# HYPERGEOMETRIC FUNCTIONS AND A FAMILY OF ALGEBRAIC CURVES 

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#### Abstract

Let $\lambda \in \mathbb{Q} \backslash\{0,1\}$ and $l \geq 2$, and denote by $C_{l, \lambda}$ the nonsingular projective algebraic curve over $\mathbb{Q}$ with affine equation given by $$
y^{l}=x(x-1)(x-\lambda) .
$$

In this paper we define $\Omega\left(C_{l, \lambda}\right)$ analogous to the real periods of elliptic curves and find a relation with ordinary hypergeometric series. We also give a relation between the number of points on $C_{l, \lambda}$ over a finite field and Gaussian hypergeometric series. Finally we give an alternate proof of a result of [13].


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## 1. Introduction

Hypergeometric functions and their relations with algebraic curves have been studied by many mathematicians. For $a_{0}, a_{1}, \ldots, a_{r}, b_{1}, b_{2}, \ldots, b_{r} \in \mathbb{C}$, the ordinary hypergeometric series ${ }_{r+1} F_{r}$ is defined as

$$
{ }_{r+1} F_{r}\left(\left.\begin{array}{cccc}
a_{0}, & a_{1}, & \cdots, & a_{r} \\
& b_{1}, & \cdots, & b_{r}
\end{array} \right\rvert\, z\right):=\sum_{n=0}^{\infty} \frac{\left(a_{0}\right)_{n}\left(a_{1}\right)_{n} \cdots\left(a_{r}\right)_{n}}{\left(b_{1}\right)_{n}\left(b_{2}\right)_{n} \cdots\left(b_{r}\right)_{n}} \frac{z^{n}}{n!},
$$

where $(a)_{0}=1,(a)_{n}:=a(a+1)(a+2) \ldots(a+n-1)$ for $n \geq 1$, and none of the $b_{i}$ is a negative integer or zero. This hypergeometric series converges absolutely for $|z|<1$. The series also converges absolutely for $|z|=1$ if $\operatorname{Re}\left(\sum b_{i}-\sum a_{i}\right)>0$ and converges conditionally for $|z|=1, z \neq 1$ if $0 \geq \operatorname{Re}\left(\sum b_{i}-\sum a_{i}\right)>-1$. For details see [1, chapter $2]$.

In [5] Greene introduced the notion of Gaussian hypergeometric series over finite fields. Since then, the interplay between ordinary hypergeometric series and Gaussian hypergeometric series has played an important role in character sum evaluation [7], the representation theory of $S L(2, \mathbb{R})$ [6] and finding the number of points on an algebraic curve over finite fields [12]. Recently, J. Rouse [13] and D. McCarthy [11] provided an expression for the real period of certain families of elliptic curves in terms of ordinary hypergeometric series. They also provided an analogous expression for the trace of Frobenius of the same family of curves in terms of Gaussian hypergeometric series, which developed the interplay between the two hypergeometric series more fully.

We will now restate some definitions from [5] which are analogous to the binomial coefficient and ordinary hypergeometric series respectively. Throughout the paper $p$ is an odd prime. We also let $\mathbb{F}_{p}$ denote the finite field with $p$ elements and we extend all
characters $\chi$ of $\mathbb{F}_{p}^{\times}$to $\mathbb{F}_{p}$ by setting $\chi(0)=0$. For characters $A$ and $B$ of $\mathbb{F}_{p}$, define $\binom{A}{B}$ as

$$
\begin{equation*}
\binom{A}{B}:=\frac{B(-1)}{p} J(A, \bar{B})=\frac{B(-1)}{p} \sum_{x \in \mathbb{F}_{p}} A(x) \bar{B}(1-x), \tag{1}
\end{equation*}
$$

where $J(A, B)$ is the Jacobi sum of the characters $A$ and $B$ of $\mathbb{F}_{p}$ and $\bar{B}$ is the inverse of $B$. With this notation, for characters $A_{0}, A_{1}, \ldots, A_{n}$ and $B_{1}, B_{2}, \ldots, B_{n}$ of $\mathbb{F}_{p}$, the Gaussian hypergeometric series ${ }_{n+1} F_{n}\left(\left.\begin{array}{cccc}A_{0}, & A_{1}, & \cdots, & A_{n} \\ & B_{1}, & \cdots, & B_{n}\end{array} \right\rvert\, x\right)$ over $\mathbb{F}_{p}$ is defined as

$$
{ }_{n+1} F_{n}\left(\left.\begin{array}{cccc}
A_{0}, & A_{1}, & \cdots, & A_{n}  \tag{2}\\
& B_{1}, & \cdots, & B_{n}
\end{array} \right\rvert\, x\right):=\frac{p}{p-1} \sum_{\chi}\binom{A_{0} \chi}{\chi}\binom{A_{1} \chi}{B_{1} \chi} \cdots\binom{A_{n} \chi}{B_{n} \chi} \chi(x)
$$

where the sum is over all characters $\chi$ of $\mathbb{F}_{p}$.
Let $\lambda \in \mathbb{Q} \backslash\{0,1\}$ and $l \geq 2$, and denote by $C_{l, \lambda}$ the nonsingular projective algebraic curve over $\mathbb{Q}$ with affine equation given by

$$
\begin{equation*}
y^{l}=x(x-1)(x-\lambda) \tag{3}
\end{equation*}
$$

The change of variables $(x, y) \mapsto\left(x+\frac{1+\lambda}{3}, \frac{y}{2}\right)$ takes (3) to

$$
\begin{equation*}
y^{l}=2^{l}(x-a)(x-b)(x-c) \tag{4}
\end{equation*}
$$

where $a=-\frac{1+\lambda}{3}, b=\frac{2 \lambda-1}{3}$, and $c=\frac{2-\lambda}{3}$.
We now define an integral for the family of curves (3) analogous to the real period of elliptic curves.
Definition 1. The complex number $\Omega\left(C_{l, \lambda}\right)$ is defined as

$$
\begin{equation*}
\Omega\left(C_{l, \lambda}\right):=2 \int_{a}^{b} \frac{d x}{y^{l-1}} \tag{5}
\end{equation*}
$$

where $x$ and $y$ are related as in (4).
Definition 2. Suppose $p$ is a prime of good reduction for $C_{l, \lambda}$. Define the integer $a_{p}\left(C_{l, \lambda}\right) b y$

$$
\begin{equation*}
a_{p}\left(C_{l, \lambda}\right):=1+p-\# C_{l, \lambda}\left(\mathbb{F}_{p}\right), \tag{6}
\end{equation*}
$$

where $\# C_{l, \lambda}\left(\mathbb{F}_{p}\right)$ denotes the number of points that the curve $C_{l, \lambda}$ has over $\mathbb{F}_{p}$.
It is clear that a prime $p$ not dividing $l$ is of good reduction for $C_{l, \lambda}$ if and only if $\operatorname{ord}_{p}(\lambda(\lambda-1))=0$.

Remark 1.1. Let $l \neq 3$. Then

$$
\# C_{l, \lambda}\left(\mathbb{F}_{p}\right)=1+\#\left\{(x, y) \in \mathbb{F}_{p}^{2}: y^{l}=x(x-1)(x-\lambda)\right\}
$$

Indeed, for $l \geq 4$, the point $[1: 0: 0]$ is the only point at infinity. Similarly, if $l=2$, the point at infinity is $[0: 1: 0]$.

Let $l=3$ and $p \equiv 1(\bmod 3)$. Let $\omega \in \mathbb{F}_{p}^{\times}$be of order 3 . Then there are three points at infinity, namely, $[1: 1: 0],[1: \omega: 0]$, and $\left[1: \omega^{2}: 0\right]$. Hence, in this case,

$$
\# C_{l, \lambda}\left(\mathbb{F}_{p}\right)=3+\#\left\{(x, y) \in \mathbb{F}_{p}^{2}: y^{l}=x(x-1)(x-\lambda)\right\}
$$

Again, if $l=3$ and $p \equiv 2(\bmod 3)$, then the point at infinity is $[1: 1: 0]$.
Remark 1.2. If $l=3, C_{l, \lambda}$ is an elliptic curve. Dehomogenizing the projective curve $C_{3, \lambda}: Y^{3}=X(X-Z)(X-\lambda Z)$ by putting $X=1$ and then making the substitution

$$
Y \rightarrow \lambda x, Z \rightarrow \lambda\left(y+\frac{1+\lambda}{2 \lambda^{2}}\right)
$$

we find that $C_{3, \lambda}$ is isomorphic over $\mathbb{Q}$ to the elliptic curve

$$
\begin{equation*}
y^{2}=x^{3}+\left(\frac{\lambda-1}{2 \lambda^{2}}\right)^{2} \tag{7}
\end{equation*}
$$

Remark 1.3. If $l=2$, then equation (3) gives an elliptic curve in Legendre normal form with real period $\Omega\left(C_{2, \lambda}\right)$, and $a_{p}\left(C_{2, \lambda}\right)$ is the trace of the Frobenius endomorphism on the curve over $\mathbb{F}_{p}$.

We now recall two results relating $\Omega\left(C_{2, \lambda}\right)$ and $a_{p}\left(C_{2, \lambda}\right)$ to hypergeometric series.
Theorem 1.4. [8, 13] If $0<\lambda<1$, then the real period $\Omega\left(C_{2, \lambda}\right)$ satisfies

$$
\frac{\Omega\left(C_{2, \lambda}\right)}{\pi}={ }_{2} F_{1}\left(\begin{array}{ccc}
1 / 2, & 1 / 2 & \mid \lambda \\
& 1 &
\end{array}\right) .
$$

Theorem 1.5. [10, 12] If $\operatorname{ord}_{p}(\lambda(\lambda-1))=0$, then

$$
-\frac{\phi(-1) a_{p}\left(C_{2, \lambda}\right)}{p}={ }_{2} F_{1}\left(\left.\begin{array}{cc}
\phi, & \phi \\
& \epsilon
\end{array} \right\rvert\, \lambda\right)
$$

where $\phi$ and $\epsilon$ are the quadratic and trivial characters of $\mathbb{F}_{p}$ respectively.
In this paper we generalize these results to the algebraic curves $C_{l, \lambda}$. The aim of this paper is to prove the following main results.

Theorem 1.6. If $0<\lambda<1$, then $\Omega\left(C_{l, \lambda}\right)$ is given by

$$
\Omega\left(C_{l, \lambda}\right)=\frac{\left(\Gamma\left(\frac{1}{l}\right)\right)^{2}}{2^{l-2} \lambda^{\frac{l-2}{l}} \Gamma\left(\frac{2}{l}\right)} \cdot{ }_{2} F_{1}\left(\left.\begin{array}{ll}
(l-1) / l, & 1 / l \\
& 2 / l
\end{array} \right\rvert\, \lambda\right) .
$$

Theorem 1.7. If $p \equiv 1(\bmod l)$ and $\operatorname{ord}_{p}(\lambda(\lambda-1))=0$, then $a_{p}\left(C_{l, \lambda}\right)$ satisfies

$$
-a_{p}\left(C_{l, \lambda}\right)= \begin{cases}p \cdot \sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right)_{2} F_{1}\left(\left.\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
\chi^{2 i}
\end{array} \right\rvert\, \lambda\right), & \text { if } l \neq 3 \\
2+p \cdot \sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right)_{2} F_{1}\left(\left.\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
& \chi^{2 i}
\end{array} \right\rvert\, \lambda\right), & \text { if } l=3\end{cases}
$$

where $\chi$ is a character of $\mathbb{F}_{p}$ of order $l$.
Theorem 1.8. For $\lambda=\frac{1}{2}$, we have

$$
\begin{equation*}
\frac{2^{\frac{(l-3)(l-1)}{l}} \Gamma\left(\frac{2}{l}\right)}{\left(\Gamma\left(\frac{1}{l}\right)\right)^{2}} \cdot \Omega\left(C_{l, \lambda}\right)=\frac{\binom{\frac{1}{2 l}}{\frac{1}{l}}}{\binom{\frac{3-2 l}{2 l}}{\frac{2-}{l}}} . \tag{8}
\end{equation*}
$$

Moreover, if $p \equiv 1(\bmod l)$, then

$$
-a_{p}\left(C_{l, \lambda}\right)=\left\{\begin{array}{cl}
p \cdot \sum_{i=1}^{\left\lfloor\frac{l-1}{2}\right\rfloor} \chi^{-2 i}(8)\left[\binom{\chi^{i}}{\chi^{-2 i}}+\binom{\phi \chi^{i}}{\chi^{-2 i}}\right], & \text { if } \frac{p-1}{l} \text { is odd and } l \neq 3  \tag{9}\\
p \cdot \sum_{i=1}^{l-1} \chi^{-i}(8)\left[\binom{\sqrt{\chi^{i}}}{\chi^{-i}}+\binom{\phi \sqrt{\chi^{i}}}{\chi^{-i}}\right], & \text { if } \frac{p-1}{l} \text { is even and } l \neq 3 \\
2+p \cdot \sum_{i=1}^{2}\left[\binom{\sqrt{\chi^{i}}}{\chi^{-i}}+\binom{\phi \sqrt{\chi^{i}}}{\chi^{-i}}\right], & \text { if } l=3
\end{array}\right.
$$

where $\chi$ is a character of $\mathbb{F}_{p}$ of order $l$ and $\phi$ is the quadratic character.
Here we extend the definition of binomial coefficient to include rational arguments via

$$
\binom{n}{k}=\frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)} .
$$

We also give a simple proof of the following result of J. Rouse (note that $\binom{1 / 4}{1 / 2}$ is real).
Theorem 1.9. [13, Theorem 3] If $\lambda=1 / 2$, then

$$
\frac{\sqrt{2}}{2 \pi} \cdot \Omega\left(C_{2, \lambda}\right)=\binom{1 / 4}{1 / 2}
$$

If $p \equiv 1(\bmod 4)$, then

$$
\frac{-\phi(-2)}{2 p} \cdot a_{p}\left(C_{2, \lambda}\right)=\operatorname{Re}\binom{\chi_{4}}{\phi}
$$

where $\chi_{4}$ is a character on $\mathbb{F}_{p}$ of order 4 and $\phi$ is the quadratic character.

## 2. Preliminaries

We start with a result which enables us to count the number of points on a curve using multiplicative characters on $\mathbb{F}_{p}$ (see [9, Proposition 8.1.5]).
Lemma 2.1. Let $a \in \mathbb{F}_{p}^{\times}$. If $n \mid(p-1)$, then

$$
\#\left\{x \in \mathbb{F}_{p}: x^{n}=a\right\}=\sum \chi(a)
$$

where the sum runs over all characters $\chi$ on $\mathbb{F}_{p}$ of order dividing $n$.
Now we recall some standard facts regarding ordinary and Gaussian hypergeometric series. First, the ordinary ${ }_{2} F_{1}$ hypergeometric series has the following integral representation [4, p. 115]:

$$
{ }_{2} F_{1}\left(\left.\begin{array}{cc}
a, & b  \tag{10}\\
& c
\end{array} \right\rvert\, x\right):=\frac{2 \Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{\pi / 2} \frac{(\sin t)^{2 b-1}(\cos t)^{2 c-2 b-1}}{\left(1-x \sin ^{2} t\right)^{a}} d t
$$

where $\operatorname{Re} c>\operatorname{Re} b>0$. Also, by (11) and (2) the Gaussian ${ }_{2} F_{1}$ hypergeometric series over $\mathbb{F}_{p}$ takes the form

$$
{ }_{2} F_{1}\left(\begin{array}{ccc}
A, & B  \tag{11}\\
& C & \mid x
\end{array}\right)=\epsilon(x) \frac{B C(-1)}{p} \sum_{y \in \mathbb{F}_{p}} B(y) \bar{B} C(1-y) \bar{A}(1-x y),
$$

where $\epsilon$ denotes the trivial character.
Next we note two transformation properties of ordinary hypergeometric series. Kummer's Theorem [2, p. 9] is given by

$$
{ }_{2} F_{1}\left(\left.\begin{array}{cc}
a, & b  \tag{12}\\
& 1+b-a
\end{array} \right\rvert\,-1\right)=\frac{\Gamma(1+b-a) \Gamma\left(1+\frac{b}{2}\right)}{\Gamma(1+b) \Gamma\left(1+\frac{b}{2}-a\right)},
$$

while Pfaff's transformation [15, p. 31] can be stated as

$$
{ }_{2} F_{1}\left(\left.\begin{array}{cc}
a, & b  \tag{13}\\
& c
\end{array} \right\rvert\, x\right)=(1-x)^{-a}{ }_{2} F_{1}\left(\begin{array}{cc|c}
a, & c-b & x \\
& c & x-1
\end{array}\right) .
$$

Greene [5, p. 91] proved the following Gaussian analogs of these transformations:

$$
{ }_{2} F_{1}\left(\left.\begin{array}{cc}
A, & B  \tag{14}\\
& \bar{A} B
\end{array} \right\rvert\,-1\right)= \begin{cases}0, & \text { if } B \text { is not a square; } \\
\binom{C}{A}+\binom{\phi C}{A}, & \text { if } B=C^{2}\end{cases}
$$

and

$$
{ }_{2} F_{1}\left(\left.\begin{array}{cc}
A, & \bar{A}  \tag{15}\\
& \bar{A} B
\end{array} \right\rvert\, \frac{1}{2}\right)=A(-2) \begin{cases}0, & \text { if } B \text { is not a square; } \\
\binom{C}{A}+\binom{\phi C}{A}, & \text { if } B=C^{2}\end{cases}
$$

where $\phi$ is the quadratic character of $\mathbb{F}_{p}$.

## 3. Proof of the results

Proof of Theorem 1.6. Recalling (4), from the definition of $\Omega\left(C_{l, \lambda}\right)$, we have

$$
\begin{aligned}
\Omega\left(C_{l, \lambda}\right) & =2 \int_{a}^{b} \frac{d x}{y^{l-1}} \\
& =2 \int_{a}^{b} \frac{d x}{2^{l-1}\{(x-a)(x-b)(x-c)\}^{\frac{l-1}{l}}} .
\end{aligned}
$$

Note that for $a<x<b$ and $0<\lambda<1,(x-a)$ is positive, while $(x-b)$ and $(x-c)$ are negative. Hence $\Omega\left(C_{l, \lambda}\right)$ is real.

Putting $(x-a)=(b-a) \sin ^{2} \theta$, we obtain

$$
\begin{aligned}
\Omega\left(C_{l, \lambda}\right) & =2 \int_{0}^{\pi / 2} \frac{2(b-a) \sin \theta \cos \theta}{2^{l-1}\left[(b-a) \sin ^{2} \theta(b-a) \cos ^{2} \theta\left\{(c-a)-(b-a) \sin ^{2} \theta\right\}\right]^{\frac{l-1}{l}}} d \theta \\
& =\frac{1}{2^{l-3}} \int_{0}^{\pi / 2} \frac{(b-a)^{\frac{2-l}{l}}(\sin \theta)^{\frac{2-l}{l}}(\cos \theta)^{\frac{2-l}{l}}}{\left\{(c-a)-(b-a) \sin ^{2} \theta\right\}^{\frac{l-1}{l}}} d \theta .
\end{aligned}
$$

Using $(b-a)=\lambda$ and $(c-a)=1$ yields

$$
\begin{aligned}
\Omega\left(C_{l, \lambda}\right) & =\frac{1}{2^{l-3} \lambda^{\frac{l-2}{l}}} \int_{0}^{\pi / 2} \frac{(\sin \theta)^{2 \frac{1}{l}-1}(\cos \theta)^{2 \frac{2}{l}-2 \frac{1}{l}-1}}{\left(1-\lambda \sin ^{2} \theta\right)^{\frac{l-1}{l}}} d \theta \\
& =\frac{\left(\Gamma\left(\frac{1}{l}\right)\right)^{2}}{2^{l-2} \lambda^{\frac{l-2}{l}} \Gamma\left(\frac{2}{l}\right)} \cdot{ }_{2} F_{1}\left(\left.\begin{array}{cc}
(l-1) / l, & 1 / l \\
2 / l
\end{array} \right\rvert\, \lambda\right)
\end{aligned}
$$

where the last equality follows from (10). This completes the proof of the theorem.
Remark 3.1. If we put $l=2$ in Theorem 1.6, we obtain Theorem 1.4.
Remark 3.2. As mentioned in Remark[1.2, $C_{3, \lambda}$ is isomorphic over $\mathbb{Q}$ to the elliptic curve (7). It would be interesting to know if there is any relation between $\Omega\left(C_{3, \lambda}\right)$ and the real period of (7).

Proof of Theorem 1.7. Since $p \equiv 1(\bmod l)$, there exists a character $\chi$ of order $l$ on $\mathbb{F}_{p}$. Using (11), we have

$$
\begin{aligned}
& \sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right)_{2} F_{1}\left(\left.\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
& \chi^{2 i}
\end{array} \right\rvert\, \lambda\right) \\
& =\sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right) \frac{\chi^{i} \chi^{2 i}(-1)}{p} \sum_{t \in \mathbb{F}_{p}} \chi^{i}(t) \overline{\chi^{i}} \chi^{2 i}(1-t) \overline{\overline{\chi^{i}}}(1-\lambda t) \\
& =\sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right) \frac{\chi^{3 i}(-1)}{p} \sum_{t \in \mathbb{F}_{p}} \chi^{i}(t) \chi^{i}(1-t) \chi^{i}(1-\lambda t)
\end{aligned}
$$

Replacing $t$ by $\frac{t}{\lambda}$, we get

$$
\begin{align*}
p \cdot \sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right)_{2} F_{1}\left(\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
& \chi^{2 i} \mid \lambda
\end{array}\right) & =\sum_{i=1}^{l-1} \sum_{t \in \mathbb{F}_{p}} \chi^{i}(t(t-1)(t-\lambda)) \\
& =\sum_{t \in \mathbb{F}_{p}} \sum_{i=1}^{l-1} \chi^{i}(t(t-1)(t-\lambda)) \tag{16}
\end{align*}
$$

Moreover,

$$
\begin{aligned}
& \#\left\{(x, y) \in \mathbb{F}_{p}^{2}: y^{l}=x(x-1)(x-\lambda)\right\} \\
& =\sum_{t \in \mathbb{F}_{p}} \#\left\{y \in \mathbb{F}_{p}: y^{l}=t(t-1)(t-\lambda)\right\} \\
& =\sum_{t \in \mathbb{F}_{p}, t(t-1)(t-\lambda) \neq 0} \#\left\{y \in \mathbb{F}_{p}: y^{l}=t(t-1)(t-\lambda)\right\}+\#\left\{t \in \mathbb{F}_{p}: t(t-1)(t-\lambda)=0\right\} .
\end{aligned}
$$

Now applying Lemma 2.1 and (16), we obtain

$$
\begin{aligned}
& \#\left\{(x, y) \in \mathbb{F}_{p}^{2}: y^{l}=x(x-1)(x-\lambda)\right\} \\
& =\sum_{t \in \mathbb{F}_{p}} \sum_{i=0}^{l-1} \chi^{i}(t(t-1)(t-\lambda))+\#\left\{t \in \mathbb{F}_{p}: t(t-1)(t-\lambda)=0\right\} \\
& =p+\sum_{t \in \mathbb{F}_{p}} \sum_{i=1}^{l-1} \chi^{i}(t(t-1)(t-\lambda)) \\
& =p+p \cdot \sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right)_{2} F_{1}\left(\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
& \chi^{2 i}
\end{array}\right)
\end{aligned}
$$

Since $\operatorname{ord}_{p}(\lambda(\lambda-1))=0$, using (6) we complete the proof of the result.
Remark 3.3. Theorem 1.5 can be obtained from Theorem 1.7 by putting $l=2$. Note that for the quadratic character $\phi$ of $\mathbb{F}_{p}$, we have $\phi\left(-\lambda^{2}\right)=\phi(-1)$.

For $l \geq 3$, the genus of the curve $C_{l, \lambda}$ is $\frac{(l-1)(l-2)}{2}$. The Hasse-Weil bound therefore yields the following result.
Corollary 3.4. Suppose $l \geq 4$. If $p \equiv 1(\bmod l)$ and $\operatorname{ord}_{p}(\lambda(\lambda-1))=0$, then

$$
\left|\sum_{i=1}^{l-1} \chi^{i}\left(-\lambda^{2}\right)_{2} F_{1}\left(\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
& \chi^{2 i}
\end{array}\right)\right| \leq \frac{(l-1)(l-2)}{\sqrt{p}}
$$

where $\chi$ is a character of $\mathbb{F}_{p}$ of order $l$.
If $l=3$, then

$$
\left|2+p \cdot \sum_{i=1}^{2} \chi^{i}\left(-\lambda^{2}\right)_{2} F_{1}\left(\left.\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
& \chi^{2 i}
\end{array} \right\rvert\, \lambda\right)\right| \leq 2 \sqrt{p}
$$

where $\chi$ is a character of $\mathbb{F}_{p}$ of order 3 .
Corollary 3.5. If $p \equiv 1(\bmod 3)$ and $x^{2}+3 y^{2}=p$, then

$$
p \cdot \sum_{i=1}^{2}{ }_{2} F_{1}\left(\left.\begin{array}{cc}
\overline{\chi^{i}}, & \chi^{i} \\
& \chi^{2 i}
\end{array} \right\rvert\,-1\right)=(-1)^{x+y}\left(\frac{x}{3}\right) \cdot 2 x-2,
$$

where $\chi$ is a character of $\mathbb{F}_{p}$ of order 3 .
Proof. As mentioned in Remark 1.2, $C_{3,-1}$ is isomorphic over $\mathbb{Q}$ to the elliptic curve $y^{2}=x^{3}+1$. By [12, Proposition 2], it is known that $a_{p}\left(C_{3,-1}\right)=(-1)^{x+y-1}\left(\frac{x}{3}\right) \cdot 2 x$. Now the result follows from Theorem 1.7.

Remark 3.6. The formula for $a_{p}\left(C_{3, \lambda}\right)$ in Theorem 1.7 gives the trace of Frobenius of the family of elliptic curves (7).

Proof of Theorem 1.8. By (12), we have

$$
\begin{align*}
{ }_{2} F_{1}\left((l-1) / l, \left.\frac{1 / l}{2 / l} \right\rvert\,-1\right) & =\frac{\Gamma\left(\frac{2}{l}\right) \Gamma\left(\frac{2 l+1}{2 l}\right)}{\Gamma\left(\frac{l+1}{l}\right) \Gamma\left(\frac{3}{2 l}\right)} \\
& =\frac{\frac{\Gamma\left(\frac{2 l+1}{2 l}\right)}{\Gamma\left(\frac{l+1}{l}\right) \Gamma\left(\frac{2 l-1}{2 l}\right)}}{\frac{\Gamma\left(\frac{3}{2 l}\right)}{\Gamma\left(\frac{2}{l}\right) \Gamma\left(\frac{2 l-1}{2 l}\right)}} \\
& =\frac{\binom{\frac{1}{2 l}}{\frac{1}{l}}}{\binom{\frac{3-2 l}{2 l}}{\frac{2-l}{l}}} . \tag{17}
\end{align*}
$$

Putting $\lambda=1 / 2$ in Theorem 1.6, we obtain the relation

$$
\frac{2^{\frac{l^{2}-3 l+2}{l}} \Gamma\left(\frac{2}{l}\right)}{\left(\Gamma\left(\frac{1}{l}\right)\right)^{2}} \cdot \Omega\left(C_{l, \frac{1}{2}}\right)={ }_{2} F_{1}\left(\begin{array}{ll|}
(l-1) / l, & 1 / l \left\lvert\, \frac{1}{2}\right. \\
& 2 / l
\end{array}\right) .
$$

Then using (13), we find that

$$
\frac{2^{\frac{l^{2}-3 l+2}{l}} \Gamma\left(\frac{2}{l}\right)}{\left(\Gamma\left(\frac{1}{l}\right)\right)^{2}} \cdot \Omega\left(C_{l, \frac{1}{2}}\right)=2^{\frac{l-1}{l}}{ }_{2} F_{1}\left(\left.\begin{array}{ll}
(l-1) / l, & 1 / l  \tag{18}\\
& 2 / l
\end{array} \right\rvert\,-1\right) .
$$

From (17) and (18), we complete the proof of (8).
Now, we shall prove the second part of the result. Note that $p$ is an odd prime. Write $\chi=w^{k}$, where $w$ is a generator of the group of Dirichlet characters mod $p$. Let $o(w)$ denote the order of $w$. Then $o(w)=p-1$ and $l=o\left(w^{k}\right)=(p-1) / \operatorname{gcd}(k, p-1)$. So $(p-1) / l=\operatorname{gcd}(k, p-1)$. If $(p-1) / l$ is even, then $k$ is also even, hence $\chi$ is a square. Conversely, if $\chi$ is a square, it is an even power of the generator $w$, hence $k$ is even, and $(p-1) / l=\operatorname{gcd}(k, p-1)$ is even. This implies that $\chi$ is a square if and only if $(p-1) / l$ is even. Moreover, $\chi^{i}$ is a square for even values of $i$, and for odd values of $i, \chi^{i}$ is a square if and only if $\chi$ is a square. Using these, from Theorem 1.7 and (15), we complete the proof of (9).

In [13], J. Rouse gave an analogy between ordinary hypergeometric series and Gaussian hypergeometric series by evaluating $\Omega\left(C_{2, \frac{1}{2}}\right)$ and $a_{p}\left(C_{2, \frac{1}{2}}\right)$ in terms of hypergeometric series. We now give an alternate proof of [13, Theorem 3, p. 3].

Proof of Theorem 1.9, Putting $l=2$ in (8), we obtain

$$
\frac{2^{-\frac{1}{2}}}{\left(\Gamma\left(\frac{1}{2}\right)\right)^{2}} \cdot \Omega\left(C_{2, \frac{1}{2}}\right)=\frac{\binom{1 / 4}{1 / 2}}{\binom{-1 / 4}{0}}
$$

which yields

$$
\frac{\sqrt{2}}{2 \pi} \cdot \Omega\left(C_{2, \frac{1}{2}}\right)=\binom{1 / 4}{1 / 2}
$$

since $\binom{-1 / 4}{0}=1$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.
For the second part, recall that $p \equiv 1(\bmod 4)$. Putting $l=2$ in (9), we find that

$$
\frac{-\phi(8)}{p} \cdot a_{p}\left(C_{2, \frac{1}{2}}\right)=\binom{\chi_{4}}{\phi}+\binom{\phi \chi_{4}}{\phi}
$$

since $\chi_{4}^{2}=\phi$. Clearly $\phi \chi_{4}=\overline{\chi_{4}}$, and this implies that $\binom{\phi \chi_{4}}{\phi}=\overline{\binom{\chi_{4}}{\phi}}$. Also, observing that $\phi(8)=\phi(2)$, we obtain

$$
\frac{-\phi(2)}{2 p} \cdot a_{p}\left(C_{2, \frac{1}{2}}\right)=\operatorname{Re}\binom{\chi_{4}}{\phi}
$$

Since $p \equiv 1(\bmod 4)$, we have that $\phi(-1)=1$ and the result follows.

Simplifying the expressions for $a_{p}\left(C_{l, \frac{1}{2}}\right)$ given in Theorem 1.8, we obtain the following result which generalizes the case $l=2, p \equiv 1(\bmod 4)$ treated in Theorem 1.9,

Corollary 3.7. Suppose that $p \equiv 1(\bmod l)$. Then we have

$$
-a_{p}\left(C_{l, \frac{1}{2}}\right)=\left\{\begin{array}{c}
2 p \cdot\left[\phi(2) \operatorname{Re}\binom{\chi_{4}}{\phi}+\sum_{i=1}^{\frac{l-4}{4}} \operatorname{Re}\left\{\chi^{-2 i}(8)\left(\binom{\chi^{i}}{\chi^{-2 i}}+\binom{\phi \chi^{i}}{\chi^{-2 i}}\right)\right\}\right] \\
\text { if } \frac{p-1}{l} \text { is odd and } l \equiv 0(\bmod 4) \\
2 p \cdot \sum_{i=1}^{\frac{l-2}{4}} \operatorname{Re}\left[\chi^{-2 i}(8)\left(\binom{\chi^{i}}{\chi^{-2 i}}+\binom{\phi \chi^{i}}{\chi^{-2 i}}\right)\right] \\
\text { if } \frac{p-1}{l} \text { is odd and } l \equiv 2(\bmod 4)
\end{array}, \begin{array}{c}
{\left[\phi(2) \operatorname{Re}\binom{\chi_{4}}{\phi}+\sum_{i=1}^{\frac{l-2}{2}} \operatorname{Re}\left\{\begin{array}{c}
\left.\psi^{-2 i}(8)\left(\binom{\psi^{i}}{\psi^{-2 i}}+\binom{\phi \psi^{i}}{\psi^{-2 i}}\right)\right\}
\end{array}\right.\right.} \\
2 p \\
2 p \cdot \sum_{i=1}^{\frac{l-1}{2}} \operatorname{Re}\left[\begin{array}{c}
\left.\psi^{-2 i}(8)\left(\binom{\psi^{i}}{\psi^{-2 i}}+\binom{\phi \psi^{i}}{\psi^{-2 i}}\right)\right] \\
\text { if } \frac{p-1}{l} \text { and } l \text { are even }
\end{array}\right. \\
2+2 p \cdot \operatorname{Re}\left[\binom{\chi}{\chi}+\binom{\phi \chi}{\chi}\right], \quad \begin{array}{c}
\text { if } l=3 ;
\end{array}
\end{array}\right.
$$

where $\psi, \chi, \chi_{4}$ are characters of $\mathbb{F}_{p}$ of order $2 l, l, 4$ respectively and $\phi$ is the quadratic character.

Corollary 3.8. If $p \equiv 1(\bmod 3)$ and $x^{2}+3 y^{2}=p$, then

$$
p \cdot \operatorname{Re}\left[\binom{\chi}{\chi}+\binom{\phi \chi}{\chi}\right]=(-1)^{x+y}\left(\frac{x}{3}\right) \cdot x-1
$$

where $\chi$ is a character of order 3 on $\mathbb{F}_{p}$ and $\phi$ is the quadratic character.
Proof. As mentioned in Remark [1.2, $C_{3,-1}$ and $C_{3, \frac{1}{2}}$ are isomorphic over $\mathbb{Q}$ to the elliptic curve $y^{2}=x^{3}+1$. By [12, Proposition 2], it is known that $a_{p}\left(C_{3,-1}\right)=(-1)^{x+y-1}\left(\frac{x}{3}\right) \cdot 2 x$. From Corollary 3.7, we have

$$
-a_{p}\left(C_{3, \frac{1}{2}}\right)=2+2 p \cdot \operatorname{Re}\left[\binom{\chi}{\chi}+\binom{\phi \chi}{\chi}\right] .
$$

Since $a_{p}\left(C_{3,-1}\right)=a_{p}\left(C_{3, \frac{1}{2}}\right)$, the result follows.

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