

On rate of convergence for infinite server Erlang–Sevastyanov’s problem

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Abstract

Polynomial convergence rates in total variation are established in Erlang–Sevastyanov’s type problem with an infinite number of servers and a general distribution of service under assumptions on the intensity of serving.

1 Introduction

The formulae by Erlang provided explicit expressions for percentages of lost customers in certain queueing systems in the stationary regime [14]. Erlang models still remain highly important in the modern world. However, what is crucial for applications and what is lacking in Erlang’s old results and some further studies is a knowledge of rate of convergence to a stationary regime. In fact, this “extended” Erlang’s problem – with estimated convergence rates – is not fully solved even nowadays. For a long time, estimations of convergence rate (mostly of exponential decay) were known only for the cases where service times have exponential distributions and under some additional assumptions, cf. [2], [50], et al. Bounds for the rates of convergence to stationary regimes for close systems – but not precisely Erlang’s ones – were a subject of study in many papers, see below. It is widely accepted that any important characteristic of quality of any queueing system in practice is computed in a stationary regime and it is, of course, a rare case where this characteristics is available in a more or less explicit form, cf., for example, [42]. However, if the rate of convergence is unknown, then the error is unknown either. Modelling may be some alternative to theoretical bounds, nevertheless, it cannot fully replace a rigorous theoretical analysis.

Our main goal is to attack the general non-Markov case with non-exponential service times for classical telephone systems. The key system to be studied is similar to one investigated in 50s by Sevastyanov and in the following three decades by other researchers. This system consists of a finite (as in Sevastyanov’s works), or

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infinite (as in some other works) number of servers; the incoming flow of customers is a conditional Poisson process with intensity that may depend on the number of customers in the system; in particular, it may increase linearly if this number is large. Each customer upon his arrival goes to one of the free servers, or – in the finite case – it may be lost if all servers are busy. All service times are random variables independent of each other and all have the same distribution function. Such models even with a finite number of servers usually do not satisfy conditions of Doeblin–Doob’s ergodic theorem about *uniform convergence* [11], [12, Ch.5-6].

The celebrated B. A. Sevastyanov’s ergodic theorem [45] (see also [44]) for Markov processes in general state spaces provided for the first time not only existence and uniqueness of stationary distribution for “telephone systems”, but also convergence in total variation. This was a pioneering result where such convergence is *non-uniform* with respect to the initial data or distribution and does not follow directly from Doeblin–Doob’s “uniform” ergodic theory. The corollary of Sevastyanov’s ergodic theorem for queueing (“telephone”) model will be briefly recalled below. Practically simultaneously with [44]–[45], T. E. Harris [21] suggested his method to study stationary measures of recurrent Markov processes; a presentation of his results and ideas, as well as of their further development – including studies of convergence rates – may be found in [4]. It may be noted that one of the basic ideas of this theory – to exploit moments of “regeneration” of the process – was proposed in the forties in [11] and further developed in [32] in relation to a very close issue of local limit theorems, which may serve as a background for coupling. A few years earlier than Sevastyanov and shortly after [32], Fortet proved [18] that a stationary distribution exists in “Sevastyanov’s case” under a bit stronger assumptions than eventually in [45] (existence of a density was assumed), along with the form of this distribution; however, he did not study uniqueness nor convergence. Some special important part of the main result of [45] – related to the property of “insensitivity” (see below) – was also rigorously obtained in [29]. The latter paper was published only in 1963, however, as quite reasonably suggested by the Editor (B. V. Gnedenko) of the volume of A. Ya. Khinchin’s works in [30, The Editor’s Introduction], the paper was, in fact, fully prepared to publication in 1954–1955. Earlier, the original Erlang’s formulae with exponential service time distribution were extended on systems with an infinite number of servers [28], and later (1965) this result was tackled by a different method in [38]. Among all these results, [45] remains the most advanced achievement in that period.

More general systems – with infinitely many servers and/or with more involved disciplines of serving – were studied further in [16], [17], [22], [31], [33], [37], [43], [51], [52], et al. Even quite recently, results in this direction were still under investigation under the name of “insensitivity” of a stationary regime (i.e., where there are some general invariants of a stationary distribution, which depend on the service time distribution only through its mean value) for advanced versions of Erlang type models in [1], [5], [36], [58], [60]. Note that most of these papers – with the exception of [22] and [52] – do not cite two other pioneering publications [33]–[34] and none of them

except [16] tackles convergence rates; in the latter paper, the result about convergence rate bounds could be called partial in comparison to our Theorem 1 below.

Sevastyanov’s version of ergodic theorem [44]–[45] also proved to be useful in some extensions, in particular, in the case $N = \infty$, see [51]. For other versions of such extensions see [37], [43], et al. (regrettably, the former publication [37] is still hardly easily available even nowadays).

Exponential convergence rate for infinite server systems of Sevastyanov’s type (and a little more general) with *non-exponential* service time distributions may be found in [26]; however, the method used there was not suitable for weaker sub-exponential rates under weaker assumptions. Establishing such weaker convergence rates for a wider class of queueing systems of Sevastyanov’s type is the main goal of the paper.

It would be an extremely hard task to mention all important publications where convergence rates for general Markov processes – or, indeed, just for applications to queueing models – were studied; some of them may be found among the references below, or in the literature provided in these references. A very incomplete list of names of major contributors includes Kalashnikov [23], [25], Borovkov [6], [7], Tuominen and Tweedie [48], [49], Thorisson [46], [47], et al. Results about convergence rates close to the Theorem 1 below for similar but yet a bit different systems may be found in the fundamental monograph [47], where, in particular, the Theorem 7.2 establishes convergence in total variation,

$$\varphi(t) \|\mu_t - \mu\|_{TV} \rightarrow 0, \quad t \rightarrow \infty. \quad (1)$$

Recall that the total variation distance between two probability measures on a measurable space (Ω, \mathcal{F}) is defined as to

$$\|\mu - \nu\|_{TV} := 2 \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|.$$

For certain more particular models see also [8] and [48]. The background idea of the approach in [47] is to use estimates of the rate of convergence in the law of large numbers (LLN); its implementation is involved and uses regeneration technique. In our model with an infinite number of servers ($N = \infty$) it is unclear how to use LLN directly and we use another method (eventually leading to LLN, too) based on a “markovisation” of the system – due to Sevastyanov – and on a local “infinitesimal” condition on the basis of *service intensity*, $h(t)$, which allows to construct Lyapunov functions. Regeneration is also in use in this paper, which gives more precise *bounds* for the distance in (1) and continues studies of various rates of convergence and mixing for a variety of Erlang–Sevastyanov’s type models commenced recently in [54] – [56]. The model in this paper is *non-Markov*.

The paper is arranged as follows. Section one is introduction. Section two contains the setting and a brief reminder of Sevastyanov’s result. Section three is devoted to the main result of this paper – polynomial convergence – Section four to some auxiliaries and Section five to the proof of the main result.

2 The setting: Erlang – Sevastyanov system

We consider the model with $1 \leq N \leq \infty$ (including ∞) identical servers working independently, with a distribution function G of service time. The incoming flow of customers is conditionally Poisson with intensity λ_n given that n servers are busy at the moment ($0 \leq n \leq N$). All service times on all servers are independent on each other and on the incoming flow. A newly arrived customer chooses any server which is not busy and its serving immediately starts. If $N < \infty$ and all servers are busy, then a new customer is lost or blocked; if $N = \infty$, then under reasonable assumptions the number of customers is finite at all times and no loss is possible. A customer which was served, immediately quits the system. We assume that at any moment t the elapsed service times of all customers in the system, say, X_t^1, \dots, X_t^n are known; the process $X_t = (X_t^1, \dots, X_t^n)$, $t \geq 0$ is Markov (cf. the Lemma 1 below); if there is no customers at t , then we denote $X_t = \Delta_0$ (note that $X_t = 0$ and $X_t = \Delta_0$ have different meanings). At $t = 0$, only finitely many servers may be busy. Following [45], we assume that a newly arrived customer is assigned a coordinate $X^k = 0$ with any $k = 0, \dots, n + 1$ with equal probabilities $(n + 1)^{-1}$ if at his arrival n servers are busy.

The “non-Markov property” of this system signifies that the number $n = n_t \in Z_+$ of customers at any time t is, generally speaking, *not* a Markov process (of course, unless the *intensity* of serving only depends on n_t). However, we make it Markov by considering it as (X_t) in the following extended state space \mathcal{X} of a variable dimension, as in [45]: \mathcal{X} is a union of finitely many (if $N < \infty$), or countably many (if $N = \infty$) subspaces,

$$\mathcal{X}_0 = \Delta_0; \quad \mathcal{X}_1 = R_+, \quad \dots, \quad \mathcal{X}_n = R_+^n, \quad \dots, \quad 0 \leq n \leq N.$$

To any $x \in \mathcal{X}_n$ with $n > 0$ there correspond n non-negative coordinates (x^1, \dots, x^n) , which signify the elapsed time of service of any of existing n customers in the system. If there is at least one customer in the system and $x = (x^1, \dots, x^n)$ is a vector of the elapsed service times, then by $n(x)$ we denote this number n ; if $x = \Delta_0$, then $n(x) := 0$.

In [45] (see also [44]) it is proved that for $N < \infty$ under the only condition

$$q^{-1} := \int_0^\infty x dG(x) < \infty,$$

there is a (unique) stationary distribution μ with a density ($1 \leq k \leq N$),

$$p_k(x^1, \dots, x^k) = p_0 \frac{\prod_{i=0}^{k-1} \lambda_i}{k!} \prod_{j=1}^k (1 - G(x^j)), \quad p_0^{-1} = \sum_{j=0}^{N-1} \frac{\prod_{i=0}^{j-1} \lambda_i}{q^j j!},$$

and, moreover, for any initial distribution the following convergence holds true,

$$\|\mu_t - \mu\|_{TV} \rightarrow 0, \quad t \rightarrow \infty, \quad (2)$$

where μ_t is a marginal distribution of the (Markov) process at t ; below μ_t^x will stand for the marginal distribution given initial state x .

Remark 1. In [51] this result was extended to $N = \infty$ under the condition $q^{-1} \limsup_{n \rightarrow \infty} \lambda_n / (n + 1) < 1$. By integrating $\int p_k(x) dx$, we obtain the stationary probabilities p_k of k customers in the system, which depend on G only through its mean value; this property is called “insensitivity”,

$$p_k = p_0 \frac{\prod_{i=0}^{k-1} \lambda_i}{q^k k!}, \quad 1 \leq k \leq N,$$

and it is an object of studies for various queueing models until nowadays.

3 Convergence rate bounds: Main result ($N = \infty$)

We consider Erlang–Sevastyanov’s system with $N = \infty$. The *service intensity* $h(t)$ is defined as follows, which we will assume to be *bounded* ($h \in B(R_+)$)

$$h(t) := \frac{g(t)}{1 - G(t)}, \quad t \geq 0, \quad \text{where } g(t) = G'(t).$$

Notice that $h \equiv \text{const}$ means that the service time has an exponential distribution, in which case (and in a bit more general one) a sufficient condition for exponentially fast convergence to the stationary distribution has been established in [26]. In all cases, $1 - G(t) = \exp\left(-\int_0^t h(s) ds\right)$. Let for $a, m > 0$,

$$L_{m,a}(x) := \left(\sum_{j=1}^{n(x)} (1 + x^j)^m \right)^a \quad (x \neq \Delta_0), \quad \& \quad L_{m,a}(\Delta_0) := 0.$$

To avoid triviality due to a degeneracy, we impose a condition

$$\lambda_0 > 0. \tag{3}$$

Theorem 1 *Assume (3), $h \in B(R_+)$ and existence of $C_0 > 0$ such that*

$$h(t) \geq \frac{C_0}{1+t}, \quad t \geq 0, \tag{4}$$

and

$$C_0 - 2(1 + 2\Lambda) > 0, \tag{5}$$

where

$$\Lambda := \sup_{n \geq 1} \left(\frac{\lambda_n}{n} \right) < \infty. \tag{6}$$

Then for any $k > 0$ small enough there exist real values $C > 0$, $a > 1$ and $m > 1$ such that for any $X_0 = x$,

$$\|\mu_t^x - \mu\|_{TV} \leq C \frac{1 + L_{m,a}(x)}{(1+t)^{k+1}}. \quad (7)$$

For Λ fixed, k may be chosen large if C_0 is large enough, see (9) below.

Remark 2. Without (3) – i.e., for $\lambda_0 = 0$ – the Theorem 1 is still valid with a trivial stationary distribution, δ_{Δ_0} and it does follow from the proof below with minimal changes.

Remark 3. As we shall see in the proof, for a substantial part of the proof it suffices to assume a slightly weaker assumption

$$C_0 - (1 + \Lambda) > 0. \quad (8)$$

However, in the end of the calculus the full version of (5) will be needed. More precisely, we will, actually, use

$$C_0 > (a + (k \vee 1)/m)(m + \Lambda 2^{a-1+(k \vee 1)/m}). \quad (9)$$

The latter bound is available with *some* $a, m > 1$, at least, for small $k > 0$ – in fact, for $k \leq m$ – under the assumption (5). As one more example, with $m = k$, the latter sufficient condition only in terms of C_0, Λ and k reads,

$$C_0 > 2(k + 2\Lambda),$$

as in this case there exists $a > 1$ for which (9) holds. Also notice that large values of k in (9) are available under C_0 large enough, or under just $C_0 > 1$ but with Λ small enough, which agrees with the intuitive idea that stability is stronger if intensity of service is in some sense significantly greater than intensity of arrivals. However, emphasize that C_0 itself is *not* intensity of serving itself, but only a multiplier in a lower bound for this function (4).

Remark 4. Of course, the greater C_0 , the more moments has the distribution of serving G . However, the method requires existence of intensity h . It would be interesting to relax the latter assumption.

4 Construction, martingales, estimates, strong Markov property

We will use notations $x = (x_1, \dots, x_n)$ and $X = (n, x)$ – and also $E_X \equiv E_x$ – and for any such $x \in \mathcal{X}_n$ with $n \geq 1$ define with any $1 \leq j \leq n$,

$$x^{(+j)} := (x_1, \dots, x_j, 0, x_{j+1}, \dots, x_n), \quad \& \quad x^{(-j)} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

To work with Lyapunov functions, it is very useful – if not compulsory – to know that the process is *strong Markov*. In continuous time it is not automatic and should be justified. About *Markov* property for finite N , cf. [18] and [45].

Some preliminaries: generators and martingales. Suppose for a while that $h \in C_b(\mathbb{R}_+)$; a bit later we will show how to relax this assumption. Before the next Lemmae we recall some well-known links between Markov processes and martingales, which seem to be a bit less popular language in queueing theory (e.g., cf. [6]). The generator (infinitesimal operator) \mathcal{G} of the process X in the space \mathcal{X} with a Borel topology \mathcal{B} on all subspaces \mathcal{X}_n (with a convention that \mathcal{X}_n is open and closed for each n) and sup-norm for $C(\mathcal{X}, \mathcal{B})$ is an operator \mathcal{G} such that (see [13])

$$\sup_{x \in \mathcal{X}} \left| \frac{E_x f(X_t) - f(x)}{t} - \mathcal{G}f(x) \right| \rightarrow 0, \quad t \rightarrow 0, \quad (10)$$

for all f from the *domain* $\mathcal{D}_{\mathcal{G}}$ of \mathcal{G} , which is usually a hard task to determine precisely and which is usually enough to have a wide enough sub-class of. In our case, it follows from (9)–(10) and continuity of h that for $f \in C_0^1(\mathcal{X})$ – with one continuous derivative and compact support – i.e., $f(x)$ vanishes if $n \geq N$ or if $\sup_i x^i \geq N$ for some N – (10) holds with

$$\begin{aligned} \mathcal{G}f(x) = & \sum_{i=1}^{n(x)} \left(\frac{\lambda_{n(x)}}{n(x)} (f(x^{(+i)}) - f(x)) + h(x^i) (f(x^{(-i)}) - f(x)) \right. \\ & \left. + \frac{\partial}{\partial x^i} f(x) \right) \times 1(n(x) > 0) + \lambda_0 (f(0) - f(x)) 1(n(x) = 0). \end{aligned} \quad (11)$$

By Dynkin's formula [13, see, e.g., corollary from the formula (1.36) as $\lambda \rightarrow \infty$],

$$E_{X_0} f(X_t) - f(X_0) = E \int_0^t \mathcal{G}f(X_s) ds \quad (12)$$

for any $f \in C_0^1$. For functions $(f(t, X), t \geq 0, X \in \mathcal{X})$ of class C_0^1 with respect to (t, X) – which vanish for large $n(x)$ and for large $X \in \mathcal{X}_n$ for any fixed n – Dynkin's formula for the process (t, X_t) reads,

$$E_{X_0} f(t, X_t) - f(0, X_0) = E \int_0^t \left(\frac{\partial}{\partial s} f(s, X_s) + \mathcal{G}f(s, X_s) \right) ds. \quad (13)$$

Note that, at least, intuitively this equality as well as (12) may be regarded as a complete probability formula, as the right hand side presents all possible developments of the trajectory from 0 to t . In terms of martingales (cf. [35]), (13) is equivalent to saying that the process

$$M_t = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} f(s, X_s) + \mathcal{G}f(s, X_s) \right) ds, \quad t \geq 0 \quad (14)$$

is a martingale. Recall for the sequel that a process is called a *local* martingale if there exists a sequence of stopping times $\tau_n \rightarrow \infty$ a.s. such that the stopped process $M_{t \wedge \tau_n}$ is a martingale for each n (cf. [35]). In turn, the statement (14) may be equivalently (by definition) rewritten in the differential form as

$$df(t, X_t) = \left(\frac{\partial}{\partial t} f(t, X_t) + \mathcal{G}f(t, X_t) \right) dt + dM_t \quad (15)$$

for any $f \in C_0^1$ (\mathcal{G} is defined above). The latter formula itself is *local* – i.e., written at any given (t, X_t) and its small neighbourhood – so it may be extended from C_0^1 to C^1 , under a natural convention that the process is stopped on the exit from some neighbourhood of the state X_t ; of course, this may require a localizing stopping time procedure if using the integral form (14) and possibly some justification that M is a martingale (and not just a local martingale).

All the above starting from the formula (10) work well if the intensities are continuous. If this is wrong, the limit in (10) may just not exist. Nevertheless, following [10] or [20] it is possible to define the process by using a notion of *extended generator*, that is, an operator for which Dynkin's formulae (12) and (13) hold. The action of extended generator on functions is given by the same expression in (11).

Lemma 1 *Under the assumptions (6) and $h \in B(R_+)$ the process X exists, has a unique distribution and is Markov and strong Markov. The Dynkin's formulae (12) and (13) hold for any $f(x) \in C_0^1$ and $f(t, x) \in C_0^1$.*

Proof. Existence (for possibly discontinuous h) follows from the results on piecewise linear or piecewise deterministic Markov processes in [20], [24], [10], as well as do uniqueness and Markov and strong Markov properties. The non-explosion is implied by the condition (6), for example, due to [20, Ch. 1.3.3]. Both Dynkin's formulae follow from [10]. Another way to show Dynkin's formula for a slightly different model was suggested in [57]. The Lemma 1 is proved.

We admit the following convention for stochastic differentials:

$$A_t dt + dM_t \leq B_t dt + dM_t \quad \text{iff} \quad A_t \leq B_t, \quad \forall t \geq 0, \quad \text{a.s.}$$

Recall that the process is called *cadlag* iff it is right continuous with left limits at any t . Note that M_t in (14) is cadlag because the right hand side is. Below we use convention $n^{-1} \sum_{i=1}^n a_i \equiv 1$ for any real values (a_i) if $n = 0$.

Lemma 2 *Under the assumptions (6) and $h \in C_b(R_+)$,*

$$\begin{aligned} L_{m,a}(X_t) - L_{m,a}(X_0) &= \int_0^t \lambda_{n(X_s)} \left(\frac{1}{n(X_s)} \sum_{i=1}^{n(X_s)} (L_{m,a}(X_s^{(+i)}) - L_{m,a}(X_s)) \right) ds \\ &+ \int_0^t \left[\sum_{i=1}^{n(X_s)} h(X_s^i) (L_{m,a}(X_s^{(-i)}) - L_{m,a}(X_s)) + \sum_{i=1}^{n(X_s)} \frac{\partial}{\partial x^i} L_{m,a}(X_s) \right] ds + M_t, \end{aligned} \quad (16)$$

with some martingale M_t .

Proof follows from (15) with $f(t, X) \equiv L_{m,a}(X)$ (see the remark above about extension of (15) to C^1), due to

$$\begin{aligned}
dL_{m,a}(X_t) &= \mathcal{G}(X_t) dt + dM_t = \lambda_n \left(\frac{1}{n} \sum_{i=1}^n L_{m,a}(X_t^{(+i)}) - L_{m,a}(X_t) \right) dt \\
&+ \sum_{i=1}^n h(X_t^i) \left(L_{m,a}(X_t^{(-i)}) - L_{m,a}(X_t) \right) dt + \sum_{i=1}^n \frac{\partial}{\partial x^i} L_{m,a}(X_t) dt + dM_t \\
&\equiv (I_1 - I_2 + I_3) dt + dM_t,
\end{aligned} \tag{17}$$

with $n = n(X_t)$. We shall see below in the Lemma 4 (without a vicious circle) that no localization is needed here as all terms in (16) will turn out to be integrable and M is, indeed, a martingale. The Lemma 2 is proved.

Lemma 3 *Under the assumptions (6), $h \in B(R_+)$ and $m, a > 1$, the following bounds or equalities hold true:*

$$I_1 \leq \Lambda a 2^{a-1} L_{m-1,1}(X_t) L_{m,a-1}(X_t) 1(X_t \neq \Delta_0) + \lambda_0 1(X_t = \Delta_0); \tag{18}$$

$$I_2 \leq a \|h\|_B L_{m,a+1}(X_t) 1(X_t \neq \Delta_0); \tag{19}$$

$$I_3 = am L_{m-1,1}(X_t) L_{m,a-1}(X_t) 1(X_t \neq \Delta_0); \tag{20}$$

$$E_x L_{m,a}(X_t) \leq (L_{m,a}(x) + \lambda_0 t) \exp((\Lambda a 2^{a-1} + ma)t). \tag{21}$$

If in addition

$$C_0 > a(m + \Lambda 2^{a-1}), \tag{22}$$

then also

$$I_2 \geq 1(X_t \neq \Delta_0) C_0 L_{m-1,1}(X_t) L_{m,a-1}(X_t). \tag{23}$$

Proof. Let us establish the bound for I_1 . Notice that for $y = x + 1 \geq 2$ and $a > 1$ we have $y^{a-1} = (x + 1)^{a-1} \leq (2x)^{a-1}$, and, hence, $y^a - x^a \leq a(y - x)y^{a-1} \leq a 2^{a-1} x^{a-1}$. Indeed, the first bound here follows for $y \geq x > 0$ and $a > 0$ from

$$\frac{d}{dx}(y^a - x^a) = -ax^{a-1} \geq -ay^{a-1} = \frac{d}{dx}(a(y - x)y^{a-1}).$$

So, we estimate,

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n L_{m,a}(X_t^{(+i)}) - L_{m,a}(X_t) \\
&= \left(\left((1+0)^m + \sum_{j=1}^n (1+X_t^j)^m \right)^a - \left(\sum_{j=1}^n (1+X_t^j)^m \right)^a \right) \\
&\leq a 2^{a-1} \left(\sum_{j=1}^n (1+X_t^j)^m \right)^{a-1} = a 2^{a-1} L_{m,a-1}(X_t).
\end{aligned}$$

Hence, due to the inequality $n(X_t) \leq L_{m-1,1}(X_t)$, we get,

$$\begin{aligned}
I_1 &= \lambda_n (L_{m,a}(X'_t) - L_{m,a}(X_t)) \leq \lambda_n a 2^{a-1} L_{m,a-1}(X_t) dt \\
&\leq \Lambda n a 2^{a-1} L_{m,a-1}(X_t) \leq \Lambda a 2^{a-1} L_{m-1,1}(X_t) L_{m,a-1}(X_t).
\end{aligned} \tag{24}$$

Further, by taking derivatives, we find,

$$\begin{aligned}
I_3 &= \sum_{i=1}^n \frac{\partial}{\partial x^i} L_{m,a}(X_t) = a \left(\sum_{\ell=1}^n (1+X_t^\ell)^m \right)^{a-1} \times \sum_{j=1}^n m (1+X_t^j)^{m-1} \\
&= am L_{m-1,1}(X_t) L_{m,a-1}(X_t).
\end{aligned} \tag{25}$$

The lower bound for I_2 under the additional (22),

$$I_2 \geq C_0 \sum_{i=1}^n (1+X_t^i)^{m-1} L_{m,a-1}(X_t) = C_0 L_{m-1,1}(X_t) L_{m,a-1}(X_t). \tag{26}$$

Emphasize that $L_{m,a-1}(X_t)$ stands here in the middle term and not $L_{m,a-1}(X_t^{(-i)})$ – the latter would be a little bit insufficient for our aims – which is justified in the next few lines. We used here the elementary inequality for real values $0 < x \leq y$ and $a > 1$,

$$y^a - x^a \geq (y-x)y^{a-1} \tag{27}$$

(instead of also correct $y^a - x^a \geq a(y-x)x^{a-1}$), for $y = \sum_{j=1}^n (1+X_t^j)^m$ and $x = \sum_{1 \leq j \leq n, j \neq i} (1+X_t^j)^m$. The bound (27) follows from the observation that both sides in (27) vanish at $y = x$ and the derivative function of the right hand side is less than that of the left hand side for $y > x (> 0)$:

$$\frac{d}{dy} (y-x)y^{a-1} = ay^{a-1} - (a-1)xy^{a-2} < ay^{a-1} = \frac{d}{dy} (y^a - x^a).$$

Hence,

$$\begin{aligned}
I_2 &= \sum_{i=1}^n h(X_t^i) \left(\left(\sum_{j=1}^n (1 + X_t^j)^m \right)^a - \left(\sum_{1 \leq j \leq n, j \neq i} (1 + X_t^j)^m \right)^a \right) \\
&\geq \sum_{i=1}^n \frac{C_0}{(1 + X_t^i)} \left(\sum_{j=1}^n (1 + X_t^j)^m - \sum_{1 \leq j \leq n, j \neq i} (1 + X_t^j)^m \right) \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^{a-1} \\
&= C_0 \sum_{i=1}^n (1 + X_t^i)^{m-1} \left(\sum_{j=1}^n (1 + X_t^j)^m \right)^{a-1} = C_0 L_{m-1,1}(X_t) L_{m,a-1}(X_t).
\end{aligned}$$

The upper bound for I_2 follows from its definition and from the remark that $n(x) \leq L_{m,1}(x)$ and $L_{m,1}L_{m,a} = L_{m,a+1}$.

Further, for any $t \geq 0$,

$$I_1 - I_2 + I_3 \leq (\Lambda a 2^{a-1} + ma) L_{m-1,1}(X_t) L_{m,a-1}(X_t) + \lambda_0 t.$$

So, from (17) and by virtue of Fatou's lemma – and using a localization for M if needed so as to vanish expectation of the (local) martingale term – we get,

$$\begin{aligned}
E_x L_{m,a}(X_{t \wedge \tau_n}) &\leq L_{m,a}(x) + \lambda_0 t \\
&+ (\Lambda a 2^{a-1} + ma) E_x \int_0^{t \wedge \tau_n} L_{m-1,1}(X_s) L_{m,a-1}(X_s) ds \\
&\leq L_{m,a}(x) + \lambda_0 t + (\Lambda a 2^{a-1} + ma) E_x \int_0^t L_{m,a}(X_{s \wedge \tau_n}) ds.
\end{aligned}$$

By Gronwall's inequality (note that $E_x L_{m,a}(X_{t \wedge \tau_n})$ is bounded),

$$E_x L_{m,a}(X_{t \wedge \tau_n}) \leq (L_{m,a}(x) + \lambda_0 t) \exp((\Lambda a 2^{a-1} + ma)t).$$

and, as $\tau_n \rightarrow \infty$, by Fatou's Lemma, also

$$E_x L_{m,a}(X_t) \leq (L_{m,a}(x) + \lambda_0 t) \exp((\Lambda a 2^{a-1} + ma)t).$$

The Lemma 3 is proved.

Lemma 4 *Under the assumptions (6), $h \in C_b(R_+)$, $m, a > 1$, for any $t > 0$,*

$$E_x \sup_{0 \leq s \leq t} L_{m,a}(X_s) < \infty, \tag{28}$$

and the local martingale M in (16) is, in fact, a martingale.

Proof. We estimate, for any $a, m > 0$,

$$E_x \sup_{0 \leq s \leq t} L_{m,a}(X_s) \leq L_{m,a}(x) + \int_0^t E_x L_{m,a}(X_s) ds + E_x \sup_{0 \leq s \leq t} |M_s|.$$

In turn, $E_x \sup_{0 \leq s \leq t} |M_s| \leq C_p (E_x |M_t|^p)^{1/p}$ for any $p > 1$ by Doob's inequality (recall that M is cadlag) and further,

$$\begin{aligned} |M_t| &\leq L_{m,a}(X_t) + L_{m,a}(x) + \left| \int_0^t \mathcal{G}(X_s) ds \right| \\ &\leq L_{m,a}(X_t) + L_{m,a}(x) + \left| \int_0^t I_1 ds \right| + \left| \int_0^t I_2 ds \right| + \left| \int_0^t I_3 ds \right|. \end{aligned} \tag{29}$$

So, due to (21), which is valid for any $m, a > 0$,

$$\begin{aligned} E_x |M_t|^p &\leq C_p L_{m,a}(x)^p + C_p E_x L_{m,a}(X_t)^p + C_p \int_0^t E_x L_{m,a+1}(X_s)^p ds \\ &= C_p L_{m,ap}(x) + C_p E_x L_{m,ap}(X_t) + C_p \int_0^t E L_{m,ap+p}(X_s) ds \\ &\leq C'(L_{m,ap+p}(x) + \lambda_0 t) \exp(Ct) < \infty. \end{aligned}$$

By virtue of Hölder's inequality, this implies (28). The Lemma 4 is proved.

5 Proof of Theorem 1

1. Consider a Lyapunov function $L_{m,a}$ at any $X_t \neq \Delta_0$ and $m, a > 1$, satisfying also (22) (compare with (5) and (9)). The idea of Lyapunov functions in a stochastic context is to verify that the 'main' negative term prevails and that 'on average' the process $L_{m,a}(X_t)$ decreases, as long as $X_t \neq \Delta_0$. From the bounds on I_1, I_2 and I_3 of the Lemma 3 it follows that I_1 and I_3 are *dominated* by I_2 . Then it would imply that the stationary measure integrates some polynomial. In turn, this would allow to extend our Lyapunov function so as to include some multiplier that depends on time. The latter would help obtain the crucial bound $E_x \tau_0^{k+1} < \infty$ for some $k > 0$ (see (37) below). Finally, the latter bound would imply "coupling" between the original process and its stationary version with a certain rate of convergence. For any $t < \tau_0$ we have,

$$I_1 - I_2 + I_3 \leq -(C_0 - \Lambda a 2^{a-1} - ma) L_{m-1,1}(X_t) L_{m,a-1}(X_t) < 0.$$

Denote

$$C_{m,\Lambda,a} := C_0 - \Lambda a 2^{a-1} - ma > 0.$$

By Fatou's lemma we get,

$$E_x L_{m,a}(X_{t \wedge \tau_0}) + C_{m,\Lambda,a} E_x \int_0^{t \wedge \tau_0} L_{m-1,1}(X_s) L_{m,a-1}(X_s) ds \leq L_{m,a}(x), \quad (30)$$

and, as $t \rightarrow \infty$,

$$E_x L_{m,a}(X_{\tau_0}) + C_{m,\Lambda,a} E_x \int_0^{\tau_0} L_{m-1,1}(X_s) L_{m,a-1}(X_s) ds \leq L_{m,a}(x).$$

In particular, $E_x \tau_0 < \infty$ for any x and also $E_0 \hat{\tau}_0 < \infty$ with $\hat{\tau}_0 := \inf(t > 0 : X_t = \Delta_0; \exists s \in (0, t) : X_s \neq \Delta_0)$. In other words, the process X is positive recurrent. According to the Harris–Khasminskii principle about invariant measures (cf., for example, [53]), there exists a (unique in our model) invariant measure μ , $\mu(A) = c E_0 \int_0^{\hat{\tau}_0} 1(X_s \in A) ds$, which integrates the function $L_{m-1,1}(x) L_{m,a-1}(x)$. As noticed by the Referee, under the accepted assumptions both existence and uniqueness of this measure also follow straightforward from [51].

In a moment, we will show one more elementary inequality

$$L_{m,1}(x)^{(m-1)/m} \leq L_{m-1,1}(x), \quad (31)$$

so that (notice that $L_{m,a}(x) L_{m,b}(x) = L_{m,a+b}(x)$ and $L_{m,1}(x)^a = L_{m,a}(x)$)

$$L_{m-1,1}(x) L_{m,a-1}(x) \geq L_{m,a-1+\frac{m-1}{m}}(x) = L_{m,a-1/m}(x),$$

and

$$E_x L_{m,a}(X_{t \wedge \tau_0}) + C_{m,\Lambda,a} E_x \int_0^{t \wedge \tau_0} L_{m,a-1/m}(X_s) ds \leq L_{m,a}(x). \quad (32)$$

So, due to Fatou's lemma,

$$E_x L_{m,a}(X_{\tau_0}) + C_{m,\Lambda,a} E_x \int_0^{\tau_0} L_{m,a-1/m}(X_s) ds \leq L_{m,a}(x). \quad (33)$$

Emphasize that both inequalities (32) and (33) have been established under the assumption $C_{m,\Lambda,a} > 0$, that is,

$$C_0 - \Lambda a 2^{\alpha-1} - m a > 0. \quad (34)$$

2. The inequality (31) follows from the inequalities with $a, b > 0$, $\alpha \in (0, 1)$ and $c = b/a$

$$(a+b)^\alpha \leq a^\alpha + b^\alpha \quad \sim \quad (1+c)^\alpha \leq 1 + c^\alpha,$$

where the latter, in turn, follows from the valid inequality for the derivatives,

$$\alpha(1+c)^{\alpha-1} \leq \alpha c^{\alpha-1}.$$

3. We are now prepared for considering a Lyapunov function which depends on time. Let $k > 0$ (*not* necessarily $k > 1$), $a, m > 0$ and $L_{m,a,k}(t, x) := (1+t)^k L_{m,a}(x)$. Similarly to the above and choosing $a, m > 1$, we have,

$$\begin{aligned} dL_{m,a,k}(t, X_t) &= L_{m,a,k}(t+dt, X_{t+dt}) - L_{m,a,k}(t, X_t) \\ &= (1+t)^k [I_1 - I_2 + I_3] dt + k(1+t)^{k-1} L_{m,a}(X_t) dt + d\tilde{M}_t \\ &\leq -(1+t)^k (C_0 - \Lambda a 2^{a-1} - ma) L_{m, a - \frac{1}{m}}(X_t) dt \\ &\quad + k(1+t)^{k-1} L_{m,a}(X_t) dt + d\tilde{M}_t, \end{aligned}$$

with some new local martingale \tilde{M} . Now the task is again to ensure that the negative part in the right hand side of the last expression prevails. We will be using the inequality established in the step 1 above. The second term may be split into two parts,

$$\begin{aligned} I &:= k(1+t)^{k-1} L_{m,a}(X_t) \tag{35} \\ &= I \times 1(k(1+t)^{k-1} L_{m,a}(X_t) \leq \epsilon(1+t)^k L_{m, a-1/m}(X_t)) \\ &\quad + I \times 1(k(1+t)^{k-1} L_{m,a}(X_t) > \epsilon(1+t)^k L_{m, a-1/m}(X_t)). \end{aligned}$$

The first term here with ' $\leq \epsilon$ ', clearly, is dominated by the main negative expression if $\epsilon > 0$ is small enough, $\epsilon < C_0 - a(m + \Lambda 2^{a-1})$. The set of such values ϵ is non-empty as long as a and m are chosen so as to satisfy (22).

Let us estimate the second term in (35). We have, for any $\ell > 0$ (later we will choose $\ell = k + \delta$ with small $\delta > 0$),

$$\begin{aligned} &I \times 1(k(1+t)^{k-1} L_{m,a}(X_t) > \epsilon(1+t)^k L_{m, a-1/m}(X_t)) \\ &\leq I \times \frac{(k L_{m,a}(X_t))^\ell}{(\epsilon(1+t) L_{m, a-1/m}(X_t))^\ell} = I \times \frac{k^\ell}{(\epsilon(1+t))^\ell} L_{m, \ell/m}(X_t). \end{aligned}$$

Therefore, the second term in (35) does not exceed

$$k(1+t)^{k-1} \times \frac{k^\ell}{(\epsilon(1+t))^\ell} L_{m, a+\ell/m}(X_t).$$

Now let us impose conditions on ℓ : let $a' := a + \ell/m$ and assume

$$C_0 - \epsilon > a'(m + \Lambda 2^{a'-1}), \tag{36}$$

in order to use inequalities similar to (32)–(33) with a' instead of a . Note that, at least, for $\ell > 0$ – and automatically k – small enough, the latter inequality holds true due to (34). Now, let us collect all terms and their bounds, integrate and take expectations (exploiting an appropriate localization for \tilde{M} if necessary),

$$\begin{aligned} & E_x L_{m,a,k}(t \wedge \tau_0, X_{t \wedge \tau_0}) + (C_{m,\Lambda,a} - \epsilon) E_x \int_0^{t \wedge \tau_0} (1+s)^k L_{m,a-1/m}(X_s) ds \\ & \leq L_{m,a}(x) + C' E_x \int_0^\infty (1+s)^{k-1-\ell} E_x 1(s \leq t \wedge \tau_0) L_{m,a+\ell/m}(X_s) ds. \\ & \leq L_{m,a}(x) + C'' L_{m,a+(\ell-1)/m}(x). \end{aligned}$$

(This writing does not necessarily mean that $\ell \geq 1$.) Due to Fatou's lemma, with $\ell = k + \delta$ and $\delta > 0$ (i.e., $\ell > k$), this implies,

$$\begin{aligned} & E_x L_{m,a,k}(\tau_0, X_{\tau_0}) + C' E_x \int_0^{\tau_0} (1+s)^k L_{m,a-1/m}(X_s) ds \\ & \leq L_{m,a}(x) + C'' L_{m,a+(\ell-1)/m}(x). \end{aligned}$$

Since $L_{m,a-1/m}(X_s) \geq 1$ for $s < \tau_0$ (notice that $a + (\ell - 1)/m > 0$), we get,

$$E_x \tau_0^{k+1} \leq C L_{m,a}(x) + C L_{m,a+(\ell-1)/m}(x),$$

or just

$$E_x \tau_0^{k+1} \leq C L_{m,a+(\ell-1)/m}(x). \quad (37)$$

Notice that for $x = \Delta_0$, the inequality (37) also trivially holds true.

4. The bound (37) along with moment inequalities (32–33) for Markov models “usually” already suffice for establishing the desired rate of convergence and there are several standard ways to accomplish the proof. So, in principle, we may claim our result at this point. However, we give a sketch of the remaining proof for completeness of the presentation and to address some specifics of the models. It is due to this specifics that while considering a couple of processes we need to take some additional care so as to tackle the hitting time of the “origin” for this couple, while “usually” it is enough to estimate moments of the hitting time of some neighbourhood of the origin. In this *second part of the proof*, we consider two independent versions X and \tilde{X} of our Markov process, one starting at x and another at the stationary measure μ . We are going to show how arrange coupling. Recall that the stationary version exists due to the Harris-Khasminskii principle, see the remark above. Denote $\bar{\tau}_0 := \inf(t \geq 0 : X_t = \tilde{X}_t = \Delta_0)$. Given $\tilde{X}_0 = y$, it may be proved that also

$$E_{x,y} \bar{\tau}_0^{k+1} \leq C \bar{L}_{m,a+\ell/m}(x, y). \quad (38)$$

Indeed, let $R > 0$ and for given m, a and ℓ , denote

$$\begin{aligned} \bar{\tau}_{0,R} := \inf(t \geq 0 : X_t = \Delta_0 \text{ and } L_{m,a-1/m}(Y_t) \leq R, \\ \text{or } Y_t = \Delta_0 \text{ and } L_{m,a-1/m}(X_t) \leq R). \end{aligned}$$

The idea of evaluating $\bar{\tau}_0$ is to establish a bound for $\bar{\tau}_{0,R}$ and then to use it for managing $\bar{\tau}_0$ with the help of (37). For this goal, consider Lyapunov functions

$$\bar{L}_{m,a}(X_t, Y_t) := L_{m,a}(X_t) + L_{m,a}(Y_t), \quad \bar{L}_{m,a,k}(t, X_t, Y_t) := (1+t)^k \bar{L}_{m,a}(X_t, Y_t),$$

with the same values of m, a, k as above for a single component. We notice that at any moment t when $(X_t, Y_t) \neq (\Delta_0, \Delta_0)$, the Lyapunov function $\bar{L}_{m,a,k}$ serves well in the sense that it decreases on average at least as fast as a single component one, $(1+t)^k L_{m,a}(X_t)$, say (if $X_t \neq \Delta_0$). If it occurs for the first time that $X_t = Y_t = \Delta_0$, then it means that $t = \bar{\tau}_0$. So, we have to inspect what happens at t when $X_t = \Delta_0$ and $\Delta_0 \neq Y_t$, but $L(Y_t) \geq R$, say. In this case the idea is that the average increment of $L_{m,a-1/m}(X_t)$ is, of course, positive but equals just $\lambda_0 dt$ and, hence, may be easily compensated by a large negative on average increment of the other component $L_{m,a-1/m}(Y_t)$. In this way we will establish below the bound

$$E_{x,y} \bar{\tau}_{0,R}^{k+1} \leq C \bar{L}_{m,a+\ell/m}(x, y), \quad (39)$$

under the condition (36). Then, once $\bar{\tau}_{0,R}$ occurred, we may wait some fixed time t_1 sufficient for Y to achieve Δ_0 with a large probability, say, at least $1/2$, while X remains at Δ_0 all that time with probability $\exp(-\lambda_0 t_1)$. If this scenario is *not* realised – which probability does not exceed some constant $\nu < 1$ – then we stop at $\bar{\tau}_{0,R} + t_1$ or a bit earlier if either X exits from Δ_0 , or $L_{m,a-1/m}(Y)$ exceeds level $R + 1$ (say). Then we wait again until the “next” moment $\bar{\tau}_{0,R}$ and repeat the whole procedure of the “attempt” to meet both components at Δ_0 . Thus, we will evaluate $\bar{\tau}_0$ by means of some geometric series, which would guarantee the desired inequality (38). Hence, let us show the bound (39) first. Recall that we have $C_{m,\Lambda,a} > 0$ and even $C_{m,\Lambda,a'} > 0$ ($a' = a + \ell/m$) due to (34) and (34), and choose ϵ and R so that

$$(C_{m,\Lambda,a} - \epsilon)R > \lambda_0. \quad (40)$$

Then there exists $C' > 0$ such that $(C_{m,\Lambda,a} - \epsilon - C')R \geq \lambda_0$. Denote

$$\begin{aligned} e_t^1 &:= 1(X_t \neq \Delta_0, Y_t \neq \Delta_0), & e_t^2 &:= 1(X_t = \Delta_0, L_{m,a-1/m}(Y_t) \geq R), \\ e_t^3 &:= 1(Y_t = \Delta_0, L_{m,a-1/m}(X_t) \geq R). \end{aligned}$$

5. We start with the function $\bar{L}_{m,a}(X_t, Y_t)$ on $t < \bar{\tau}_{0,R}$. Repeating the calculus at the step 1, we obtain the following bounds,

$$\begin{aligned} E_{x,y} \bar{L}_{m,a}(X_{t \wedge \bar{\tau}_{0,R}}, Y_{t \wedge \bar{\tau}_{0,R}}) + E_x \int_0^{t \wedge \bar{\tau}_{0,R}} \{[(e_s^1 + e_s^3)C_{m,\Lambda,a}L_{m,a-1/m}(X_s) - e_s^3\lambda_0] \\ + [(e_s^1 + e_s^2)C_{m,\Lambda,a}L_{m,a-1/m}(Y_s) - e_s^2\lambda_0]\} ds \leq \bar{L}_{m,a}(x, y), \end{aligned} \quad (41)$$

and, due to Fatou's lemma, also

$$\begin{aligned}
& E_{x,y} \bar{L}_{m,a}(X_{\bar{\tau}_{0,R}}, Y_{\bar{\tau}_{0,R}}) + E_x \int_0^{\bar{\tau}_{0,R}} \{[(e_s^1 + e_s^3)C_{m,\Lambda,a}L_{m,a-1/m}(X_s) - e_s^3\lambda_0] \\
& + [(e_s^1 + e_s^2)C_{m,\Lambda,a}L_{m,a-1/m}(Y_s) - e_s^2\lambda_0]\} ds \leq \bar{L}_{m,a}(x, y).
\end{aligned} \tag{42}$$

Due to the condition (40), all integrands “[...]” in (41) and (42) are non-negative, so, in particular, for any $t \geq 0$,

$$E_{x,y} \bar{L}_{m,a}(X_{t \wedge \bar{\tau}_{0,R}}, Y_{t \wedge \bar{\tau}_{0,R}}) \vee E_{x,y} \bar{L}_{m,a}(X_{\bar{\tau}_{0,R}}, Y_{\bar{\tau}_{0,R}}) \leq \bar{L}_{m,a}(x, y). \tag{43}$$

6. Now we are ready to consider the Lyapunov function $\bar{L}_{m,a,k}(t, X_t, Y_t)$ depending also on time. Similarly to the step 3 – see the formula (35) – we have on $t < \bar{\tau}_{0,R}$ with some new local martingale \hat{M}_t ,

$$\begin{aligned}
d\bar{L}_{m,a,k}(t, X_t, Y_t) &= \bar{L}_{m,a,k}(t + dt, X_{t+dt}, Y_{t+dt}) - \bar{L}_{m,a,k}(t, X_t, Y_t) \\
&\leq e_t^1(1+t)^k \left[(I_1^Y - I_2^Y + I_3^Y + \frac{kL_{m,a}(Y_t)}{1+t}) dt \right. \\
&\quad \left. + (I_1^Y - I_2^Y + I_3^Y + \frac{kL_{m,a}(X_t)}{1+t}) dt + d\hat{M}_t \right] \\
&+ e_t^2(1+t)^k \left[(I_1^Y - I_2^Y + I_3^Y + \lambda_0 + \frac{kL_{m,a}(Y_t)}{1+t}) dt + d\hat{M}_t \right] \\
&+ e_t^3(1+t)^k \left[(I_1^X - I_2^X + I_3^X + \lambda_0 + \frac{kL_{m,a}(X_t)}{1+t}) dt + d\hat{M}_t \right] \\
&=: (J_1 + J_2 + J_3)dt + d\tilde{M}_t,
\end{aligned}$$

again with a new local martingale \tilde{M} and with

$$\begin{aligned}
J_1 &:= e_t^1(1+t)^k \left[I_1^Y - I_2^Y + I_3^Y + \frac{kL_{m,a}(Y_t)}{1+t} + I_1^Y - I_2^Y + I_3^Y + \frac{kL_{m,a}(X_t)}{1+t} \right], \\
J_2 &:= e_t^2(1+t)^k \left[I_1^Y - I_2^Y + I_3^Y + \lambda_0 + \frac{kL_{m,a}(Y_t)}{1+t} \right], \\
J_3 &:= e_t^3(1+t)^k \left[I_1^X - I_2^X + I_3^X + \lambda_0 + \frac{kL_{m,a}(X_t)}{1+t} \right].
\end{aligned}$$

Here the first term J_1 is estimated identically to what was done at the step 3 for the only component X , and this gives

$$\begin{aligned} J_1 &\leq e_t^1 (-(1+t)^k (C_{m,\Lambda,a} - \epsilon) \bar{L}_{m,a-1/m}(X_t, Y_t)) \\ &\quad + e_t^1 k(1+t)^{k-1} \times \frac{k^\ell}{(\epsilon(1+t))^\ell} \bar{L}_{m,a+\ell/m}(X_t, Y_t). \end{aligned}$$

The second and the third terms allow the bounds,

$$\begin{aligned} J_2 &\leq e_t^2 (-(1+t)^k C_{m,\Lambda,a} L_{m,a-1/m}(Y_t) - \lambda_0) \\ &\quad + e_t^2 k(1+t)^{k-1} \times \frac{k^\ell}{(\epsilon(1+t))^\ell} L_{m,a+\ell/m}(Y_t), \end{aligned}$$

$$\begin{aligned} J_3 &\leq e_t^3 (-(1+t)^k C_{m,\Lambda,a} L_{m,a-1/m}(X_t) - \lambda_0) \\ &\quad + e_t^3 k(1+t)^{k-1} \times \frac{k^\ell}{(\epsilon(1+t))^\ell} L_{m,a+\ell/m}(X_t), \end{aligned}$$

Now, let us collect all terms and their bounds, integrate and take expectation, also using localization for the martingale term if necessary. Notice that $1(s < \bar{\tau}_0)(e_s^1 + e_s^2 + e_s^3) = 1(s < \bar{\tau}_0)$ and

$$\begin{aligned} 1(s < \bar{\tau}_0)[(e_s^1 + e_s^3)L_{m,a-1/m}(X_s) + (e_s^1 + e_s^2)L_{m,a-1/m}(Y_s)] \\ = 1(s < \bar{\tau}_0)\bar{L}_{m,a-1/m}(X_s, Y_s). \end{aligned}$$

So, we have,

$$\begin{aligned} &E_{x,y} \bar{L}_{m,a,k}(t \wedge \bar{\tau}_{0,R}, X_{t \wedge \tau_0}, Y_{t \wedge \tau_0}) \\ &+ (C_{m,\Lambda,a} - \epsilon - \frac{\lambda_0}{R}) E_{x,y} \int_0^{t \wedge \bar{\tau}_{0,R}} (1+s)^k \bar{L}_{m,a-1/m}(X_s, Y_s) ds \\ &\leq \bar{L}_{m,a}(x, y) + C' \int_0^\infty E_{x,y} 1(s \leq t \wedge \bar{\tau}_{0,R}) (1+s)^{k-1-\ell} \bar{L}_{m,a+\ell/m}(X_s, Y_s) ds. \end{aligned}$$

Further, recall that $a' = a + \ell/m$ and (36) holds true, whence,

$$E_{x,y} 1(s \leq t \wedge \bar{\tau}_{0,R}) \bar{L}_{m,a+\ell/m}(X_s, Y_s) \leq \bar{L}_{m,a+\ell/m}(x, y).$$

From here we conclude,

$$\begin{aligned} &E_{x,y} \bar{L}_{m,a,k}(t \wedge \bar{\tau}_{0,R}, X_{t \wedge \tau_0}, Y_{t \wedge \tau_0}) + C' E_{x,y} \int_0^{t \wedge \bar{\tau}_{0,R}} (1+s)^k \bar{L}_{m,a-1/m}(X_s, Y_s) ds \\ &\leq C \bar{L}_{m,a+\ell/m}(x, y). \end{aligned}$$

Due to Fatous's lemma, with $\ell = k + \delta$ (i.e., $\ell > k$) this implies,

$$\begin{aligned} E_{x,y} \bar{L}_{m,a,k}(\bar{\tau}_{0,R}, X_{\bar{\tau}_{0,R}}, Y_{\bar{\tau}_{0,R}}) + C' E_{x,y} \int_0^{\bar{\tau}_{0,R}} (1+s)^k \bar{L}_{m,a-1/m}(X_s, Y_s) ds \\ \leq C \bar{L}_{m,a+\ell/m}(x, y). \end{aligned}$$

Since $\bar{L}_{m,a-1/m}(X_s) \geq 1$ on $s < \bar{\tau}_0$ (and on $s < \bar{\tau}_{0,R}$), we get

$$E_{x,y} \bar{\tau}_{0,R}^{k+1} \leq C \bar{L}_{m,a+\ell/m}(x, y),$$

so that (39) is established.

7. Now let us show (38). As explained above, to this aim we choose t_1 so that

$$\sup_{u: L_{m,a-1/m}(u) \leq R+1} P_u(\tau_0 > t_1) \leq t_1^{-(k+1)} \sup_{u: L_{m,a-1/m}(u) \leq R+1} E_u \tau_0^{k+1} \leq \frac{1}{2}.$$

Recall that $\nu < 1$ is defined above in the step 4 as follows,

$$\begin{aligned} 1 - \nu := \inf_{L_{m,a-1/m}(u) \leq R+1} P_{\Delta_0, u}(X_s \equiv \Delta_0, 0 \leq s \leq t_1, \& \exists t \in [0, t_1] : Y_t = \Delta_0) \\ \geq \frac{1}{2} \exp(-\lambda_0 t_1) > 0. \end{aligned}$$

Let $\bar{\tau}(1) := \bar{\tau}_{0,R}$, $\bar{\tau}(n+1) := \theta_{\bar{\tau}(n)+t_1} \bar{\tau}_{0,R}$, $n = 1, 2, \dots$, where θ_t is a shift operator for the process $((X_t, Y_t), t \geq 0)$ (see [13]). Then we estimate,

$$E_{x,y} \bar{\tau}_0^{k+1} \leq \sum_{n \geq 1} (E_{x,y}(\bar{\tau}(n) + t_1)^{k+1} \nu^{n-1} (1 - \nu)). \quad (44)$$

Whence,

$$E_{x,y}(\bar{\tau}(1) + t_1)^{k+1} \leq 2^k (E_{x,y} \bar{\tau}(1)^{k+1} + t_1^{k+1}) \leq C \bar{L}_{m,a+\ell/m}(x, y) + C.$$

By induction and using $\bar{\tau}(n) = \sum_{k=1}^n (\bar{\tau}(k) - \bar{\tau}(k-1))$ with $\bar{\tau}(0) := 0$, we have,

$$\begin{aligned} E_{x,y}(\bar{\tau}(n) + t_1)^{k+1} &\leq C n^k ((n-1) + \bar{L}_{m,a+\ell/m}(x, y) + 1) \\ &\leq C n^{k+1} (\bar{L}_{m,a+\ell/m}(x, y) + 1). \end{aligned}$$

Substitution of this into (44) shows that, indeed, (39) implies (38), under the assumption (40), the latter being guaranteed by (36).

8. From (38) we conclude, with invariant distribution μ ,

$$E_{x,\mu} \bar{\tau}_0^{k+1} \leq C L_{m,a+\ell/m}(x) + C \int L_{m,a+\ell/m}(y) \mu(dy). \quad (45)$$

Recall that from (32) it follows that μ integrates the function $L_{m,a-1/m}(x)$ for any couple (a, m) satisfying $a, m > 1$ and (9): $C_0 > a(m + \Lambda 2^{a-1})$. Hence, for the integral in (45) to converge, it suffices $C_0 > (a + \ell/m)(m + \Lambda 2^{a-1 + \ell/m})$. The latter is guaranteed by (36), that is, by (34) and by the choice of ℓ close enough to k . In turn, (34) is guaranteed by the assumption (5) if $k > 0$ and $\ell > 0$ are sufficiently small. Then, for any $k > 0$ small enough, there exist $a > 1$, $m > 1$ and $\ell > k$ such that the integral in (45) converges and we get

$$E_{x,\mu} \bar{r}_0^{k+1} \leq CL_{m,a+\ell/m}(x) + C. \quad (46)$$

9. Now, we may estimate the right hand side in the coupling inequality,

$$\|\mu_t^x - \mu\|_{TV} = 2 \sup_A (\mu_t^x - \mu)(A) \leq 2P_{x,\mu}(T > t),$$

where $T := \inf(t \geq 0 : X_t = \tilde{X}_t = 0)$. It follows from (46) in a standard way (cf. [25], [53], et al.) that for any $\nu > 0$ there exists $C > 0$ such that

$$P_{x,\mu}(T > t) \leq C(1 + L_{m,a+\ell/m}(x))(1 + t)^{-(k+1)+\nu}. \quad (47)$$

This is equivalent to (7). The Theorem 1 is proved.

Remark 5. The main result may be extended to the case where both λ and h depend on the whole state of the process X satisfying the same generic assumptions (4), (5) and (6), with $h(t)$ replaced by $h(t, x)$ and $\Lambda := \sup_{n \geq 1} (\lambda_n/n)$ by $\Lambda := \sup_{n \geq 1, x} (\lambda_{n,x}/n)$. Similar convergence rate *independent of N* may be proved in the same way for the model with any $N < \infty$; here “usual” bounds could easily depend on this parameter. Similar bounds may be established for *mixing rates* by using the approach from [53]. For a *random* initial distribution μ_0 , similar or weaker bounds may be proved depending on moments of μ_0 .

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