Littlewood-Paley Characterizations of Hardy-type Spaces Associated with Ball Quasi-Banach Function Spaces

In Memory of Professor Carlos Berenstein

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Abstract Let X be a ball quasi-Banach function space on $\mathbb{R}^n$. In this article, assuming that the powered Hardy–Littlewood maximal operator satisfies some Fefferman–Stein vector-valued maximal inequality on X and is bounded on the associated space, the authors establish various Littlewood–Paley function characterizations of the Hardy space $H_X(\mathbb{R}^n)$ associated with X, under some weak assumptions on the Littlewood–Paley functions. To this end, the authors also establish a useful estimate on the change of angles in tent spaces associated with X. All these results have wide applications. Particularly, when $X := M_r^p(\mathbb{R}^n)$ (the Morrey space), $X := L^{\vec{p}}(\mathbb{R}^n)$ (the mixed-norm Lebesgue space), $X := L^{p(\cdot)}(\mathbb{R}^n)$ (the variable Lebesgue space), $X := L^p_\omega(\mathbb{R}^n)$ (the weighted Lebesgue space) and $X := (E^r_\Phi)_t(\mathbb{R}^n)$ (the Orlicz-slice space), the Littlewood–Paley function characterizations of $H_X(\mathbb{R}^n)$ obtained in this article improve the existing results via weakening the assumptions on the Littlewood–Paley functions and widening the range of λ in the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$.

1 Introduction

The real-variable theory of the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ was originally initiated by Stein and Weiss [63] and further developed by Fefferman and Stein [24]. It is well known that the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ plays a key role in harmonic analysis, partial differential equations and other analysis subjects. In particular, when $p \in (0,1]$, $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ in the study on the boundedness of Calderón–Zygmund operators. In recent decades, in order to meet the requirements arising in the study on the boundedness of operators, partial differential equations and some other analysis subjects, various variants of Hardy spaces have been introduced and their real-variable theories have been well developed; these variants of Hardy spaces were built on some elementary function spaces such as weighted Lebesgue spaces (see [61]), (weighted) Herz spaces (see, for instance, [15, 27,

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28, 47, 48]), (weighted) Morrey spaces (see, for instance [39, 58, 32]), Orlicz spaces (see, for instance, [38, 62, 67, 52, 73]), Lorentz spaces (see, for instance, [1]), Musielak–Orlicz spaces (see, for instance, [41, 71]) and variable function spaces (see, for instance, [21, 51, 74]). Observe that these aforementioned elementary function spaces are all included in a generalized framework called ball quasi-Banach function spaces which were introduced, very recently, by Sawano et al. [59]. Moreover, Sawano et al. [59], Wang et al. [68] and Zhang et al [69, 77] established a unified real-variable theory for Hardy spaces and weak Hardy spaces associated with ball quasi-Banach function spaces on $\mathbb{R}^n$ and gave some applications of these Hardy-type spaces to the boundedness of Calderón–Zygmund operators and pseudo-differential operators. More function spaces based on ball quasi-Banach function spaces can be found in Sawano [57].

Recall that the original work of the Littlewood–Paley theory should be owned to Littlewood and Paley [44]. Moreover, the Littlewood–Paley theory of Hardy spaces was further developed by Calderón [13] and Fefferman and Stein [24]. In recent decades, the Littlewood–Paley theory of various variants of Hardy spaces has been well developed; see, for instances, [25, 34, 35, 42, 43, 45, 69, 70, 76, 78] and the related references. Particularly, Folland and Stein [25] obtained the Littlewood–Paley function characterizations of Hardy spaces on homogeneous groups. Observe that, in the case of $H^p(\mathbb{R}^n)$ with $p \in (0, \infty)$, the best known range of the parameter λ in the g_λ^*-function characterization of Hardy spaces in [25] is $(\max\{1, 2/p\}, \infty)$ and the function φ appearing in the definitions of the Littlewood–Paley functions in [25] only need to satisfy zero order vanishing moment. However, compared with the results in [25] on the Littlewood–Paley function characterizations in the case of $H^p(\mathbb{R}^n)$ with $p \in (0, \infty)$, the range of λ in the g_λ^*-function characterization appearing in [35, 76, 68] does not coincide with the range in [25], namely, $\lambda \in (\max\{1, 2/p\}, \infty)$, and, in [34, 35, 42, 43, 45, 70], the function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ appearing in the definitions of Littlewood–Paley functions is supposed to be supported in the unit ball and have at least vanishing moments up to order $\lfloor n(\frac{1}{p} - 1)\rfloor$, which is much stronger than the corresponding assumptions on φ appearing in the definitions of Littlewood–Paley functions in [25]. Here and thereafter, the symbol $\lfloor s\rfloor$ for any $s \in \mathbb{R}$ denotes the largest integer not greater than s.

Let X be a ball quasi-Banach function space on $\mathbb{R}^n$ introduced in [59]. Assuming that the powered Hardy–Littlewood maximal operator satisfies some Fefferman–Stein vector-valued maximal inequality on X as well as it is bounded on the associated space, Sawano et al. [59] established the Lusin area function characterization of $H_X(\mathbb{R}^n)$. Recently, Wang et al. [68] obtained the Littlewood–Paley g function and the Littlewood–Paley g_λ^*-function characterizations of $H_X(\mathbb{R}^n)$. We should point out that, to characterize $H_X(\mathbb{R}^n)$, Sawano et al. [59] and Wang et al. [68] required that the function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ appearing in the definitions of Littlewood–Paley functions satisfies $\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \leq \widehat{\varphi} \leq \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$. Although, when $p \in (0, 1]$ and $X := L^p(\mathbb{R}^n)$, the range of λ in the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$ obtained by Wang et al. [68] coincides with the best known one in [25], namely, $\lambda \in (2/p, \infty)$, the range of λ in [68] is not optimal even when $p \in (1, \infty)$ and $X := L^p(\mathbb{R}^n)$. This point motivates us to optimize the range of the parameter λ appearing in the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$ in [68]. Recently, when studying the Littlewood–Paley function characterizations of weak Hardy type spaces associated with ball quasi-Banach function spaces, Wang et al. [69] found that it suffices to require that the function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies that $\widehat{\varphi}(\vec{0}_n) = 0$ and, for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, there exists a $t \in (0, \infty)$ such that $\widehat{\varphi}(tx) \neq 0$. This motivates us to improve the existing results of the Littlewood–Paley function characterizations of $H_X(\mathbb{R}^n)$ in [59, 68] by weakening the assumption on φ. In this article, we re-establish various Littlewood–Paley function characterizations of $H_X(\mathbb{R}^n)$ via weakening the assumptions on the Littlewood–Paley functions and widening the range of λ in the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$. To this end, we also establish a useful estimate on the change of angles in tent spaces associated with X (see Theorem 3.3 below). Particularly, the assumptions on the Littlewood–Paley functions in this article are much weaker than the corresponding assumptions in [34, 35, 42, 43, 45, 68, 59, 70] (see Remark 4.3 below). Besides, under these weaker assumptions on φ in the definition of Littlewood–Paley functions, the g_λ^*-function characterization of $H_X(\mathbb{R}^n)$ obtained in this article improves the existing results via widening the range of λ in [68] (see Theorem 4.11 below). We point out that the φ appearing in the definition of the Littlewood–Paley functions only need to satisfy a zero order vanishing moment, which coincides with the corresponding assumptions in [25] and, when $X := L^p(\mathbb{R}^n)$ with $p \in (0,\infty)$, the range of λ in the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$ in Theorem 4.11 below coincides with the best known one in [25], namely, $(\max\{1, 2/p\}, \infty)$. In [68], the estimate on the change of angles in tent spaces associated with X plays a key role in the proof of the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$. However, to optimize the range of λ in the Littlewood–Paley g_λ^* function characterization of $H_X(\mathbb{R}^n)$ in [68], the estimate on the change of angles in tent spaces associated with X in [68] is no longer feasible. To establish a more precise estimate on the change of angles (see Theorem 3.3 below), instead of applying the atomic characterization of the tent space which was used in [68], we employ a method different from [68], namely, we now use an extrapolation theorem over ball Banach function spaces (see Lemma 2.12 below) which was proved in [77, Lemma 7.34], and then establish a more refined estimate on the change of angles (see Theorem 3.3 below). The assumptions in this estimate on the ball quasi-Banach function space X are much weaker than the corresponding assumptions in [68, Lemma 2.20]. All of these results have wide applications. When $X := M_r^p(\mathbb{R}^n)$ (the Morrey space), $X := L^{\vec{p}}(\mathbb{R}^n)$ (the mixed-norm Lebesgue space), $X := L^{p(\cdot)}(\mathbb{R}^n)$ (the variable Lebesgue space), $X := L^p_\omega(\mathbb{R}^n)$ (the weighted Lebesgue space) and $X := (E^r_\Phi)_t(\mathbb{R}^n)$ (the Orlicz-slice space), the Littlewood–Paley function characterizations of $H_X(\mathbb{R}^n)$ obtained in this article improve the existing results in [35, 59, 68, 70, 76] via weakening the assumptions on the Littlewood–Paley functions and widening the range of λ in the Littlewood–Paley g_λ^*-function characterization.

To be precise, this article is organized as follows.

In Section 2, we recall some notions concerning the ball (quasi)-Banach function space X. Then we state the assumptions of the Fefferman–Stein vector-valued maximal inequality on X (see Assumption 2.6 below) and the boundedness on the r-convexification of its associated space for the Hardy–Littlewood maximal operator (see Assumption 2.7 below). Finally, we recall the extrapolation theorem over ball quasi-Banach function spaces proved in [77] and some notions about the Hardy space $H_X(\mathbb{R}^n)$ introduced in [59].

In Section 3, via [49, Proposition 3.2] (see Lemma 3.2 below) and the extrapolation theorem (see Lemma 2.12 below) which was proved in [77, Lemma 7.34], we establish an estimate on the change of angles in tent spaces associated with a ball quasi-Banach function space X (see Theorem 3.3 below), which plays a key role in the proof of the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$ (see Theorem 4.11 below).

Section 4 contains some square function characterizations of $H_X(\mathbb{R}^n)$, including its characterizations via the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function, respectively, in Theorems 4.9, 4.11 and 4.13 below. We first prove Theorem 4.9, the

Lusin area function characterization of $H_X(\mathbb{R}^n)$. To this end, via borrowing some ideas from [69], we use the atomic characterization of the tent space associated to X (see Lemma 4.5 below) to decompose a distribution f into a sequence of molecules; then, applying some ideas used in the proof of [69, Theorem 3.7], we prove Theorem 4.9 under some even weaker assumptions on the Lusin area function. After we obtain the Lusin area function characterization of $H_X(\mathbb{R}^n)$, using the estimate on the change of angles (see Theorem 3.3 below), we establish the Littlewood-Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$, namely, we prove Theorem 4.11. Finally, applying an estimate initiated by Ullrich [66] and further improved by Wang et al. [69] (see Lemma 4.14 below), we obtain the Littlewood–Paley g-function characterization of $H_X(\mathbb{R}^n)$, namely, we show Theorem 4.13.

In Section 5, we apply the above results, respectively, to the Morrey space, the mixed-norm Lebesgue space, the variable Lebesgue space, the weighted Lebesgue space and the Orlicz-slice space. Recall that, in these five examples, only variable Lebesgue spaces are quasi-Banach function spaces and the others are only ball quasi-Banach function spaces.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$. We always denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha,\beta,\ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \beta, \ldots$. The *symbol* $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. The *symbol* $\lfloor s \rfloor$ for any $s \in \mathbb{R}$ denotes the largest integer not greater than s. We use $\vec{0}_n$ to denote the *origin* of $\mathbb{R}^n$ and let $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$. If E is a subset of $\mathbb{R}^n$, we denote by $\mathbf{1}_E$ its characteristic function and by $E^\complement$ the set $\mathbb{R}^n \setminus E$. For any $\theta := (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n$, let $|\theta| := \theta_1 + \cdots + \theta_n$. Furthermore, for any ball B in $\mathbb{R}^n$ and $j \in \mathbb{Z}_+$, let $S_j(B) := (2^{j+1}B) \setminus (2^j B)$ with $j \in \mathbb{N}$ and $S_0(B) := 2B$. Finally, for any $q \in [1, \infty]$, we denote by q' its *conjugate exponent*, namely, $1/q + 1/q' = 1$.

2 Preliminaries

In this section, we first present some preliminary known facts on the ball quasi-Banach function space X in §2.1. Then we state the assumptions of the Fefferman–Stein vector-valued maximal inequality on X and the boundedness on the s-convexification of X for the Hardy–Littlewood maximal operator in §2.2. In §2.3, we recall the extrapolation theorem associated with the ball quasi-Banach function space X. Finally, the notion of the Hardy type space $H_X(\mathbb{R}^n)$ associated with X was recalled in §2.4.

2.1 Ball quasi-Banach function spaces

In this subsection, we recall some preliminary known facts on ball quasi-Banach function spaces introduced in [59].

Denote by the *symbol* $\mathscr{M}(\mathbb{R}^n)$ the set of all measurable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : \ |x - y| < r\}$ and

$$\mathbb{B} := \{B(x, r) : \ x \in \mathbb{R}^n \quad \text{and} \quad r \in (0, \infty)\}. \tag{2.1}$$

Definition 2.1. A quasi-Banach space $X \subset \mathscr{M}(\mathbb{R}^n)$ is called a *ball quasi-Banach function space* if it satisfies

(i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;

(ii) $|g| \le |f|$ almost everywhere implies that $\|g\|_X \le \|f\|_X$;

(iii) $0 \le f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;

(iv) $B \in \mathbb{B}$ implies that $\mathbf{1}_B \in X$, where $\mathbb{B}$ is as in (2.1).

Moreover, a ball quasi-Banach function space X is called a *ball Banach function space* if the norm of X satisfies the triangle inequality: for any $f,\ g \in X$,

$$\|f + g\|_X \le \|f\|_X + \|g\|_X \tag{2.2}$$

and, for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on B, such that, for any $f \in X$,

$$\int_B |f(x)|\, dx \le C_{(B)}\|f\|_X. \tag{2.3}$$

For any ball Banach function space X, the *associate space* (also called the *Köthe dual*) X' is defined by setting

$$X' := \left\{f \in \mathscr{M}(\mathbb{R}^n) :\ \|f\|_{X'} := \sup\left\{\|fg\|_{L^1(\mathbb{R}^n)} :\ g \in X,\ \|g\|_X = 1\right\} < \infty\right\}, \tag{2.4}$$

where $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$ (see, for instance, [10, Chapter 1, Definitions 2.1 and 2.3]).

Remark 2.2. (i) By [59, Proposition 2.3], we know that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

(ii) Recall that a quasi-Banach space $X \subset \mathscr{M}(\mathbb{R}^n)$ is called a *quasi-Banach function space* if it is a ball quasi-Banach function space and it satisfies Definition 2.1(iv) with ball replaced by any measurable set of finite measure (see, for instance, [10, Chapter 1, Definitions 1.1 and 1.3]). It is easy to see that every quasi-Banach function space is a ball quasi-Banach function space. As was mentioned in [59, p. 9] and [69, Section 5], the family of ball Banach function spaces includes Morrey spaces, mixed-norm Lebesgue spaces, variable Lebesgue spaces, weighted Lebesgue spaces and Orlicz-slice space, which are not necessary to be Banach function spaces.

The following lemma is just [77, Lemma 2.6].

Lemma 2.3. *Every ball Banach function space X coincides with its second associate space X''. In other words, a function f belongs to X if and only if it belongs to X'' and, in that case,*

$$\|f\|_X = \|f\|_{X''}.$$

We still need to recall the notion of the convexity of ball quasi-Banach spaces, which is a part of [59, Definition 2.6].

Definition 2.4. Let X be a ball quasi-Banach function space and $p \in (0, \infty)$. The *p-convexification* X^p of X is defined by setting $X^p := \{f \in \mathscr{M}(\mathbb{R}^n) : |f|^p \in X\}$ equipped with the quasi-norm $\|f\|_{X^p} := \||f|^p\|_X^{1/p}$.

Lemma 2.5. *Let X be a ball Banach function space and $p \in [1, \infty)$. Then X^p is a ball Banach function space.*

Proof. Let $p \in [1, \infty)$. From the fact that X is a ball Banach function space and the definition of X^p, it easily follows that X^p is a ball quasi-Banach function space. Thus, to prove that X^p is a ball Banach function space, it suffices to show that, for any $f,\ g \in X^p$,

$$\|f + g\|_{X^p} \le \|f\|_{X^p} + \|g\|_{X^p} \tag{2.5}$$

and, for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on B, such that, for any $f \in X$,

$$\int_B |f(x)|\, dx \le C_{(B)}\|f\|_X. \tag{2.6}$$

We first prove (2.5). By Definition 2.4, Lemma 2.3, (2.4) and the Minkowski inequality, we conclude that

$$\begin{aligned}
\|f + g\|_{X^p} &= \left\||f + g|^p\right\|_X^{1/p} = \left\||f + g|^p\right\|_{X''}^{1/p} \\
&= \left[\sup\left\{\left\|(f + g)^p h\right\|_{L^1(\mathbb{R}^n)} :\ h \in X',\ \|h\|_{X'} = 1\right\}\right]^{\frac{1}{p}} \\
&= \sup\left\{\left\|(f + g)h^{\frac{1}{p}}\right\|_{L^p(\mathbb{R}^n)} :\ h \in X',\ \|h\|_{X'} = 1\right\} \\
&\le \sup\left\{\left\|fh^{\frac{1}{p}}\right\|_{L^p(\mathbb{R}^n)} + \left\|gh^{\frac{1}{p}}\right\|_{L^p(\mathbb{R}^n)} :\ h \in X',\ \|h\|_{X'} = 1\right\} \\
&\le \left[\sup\left\{\left\||f|^p h\right\|_{L^1(\mathbb{R}^n)} :\ h \in X',\ \|h\|_{X'} = 1\right\}\right]^{\frac{1}{p}} \\
&\quad + \left[\sup\left\{\left\||g|^p h\right\|_{L^1(\mathbb{R}^n)} :\ h \in X',\ \|h\|_{X'} = 1\right\}\right]^{\frac{1}{p}} \\
&= \left\||f|^p\right\|_X^{\frac{1}{p}} + \left\||g|^p\right\|_X^{\frac{1}{p}} = \|f\|_{X^p} + \|g\|_{X^p},
\end{aligned}$$

which implies that (2.5) holds true.

Now we show (2.6). From the Hölder inequality and (2.3), we deduce that, for any $B \in \mathbb{B}$ and $f \in X^p$,

$$\int_B |f(x)|\, dx \le \left[\int_B |f(x)|^p\, dx\right]^{1/p} |B|^{1-1/p} \le C_{(B)} \left\||f|^p\right\|_X^{1/p} = C_{(B)} \|f\|_{X^p},$$

where the positive constant $C_{(B)}$ is independent of f but depending on B. Thus, (2.6) holds true, which, combined with (2.5), then completes the proof of Lemma 2.5. □

2.2 Assumptions on the Hardy–Littlewood maximal operator

Denote by the *symbol* $L^1_{\rm loc}(\mathbb{R}^n)$ the set of all locally integrable functions on $\mathbb{R}^n$. The *Hardy–Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \tag{2.7}$$

where the supremum is taken over all balls $B \in \mathbb{B}$ containing x.

For any $\theta \in (0, \infty)$, the *powered Hardy–Littlewood maximal operator* $\mathcal{M}^{(\theta)}$ is defined by setting, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^{(\theta)}(f)(x) := \left\{\mathcal{M}\left(|f|^\theta\right)(x)\right\}^{1/\theta}. \tag{2.8}$$

The approach used in this article heavily depends on the following assumptions on the boundedness of the Hardy–Littlewood maximal function on X, which is just [59, (2.8)].

Assumption 2.6. *Let X be a ball quasi-Banach function space. For some $\theta,\ s \in (0, 1]$ and $\theta < s$, there exists a positive constant C such that, for any $\{f_j\}_{j=1}^\infty \subset \mathscr{M}(\mathbb{R}^n)$,*

$$\left\|\left\{\sum_{j=1}^\infty \left[\mathcal{M}^{(\theta)}(f_j)\right]^s\right\}^{\frac{1}{s}}\right\|_X \le C\left\|\left\{\sum_{j=1}^\infty |f_j|^s\right\}^{\frac{1}{s}}\right\|_X. \tag{2.9}$$

Assumption 2.7. *Let X be a ball quasi-Banach function space. Assume that there exist $s \in (0, \infty)$ and $q \in (s, \infty]$ such that $X^{1/s}$ is a ball Banach function space and, for any $f \in (X^{1/s})'$,*

$$\left\|\mathcal{M}^{((q/s)')}(f)\right\|_{(X^{1/s})'} \le C\, \|f\|_{(X^{1/s})'}\,, \tag{2.10}$$

where the positive constant C is independent of f.

Lemma 2.8. *Let X be a ball Banach function space and $p \in [1, \infty)$. If $\mathcal{M}$ is bounded on X, then $\mathcal{M}$ is also bounded on X^p.*

Proof. By the Hölder inequality, we know that, for any $p \in [1, \infty)$ and any locally integrable function f, $[\mathcal{M}(f)]^p \le \mathcal{M}(|f|^p)$. From this and the fact that $\mathcal{M}$ is bounded on X, we deduce that

$$\|\mathcal{M}(f)\|_{X^p} = \left\|[\mathcal{M}(f)]^p\right\|_X^{\frac{1}{p}} \le \left\|\mathcal{M}(|f|^p)\right\|_X^{\frac{1}{p}} \lesssim \left\||f|^p\right\|_X^{\frac{1}{p}} \sim \|f\|_{X^p}. \tag{2.11}$$

This finishes the proof of Lemma 2.8. □

Lemma 2.9. *Let X be a ball quasi-Banach function space. Assume that there exists a $\theta \in (1, \infty)$ such that, for any $f \in \mathscr{M}(\mathbb{R}^n)$,*

$$\left\|\mathcal{M}^{(\theta)}(f)\right\|_X \le C\, \|f\|_X\,,$$

where the positive constant C is independent of f. Then $\mathcal{M}$ is bounded on X, namely, there exists a positive constant C such that, for any $f \in X$, $\|\mathcal{M}(f)\|_X \le C\|f\|_X$.

Proof. From Definition 2.4, (2.8) and (2.10), it follows that, for any $f \in \mathscr{M}(\mathbb{R}^n)$,

$$\|\mathcal{M}(f)\|_{X^{1/\theta}} = \left\|[\mathcal{M}(f)]^{1/\theta}\right\|_X^\theta = \left\|\mathcal{M}^{(\theta)}(f^{1/\theta})\right\|_X^\theta \lesssim \|f\|_{X^{1/\theta}}\,,$$

which, together with Lemma 2.8 and $\theta \in (1, \infty)$, further implies

$$\|\mathcal{M}(f)\|_X \le C\, \|f\|_X\,.$$

This finishes the proof of Lemma 2.9. □

2.3 Extrapolation theorem on ball Banach function spaces

Now, we recall the notions of Muckenhoupt weights $A_p(\mathbb{R}^n)$ (see, for instance, [30]).

Definition 2.10. An $A_p(\mathbb{R}^n)$-*weight* ω, with $p \in [1, \infty)$, is a locally integrable and nonnegative function on $\mathbb{R}^n$ satisfying that, when $p \in (1, \infty)$,

$$\sup_{B\in\mathbb{B}}\left[\frac{1}{|B|}\int_B \omega(x)\,dx\right]\left[\frac{1}{|B|}\int_B \{\omega(x)\}^{\frac{1}{1-p}}\,dx\right]^{p-1} < \infty$$

and, when $p = 1$,

$$\sup_{B\in\mathbb{B}}\frac{1}{|B|}\int_B \omega(x)\,dx\left[\left\|\omega^{-1}\right\|_{L^\infty(B)}\right] < \infty,$$

where $\mathbb{B}$ is as in (2.1). Define $A_\infty(\mathbb{R}^n) := \bigcup_{p\in[1,\infty)} A_p(\mathbb{R}^n)$.

Definition 2.11. Let $p \in (0, \infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. The *weighted Lebesgue space* $L^p_\omega(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx\right]^{\frac{1}{p}} < \infty.$$

The following extrapolation theorem is just [77, Lemma 7.34], which is a slight variant of a special case of [20, Theorem 4.6] via replacing Banach function spaces by ball Banach function spaces.

Lemma 2.12. *Let X be a ball Banach function space and $p_0 \in (0, \infty)$. Let $\mathcal{F}$ be the set of all pairs of nonnegative measurable functions (F, G) such that, for any given $\omega \in A_1(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} [F(x)]^{p_0}\omega(x)\,dx \le C_{(p_0,[\omega]_{A_1(\mathbb{R}^n)})}\int_{\mathbb{R}^n} [G(x)]^{p_0}\omega(x)\,dx,$$

where $C_{(p_0,[\omega]_{A_1(\mathbb{R}^n)})}$ is a positive constant independent of (F, G), but depends on p_0 and $[\omega]_{A_1(\mathbb{R}^n)}$. Assume that there exists a $q_0 \in [p_0, \infty)$ such that X^{1/q_0} is a Banach function space and $\mathcal{M}$ is bounded on $(X^{1/q_0})'$. Then there exists a positive constant C such that, for any $(F, G) \in \mathcal{F}$,

$$\|F\|_X \le C\|G\|_X.$$

2.4 Hardy type spaces

Now we recall the notion of Hardy type spaces associated with ball quasi-Banach function spaces introduced in [59].

Definition 2.13. Let X be a ball quasi-Banach function space. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n}\psi(x)\,dx \neq 0$ and $b \in (0, \infty)$ sufficiently large. Then the *Hardy space* $H_X(\mathbb{R}^n)$ associated with X is defined by setting

$$H_X(\mathbb{R}^n) := \left\{f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{H_X(\mathbb{R}^n)} := \left\|M_b^{**}(f, \psi)\right\|_X < \infty\right\},$$

where $M_b^{**}(f, \psi)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_b^{**}(f, \psi)(x) := \sup_{(y,t)\in\mathbb{R}^{n+1}_+}\frac{|\psi_t * f(x - y)|}{(1 + |y|/t)^b}. \tag{2.12}$$

Definition 2.14. Let X be a ball quasi-Banach function space, $\epsilon\in(0,\infty)$, $q\in[1,\infty]$ and $d\in\mathbb{Z}_+$. A measurable function m is called an $(X,\,q,\,d,\,\epsilon)$-*molecule* associated with some ball $B\subset\mathbb{R}^n$ if

(i) for any $j\in\mathbb{N}$, $\|m\|_{L^q(S_j(B))}\le 2^{-j\epsilon}|S_j(B)|^{\frac{1}{q}}\|\mathbf{1}_B\|_X^{-1}$, where $S_0:=B$ and, for any $j\in\mathbb{N}$, $S_j(B):=(2^jB)\setminus(2^{j-1}B)$;

(ii) $\int_{\mathbb{R}^n}m(x)x^\beta\,dx=0$ for any $\beta:=(\beta_1,\ldots,\beta_n)\in\mathbb{Z}_+^n$ with $|\beta|:=\beta_1+\cdots+\beta_n\le d$, here and thereafter, for any $x:=(x_1,\ldots,x_n)\in\mathbb{R}^n$, $x^\beta:=x_1^{\beta_1}\cdots x_n^{\beta_n}$.

We also need the following molecular characterization of $H_X(\mathbb{R}^n)$, which is just [59, Theorem 3.9].

Lemma 2.15. *Assume that X is a ball quasi-Banach function space satisfying Assumption 2.6 with $0<\theta<s\le1$ and Assumption 2.7 with some $q\in(1,\infty]$ and the same s as in Assumption 2.6. Let $d\in\mathbb{Z}_+$ with $d\ge\lfloor n(1/\theta-1)\rfloor$ and $\epsilon\in(0,\infty)$ satisfy $\epsilon>n(1/\theta-1/q)$. Then $f\in H_X(\mathbb{R}^n)$ if and only if there exist a sequence $\{m_j\}_{j=1}^\infty$ of $(X,\,q,\,d,\,\epsilon)$- molecules associated, respectively, with the balls $\{B_j\}_{j=1}^\infty\subset\mathbb{B}$, and $\{\lambda_j\}_{j=1}^\infty\subset[0,\infty)$ satisfying*

$$\left\|\left\{\sum_{j=1}^\infty\left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X}\right)^s\mathbf{1}_{B_j}\right\}^{1/s}\right\|_X<\infty$$

such that $\sum_{j=1}^\infty\lambda_jm_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Moreover,

$$\|f\|_{H_X(\mathbb{R}^n)}\sim\left\|\left\{\sum_{j=1}^\infty\left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X}\right)^s\mathbf{1}_{B_j}\right\}^{1/s}\right\|_X,$$

where the positive equivalence constants are independent of f.

3 Change of angles in X-tent spaces

In this section, we establish an estimate on the change of angles in X-tent spaces. Now we recall the notion of tent spaces associated with X.

Definition 3.1. For any $\alpha\in(0,\infty)$ and $x\in\mathbb{R}^n$, let $\Gamma_\alpha(x):=\{(y,t)\in\mathbb{R}_+^{n+1}:\ |x-y|<\alpha t\}$, which is called the *cone* of aperture α with vertex $x\in\mathbb{R}^n$.

Let $\alpha\in(0,\infty)$. For any measurable function $F:\ \mathbb{R}_+^{n+1}:=\mathbb{R}^n\times(0,\infty)\to\mathbb{C}$ and $x\in\mathbb{R}^n$, define

$$\mathcal{A}^{(\alpha)}(F)(x):=\left[\int_{\Gamma_\alpha(x)}|F(y,t)|^2\,\frac{dy\,dt}{t^{n+1}}\right]^{\frac{1}{2}},\tag{3.1}$$

where $\Gamma_\alpha(x)$ is as in Definition 3.1. A measurable function F is said to belong to the *tent space* $T_2^{p,\alpha}(\mathbb{R}_+^{n+1})$, with $p\in(0,\infty)$, if $\|F\|_{T_2^{p,\alpha}(\mathbb{R}_+^{n+1})}:=\|\mathcal{A}^{(\alpha)}(F)\|_{L^p(\mathbb{R}^n)}<\infty$. Recall that Coifman et al. [18] introduced the tent space $T_2^{p,\alpha}(\mathbb{R}_+^{n+1})$ for any $p\in(0,\infty)$ and $\alpha:=1$. For any given ball quasi-Banach function space X, the X-*tent space* $T_X^\alpha(\mathbb{R}_+^{n+1})$, with aperture α, is defined to be the set of all measurable functions F such that $\mathcal{A}^{(\alpha)}(F) \in X$ and naturally equipped with the quasi-norm $\|F\|_{T_X^\alpha(\mathbb{R}_+^{n+1})} := \|\mathcal{A}^{(\alpha)}(F)\|_X$.

To prove Theorem 3.3, we need the following inequality on the change of angles in weighted Lebesgue spaces, which is a part of [49, Proposition 3.2].

Lemma 3.2. *Let $\alpha,\ \beta \in (0,\infty)$ with $\alpha \le \beta$ and $q \in [1,\infty)$. If $\omega \in A_q(\mathbb{R}^n)$ and $p \in (0,2q]$, then, for any measurable function F on $\mathbb{R}_+^{n+1}$,*

$$\int_{\mathbb{R}^n} \left|\mathcal{A}^{(\beta)}(F)(x)\right|^p \omega(x)\,dx \le C\left(\frac{\beta}{\alpha}\right)^{nq} \int_{\mathbb{R}^n} \left|\mathcal{A}^{(\alpha)}(F)(x)\right|^p \omega(x)\,dx,$$

where the positive constant C is independent of α, β and F.

Using Lemma 3.2 and Theorem 2.12, we have the following estimate on the change of angles in X-tent spaces, which plays a key role in the proof of Theorem 4.11 below.

Theorem 3.3. *Let X be a ball quasi-Banach function space. Assume that there exists an $s \in (0,\infty)$ such that $X^{1/s}$ is a ball Banach function space and $\mathcal{M}$ is bounded on $(X^{1/s})'$. Then there exists a positive constant C such that, for any $\alpha \in [1,\infty)$ and any measurable function F on $\mathbb{R}_+^{n+1}$,*

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_X \le C\alpha^{\max\{\frac{n}{2},\frac{n}{s}\}} \left\|\mathcal{A}^{(1)}(F)\right\|_X. \tag{3.2}$$

Remark 3.4. Assume that X is a ball quasi-Banach function space satisfying Assumption 2.6 with $0 < \theta < s \le 1$ and Assumption 2.7 with some $q \in (1,\infty]$ and the same s as in Assumption 2.6. In this case, Wang et al. [68, Lemma 2.20] proved that there exists a positive constant C such that, for any $\alpha \in [1,\infty)$ and any measurable function F on $\mathbb{R}_+^{n+1}$,

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_X \le C\alpha^{\max\{\frac{n}{2}-\frac{n}{q}+\frac{n}{\theta},\frac{n}{\theta}\}} \left\|\mathcal{A}^{(1)}(F)\right\|_X.$$

Compared with the assumptions and the conclusions of [68, Lemma 2.20], the assumptions in Theorem 3.3 are much weaker and the conclusions in Theorem 3.3 are more refined. We should also point out that, in the case of $X := L^p(\mathbb{R}^n)$ with $p \in (0,\infty)$, Theorem 3.3 coincides with the classical conclusions in [6, Theorem 1.1].

Proof of Theorem 3.3. Let X be a ball quasi-Banach function space and $s \in (0,\infty)$. Assume that $X^{1/s}$ is a ball Banach function space and $\mathcal{M}$ is bounded on $(X^{1/s})'$. To show (3.2), we consider the following two cases on s.

If $s \in (0,2]$, let

$$\mathcal{F} := \left\{\left(\mathcal{A}^{(\alpha)}(F), \alpha^{\frac{n}{s}}\mathcal{A}^{(1)}(F)\right) :\ \alpha \in [1,\infty),\ F \in \mathscr{M}(\mathbb{R}_+^{n+1})\right\}.$$

Then, by Lemma 3.2, we know that, for any given $\omega \in A_1(\mathbb{R}^n)$ and for any $(\mathcal{A}^{(\alpha)}(F), \alpha^{\frac{n}{s}}\mathcal{A}^{(1)}(F)) \in \mathcal{F}$,

$$\int_{\mathbb{R}^n} \left|\mathcal{A}^{(\alpha)}(F)(x)\right|^s \omega(x)\,dx \lesssim \alpha^n \int_{\mathbb{R}^n} \left|\mathcal{A}^{(1)}(F)(x)\right|^s \omega(x)\,dx \sim \int_{\mathbb{R}^n} \left|\alpha^{\frac{n}{s}}\mathcal{A}^{(1)}(F)(x)\right|^s \omega(x)\,dx,$$

which, together with the assumptions that $X^{1/s}$ is a ball Banach function space and $\mathcal{M}$ is bounded on $(X^{1/s})'$, and Lemma 2.12, further implies that, for any $\alpha \in [1, \infty)$ and $F \in \mathscr{M}(\mathbb{R}^{n+1}_+)$,

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_X \lesssim \alpha^{\frac{n}{s}} \left\|\mathcal{A}^{(1)}(F)\right\|_X . \tag{3.3}$$

This is the desired estimate.

If $s \in (2, \infty)$, let

$$\mathcal{F} := \left\{\left(\mathcal{A}^{(\alpha)}(F), \alpha^{\frac{n}{2}} \mathcal{A}^{(1)}(F)\right) : \ \alpha \in (1, \infty), \ F \in \mathscr{M}(\mathbb{R}^{n+1}_+)\right\}.$$

Since $s/2 > 1$, by the definition of $A_p(\mathbb{R}^n)$ weights, we know that, for any $\omega \in A_1(\mathbb{R}^n)$, $\omega \in A_{s/2}(\mathbb{R}^n)$ holds true. From this and Lemma 3.2, we easily deduce that, for any given $\omega \in A_1(\mathbb{R}^n)$ and for any $(\mathcal{A}^{(\alpha)}(F), \alpha^{\frac{n}{2}} \mathcal{A}^{(1)}(F)) \in \mathcal{F}$,

$$\int_{\mathbb{R}^n} \left|\mathcal{A}^{(\alpha)}(F)(x)\right|^s \omega(x)\, dx \lesssim \alpha^{\frac{ns}{2}} \int_{\mathbb{R}^n} \left|\mathcal{A}^{(1)}(F)(x)\right|^s \omega(x)\, dx \sim \int_{\mathbb{R}^n} \left|\alpha^{\frac{n}{2}} \mathcal{A}^{(1)}(F)(x)\right|^s \omega(x)\, dx,$$

which, combined with the assumptions that $X^{1/s}$ is a ball Banach function space and $\mathcal{M}$ is bounded on $(X^{1/s})'$, and Lemma 2.12, further implies that, for any $\alpha \in [1, \infty)$ and $F \in \mathscr{M}(\mathbb{R}^{n+1}_+)$,

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_X \lesssim \alpha^{\frac{n}{2}} \left\|\mathcal{A}^{(1)}(F)\right\|_X . \tag{3.4}$$

By (3.3) and (3.4), we conclude that, for any $\alpha \in [1, \infty)$ and $F \in \mathscr{M}(\mathbb{R}^{n+1}_+)$,

$$\left\|\mathcal{A}^{(\alpha)}(F)\right\|_X \lesssim \alpha^{\max\{\frac{n}{2}, \frac{n}{s}\}} \left\|\mathcal{A}^{(1)}(F)\right\|_X ,$$

which is also the desired estimate and hence then completes the proof of Theorem 3.3. □

4 Littlewood–Paley function characterizations

In this section, we establish various Littlewood–Paley function characterizations of $H_X(\mathbb{R}^n)$, including its characterizations via the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function, respectively, in §4.1, §4.2 and §4.3 below.

In what follows, the symbol $\vec{0}_n$ denotes the *origin* of $\mathbb{R}^n$ and, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\phi}$ denotes its *Fourier transform* which is defined by setting, for any $\xi \in \mathbb{R}^n$,

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \xi} \phi(x)\, dx.$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f}$ is defined by setting, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \widehat{f}, \varphi\rangle := \langle f, \widehat{\varphi}\rangle$; also, for any $f \in \mathcal{S}(\mathbb{R}^n)$ [resp., $\mathcal{S}'(\mathbb{R}^n)$], $f^\vee$ denotes its *inverse Fourier transform* which is defined by setting, for any $\xi \in \mathbb{R}^n$, $f^\vee(\xi) := \widehat{f}(-\xi)$ [resp., for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle f^\vee, \varphi\rangle := \langle f, \varphi^\vee\rangle$].

Definition 4.1. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\widehat{\varphi}(\vec{0}_n) = 0$ and assume that, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, there exists a $t \in (0, \infty)$ such that $\widehat{\varphi}(t\xi) \neq 0$. For any distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, the *Lusin-area function* $S(f)$

and the *Littlewood-Paley g_λ^*-function* $g_\lambda^*(f)$ with any given $\lambda \in (0,\infty)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$S(f)(x) := \left\{ \int_{\Gamma(x)} |\varphi_t * f(y)|^2 \, \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}} \tag{4.1}$$

and

$$g_\lambda^*(f)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} |\varphi_t * f(x)|^2 \, \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}}, \tag{4.2}$$

where, for any $x \in \mathbb{R}^n$, $\Gamma(x)$ is as in Definition 3.1 and, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^n$, $\varphi_t(x) := t^{-n}\varphi(x/t)$.

Definition 4.2. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\widehat{\varphi}(\vec{0}_n) = 0$ and assume that, for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, there exists a $j \in \mathbb{Z}$ such that $\widehat{\varphi}(2^j x) \neq 0$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the *Littlewood–Paley g-function* $g(f)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$g(f)(x) := \left[\int_0^\infty |f * \varphi_t(x)|^2 \, \frac{dt}{t} \right]^{1/2}. \tag{4.3}$$

Remark 4.3. (i) The way to define the Littlewood–Paley functions in Definitions 4.1 and 4.2 is the same as in [69, (3.2), (3.3) and (3.4)]. Observe that, in Definitions 4.1 and 4.2, we did not assume that φ is radial and has compact support and hence, compared with the assumptions required in [34, 35, 43, 51, 78, 76], the assumptions here in both cases are quite weaker.

(ii) In all these Littlewood–Paley function characterizations of $H_X(\mathbb{R}^n)$, we only need $\widehat{\varphi}(\vec{0}_n) = 0$, namely, φ has a zero order vanishing moment. Compared with all the known results on the Littlewood–Paley function characterizations on function spaces (see, for instance, [34, 35, 43, 51, 78, 76]), this assumption on the vanishing moment of φ is also minimal.

4.1 Characterization by the Lusin area function

In this subsection, borrowing some ideas from the proof of [69, Theorem 3.12], we characterize the Hardy type space $H_X(\mathbb{R}^n)$ by the Lusin-area function.

Let $\alpha \in (0,\infty)$. For any ball $B(x,r) \subset \mathbb{R}^n$ with $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, let

$$T_\alpha(B) := \left\{ (y,t) \in \mathbb{R}^{n+1}_+ : \ 0 < t < \frac{r}{\alpha}, \ |y-x| < r - \alpha t \right\}.$$

When $\alpha = 1$, we denote $T_\alpha(B)$ simply by $T(B)$.

Definition 4.4. Let X be a ball quasi-Banach function space, $p \in (1,\infty)$ and $\alpha \in (0,\infty)$. A measurable function $a : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ is called a *(T_X, p)-atom*, supported in $T(B)$, if there exists a ball $B \subset \mathbb{R}^n$ such that

(i) $\operatorname{supp}(a) := \{(x,t) \in \mathbb{R}^{n+1}_+ : \ a(x,t) \neq 0\} \subset T(B)$,

(ii) $\|a\|_{T_2^{p,1}(\mathbb{R}^{n+1}_+)} \le |B|^{1/p}/\|\mathbf{1}_B\|_X$.

Moreover, if a is a (T_X, p)-atom for any $p \in (1,\infty)$, then a is called a *(T_X,∞)-atom*.

To establish the Lusin area function characterization of $H_X(\mathbb{R}^n)$, we need the following lemma which is just [59, Proposition 4.9].

Lemma 4.5. *Let $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ be a measurable function. Assume that X is a ball quasi-Banach function space satisfying Assumptions 2.6 and 2.7 with the same $s \in (0,1]$. Then $f \in T_X^1(\mathbb{R}^{n+1}_+)$ if and only if there exist a sequence $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ and a sequence $\{A_j\}_{j=1}^\infty$ of (T_X^1,∞)-atoms supported, respectively, in $\{T(B_j)\}_{j=1}^\infty$ such that, for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,*

$$F(x,t) = \sum_{j=1}^\infty \lambda_j A_j(x,t) \quad \text{and} \quad |F(x,t)| = \sum_{j=1}^\infty \lambda_j |A_j(x,t)|$$

pointwisely, and

$$\left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X} \right)^s \mathbf{1}_{B_j} \right\}^{1/s} \right\|_X < \infty.$$

Moreover,

$$\|f\|_{T_X^1(\mathbb{R}^{n+1}_+)} \sim \Lambda\left(\{\lambda_j A_j\}_{j\in\mathbb{N}}\right)$$

where the positive equivalence constants are independent of f, but depend on s.

Combining Calderón [12, Lemma 4.1] and Folland and Stein [25, Theorem 1.64] (see also [70, Lemma 4.6]), the following Calderón reproducing formula was obtained in [77, Lemma 4.4]. Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$ (see, for instance, [25, p. 50]), here and thereafter, for any $t \in (0,\infty)$ and function ϕ on $\mathbb{R}^n$, we always let $\phi_t(\cdot) := t^{-n}\phi(\cdot/t)$. In what follows, the *symbol* $\epsilon \to 0^+$ means that $\epsilon \in (0,\infty)$ and $\epsilon \to 0$, and the *symbol* $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions with compact supports.

Lemma 4.6. *Let ϕ be a Schwartz function and assume that, for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, there exists a $t \in (0,\infty)$ such that $\widehat{\phi}(tx) \neq 0$. Then there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi} \in C_c^\infty(\mathbb{R}^n)$ with its support away from $\vec{0}_n$, $\widehat{\phi}\widehat{\psi} \ge 0$ and, for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,*

$$\int_0^\infty \widehat{\phi}(tx)\widehat{\psi}(tx)\,\frac{dt}{t} = 1.$$

Moreover, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, if f vanishes weakly at infinity, then

$$f = \int_0^\infty f * \phi_t * \psi_t \,\frac{dt}{t} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

namely,

$$f = \lim_{\substack{\epsilon\to 0^+ \\ A\to\infty}} \int_\epsilon^A f * \phi_t * \psi_t \,\frac{dt}{t} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).$$

To obtain the Lusin area function characterization of $H_X(\mathbb{R}^n)$, we also need the following technical lemma.

Lemma 4.7. *Let X be a ball quasi-Banach function space. Assume that there exists an $s \in (0,\infty)$ such that $X^{1/s}$ is a ball Banach function space and $\mathcal{M}$ is bounded on $(X^{1/s})'$. Then there exists an $\epsilon \in (0,1)$ such that X continuously embeds into $L^s_\omega(\mathbb{R}^n)$ with $\omega := [\mathcal{M}(\mathbf{1}_{B(\vec{0}_n,1)})]^\epsilon$.*

Proof. By [59, Lemma 2.15(ii)] and the fact that $\mathcal{M}$ is bounded on $(X^{1/s})'$, we know that there exists an $\eta \in (1,\infty)$ such that $\mathcal{M}^{(\eta)}$ is bounded on $(X^{1/s})'$. Let $\epsilon \in (1/\eta, 1)$ and $\omega := [\mathcal{M}(\mathbf{1}_{B(\vec{0}_n,1)})]^\epsilon$. To show this lemma, it suffices to prove that, for any $f \in X$,

$$\|f\|_{L^s_\omega(\mathbb{R}^n)} \lesssim \|f\|_X. \tag{4.4}$$

Indeed, from [30, (2.1.6)], we deduce that, for any $x \in \mathbb{R}^n$, $\mathcal{M}(\mathbf{1}_{B(\vec{0}_n,1)})(x) \sim (|x|+1)^{-n}$, which implies that, for any $x \in B(\vec{0}_n, 2)$,

$$\omega(x) = \left[\mathcal{M}\left(\mathbf{1}_{B(\vec{0}_n,1)}\right)(x)\right]^\epsilon \lesssim 1 \tag{4.5}$$

and, for any $k \in \mathbb{N}$ and $x \in B(\vec{0}_n, 2^{k+1}) \setminus B(\vec{0}_n, 2^k)$,

$$\omega(x) = \left[\mathcal{M}\left(\mathbf{1}_{B(\vec{0}_n,1)}\right)(x)\right]^\epsilon \lesssim 2^{-\epsilon kn}. \tag{4.6}$$

Combining (4.5), (4.6), the Hölder inequality and the fact that $X^{1/s}$ is a ball Banach function space, we conclude that, for any $f \in X$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^s \,\omega(x)\,dx &= \int_{B(\vec{0}_n,2)} |f(x)|^s \,\omega(x)\,dx + \sum_{k=1}^\infty \int_{B(\vec{0}_n,2^{k+1})\setminus B(\vec{0}_n,2^k)} \cdots \\
&\lesssim \left\||f|^s\right\|_{X^{1/s}} \left\|\mathbf{1}_{B(\vec{0}_n,2)}\right\|_{(X^{1/s})'} + \sum_{k=1}^\infty \int_{B(\vec{0}_n,2^{k+1})\setminus B(\vec{0}_n,2^k)} |f(x)|^s 2^{-\epsilon kn}\,dx \\
&\lesssim \left\||f|^s\right\|_{X^{1/s}} \left\|\mathbf{1}_{B(\vec{0}_n,2)}\right\|_{(X^{1/s})'} + \sum_{k=1}^\infty 2^{-\epsilon kn} \left\||f|^s\right\|_{X^{1/s}} \left\|\mathbf{1}_{B(\vec{0}_n,2^{k+1})}\right\|_{(X^{1/s})'},
\end{aligned}$$

which, together with $\mathbf{1}_{B(\vec{0}_n,2^k)} \lesssim 2^{kn/\eta}\mathcal{M}^{(\eta)}(1_{B(\vec{0}_n,1)})$ for any $k \in \mathbb{N}$ and the fact that $\mathcal{M}^{(\eta)}$ is bounded on $(X^{1/s})'$, further implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^s \,\omega(x)\,dx &\lesssim \sum_{k=1}^\infty 2^{-\epsilon kn} \|f\|_X^s \left\|2^{kn/\eta}\mathcal{M}^{(\eta)}(1_{B(\vec{0}_n,1)})\right\|_{(X^{1/s})'} \\
&\lesssim \sum_{k=1}^\infty 2^{-(\epsilon-1/\eta)kn} \|f\|_X^s \left\|\mathbf{1}_{B(\vec{0}_n,1)}\right\|_{(X^{1/s})'} \lesssim \|f\|_X^s,
\end{aligned}$$

which implies that (4.4) holds true and hence completes the proof of Lemma 4.7. □

Now we recall the notion of atoms associated with X, which is just [59, Definition 3.5].

Definition 4.8. Let X be a ball quasi-Banach function space, $q \in (1, \infty]$ and $d \in \mathbb{Z}_+$. Then a measurable function a on $\mathbb{R}^n$ is called an *(X, q, d)-atom* if there exists a ball $B \in \mathbb{B}$ such that

(i) $\operatorname{supp} a := \{x \in \mathbb{R}^n : \ a(x) \neq 0\} \subset B$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq \frac{|B|^{1/q}}{\|\mathbf{1}_B\|_X}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\, dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq d$.

In what follows, for any $t \in (0, \infty)$, the *symbol $e^{t\Delta} f$* denotes the heat extension of $f \in \mathcal{S}'(\mathbb{R}^n)$, namely, for any $x \in \mathbb{R}^n$,

$$e^{t\Delta} f(x) := \left\langle f, \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - \cdot|^2}{4t}\right)\right\rangle.$$

Theorem 4.9. *Assume that X is a ball quasi-Banach function space satisfying Assumptions 2.6 and 2.7 with the same $s \in (0, 1]$. Then $f \in H_X(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|S(f)\|_X < \infty$. Moreover, for any $f \in H_X(\mathbb{R}^n)$,*

$$\|f\|_{H_X(\mathbb{R}^n)} \sim \|S(f)\|_X,$$

where the positive equivalence constants are independent of f.

Remark 4.10. If φ appearing in the definition of $S(f)$ as in (4.1) satisfies that $\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \leq \widehat{\varphi} \leq \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$, then, in this case, Theorem 4.9 coincides with [59, Theorem 3.21]. We point out that there exists a gap in lines 1-17 of [59, p. 52] which appears in the proof of [59, Theorem 3.21] when Sawano et al. proved that f vanishes weakly at infinity. We seal this gap in the below proof of Theorem 4.9 by using Lemma 4.7.

Proof of Theorem 4.9. Let θ and s be as in Assumption 2.6.

We first prove the necessity. To this end, let $f \in H_X(\mathbb{R}^n)$ and we need to show that f vanishes weakly at infinity, which seals the gap mentioned in Remark 4.10. From Lemma 4.7, we deduce that there exists an $\epsilon \in (0, 1)$ such that X continuously embeds into $L^s_\omega(\mathbb{R}^n)$ with $\omega := [\mathcal{M}(\mathbf{1}_{B(\vec{0}_n,1)})]^\epsilon$, which implies that $f \in H^s_\omega(\mathbb{R}^n)$, where $H^s_\omega(\mathbb{R}^n)$ is the weighted Hardy space as in Definition 2.13 with X replaced by $L^s_\omega(\mathbb{R}^n)$. By [30, Theorem 7.2.7], we know that $\omega \in A_1(\mathbb{R}^n)$, which, combined with [68, Remark 2.4(b) and Remark 2.6(b)], implies that $L^s_\omega(\mathbb{R}^n)$ satisfies all the assumptions of [59, Theorems 3.7]. Let $d \geq \lfloor n(1/\theta - 1)\rfloor$. Then, using [59, Theorem 3.7] and the fact that $f \in H^s_\omega(\mathbb{R}^n)$, we conclude that there exist a sequence $\{a_j\}_{j\in\mathbb{N}}$ of $(L^s_\omega(\mathbb{R}^n), \infty, d)$-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbb{N}}$ and a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0, \infty)$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad H^s_\omega(\mathbb{R}^n) \tag{4.7}$$

and

$$\left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{\lambda_j \mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^s_\omega(\mathbb{R}^n)}} \right]^s \right\}^{\frac{1}{s}} \right\|_{L^s_\omega(\mathbb{R}^n)} \lesssim \|f\|_{H^s_\omega(\mathbb{R}^n)} < \infty, \tag{4.8}$$

where an $(L^s_\omega(\mathbb{R}^n),\infty,d)$-atom is as in Definition 4.8 with X replaced by $L^s_\omega(\mathbb{R}^n)$. Take $\varphi,\ \psi\in\mathcal{S}(\mathbb{R}^n)$. Then, by the fact that $f=\sum_{j\in\mathbb{N}}\lambda_j a_j$ in $H^s_\omega(\mathbb{R}^n)$, we find that, for any $t\in(0,\infty)$ and finite set $\mathfrak{F}\subset\mathbb{N}$,

$$\int_{\mathbb{R}^n}\psi_t*f(x)\varphi(x)\,dx=\sum_{j\in\mathfrak{F}}\lambda_j\int_{\mathbb{R}^n}\psi_t*a_j(x)\varphi(x)\,dx+\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\cdots. \tag{4.9}$$

Notice that, for any $j\in\mathbb{N}$, $a_j\in L^\infty(\mathbb{R}^n)$, which implies that a_j vanishes weakly at infinity. Thus, for any given $\varepsilon\in(0,\infty)$ and finite set $\mathfrak{F}\subset\mathbb{N}$, there exists $t_{\varepsilon,\mathfrak{F}}\in(0,\infty)$ such that, for any $t\in(t_{\varepsilon,\mathfrak{F}},\infty)$,

$$\left|\sum_{j\in\mathfrak{F}}\lambda_j\int_{\mathbb{R}^n}\psi_t*a_j(x)\varphi(x)\,dx\right|<\varepsilon. \tag{4.10}$$

Moreover, for any $j\in\mathbb{N}$, using the fact that a_j is an $(L^s_\omega(\mathbb{R}^n),\infty,d)$-atom, similarly to the proof of [59, (4.16)], we conclude that, for any $t,\ t_1\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$\left|e^{-t_1\Delta}\psi_t*a_j(x)\right|\lesssim\frac{1}{\|\mathbf{1}_{B_j}\|_{L^s_\omega(\mathbb{R}^n)}}\mathcal{M}^{(\theta)}(\mathbf{1}_{B_j})(x),$$

which, together with [59, Corollary 3.2], $s\in(0,1]$ and Assumption 2.6, implies that

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\lambda_j\int_{\mathbb{R}^n}\psi_t*a_j(x)\varphi(x)\,dx\right|\\
&\quad=\left|\left(\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\lambda_j\psi_t*a_j\right)*[\varphi(-\cdot)](\vec{0}_n)\right|\lesssim\left\|\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\lambda_j\psi_t*a_j\right\|_{H^s_\omega(\mathbb{R}^n)}\\
&\quad\sim\left\|\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\lambda_j\sup_{t_1\in(0,\infty)}\left|e^{-t_1\Delta}\psi_t*a_j\right|\right\|_{L^s_\omega(\mathbb{R}^n)}\lesssim\left\|\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_{L^s_\omega(\mathbb{R}^n)}}\mathcal{M}^{(\theta)}(\mathbf{1}_{B_j})\right\|_{L^s_\omega(\mathbb{R}^n)}\\
&\quad\lesssim\left\|\left\{\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\left[\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_{L^s_\omega(\mathbb{R}^n)}}\mathcal{M}^{(\theta)}(\mathbf{1}_{B_j})\right]^s\right\}^{1/s}\right\|_{L^s_\omega(\mathbb{R}^n)}\lesssim\left\|\left\{\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_{L^s_\omega(\mathbb{R}^n)}}\right)^s\mathbf{1}_{B_j}\right\}^{1/s}\right\|_{L^s_\omega(\mathbb{R}^n)}.
\end{aligned}$$

From this, the fact that

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_{L^s_\omega(\mathbb{R}^n)}}\right)^s\mathbf{1}_{B_j}\right\}^{\frac{1}{s}}\right\|_{L^s_\omega(\mathbb{R}^n)}<\infty,$$

and the dominated convergence theorem, we deduce that, for $\varepsilon\in(0,\infty)$ as in (4.10), there exists a finite set $\mathfrak{F}\subset\mathbb{N}$ such that, for any $t\in(0,\infty)$,

$$\left|\sum_{j\in\mathbb{N}\setminus\mathfrak{F}}\lambda_j\int_{\mathbb{R}^n}\psi_t*a_j(x)\varphi(x)\,dx\right|<\varepsilon,$$

which, combined with (4.9) and (4.10), further implies that f vanishes weakly at infinity.

Then, by an argument similar to that used in the proof of [59, Theorem 3.21], we obtain the necessity of this theorem and we omit the details.

Now we prove the sufficiency. To this end, let $f \in \mathcal{S}'(\mathbb{R}^n)$ vanish weakly at infinity and satisfy that $S(f) \in X$. By this and Lemma 4.6, we conclude that there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\widehat{\psi}) \subset B(\vec{0}_n, b) \setminus B(\vec{0}_n, a), \tag{4.11}$$

where $b,\ a \in (0, \infty)$ and $b > a$, and

$$f = \int_0^\infty f * \varphi_t * \psi_t \frac{dt}{t} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \tag{4.12}$$

with φ as in Definition 4.1. It remains to prove that $f \in H_X(\mathbb{R}^n)$. For any $(x, t) \in \mathbb{R}^{n+1}_+$, let $F(x, t) := f * \varphi_t(x)$. Then, by the fact that $\|S(f)\|_X < \infty$, we know that $F \in T^1_X(\mathbb{R}^{n+1}_+)$. From this and Lemma 4.5, it follows that there exist a sequence $\{A_j\}_{j=1}^\infty$ of (T^1_X, ∞)-atoms and $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ such that, for almost every $(x, t) \in \mathbb{R}^{n+1}_+$,

$$F(x, t) = \sum_{j\in\mathbb{N}} \lambda_j A_j(x, t) \quad \text{and} \quad |F(x, t)| = \sum_{j\in\mathbb{N}} \lambda_j |A_j(x, t)| \tag{4.13}$$

pointwisely on $\mathbb{R}^{n+1}_+$ and, for some $s \in (0, 1]$,

$$\left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X} \right)^s \mathbf{1}_{B_j} \right\}^{1/s} \right\|_X \lesssim \|F\|_{T^1_X(\mathbb{R}^{n+1}_+)} \sim \|S(f)\|_X . \tag{4.14}$$

For any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$A_j(x) := \int_0^\infty \int_{\mathbb{R}^n} A_j(y, t)\psi_t(x - y) \frac{dy\,dt}{t}. \tag{4.15}$$

Similarly to the proof of [34, Lemma 4.8], we conclude that, up to a harmless constant multiple, $\{A_j\}_{j=1}^\infty$ is a sequence of (X, q, d, ϵ)- molecules associated, respectively, with balls $\{B_j\}_{j=1}^\infty$, where q, d and ϵ are as in Lemma 2.15. Repeating the argument used in [69, (3.27)], we find that $\sum_{j=1}^\infty \lambda_j A_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Using this, (4.14) and Lemma 2.15, we then obtain $\sum_{j=1}^\infty \lambda_j A_j \in H_X(\mathbb{R}^n)$ and

$$\left\| \sum_{j=1}^\infty \lambda_j A_j \right\|_{H_X} \lesssim \left\| \left\{ \sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X} \right)^s \mathbf{1}_{B_j} \right\}^{1/s} \right\|_X \lesssim \|S(f)\|_X . \tag{4.16}$$

Let $g := \sum_{j=1}^\infty \lambda_j A_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Then $g \in H_X(\mathbb{R}^n)$. From this and the necessity of this theorem, it follows that g vanishes weakly at infinity. Using this and repeating the proof of [69, (3.30)] , we find that

$$f = g \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \tag{4.17}$$

holds true. By (4.17) and (4.16), we conclude that $f \in H_X(\mathbb{R}^n)$ and $\|f\|_{WH_X(\mathbb{R}^n)} \lesssim \|S(f)\|_{WX}$, which completes the proof of the sufficiency and hence of Theorem 4.9. $\square$

4.2 Characterization by the Littlewood–Paley g_λ^*-Function

In this subsection, we establish the Littlewood–Paley g_λ^*-function characterization of $H_X(\mathbb{R}^n)$. Let X be a ball quasi-Banach function space and

$$r_+ := \sup\{s \in (0,\infty) : \ X \text{ satisfies Assumption 2.7 for this } s \text{ and some } q \in (s,\infty)\}. \tag{4.18}$$

Theorem 4.11. *Assume that X is a ball quasi-Banach function space satisfying Assumptions 2.6 and 2.7 with the same $s \in (0,1]$. Let r_+ be as in* (4.18) *and $\lambda \in (\max\{1, 2/r_+\},\infty)$. Then $f \in H_X(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|g_\lambda^*(f)\|_X < \infty$. Moreover, for any $f \in H_X(\mathbb{R}^n)$,*

$$\|f\|_{H_X(\mathbb{R}^n)} \sim \|g_\lambda^*(f)\|_X,$$

where the positive equivalence constants are independent of f.

Remark 4.12. Assume that X is a ball quasi-Banach function space satisfying Assumption 2.6 with $0 < \theta < s \le 1$ and Assumption 2.7 with some $q \in (1,\infty]$ and the same s as in Assumption 2.6. Let r_+ be as in (4.18). We point out that the g_λ^*-function characterization in Theorem 4.11 widens the range of $\lambda \in (\max\{2/\theta, 2/\theta+1-2/q\},\infty)$ in [68, Theorem 2.10(ii)] into $\lambda \in (\max\{1, 2/r_+\},\infty)$. In the case of $X := L^p(\mathbb{R}^n)$ with $p \in (0,\infty)$, the range of λ in Theorem 4.11 coincides with the best known one, namely, $(\max\{1, 2/p\},\infty)$ in [25].

Proof of Theorem 4.11. By Theorem 4.9 and the fact that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, $S(f) \le g_\lambda^*(f)$, we easily obtain the sufficiency of this theorem and still need to show its necessity.

To this end, let $f \in H_X(\mathbb{R}^n)$. By Theorem 4.9, we know that f vanishes weakly at infinity. Moreover, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} g_\lambda^*(f)(x) &\le \left\{\int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |\varphi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right. \\ &\qquad \left. + \sum_{m=0}^\infty \int_0^\infty \int_{2^m t \le |x-y| < 2^{m+1} t} \cdots \right\}^{\frac12} \\ &\le \mathcal{A}^{(1)}(F)(x) + \sum_{m=0}^\infty 2^{\frac{-\lambda n m}{2}} \mathcal{A}^{(2^{m+1})}(F)(x), \end{aligned}$$

where $F(x,t) := \varphi_t * f(x)$ for any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$.

For any given $\lambda \in (\max\{1, 2/r_+\},\infty)$, there exists an $r \in (0, r_+]$ such that X satisfies Assumption 2.7 for this r and some $q \in (r,\infty)$, and $\lambda \in (\max\{1, 2/r\},\infty)$. Let $\nu \in (0, \min\{1, r\}]$. Then, by the assumption that $X^{1/r}$ is a ball Banach function space, and Lemma 2.5, we know that $X^{1/\nu}$ is also a ball Banach function space. Since X satisfies Assumption 2.7, from Lemma 2.9 and the fact that $(q/r)' \in (1,\infty)$, we deduce that $\mathcal{M}$ is bounded on $(X^{1/r})'$, which, combined with the fact that $X^{1/r}$ is a ball Banach function space, and Theorem 3.3, implies that, for any $m \in \mathbb{N}$,

$$\left\|\mathcal{A}^{(2^{m+1})}(F)\right\|_X \lesssim \max\left\{2^{\frac{mn}{2}}, 2^{\frac{mn}{r}}\right\} \left\|\mathcal{A}^{(1)}(F)\right\|_X.$$

By this, the fact that $X^{1/\nu}$ is a ball Banach function space, $\lambda \in (\max\{\frac{2}{r}, 1\}, \infty)$, and Theorem 4.9, we conclude that

$$\begin{aligned}\left\|g_\lambda^*(f)\right\|_X^\nu &= \left\|\left[g_\lambda^*(f)\right]^\nu\right\|_{X^{1/\nu}} \lesssim \left\|\left[\mathcal{A}^{(1)}(F)\right]^\nu\right\|_{X^{1/\nu}} + \sum_{m=0}^\infty 2^{\frac{-\lambda nm\nu}{2}} \left\|\left[\mathcal{A}^{(2^{m+1})}(F)\right]^\nu\right\|_{X^{1/\nu}} \\ &\lesssim \left\|\mathcal{A}^{(1)}(F)\right\|_X^\nu + \sum_{m=0}^\infty 2^{\frac{-\lambda nm\nu}{2}} \max\left\{2^{\frac{mn\nu}{2}}, 2^{\frac{mn\nu}{r}}\right\} \left\|\mathcal{A}^{(1)}(F)\right\|_X^\nu \\ &\lesssim \|S(f)\|_X^\nu \sim \|f\|_{H_X(\mathbb{R}^n)}^\nu,\end{aligned}$$

where, in the penultimate step, we used the fact that $\mathcal{A}^{(1)}(F) = S(f)$. This finishes the proof of the necessity and hence of Theorem 4.11. □

4.3 Characterization by the Littlewood–Paley g-function

We have the following Littlewood–Paley g-function characterization of $H_X(\mathbb{R}^n)$.

Theorem 4.13. *Assume that X is a ball quasi-Banach function space satisfying Assumption 2.6 with $0 < \theta < s \le 1$ and Assumption 2.7 with the same s as in Assumption 2.6. Assume that there exists a positive constant C such that, for any $\{f_j\}_{j=1}^\infty \subset \mathscr{M}(\mathbb{R}^n)$,*

$$\left\|\left\{\sum_{j=1}^\infty \left[\mathcal{M}^{(\theta)}(f_j)\right]^s\right\}^{\frac{1}{s}}\right\|_{X^{s/2}} \le C \left\|\left\{\sum_{j=1}^\infty |f_j|^s\right\}^{\frac{1}{s}}\right\|_{X^{s/2}}. \tag{4.19}$$

Then $f \in H_X(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|g(f)\|_X < \infty$. Moreover, for any $f \in H_X(\mathbb{R}^n)$,

$$\|f\|_{H_X(\mathbb{R}^n)} \sim \|g(f)\|_X,$$

where the positive equivalence constants are independent of f.

The following pointwise estimate is a slight variant of [66, (2.66)], which is just [69, Lemma 3.21].

Lemma 4.14. *Let ϕ be a Schwartz function and assume that, for any $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, there exists a $j \in \mathbb{Z}$ such that $\widehat{\phi}(2^j x) \neq 0$. Then, for any given $N_0 \in \mathbb{N}$ and $\gamma \in (0, \infty)$, there exists a positive constant $C_{(N_0,\gamma,\phi)}$, depending only on n, N_0, γ and ϕ, such that, for any $s \in [1, 2]$, $a \in (0, N_0]$, $l \in \mathbb{Z}$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\left[\left(\phi_{2^{-l}s}^* f\right)_a(x)\right]^\gamma \le C_{(N_0,\gamma,\phi)} \sum_{k=0}^\infty 2^{-kN_0\gamma} 2^{(k+l)n} \int_{\mathbb{R}^n} \frac{|(\phi_{2^{-(k+l)}})_s * f(y)|^\gamma}{(1 + 2^l|x - y|)^{a\gamma}}\, dy.$$

Proof of Theorem 4.13. By an argument similar to that used in the proof of [68, Theorem 2.10(i)], we obtain the necessity of this theorem.

Conversely, repeating the proof of [68, Theorem 2.10(i)] via replacing [68, Lemma 2.14] used therein by Lemma 4.14 here, we complete the proof of the sufficiency and hence of Theorem 4.13. □

5 Applications

In this section, we apply Theorems 4.9, 4.11 and 4.13, respectively, to five concrete examples of ball quasi-Banach function spaces, namely, Morrey spaces (see §5.1 below), mixed-norm Lebesgue spaces (see §5.2 below), variable Lebesgue spaces (see §5.3 below), weighted Lebesgue spaces (see §5.4 below) and Orlicz-slice spaces (see §5.5 below). Observe that, among these five examples, only variable Lebesgue spaces are quasi-Banach function spaces as in Remark 2.2(ii), while the other four examples are ball quasi-Banach function spaces, which are not necessary to be quasi-Banach function spaces.

5.1 Morrey spaces

Recall that, due to the applications in elliptic partial differential equations, the Morrey space $M_r^p(\mathbb{R}^n)$ with $0 < r \le p < \infty$ was introduced by Morrey [50] in 1938. In recent decades, there exists an increasing interest in applications of Morrey spaces to various areas of analysis such as partial differential equations, potential theory and harmonic analysis (see, for instance, [3, 4, 16, 39, 75]).

Definition 5.1. Let $0 < r \le p < \infty$. The *Morrey space* $M_r^p(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{M_r^p(\mathbb{R}^n)} := \sup_{B\in\mathbb{B}} |B|^{1/p-1/r}\|f\|_{L^q(B)} < \infty,$$

where $\mathbb{B}$ is as in (2.1) (the set of all balls of $\mathbb{R}^n$).

Remark 5.2. Observe that, as was pointed out in [59, p. 86], $M_r^p(\mathbb{R}^n)$ may not be a quasi-Banach function space, but it is a ball quasi-Banach function space as in Definition 2.1.

Let $0 < r \le p < \infty$. From [60, Theorem 2.4] and [65, Lemma 2.5], it follows that $M_r^p(\mathbb{R}^n)$ satisfies Assumption 2.6 for any $\theta,\ s \in (0, \min\{1, r\})$ with $\theta < s$ (see also [16, 33]). Applying [33, Lemma 5.7] and [61, Theorem 4.1], we can easily show that Assumption 2.7 holds true for any given $s \in (0, r)$ and $q \in (r, \infty]$, and $X := M_r^p(\mathbb{R}^n)$. Thus, all the assumptions of main theorems in Section 4 are satisfied. Using Theorems 4.9, 4.13 and 4.11, we obtain the following characterizations of Morrey–Hardy space $HM_r^p(\mathbb{R}^n)$, respectively, in terms of the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function.

Theorem 5.3. *Let $p,\ r \in (0,\infty)$ with $r \le p$. Then $f \in HM_r^p(\mathbb{R}^n)$ if and only if either of the following two items holds true:*

(i) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $S(f) \in M_r^p(\mathbb{R}^n)$, where $S(f)$ is as in* (4.1).

(ii) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in M_r^p(\mathbb{R}^n)$, where $g(f)$ is as in* (4.3).

Moreover, for any $f \in HM_r^p(\mathbb{R}^n)$,

$$\|f\|_{HM_r^p(\mathbb{R}^n)} \sim \|S(f)\|_{M_r^p(\mathbb{R}^n)} \sim \|g(f)\|_{M_r^p(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.4. If φ appearing in the definitions of $S(f)$ and $g(f)$ as in (4.1) and (4.3) satisfies that $\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \le \widehat{\varphi} \le \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$, then, in this case, Theorem 5.24 was obtained by Sawano et al. [59, Theorem 3.21].

Theorem 5.5. *Let $p,\ r \in (0,\infty)$ with $r \le p$, and $\lambda \in (\max\{1, 2/r\}, \infty)$. Then $f \in HM_r^p(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|g_\lambda^*(f)\|_{M_r^p(\mathbb{R}^n)} < \infty$. Moreover, for any $f \in HM_r^p(\mathbb{R}^n)$,*

$$\|f\|_{HM_r^p(\mathbb{R}^n)} \sim \|g_\lambda^*(f)\|_{M_r^p(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.6. If $p \in (0,1]$, $r \in (0,p]$ and φ appearing in the definition of $g_\lambda^*(f)$ as in (4.2) satisfies that $\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \le \widehat{\varphi} \le \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$ then, in this case, Wang et al. [68, Corollary 2.22(v)] also obtained the same result as in Theorem 5.5. When $p \in (1,\infty)$ and $r \in (0,p]$, to the best of our knowledge, the result of Theorem 5.5 is new.

5.2 Mixed-norm Lebesgue spaces

The mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ was studied by Benedek and Panzone [9] in 1961, which can be traced back to Hormander [31]. Later on, in 1970, Lizorkin [46] further developed both the theory of multipliers of Fourier integrals and estimates of convolutions in the mixed-norm Lebesgue spaces. Particularly, in order to meet the requirements arising in the study of the boundedness of operators, partial differential equations and some other analysis fields, the real-variable theory of mixed-norm function spaces, including mixed-norm Morrey spaces, mixed-norm Hardy spaces, mixed-norm Besov spaces and mixed-norm Triebel–Lizorkin spaces, has rapidly been developed in recent years (see, for instance, [17, 29, 55, 37, 35, 36]).

Definition 5.7. Let $\vec{p} := (p_1, \ldots, p_n) \in (0,\infty]^n$. The *mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$* is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1,\ldots,x_n)|^{p_1}\, dx_1\right]^{\frac{p_2}{p_1}} \cdots dx_n\right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1,\ldots,n\}$.

In this subsection, for any $\vec{p} := (p_1,\ldots,p_n) \in (0,\infty)^n$, we always let $p_- := \min\{p_1,\ldots,p_n\}$ and $p_+ := \max\{p_1,\ldots,p_n\}$.

Let $\vec{p} \in (0,\infty)^n$. Then $L^{\vec{p}}(\mathbb{R}^n)$ satisfies Assumption 2.6 for any $\theta,\ s \in (0, \min\{1, p_-\})$ with $\theta < s$ (see [35, Lemma 3.7]). Applying [35, Lemma 3.5] and [9, Theorem 1.a], we can easily show that Assumption 2.7 holds true for any given $s \in (0, p_-)$ and $q \in (p_+, \infty]$, and $X := L^{\vec{p}}(\mathbb{R}^n)$. Thus, all the assumptions of main theorems in Sections 3 and 4 are satisfied. Using Theorems 4.9, 4.13 and 4.11, we obtain the following characterizations of the mixed Hardy space $H^{\vec{p}}(\mathbb{R}^n)$, respectively, in terms of the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function.

Theorem 5.8. *Let $\vec{p} := (p_1,\ldots,p_n) \in (0,\infty)^n$. Then $f \in H^{\vec{p}}(\mathbb{R}^n)$ if and only if either of the following two items holds true:*

(i) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $S(f) \in L^{\vec{p}}(\mathbb{R}^n)$, where $S(f)$ is as in* (4.1).

(ii) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in L^{\vec{p}}(\mathbb{R}^n)$, where $g(f)$ is as in* (4.3).

Moreover, for any $f \in H^{\vec{p}}(\mathbb{R}^n)$,

$$\|f\|_{H^{\vec{p}}(\mathbb{R}^n)} \sim \|S(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \|g(f)\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.9. Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty)^n$. If φ appearing in the definitions of $S(f)$ and $g(f)$ as in (4.1) and (4.3) satisfies that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function such that, for any multi-index $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq \lfloor n(\frac{1}{\min\{p_1,\ldots,p_n\}} - 1)\rfloor$, $\int_{\mathbb{R}^n} x^\alpha \varphi(x)\,dx = 0$ and, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(2^k\xi)|^2 = 1$, then, in this case, Theorem 5.8 was obtained by Huang et al. [35, Theorems 4.1 and 4.2] as a special case.

Theorem 5.10. *Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty)^n$. Let $\lambda \in (\max\{1, \frac{2}{\min\{p_1,\ldots,p_n\}}\}, \infty)$. Then $f \in H^{\vec{p}}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|g_\lambda^*(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} < \infty$. Moreover, for any $f \in H^{\vec{p}}(\mathbb{R}^n)$,*

$$\|f\|_{H^{\vec{p}}(\mathbb{R}^n)} \sim \|g_\lambda^*(f)\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.11. Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty)^n$. If φ appearing in the definition of $g_\lambda^*(f)$ as in (4.2) is as in Remark 5.9, then, in this case, Theorem 5.10 widens the range of $\lambda \in (1 + \frac{2}{\min\{2,p_1,\ldots,p_n\}}, \infty)$ in [35, Theorem 4.3] into $\lambda \in (\max\{1, \frac{2}{\min\{p_1,\ldots,p_n\}}\}, \infty)$.

5.3 Variable Lebesgue spaces

Let $p(\cdot): \mathbb{R}^n \to [0, \infty)$ be a measurable function. Then the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} [|f(x)|/\lambda]^{p(x)}\,dx \leq 1\right\} < \infty.$$

We refer the reader to [53, 54, 40, 19, 22] for more details on variable Lebesgue spaces.

For any measurable function $p(\cdot): \mathbb{R}^n \to (0, \infty)$, in this subsection, we let

$$\widetilde{p_-} := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad \widetilde{p}_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

If $0 < \widetilde{p}_- \leq \widetilde{p}_+ < \infty$, then, similarly to the proof of [23, Theorem 3.2.13], we know that $L^{p(\cdot)}(\mathbb{R}^n)$ is a quasi-Banach function space and hence a ball quasi-Banach function space.

A measurable function $p(\cdot): \mathbb{R}^n \to (0, \infty)$ is said to be *globally log-Hölder continuous* if there exists a $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e + 1/|x - y|)}$$

and

$$|p(x) - p_\infty| \lesssim \frac{1}{\log(e + |x|)},$$

where the positive equivalence constants are independent of x and y.

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a globally log-Hölder continuous function satisfying $0 < \widetilde{p}_- \le \widetilde{p}_+ < \infty$. Adamowicz et al. [2] obtained the boundedness of the Hardy–Littlewood maximal operator on variable Lebesgue spaces. Using this and [5, Theorem 3.1], we can readily prove that, for any $\theta,\ s \in (0, \min\{1, \widetilde{p}_-\})$ with $\theta < s$, Assumption 2.6 is satisfied (see also [19, 21]). Furthermore, from [19, Lemma 2.16], we deduce that Assumption 2.7 holds true for any given $s \in (0, \widetilde{p}_-)$ and $q \in (\widetilde{p}_+, \infty]$, and $X := L^{p(\cdot)}(\mathbb{R}^n)$. Thus, all the assumptions of main theorems in Section 4 are satisfied. Using Theorems 4.9, 4.13 and 4.11, we obtain the following characterizations of variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, respectively, in terms of the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function.

Theorem 5.12. *Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a globally log-Hölder continuous function satisfying $0 < \widetilde{p}_- \le \widetilde{p}_+ < \infty$. Then $f \in H^{\vec{p}}(\mathbb{R}^n)$ if and only if either of the following two items holds true:*

(i) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $S(f) \in L^{p(\cdot)}(\mathbb{R}^n)$, where $S(f)$ is as in* (4.1).

(ii) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in L^{p(\cdot)}(\mathbb{R}^n)$, where $g(f)$ is as in* (4.3).

Moreover, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.13. If φ appearing in the definitions of $S(f)$ and $g(f)$ as in (4.1) and (4.3) satisfies that $\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \le \widehat{\varphi} \le \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$, then, in this case, Theorem 5.12 was obtained by [59, Theorem 3.21] and [68, Corollary 2.22(vi)].

Theorem 5.14. *Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a globally log-Hölder continuous function satisfying $0 < \widetilde{p}_- \le \widetilde{p}_+ < \infty$. Let $\lambda \in (\max\{1, \frac{2}{\widetilde{p}_-}\}, \infty)$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|g_\lambda^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$. Moreover, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \|g_\lambda^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.15. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a globally log-Hölder continuous function satisfying $0 < \widetilde{p}_- \le \widetilde{p}_+ < \infty$. Let $\lambda \in (\max\{1, \frac{2}{\widetilde{p}_-}\}, \infty)$. If φ appearing in the definition of $g_\lambda^*(f)$ as in (4.2) satisfies that $\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \le \widehat{\varphi} \le \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$, then, in this case, Theorem 5.26 widens the range of $\lambda \in (\max\{\frac{2}{\min\{1,\widetilde{p}_-\}}, 1 - \frac{2}{\max\{1,\widetilde{p}_+\}} + \frac{2}{\min\{1,\widetilde{p}_-\}}\}, \infty)$ in [68, Corollary 2.22(vi)] into $\lambda \in (\max\{1, \frac{2}{\widetilde{p}_-}\}, \infty)$.

5.4 Weighted Lebesgue spaces

If X is a weighted Lebesgue space $L^p_\omega(\mathbb{R}^n)$ with $p \in (0,\infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ is just the weighted Hardy space $H^p_\omega(\mathbb{R}^n)$, which was studied in [11, 26, 43, 14, 64, 72, 71].

It is worth pointing out that a weighted Lebesgue space with an $A_\infty(\mathbb{R}^n)$-weight may not be a Banach function space; see [59, Section 7.1]. From [23, Theorem 2.7.4], we deduce that, when $p \in (1,\infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$, $[L^p_\omega(\mathbb{R}^n)]' = L^{p'}_{\omega^{1-p'}}(\mathbb{R}^n)$, where $[L^p_\omega(\mathbb{R}^n)]'$ is the associated space of $L^p_\omega(\mathbb{R}^n)$ as in (2.4) with $X := L^p_\omega(\mathbb{R}^n)$. For any given $\omega \in A_\infty(\mathbb{R}^n)$, let

$$q_\omega := \inf\left\{q \in [1,\infty) :\ \omega \in A_q(\mathbb{R}^n)\right\}. \tag{5.1}$$

Let $p \in (0,\infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. Obviously, $\omega \in A_{p/\theta}(\mathbb{R}^n)$ holds true for any $\theta \in (0, p/q_\omega)$. From this and [5, Theorem 3.1(b)], we deduce that $L^p_\omega(\mathbb{R}^n)$ satisfies Assumption 2.6 for any $\theta,\ s \in (0, \min\{1, p/q_\omega\})$ with $\theta < s$. Moreover, by [30, Proposition 7.1.5(4)], we know that, for any $s \in (0, p/q_\omega)$, $\omega^{1-(p/s)'} \in A_{(p/s)'}(\mathbb{R}^n)$ holds true. By this and [30, Corollary 7.2.6], we conclude that there exists a $q \in (0,\infty)$ such that $\omega^{1-(p/s)'} \in A_{(p/s)'/(q/s)'}(\mathbb{R}^n)$. Then, from [30, Theorem 7.1.9(b)] and the fact that $L^{(p/s)'/(q/s)'}_{\omega^{1-(p/s)'}}(\mathbb{R}^n) = [L^{(p/s)'}_{\omega^{1-(p/s)'}}(\mathbb{R}^n)]^{1/(q/s)'}$, we deduce that $\mathcal{M}^{((q/s)')}$ is bounded on $L^{(p/s)'}_{\omega^{1-(p/s)'}}(\mathbb{R}^n)$, which shows that, for any given $s \in (0, p/q_\omega)$, there exists a $q \in (0,\infty)$ such that Assumption 2.7 holds true. Thus, all the assumptions of main theorems in Section 4 are satisfied. Using Theorems 4.9, 4.13 and 4.11, we immediately obtain the following characterizations of $H^p_\omega(\mathbb{R}^n)$ by means of the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g^*_λ-function.

Theorem 5.16. *Let $p \in (0,\infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. Then $f \in H^p_\omega(\mathbb{R}^n)$ if and only if either of the following two items holds true:*

(i) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $S(f) \in L^p_\omega(\mathbb{R}^n)$, where $S(f)$ is as in* (4.1).

(ii) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in L^p_\omega(\mathbb{R}^n)$, where $g(f)$ is as in* (4.3).

Moreover, for any $f \in H^p_\omega(\mathbb{R}^n)$,

$$\|f\|_{H^p_\omega(\mathbb{R}^n)} \sim \|S(f)\|_{L^p_\omega(\mathbb{R}^n)} \sim \|g(f)\|_{L^p_\omega(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.17. Let $p \in (0,1]$ and $\omega \in A_\infty(\mathbb{R}^n)$. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ appearing in the definitions of $S(f)$ and $g(f)$ as in (4.1) and (4.3) is a radial function supported in the unit ball $B(\vec{0}_n, 1)$ satisfying that, for any multi-index $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le \lfloor n(q_\omega/p - 1)\rfloor$, $\int_{\mathbb{R}^n} x^\alpha \varphi(x)\,dx = 0$ and, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, $\int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} = 1$. In this case, Thereom 5.16 was obtained by Liang et al. [34, Theorem 4.13] as a special case.

Theorem 5.18. *Let $p \in (0,\infty)$, $\omega \in A_\infty(\mathbb{R}^n)$ and $\lambda \in (\max\{1, 2q_\omega/p\}, \infty)$, where q_ω is as in* (5.1). *Then $f \in H^p_\omega(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|g^*_\lambda(f)\|_{L^p_\omega(\mathbb{R}^n)} < \infty$. Moreover, for any $f \in H^p_\omega(\mathbb{R}^n)$,*

$$\|f\|_{H^p_\omega(\mathbb{R}^n)} \sim \|g^*_\lambda(f)\|_{L^p_\omega(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.19. Let $p \in (0,1]$ and φ appearing in the definition of $g^*_\lambda(f)$ in (4.2) be as in Remark 5.17. In this case, Thereom 5.18 was obtained by Liang et al. [43, Theorem 4.8] as a special case.

5.5 Orlicz-slice spaces

First, we recall the notions of both Orlicz functions and Orlicz spaces (see, for instance, [56]).

Definition 5.20. A function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing and satisfies $\Phi(0) = 0$, $\Phi(t) > 0$ whenever $t \in (0,\infty)$, and $\lim_{t\to\infty} \Phi(t) = \infty$.

An Orlicz function Φ as in Definition 5.20 is said to be of *lower* (resp., *upper*) *type* p with $p \in (-\infty,\infty)$ if there exists a positive constant $C_{(p)}$, depending on p, such that, for any $t \in [0,\infty)$ and $s \in (0,1)$ [resp., $s \in [1,\infty)$],

$$\Phi(st) \le C_{(p)} s^p \Phi(t).$$

A function $\Phi : [0,\infty) \to [0,\infty)$ is said to be of *positive lower* (resp., *upper*) *type* if it is of lower (resp., upper) type p for some $p \in (0,\infty)$.

Definition 5.21. Let Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. The *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} < \infty.$$

Now we recall the notion of Orlicz-slice spaces.

Definition 5.22. Let $t,\ r \in (0,\infty)$ and Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. The *Orlicz-slice space* $(E_\Phi^r)_t(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{(E_\Phi^r)_t(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} \left[\frac{\|f\mathbf{1}_{B(x,t)}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{B(x,t)}\|_{L^\Phi(\mathbb{R}^n)}}\right]^r dx\right\}^{\frac{1}{r}} < \infty.$$

Remark 5.23. By [76, Lemma 2.28], we know that the Orlicz-slice space $(E_\Phi^r)_t(\mathbb{R}^n)$ is a ball quasi-Banach function space, but it may not be a quasi-Banach function space (see, for instance, [77, Remark 7.41(i)])

The Orlicz-slice space was introduced by Zhang et al. [76], which is a generalization of the slice spaces proposed by Auscher and Mourgoglou [7] and Auscher and Prisuelos-Arribas [8]. Let $t,\ r \in (0,\infty)$ and Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. Then $(E_\Phi^r)_t(\mathbb{R}^n)$ satisfies Assumption 2.6 for any $\theta,\ s \in (0,\min\{1,r,p_\Phi^-\})$ with $\theta < s$ (see [76, Lemma 4.3]). Furthermore, from [76, Lemmas 4.4], we deduce that Assumption 2.7 holds true for any given $s \in (0,\min\{r,p_\Phi^-\})$ and $q \in (\max\{r,p_\Phi^+\},\infty)$, and $X := (E_\Phi^r)_t(\mathbb{R}^n)$. Thus, all the assumptions of main theorems in Sections 3 and 4 are satisfied. Using Theorems 4.9, 4.13 and 4.11, we immediately obtain the following characterizations of $(HE_\Phi^r)_t(\mathbb{R}^n)$ in terms of the Lusin area function, the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function.

Theorem 5.24. *Let* $t \in (0,\infty)$*,* $r,\ p_\Phi^-,\ p_\Phi^+ \in (0,\infty)$ *and* Φ *be an Orlicz function with positive lower type* p_Φ^- *and positive upper type* p_Φ^+*. Then* $f \in (HE_\Phi^r)_t(\mathbb{R}^n)$ *if and only if either of the following two items holds true:*

(i) $f \in \mathcal{S}'(\mathbb{R}^n)$*,* f *vanishes weakly at infinity and* $S(f) \in (E_\Phi^r)_t(\mathbb{R}^n)$*, where* $S(f)$ *is as in* (4.1).

(ii) *$f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in (E^r_\Phi)_t(\mathbb{R}^n)$, where $g(f)$ is as in* (4.3).

Moreover, for any $f \in (HE^q_\Phi)_t(\mathbb{R}^n)$,

$$\|f\|_{(HE^r_\Phi)_t(\mathbb{R}^n)} \sim \|S(f)\|_{(E^r_\Phi)_t(\mathbb{R}^n)} \sim \|g(f)\|_{(E^r_\Phi)_t(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.25. Let φ appearing in the definitions of $S(f)$ and $g(f)$ as in (4.1) and (4.3) satisfy that $\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \le \widehat{\varphi} \le \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}$. In this case, Theorem 5.24 was obtained by Zhang et al. [76, Theorem 3.17].

Theorem 5.26. *Let $t \in (0,\infty)$, r, p^-_Φ, $p^+_\Phi \in (0,\infty)$ and Φ be an Orlicz function with positive lower type p^-_Φ and positive upper type p^+_Φ. Let $\lambda \in (\max\{1, \frac{2}{\min\{p^-_\Phi, r\}}\}, \infty)$. Then $f \in (HE^r_\Phi)_t(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $\|g^*_\lambda(f)\|_{(E^r_\Phi)_t(\mathbb{R}^n)} < \infty$, where $g^*_\lambda(f)$ is as in* (4.2). *Moreover, for any $f \in (HE^q_\Phi)_t(\mathbb{R}^n)$,*

$$\|f\|_{(HE^r_\Phi)_t(\mathbb{R}^n)} \sim \|g^*_\lambda(f)\|_{(E^r_\Phi)_t(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Remark 5.27. Let r, p^-_Φ, $p^+_\Phi \in (0,\infty)$ and Φ be an Orlicz function with positive lower type p^-_Φ and positive upper type p^+_Φ. Let φ appearing in the definition of $g^*_\lambda(f)$ as in (4.2) satisfy that

$$\mathbf{1}_{B(\vec{0}_n,4)\setminus B(\vec{0}_n,2)} \le \widehat{\varphi} \le \mathbf{1}_{B(\vec{0}_n,8)\setminus B(\vec{0}_n,1)}.$$

In this case, Theorem 5.26 widens the range of $\lambda \in (\max\{\frac{2}{\min\{1,p^-_\Phi,r\}}, 1 - \frac{2}{\max\{1,p^+_\Phi,r\}} + \frac{2}{\min\{1,p^-_\Phi,r\}}\}, \infty)$ in [76, Theorem 3.19] into $\lambda \in (\max\{1, \frac{2}{\min\{p^-_\Phi,r\}}\}, \infty)$.

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