple roots of the equation $\zeta(s) = a$ with arbitrarily large imaginary part. Besides, we give a heuristic interpretation of a certain regularity of the graph of the curve $t \mapsto \zeta(\frac{1}{2} + it)$. Moreover, we show that the curve $\mathbb{R} \ni t \mapsto (\zeta(\frac{1}{2} + it), \zeta'(\frac{1}{2} + it))$ is not dense in $\mathbb{C}^2$.

KEYWORDS: Riemann zeta-function, value-distribution

MATHEMATICAL SUBJECT CLASSIFICATION: 11M06

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The Riemann zeta-function $\zeta(s)$ is of special interest in number theory and complex analysis. The real zeros of $\zeta(s)$ are called trivial; they are located at $s = -2n, n \in \mathbb{N}$. All other zeros are called nontrivial; they lie in the critical strip $0 < \operatorname{Re} s < 1$ and are known to be relevant in many questions concerning the distribution of prime numbers. It is well-known that there are infinitely many nontrivial zeros. More precisely, for the number $N(T)$ of nontrivial zeros $\rho = \beta + i\gamma$ satisfying $0 < \gamma \leq T$ the asymptotical formula

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

holds as $T \rightarrow \infty$. Conrey [7] has shown that more than two fifths of the zeros lie on the critical line $\operatorname{Re} s = \frac{1}{2}$ and are simple; this result has been slightly improved by Bui, Conrey & Young [6]. The famous yet unproved Riemann hypothesis states that all nontrivial zeros lie on the critical line and the simplicity hypothesis claims that all (or at least almost all) zeros are simple. In this article we are concerned with the general value-distribution. A famous open problem in this direction is the question whether the values $\zeta(\frac{1}{2} + it)$ for $t \in \mathbb{R}$ are dense in the complex plane.

The zeta-function has no exceptional values (in the meaning of Nevanlinna theory) except infinity as was shown by Ye [29] (see also [25], Chapter 7). There are remarkable quantitative results. For example, it was shown by Bohr & Jessen [3] that $\log \zeta(s)$ assumes any complex value infinitely often in any vertical strip $\sigma_1 < \operatorname{Re} s < \sigma_2$ satisfying $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, and that for fixed $a \neq 0$ the number of

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such roots of the equation $\zeta(s) = a$ with imaginary part bounded by T has linear asymptotic growth as $T \rightarrow \infty$. For arbitrary complex a the roots of

$$\zeta(s) = a$$

are called a -points and are denoted by $\rho_a = \beta_a + i\gamma_a$. There is an a -point near any trivial zero $s = -2n$ for sufficiently large n and apart from these a -points there are only finitely many other a -points in the half-plane $\sigma \leq 0$ (see Lemma 6 below). The a -points with $\beta_a \leq 0$ are said to be trivial; all other a -points are called nontrivial. For any fixed a , there exist left and right half-planes free of nontrivial a -points (see formulas (12) and (20)). As in the case of zeros ($a = 0$) there is a Riemann-von Mangoldt-type formula for the number $N_a(T)$ of nontrivial a -points with imaginary part γ_a satisfying $0 < \gamma_a \leq T$, namely, as $T \rightarrow \infty$,

$$(1) \quad N_a(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e c_a} + O(\log T)$$

with the constant $c_a = 1$ if $a \neq 1$, and $c_1 = 2$. This was first proved by Landau [4]¹ (see also [25], Chapter 7). We observe that these asymptotics are essentially independent of a :

$$N_a(T) \sim N(T).$$

Levinson [19] proved that all but $O(N(T)/\log \log T)$ of the a -points with imaginary part in $T < t < 2T$ lie in

$$(2) \quad |\operatorname{Re} s - \tfrac{1}{2}| < \frac{(\log \log T)^2}{\log T},$$

and hence the a -points are clustered around the critical line. In the special case of zeros this result was obtained by several mathematicians in the beginning of the 20th century and was sometimes misleadingly interpreted as indicator for the truth of the Riemann hypothesis. Levinson's result was first proved by Landau [4] under assumption of the truth of the Riemann hypothesis.

We turn to the value-distribution on the critical line. It is known (see Corollary 3 in Spira [23]) that $\zeta'(\frac{1}{2} + it) \neq 0$ if $\zeta(\frac{1}{2} + it) \neq 0$. Hence, there are no multiple a -points of $\zeta(s)$ on the critical line $\operatorname{Re} s = \frac{1}{2}$ except for possibly $a = 0$. It follows that not all combinations of values for the zeta-function and its derivative are possible. In this direction the following theorem is true.

Theorem 1. *The set*

$$\{(\zeta(\tfrac{1}{2} + it), \zeta'(\tfrac{1}{2} + it)) : t \in \mathbb{R}\}$$

is not dense in $\mathbb{C}^2$.

It follows that the only possible singularities of the curve $t \mapsto \zeta(\frac{1}{2} + it)$ have to lie in the origin. By this result we see that the curve $\mathbb{R} \ni t \rightarrow (\zeta(\frac{1}{2} + it), \zeta'(\frac{1}{2} + it))$ fails to visit all neighborhoods of all points in $\mathbb{C}^2$. If the Riemann hypothesis is true, the values $\zeta(\sigma + it)$ for $t \in \mathbb{R}$ are not dense for any fixed $\sigma < \frac{1}{2}$ (see Proposition 5 below). As mentioned above, it is unknown whether the values of the zeta-function

¹The paper [4] of Bohr, Landau & Littlewood consists of three independent chapters, the first belonging essentially to Bohr, the second to Landau, and the third to Littlewood.

on the critical line are dense in the complex plane or not. It was shown by Voronin [28] that the multidimensional analogue is true for vertical lines in the open right half of the critical strip: the set $\{(\zeta(\sigma + it), \zeta'(\sigma + it), \dots, \zeta^{(n-1)}(\sigma + it)) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$ for all positive integers n for every fixed $\sigma \in (\frac{1}{2}, 1)$. Actually, this result was proved by Voronin previous to his famous universality theorem which may be interpreted as an infinite analogue (see [13, 25]); the case $n = 1$ is due to Bohr & Courant [2] (see Figure 1). However, the situation on the critical line is completely different as follows from Theorem 1 above; in particular we see that Voronin's universality theorem cannot be extended to any region that covers the critical line.

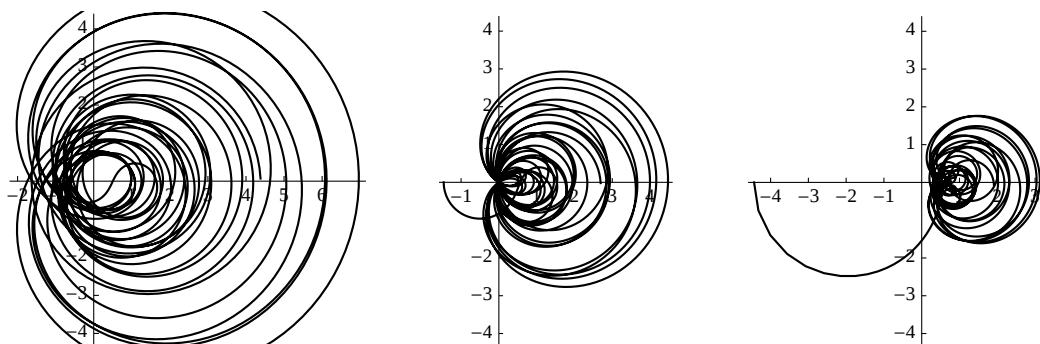


Figure 1: The curves $t \mapsto \zeta(\sigma + it)$ for $\sigma = \frac{1}{5}, \frac{1}{2}$, and $\frac{4}{5}$ from left to right, all for $t \in [0, 100]$. The curve on the right is known to be dense in the complex plane, the curve on the left is not dense if Riemann's hypothesis is true, and for the curve in the middle this question is open.

There is another related topic we want to investigate. The set of values of $\zeta(s)$ has the cardinality of the continuum. For some values of a the underlying a -points might be non-simple, namely if the derivative vanishes; however, $\zeta'(s)$ has only countably many zeros. Thus, there are only countably many numbers a such that among all a -points there is at least one non-simple a -point. We conjecture that for any fixed complex number a , almost all a -points are simple. By a rather simple method Conrey, Ghosh & Gonek [8] proved that there are infinitely many simple zeros of the zeta-function; in [9] they have shown by technical refinement that more than $\frac{19}{27}$ of the nontrivial zeros are simple provided the Riemann hypothesis for the Riemann zeta-function and the generalized Lindelöf hypothesis for all Dirichlet L -functions are true; the latter condition has been removed by Bui & Heath-Brown [5]. It is our aim to extend their method to simple a -points; however, for the sake of simplicity and unconditionality, here we are not concerned with this type of quantitative results.

Theorem 2. *Let a be any fixed complex number. As $T \rightarrow \infty$,*

$$\begin{aligned} \sum_{0 < \gamma_a \leq T} \zeta'(\rho_a) &= \left(\frac{1}{2} - a\right) \frac{T}{2\pi} \left(\log \frac{T}{2\pi}\right)^2 + (c_0 - 1 + 2a) \frac{T}{2\pi} \log \frac{T}{2\pi} \\ &\quad + (1 - c_0 - c_0^2 - 3c_1 - 2a) \frac{T}{2\pi} + E(T), \end{aligned}$$

where the summation is over nontrivial a -points $\rho_a = \beta_a + i\gamma_a$, the numbers c_n are the Stieltjes constants (defined by (3) below), and the error term is $E(T) \ll T \exp(-C(\log T)^{1/2})$ with some absolute positive constant C ; if the Riemann hypothesis is true, then $E(T) \ll T^{1/2+\epsilon}$. In any case, for any complex number a there exist infinitely many simple a -points.

It should be mentioned that Gonek, Lester & Milinovich [15] proved, subject to certain hypotheses, that a positive proportion of the a -points of $\zeta(s)$ are simple; they also obtained an unconditional result for the a -points in fixed strips to the right of the critical line.

In view of the asymptotic formula of Theorem 2 the value $a = \frac{1}{2}$ appears to be somehow special for the zeta-function since in this case the main term is of lower order. It is easy to see that for $a = \frac{1}{2}$ the second order term does not vanish. In fact, the Stieltjes constants are the coefficients of the Laurent series expansion of zeta at $s = 1$,

$$(3) \quad \zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{n!} (s-1)^n,$$

the constant term $c_0 = \lim_{N \rightarrow \infty} (\sum_{n=1}^N \frac{1}{n} - \log N) = 0.577\dots$ is the Euler-Mascheroni constant (see Ivić [18]). We do not know why the value $\frac{1}{2}$ is special in this sense. Nevertheless, the asymptotics from Theorem 2 serve very well for a *heuristic* explanation of the very regular behaviour of the curve $t \mapsto \zeta(\frac{1}{2} + it)$ as we shall explain now. Based on computations by Haselgrove [16], Shanks [22] observed that $\zeta(\frac{1}{2} + it)$ approaches its zeros most of the times from the 3rd or 4th quadrant, following Gram's law.² It was conjectured by Shanks that the values $\zeta'(\frac{1}{2} + i\gamma)$ are positive real in the mean, where γ runs through the set of positive ordinates of the nontrivial zeros. This follows from the asymptotics obtained by Conrey, Ghosh & Gonek [8] under assumption of the Riemann hypothesis. More precise asymptotical formulas were derived by Fujii [10, 12] (see (19) below). Theorem 2 extends these results to general a . By Levinson's result (2) almost all a -points lie arbitrarily close to the critical line, so we may expect that the main contribution results from these a -points. Notice that the tangent to the curve $t \mapsto \zeta(\frac{1}{2} + it)$ is given by $i\zeta'(\frac{1}{2} + it)$. In conclusion, the main term $(\frac{1}{2} - a) \frac{T}{2\pi} \log T$ describes how the values $\zeta(\frac{1}{2} + it)$ approach the value a in the complex plane on average (see Figure 2).

Finally, we shall prove another theorem of the same flavour.

²Recently, it was shown by Trudgian [27] that Gram's law fails for a positive proportion. The first failure appears at $t = 282.454\dots$

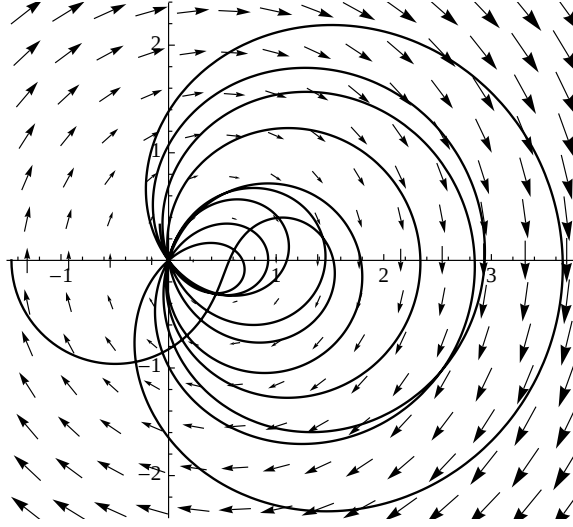


Figure 2: The curve $t \mapsto \zeta(\frac{1}{2} + it)$ for $0 \leq t \leq 50$ and the vector field $i(\frac{1}{2} - a)$; The value $\frac{1}{2}$ is the fixed point of the latter – the eye of the hurricane!

Theorem 3. For $0 \neq \delta := \frac{2\pi\alpha}{\log \frac{T}{2\pi}} \ll 1$, as $T \rightarrow \infty$,

$$\begin{aligned} & \sum_{0 < \gamma_a \leq T} (\zeta(\rho_a + i\delta) - a) \\ &= \left(1 - \frac{\sin 2\pi\alpha}{2\pi\alpha} + i\pi\alpha \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 - a \{ 1 - \cos 2\pi\alpha + i \sin 2\pi\alpha \} \right) \frac{T}{2\pi} \log \frac{T}{2\pi} \\ &+ \frac{T}{2\pi} \left(-1 + \exp(-2\pi i\alpha) \left(\frac{1}{i\delta} \left(\frac{1}{1-i\delta} - 1 \right) - \frac{1}{1-i\delta} \left(\zeta(1-i\delta) + \frac{1}{i\delta} \right) \right) \right. \\ &+ \frac{\zeta'}{\zeta}(1+i\delta) + \frac{1}{i\delta} - a \left\{ 1 + \log c_a + (\cos 2\pi\alpha - i \sin 2\pi\alpha) \times \right. \\ &\left. \left. \times \left(\frac{1}{(1-i\delta)^2} - \frac{2\pi i\alpha}{1-i\delta} \right) \right\} \right) + E(T), \end{aligned}$$

uniformly in α , where the summation is over nontrivial a -points, c_a is the constant from (1) and the error term is of the same size as in Theorem 2.

Theorem 3 generalizes another result of Fujii [10] in the special case $a = 0$:

$$\sum_{0 < \gamma \leq T} \zeta(\rho + i\delta) \sim \left(1 - \frac{\sin 2\pi\alpha}{2\pi\alpha} + i\pi\alpha \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 \right) \frac{T}{2\pi} \log T,$$

where the summation is taken over the zeros $\rho = \beta + i\gamma$; the precise asymptotical formula with remainder term is given below as (23) and (24). A similar discrete moment was considered by Gonek [14], who proved under assumption of the Riemann hypothesis that

$$\sum_{0 < \gamma \leq T} |\zeta(\frac{1}{2} + i(\gamma + \delta))|^2 = \left(1 - \left(\frac{\sin(\pi\alpha)}{\pi\alpha} \right)^2 \right) \frac{T}{2\pi} (\log T)^2 + O(T \log T)$$

uniformly in α for $|\alpha| \leq \frac{1}{2} \log T$. Fujii [11] refined Gonek's result in replacing the error term by further explicit main terms plus an error term of order $O(T^{1/2}(\log T)^3)$. Based on the idea to model the behaviour of the Riemann zeta-function on the critical line by the characteristic polynomials of certain Random Matrix ensembles, Hughes [17] conjectured, assuming the Riemann hypothesis, that

$$\sum_{0 < \gamma \leq T} |\zeta(\frac{1}{2} + i(\gamma + \delta))|^{2k} \sim F_k(2\pi\alpha) a(k) \frac{G(k+1)^2}{G(2k+1)} \frac{T}{2\pi} (\log T)^{k^2+1},$$

where F_k is defined in terms of Bessel-functions, $a(k)$ is an Euler product, and G is Barnes' double gamma-function. So far, the Random Matrix model was not used to do predictions off the critical line. It would be very interesting to have a counterpart of Theorem 3 in Random Matrix Theory.

The remaining parts of the article are organized as follows. In the next section we give the proof of Theorem 1. In Section 3 we collect some preliminary results for the proof of Theorem 2 which is given in Section 4, resp. the proof of Theorem 3 which is given in Section 5. In the final section we state some concluding remarks. For basic zeta-function theory we refer to Ivić [18], Titchmarsh [26].

2. PROOF OF THEOREM 1

Logarithmic differentiation of the functional equation

$$(4) \quad \zeta(s) = \Delta(s)\zeta(1-s), \quad \text{where} \quad \Delta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin(\frac{\pi s}{2}),$$

yields

$$(5) \quad \frac{\zeta'}{\zeta}(s) = \frac{\Delta'}{\Delta}(s) - \frac{\zeta'}{\zeta}(1-s).$$

In view of

$$(6) \quad \frac{\Delta'}{\Delta}(\sigma + it) = -\log \frac{|t|}{2\pi} + O(|t|^{-1}) \quad \text{for} \quad |t| \geq 1$$

we get

$$(7) \quad \frac{\zeta'}{\zeta}(\frac{1}{2} + it) = -\overline{\left(\frac{\zeta'}{\zeta}\right)}(\frac{1}{2} + it) - \log \frac{|t|}{2\pi} + O(|t|^{-1}).$$

For $a \in \mathbb{R}$ with $a \neq 0$ we assume that there is t_a such that

$$(8) \quad |\zeta(\frac{1}{2} + it_a) - a| < \epsilon \quad \text{and} \quad |\zeta'(\frac{1}{2} + it_a) - a| < \epsilon,$$

where $0 < \epsilon < |a|$. Then

$$\frac{\zeta'}{\zeta}(\frac{1}{2} + it_a) = \frac{a + O(\epsilon)}{a + O(\epsilon)} = 1 + O(\epsilon).$$

If a is sufficiently large then $|t_a| \geq 1$. Hence, we deduce from (7) that

$$2 = -\log \frac{|t_a|}{2\pi} + O(|t_a|^{-1} + \epsilon).$$

For sufficiently large a the latter formula is in contradiction with (8). This finishes the proof of Theorem 1.

3. PRELIMINARIES

In the sequel we write the complex variable as $s = \sigma + it$ with real σ, t . We start with the growth of the zeta-function in the left half-plane.

Lemma 4. *There is a constant $c > 0$ such that, for $\sigma \leq 0$ and $|t| \geq 2$,*

$$|\zeta(\sigma + it)| > \frac{c|t|^{1/2-\sigma}}{\log t}.$$

If Riemann's hypothesis is true, then for any $\varepsilon > 0$ and any $\sigma_0 < \frac{1}{2}$ there is $c = c(\varepsilon, \sigma_0) > 0$ such that, for $\sigma \leq \sigma_0 < \frac{1}{2}$ and $|t| \geq 2$,

$$|\zeta(\sigma + it)| > c|t|^{1/2-\sigma+\varepsilon}.$$

Proof. It is known (see Patterson [20], Exercise 4.6) that

$$\zeta(1 + it) \gg \frac{1}{\log t} \quad \text{for } |t| \geq 2.$$

Then the first part of the lemma follows by the functional equation (4) in combination with Stirling's formula

$$(9) \quad \Gamma(s) = \exp \left(\left(s - \frac{1}{2} \right) \log s - s + \frac{\log 2\pi}{2} \right) \left(1 + O_{\sigma_1} \left(\frac{1}{s} \right) \right) \quad \text{for } \sigma \geq \sigma_1 > 0.$$

Assuming the Riemann hypothesis, for fixed $\sigma > \frac{1}{2}$ we may use the bound $\zeta(\sigma + it) \gg |t|^{-\varepsilon}$ from [26], Chapter 14.2. This finishes the proof of the lemma.

We have the following application of the previous lemma.

Proposition 5. *If the Riemann hypothesis is true, then the values $\zeta(\sigma + it)$ for $t \in \mathbb{R}$ are not dense for any fixed $\sigma < \frac{1}{2}$.*

Proof. By Lemma 4, for $|t| > T_0 > 2$, the values of $|\zeta(\sigma + it)|$ are greater than some constant $C = C(T_0)$. Then the curve $\zeta(\sigma + it)$, $t \in [-T_0, T_0]$, which is continuous and of finite length, can not be dense in the disc $|z| \leq C$.

The next lemma shows that certain a -points are related to trivial zeros of the zeta-function:

Lemma 6. *For any complex number a there exists a positive integer N such that there is a simple a -point of $\zeta(s)$ in a small neighbourhood around $s = -2n$ for all positive integers $n \geq N$; apart from these there are no other a -points in the left half-plane $\operatorname{Re} s \leq 0$ except possibly finitely many near $s = 0$.*

This observation is due to Levinson [19]; for the proof one applies the functional equation for ζ in combination with Rouché's theorem and Stirling's formula. The second assertion follows from Lemma 4.

Now we investigate the order of growth of the almost entire function $\zeta(s) - a$. Note that the order of an entire function f is defined to be the infimum of all real numbers b for which the estimate

$$|f(s)| \leq \exp(|s|^b)$$

holds for all sufficiently large $|s|$. The following lemma is well-known in the case $a = 0$; the general case can be treated similarly.

Lemma 7. *For any a the function $(s-1)(\zeta(s)-a)$ is entire of order 1.*

Proof is analogous to the proof of Theorem 2.12 in Titchmarsh [26].

The next lemma generalizes the well-known partial fraction decomposition of the logarithmic derivative of ζ :

Lemma 8. *Let a be a fixed complex number. Then, for $-1 \leq \sigma \leq 2, |t| \geq 1$,*

$$\frac{\zeta'(s)}{\zeta(s)-a} = \sum_{|t-\gamma_a| \leq 1} \frac{1}{s-\rho_a} + O(\log(|t|+1)),$$

where the summation is taken over all a -points $\rho_a = \beta_a + i\gamma_a$ satisfying $|t-\gamma_a| \leq 1$.

Proof. Since $(s-1)(\zeta(s)-a)$ is an integral function of order one (by the previous lemma), Hadamard's factorization theorem yields

$$(s-1)(\zeta(s)-a) = \exp(A+Bs) \prod_{\rho_a} \left(1 - \frac{s}{\rho_a}\right) \exp\left(\frac{s}{\rho_a}\right),$$

where A and B are certain complex constants and the product is taken over *all* zeros ρ_a of $(s-1)(\zeta(s)-a)$ (trivial and nontrivial a -points). Hence, taking the logarithmic derivative, we get

$$\frac{\zeta'(s)}{\zeta(s)-a} = B - \frac{1}{s-1} + \sum_{\rho_a} \left(\frac{1}{s-\rho_a} + \frac{1}{\rho_a} \right);$$

the latter formula can be found in [4], however, for our reasoning we prefer to work with a truncated version. By Lemma 6 there exists a positive constant c such that the imaginary parts of all a -points in the left half-plane lie in the interval $[-c, c]$. Moreover, it follows that there are $\frac{1}{2}\sigma + O(1)$ many of these trivial a -points with real part greater than $-\sigma$ as $\sigma \rightarrow +\infty$. Thus, for s distant from any of these trivial a -points, we have

$$\begin{aligned} \sum_{\text{trivial } \rho_a} \left(\frac{1}{s-\rho_a} + \frac{1}{\rho_a} \right) &\ll \sum_{\text{trivial } \rho_a} \frac{\sqrt{\sigma^2 + t^2}}{\sqrt{\beta_a^2 + \gamma_a^2} \sqrt{(\sigma - \beta_a)^2 + (t - \gamma_a)^2}} \\ &\ll 1 + \int_1^\infty \frac{|t|}{x\sqrt{x^2 + t^2}} dx \ll \log t \end{aligned}$$

as $t \rightarrow \infty$. Hence, for those values of s ,

$$\frac{\zeta'(s)}{\zeta(s)-a} = B - \frac{1}{s-1} + \sum_{\text{nontrivial } \rho_a} \left(\frac{1}{s-\rho_a} + \frac{1}{\rho_a} \right) + O(\log t).$$

Note that the main term in the Riemann-von Mangoldt type formula for the number of a -points (1) does not depend on a . Therefore, the same reasoning as for $a = 0$, the case of zeros of ζ , can be applied to the latter formula (see Titchmarsh [26], §9.6). This yields the assertion of the lemma.

4. PROOF OF THEOREM 2

By the calculus of residues,

$$(10) \quad \sum_{0 < \gamma_a \leq T} \zeta'(\rho_a) = \frac{1}{2\pi i} \oint \frac{\zeta'(s)^2}{\zeta(s) - a} ds,$$

where the integration is taken over a rectangular contour in counterclockwise direction according to the location of the nontrivial a -points of $\zeta(s)$, to be specified below. In view of the Riemann-von Mangoldt-type formula (1) the ordinates of the a -points cannot lie too dense. For any large T_0 we can find a $T \in [T_0, T_0 + 1)$ such that

$$(11) \quad \min_{\rho_a} |T - \gamma_a| \geq \frac{1}{\log T},$$

where the minimum is taken over all nontrivial a -points $\rho_a = \beta_a + i\gamma_a$. We shall distinguish the cases $a \neq 1$ and $a = 1$.

First, let's assume that $a \neq 1$. We choose $B = \log T$. For $\sigma \rightarrow +\infty$ we have that

$$(12) \quad \zeta(\sigma + it) = 1 + o(1)$$

uniformly in t . Thus there are no a -points in the half-plane $\operatorname{Re} s > B - 1$. Further, define $b = 1 + \frac{1}{\log T}$. Then we may suppose that there are no a -points on the line segments $[B, B + iT]$ and $[1 - b, 1 - b + iT]$ (by varying b slightly if necessary). Moreover, in view of Lemma 6 there are only finitely many trivial a -points to the right of $\operatorname{Re} s = 1 - b$. Hence, in (10) we may choose the counterclockwise oriented rectangular contour $\mathcal{R}$ with vertices $1 - b + i, B + i, B + iT, 1 - b + iT$ at the expense of a small error for disregarding the finitely many nontrivial a -points below $\operatorname{Im} s = 1$ and for counting finitely many trivial a -points to the right of $\operatorname{Re} s = 1 - b$:

$$\sum_{0 < \gamma_a \leq T} \zeta'(\rho_a) = \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{\zeta'(s)^2}{\zeta(s) - a} ds + O(1).$$

If there is any a -point on the line segment $[1 - b + i, B + i]$, we exclude this value by a small indentation; the contribution of the integral over this interval is bounded, hence negligible.

Next we consider the integral over the upper horizontal line segment $[B + iT, 1 - b + iT]$. By the Phragmén-Lindelöf principle and by the functional equation (4), for $\sigma \geq -3$,

$$(13) \quad \zeta(\sigma + it) \ll |t|^{\max\{(1-\sigma), 0\}/2+\epsilon} \quad \text{as } |t| \rightarrow \infty$$

with an implicit constant depending only on ϵ (see Titchmarsh [26], §5.1). Hence by Cauchy's integral formula we deduce, for $\sigma \geq -2$,

$$\zeta'(\sigma + it) \ll |t|^{\max\{(1-\sigma), 0\}/2+\epsilon} \quad \text{as } |t| \rightarrow \infty.$$

Then from Lemma 8 in view of the number of nontrivial a -points (1) we get, for $\sigma \geq 1 - b$,

$$(14) \quad \frac{\zeta'(\sigma + iT)^2}{\zeta(\sigma + iT) - a} \ll T^{(1-\sigma)/2+\epsilon}.$$

Consequently, the integrals over the horizontal line segments contribute at most $O(T^{1/2+\epsilon})$.

It remains to consider the vertical integrals. For $\sigma \rightarrow +\infty$,

$$\zeta'(s) \ll 2^{-\sigma}$$

uniformly in t . This and the formula (12) give

$$\int_B^{B+iT} \frac{\zeta'(s)^2}{\zeta(s) - a} ds \ll T^{-2 \log 2} \log T.$$

Collecting together,

$$(15) \quad \sum_{0 < \gamma_a \leq T} \zeta'(\rho_a) = -\frac{1}{2\pi i} \int_{1-b}^{1-b+iT} \frac{\zeta'(s)^2}{\zeta(s) - a} ds + O(T^{1/2+\epsilon}).$$

Hence, it remains to evaluate the integral over the left vertical line segment $[1-b+iT, 1-b]$ of $\mathcal{R}$.

By Lemma 4 there exists a positive constant A , depending only on a , such that

$$\left| \frac{a}{\zeta(s)} \right| < \frac{1}{2} \quad \text{for } s = 1-b+it, \quad |t| \geq A.$$

Hence, the geometric series expansion

$$\frac{1}{\zeta(s) - a} = \frac{1}{\zeta(s)} \left(1 + \frac{a}{\zeta(s)} + \sum_{k=2}^{\infty} \left(\frac{a}{\zeta(s)} \right)^k \right)$$

is valid for s from $[1-b+iA, 1-b+iT]$. Since

$$\frac{1}{2\pi i} \int_{1-b+i}^{1-b+iA} \frac{\zeta'(s)^2}{\zeta(s) - a} ds \ll 1,$$

we deduce

$$\begin{aligned} & -\frac{1}{2\pi i} \int_{1-b}^{1-b+iT} \frac{\zeta'(s)^2}{\zeta(s) - a} ds \\ &= -\frac{1}{2\pi i} \int_{1-b+iA}^{1-b+iT} \left\{ \frac{\zeta'^2}{\zeta}(s) + a \left(\frac{\zeta'}{\zeta}(s) \right)^2 + \frac{\zeta'^2}{\zeta}(s) \sum_{k=2}^{\infty} \left(\frac{a}{\zeta(s)} \right)^k \right\} ds \\ (16) \quad & +O(1) \\ &= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + O(1), \end{aligned}$$

say. To estimate the third integral, we use Lemma 4 in combination with (5) and (6) in order to obtain

$$\begin{aligned} \mathcal{J}_3 &= -\frac{a}{2\pi i} \int_{1-b+iA}^{1-b+iT} \left(\frac{\zeta'}{\zeta}(s) \right)^2 \sum_{\ell=1}^{\infty} \left(\frac{a}{\zeta(s)} \right)^{\ell} ds \\ (17) \quad &\ll T(\log T)^2 \sum_{\ell=1}^{\infty} \left(\frac{\log T}{T^{1/2}} \right)^{\ell} \ll T^{1/2}(\log T)^3. \end{aligned}$$

Applying the functional equation in the form (5), we find for the second integral in (16)

$$\begin{aligned}\mathcal{J}_2 &= -\frac{a}{2\pi i} \int_{1-b+iA}^{1-b+iT} \left(\frac{\Delta'}{\Delta}(s) - \frac{\zeta'}{\zeta}(1-s) \right)^2 ds \\ &= -\frac{a}{2\pi i} \int_{1-b+iA}^{1-b+iT} \left\{ \left(\frac{\Delta'}{\Delta}(s) \right)^2 - 2 \frac{\Delta'}{\Delta}(s) \frac{\zeta'}{\zeta}(1-s) + \left(\frac{\zeta'}{\zeta}(1-s) \right)^2 \right\} ds \\ &= \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3,\end{aligned}$$

say. In combination with the asymptotic formula (6) we easily get

$$\begin{aligned}\mathcal{H}_1 &= -\frac{a}{2\pi} \int_A^T \left(-\log \frac{t}{2\pi} + O(t^{-1}) \right)^2 dt \\ &= -a \left\{ \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^2 - \frac{T}{\pi} \log \frac{T}{2\pi} + \frac{T}{\pi} \right\} + O(\log T).\end{aligned}$$

In a similar way we find

$$\begin{aligned}\mathcal{H}_2 &= -\frac{a}{\pi} \int_A^T \left(\log \frac{t}{2\pi} + O(t^{-1}) \right) \frac{\zeta'}{\zeta}(b-it) dt \\ &= \frac{a}{\pi} \sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^b} \int_A^T \left(\log \frac{t}{2\pi} + O(t^{-1}) \right) \exp(-it \log m) dt.\end{aligned}$$

Integration by parts shows that the integral is $O(\log T)$, hence

$$\mathcal{H}_2 \ll \log T \sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^b} = \log T \left| \frac{\zeta'}{\zeta}(b) \right| \ll (\log T)^2.$$

The same reasoning shows

$$\mathcal{H}_3 \ll \sum_{m,n=2}^{\infty} \frac{\Lambda(m)\Lambda(n)}{(mn)^b} \left| \int_A^T \exp(-it \log(mn)) dt \right| \ll (\log T)^2.$$

Hence, collecting together we find

$$(18) \quad \mathcal{J}_2 = -a \left\{ \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^2 - \frac{T}{\pi} \log \frac{T}{2\pi} + \frac{T}{\pi} \right\} + O((\log T)^2).$$

It remains to evaluate the first integral on the right-hand side of (16). It should be noted that this is essentially the integral giving the main term in the proof of Conrey, Ghosh & Gonek for the existence of infinitely many simple zeros [8], resp. [9] (apart from the mollifier used there to obtain conditional quantitative results), so

$$\mathcal{J}_1 = -\frac{1}{2\pi i} \int_b^{b+iT} \frac{\zeta'^2}{\zeta}(s) ds = \sum_{0 < \gamma \leq T} \zeta'(\rho) + O(T^{1/2+\epsilon}) \sim \frac{T}{4\pi} (\log T)^2,$$

where the summation is over all nontrivial zeros. Fujii [10, 12] obtained a more precise asymptotic formula, namely, as $T \rightarrow \infty$,

$$(19) \quad \sum_{0 < \gamma \leq T} \zeta'(\rho) = \frac{T}{4\pi} \left(\log \frac{T}{2\pi} \right)^2 + (c_0 - 1) \frac{T}{2\pi} \log \frac{T}{2\pi} + (1 - c_0 - c_0^2 - 3c_1) \frac{T}{2\pi} + \mathcal{E}(T),$$

where the numbers c_0, c_1 are the Stieltjes constants given by (3), and the error term $\mathcal{E}(T)$ is $O(T \exp(-C(\log T)^{1/2}))$ unconditionally with some absolute positive constant C , resp. $O(T^{1/2}(\log T)^{7/2})$ under assumption of the Riemann hypothesis; note that [12] contains a correction of the corresponding formula in [10].

Substituting (17), (18), and (19) into (16), leads via (15) to the desired asymptotical formula for every T satisfying condition (11). To get this uniformly in T we allow an arbitrarily T at the expense of an error $\ll T^{1/2+\epsilon}$ (by shifting the path of integration using (13)). This proves Theorem 2 in the case $a \neq 1$.

For $a = 1$ we consider the function $f(s) := 2^s(\zeta(s) - 1)$ in place of $\zeta(s) - a$. By the Dirichlet series expansion

$$(20) \quad 2^s(\zeta(s) - 1) = 1 + \sum_{n=3}^{\infty} \left(\frac{2}{n}\right)^s$$

it follows that there is a zero-free right half-plane for $f(s)$. Computing the logarithmic derivative,

$$\frac{f'}{f}(s) = \log 2 + \frac{\zeta'(s)}{\zeta(s) - 1},$$

we observe that the non-constant term corresponds to the logarithmic derivative in the case $a \neq 1$ while the constant term does not contribute by integration over a closed contour. This proves Theorem 2 for general a .

5. PROOF OF THEOREM 3

The proof is rather similar to the previous one. Let the quantities b, B and A be defined as above. We start with

$$(21) \quad \sum_{0 < \gamma_a \leq T} \zeta(\rho_a + i\delta) = \frac{1}{2\pi i} \oint \frac{\zeta'(s)}{\zeta(s) - a} \zeta(s + i\delta) ds,$$

where the integration is again over a rectangular contour with vertices $1 - b + i, B + i, B + iT, 1 - b + iT$ in counterclockwise direction. As before we assume that there are no a -points on this contour, otherwise we can circumvent these values by a small indentation at the expense of an error $O(T^{1/2+\epsilon})$. By the same reasoning as above (see (16)) the main contribution comes from the integral

$$(22) \quad -\frac{1}{2\pi i} \int_{1-b+iA}^{1-b+iT} \frac{\zeta'}{\zeta}(s) \left(1 + \frac{a}{\zeta(s)} + \sum_{k=2}^{\infty} \left(\frac{a}{\zeta(s)}\right)^k\right) \zeta(s + i\delta) ds,$$

defining a sum consisting of three terms. The first term was essentially already computed by Fujii [10], when he proved in the case $a = 0$ the asymptotical formula

$$(23) \quad \begin{aligned} \sum_{0 < \gamma \leq T} \zeta(\rho + i\delta) &= \left(1 - \frac{\sin 2\pi\alpha}{2\pi\alpha} + i\pi\alpha \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right) \frac{T}{2\pi} \log \frac{T}{2\pi} \\ &\quad + \frac{T}{2\pi} c(\alpha, T) + \mathcal{E}(T) \end{aligned}$$

where the summation is taken over the zeros $\rho = \beta + i\gamma$,

$$(24) \quad \begin{aligned} c(\alpha, T) &:= -1 + \exp(-2\pi i\alpha) \left\{ \frac{1}{i\delta} \left(\frac{1}{1-i\delta} - 1 \right) - \frac{1}{1-i\delta} \left(\zeta(1-i\delta) + \frac{1}{i\delta} \right) \right\} \\ &\quad + \frac{\zeta'}{\zeta}(1+i\delta) + \frac{1}{i\delta}, \end{aligned}$$

and the error term estimate $\mathcal{E}(T) \ll T \exp(-C(\log T)^{1/2})$ unconditionally, resp. $\ll T^{1/2}(\log T)^{5/2}$ under assumption of Riemann's hypothesis. This yields

$$(25) \quad \begin{aligned} & -\frac{1}{2\pi i} \int_{1-b+iA}^{1-b+iT} \frac{\zeta'}{\zeta}(s) \zeta(s+i\delta) ds = \sum_{0 < \gamma \leq T} \zeta(\rho+i\delta) + O(T^{1/2+\epsilon}) \\ & = \left(1 - \frac{\sin 2\pi\alpha}{2\pi\alpha} + i\pi\alpha \left(\frac{\sin \pi\alpha}{\pi\alpha} \right)^2 \right) \frac{T}{2\pi} \log \frac{T}{2\pi} + \frac{T}{2\pi} c(\alpha, T) + \mathcal{E}(T). \end{aligned}$$

The third term in (22) contributes again to the error term. Applying the functional equations (4) and (5), the second term can be rewritten as

$$\begin{aligned} & -\frac{a}{2\pi i} \int_{1-b+iA}^{1-b+iT} \frac{\zeta'}{\zeta}(s) \frac{\zeta(s+i\delta)}{\zeta(s)} ds \\ & = -\frac{a}{2\pi i} \int_{1-b+iA}^{1-b+iT} \left(\frac{\Delta'}{\Delta}(s) - \frac{\zeta'}{\zeta}(1-s) \right) \frac{\Delta(s+i\delta)}{\Delta(s)} \frac{\zeta(1-s-i\delta)}{\zeta(1-s)} ds \\ & = \mathcal{K}_1 + \mathcal{K}_2, \end{aligned}$$

say. It follows from Stirling's formula (9) that

$$\frac{\Delta(\sigma + i(\delta + t))}{\Delta(\sigma + it)} = \exp \left(-i\delta \log \frac{t}{2\pi} \right) (1 + O(t^{-1})).$$

Using (6) we have that

$$\mathcal{K}_1 = \frac{a}{2\pi} \sum_{m,n=1}^{\infty} \frac{\mu(m)n^{i\delta}}{(mn)^b} \int_A^T \left(\log \frac{t}{2\pi} + O(t^{-1}) \right) \exp \left(it \log(mn) - i\delta \log \frac{t}{2\pi} \right) dt.$$

For $mn \neq 1$ the integral can be estimated by integrating by parts; these terms contribute an error term $O((\log T)^3)$. Computing the integral for $m = n = 1$ yields

$$\begin{aligned} \mathcal{K}_1 &= \frac{a}{2\pi} \int_A^T \log \frac{t}{2\pi} \exp \left(-i\delta \log \frac{t}{2\pi} \right) dt + O((\log T)^3) \\ &= a \exp(-2\pi i\alpha) \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - \frac{1}{1-i\delta} \right) \frac{1}{1-i\delta} + O((\log T)^3). \end{aligned}$$

This gives besides the first term in (22) a further contribution to the main term.

Similarly we get $\mathcal{K}_2 \ll (\log T)^3$.

Note that

$$\frac{1}{1-i\delta} \log \frac{T}{2\pi} = \log \frac{T}{2\pi} + \frac{2\pi i\alpha}{1-i\delta}.$$

Together with (25) we get the asymptotic formula for (21). Subtracting (1) the proof of Theorem 3 is complete.

It should be noted that differentiation of the formula of Theorem 3 with respect to α leads to the formula of Theorem 2; for this purpose one has to be aware that all error terms are uniform in α .

6. CONCLUDING REMARKS

i) Similar graphs as for curves $t \mapsto \zeta(\sigma + it)$ (Figure 1) appear for other zeta- and L -functions too (see for example Akiyama & Tanigawa [1] for L -functions associated with elliptic curves). It seems that the shape of these curves depends on the type of functional equation, the location of zeros, as well as on the first coefficient of the Dirichlet series expansion.

ii) It is possible to consider short intervals $(T, T + H]$ for the imaginary parts of nontrivial a -points in place of $(0, T]$ as was done in [24] for zeros; here *short* means that $T^{1/2+\epsilon} \leq H \leq T$. Moreover, we observe that, generalizing Theorem 2 and 3,

$$\sum_{|\gamma_a| \leq T} f(\rho_a) = \sum_{|\gamma| \leq T} f(\rho) - \frac{a}{2\pi i} \int \frac{\zeta'(s)}{\zeta(s) - a} f(s) ds + \text{error},$$

where the summation on the right-hand side is taken over nontrivial zeros (and not a -points), and where f is *any sufficiently smooth* Dirichlet polynomial or series. These extensions will be considered in a sequel to this article.

iii) As already pointed out in the introduction, quite much is known about the distribution of a -points to the right of the critical line whenever a is a fixed complex number different from zero. On the contrary, on the critical line for no complex number a apart from zero it is proved that there exist infinitely many a -points. Hence, it seems to be an interesting problem to study the location of a -points to the left or to the right of curves $s = \sigma(t) + it$ with $\lim_{t \rightarrow \infty} \sigma(t) = \frac{1}{2}$. Selberg [21] was the first to obtain results on the statistical distribution of a -points in such regions (even for elements in the Selberg class).

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