# On the adiabatic limit of Hadamard states

Nicolò Drago & Christian Gérard

Abstract. We consider the adiabatic limit of Hadamard states for free quantum Klein-Gordon fields, when the background metric and the field mass are slowly varied from their initial to final values. If the Klein-Gordon field stays massive, we prove that the adiabatic limit of the initial vacuum state is the (final) vacuum state, by extending to the symplectic framework the adiabatic theorem of Avron -Seiler-Yaffe.

In cases when only the field mass is varied, using an abstract version of the mode decomposition method we can also consider the case when the initial or final mass vanishes, and the initial state is either a thermal state or a more general Hadamard state.

## 1. Introduction

In this paper we study the adiabatic limit of Hadamard states for free quantum Klein-Gordon fields. Hadamard states play nowadays a crucial role in the algebraic approach to Quantum Field Theory on curved spacetimes. They are suitable linear positive and normalized functionals on the $*$-algebra of observables [KM], which enjoy further microlocal properties [R1, R2]. They play an important role in algebraic Quantum Field Theory for several reasons, [GK, W, FV, HW], ultimately linked to the fact that the Hadamard condition is the correct criterion to single out physically relevant states. Nowadays the literature on Hadamard states is wide, ranging from existence results [FNW1, FNW2] to explicit constructing techniques [BDM, DD, DMP, FMR, GW1, GW2, WZ].

In this paper we describe another construction of Hadamard states via a deformation procedure in parameter space. This deformation procedure is obtained by considering an "intermediate" theory with a smoothly deformed parameter, which interpolates between the two values of interest (eg between two values of the mass). States for this latter theory can be thought as smooth deformations of states from one theory to the other. Actually, it is only in the final step that one really recovers a state for the theory of interest: This step consists in a limit procedure, the so-called *adiabatic limit*.

In the first part of the paper we consider massive Klein-Gordon field with an external electromagnetic potential in a globally hyperbolic spacetime. The metric, electromagnetic potential and the field mass are smoothly deformed from their initial to final values. We show in Thm. 3.4 that the adiabatic limit of the initial vacuum state is again the final vacuum state. For this we generalize the well known results of [ASY] on the adiabatic limit for a symplectic, rather than unitary, dynamics.

The previous analysis leaves out the massless case, which is typically affected by infrared divergences. The treatment of this case is the content of the second part

2010 *Mathematics Subject Classification.* 81T20, 35S05, 35S35.

*Key words and phrases.* Quantum Field Theory on curved spacetimes, Hadamard states, adiabatic limits.

The authors would like to thank the referees for their help in improving the readability of the paper.

of the paper, which specializes the model previously described to the case where only the field mass is varied. The Klein-Gordon equation is then separable and one can restrict attention to quasi-free states whose covariances are diagonal w.r.t. the spatial Laplacian. The construction of such states is known in the physics literature as the mode decomposition method. Some aspects of this analysis already appear, in a different formulation and in special cases, in [DD, DHP].

The main result proved in Prop. 5.2 is that the adiabatic limit for the mass parameter can be performed for a large class of such states, containing in particular vacuum states and thermal states. As a particular case we prove that the KMS property, which characterizes states in thermal equilibrium (see [S]), is not preserved by the adiabatic limit.

The paper is structured as follows: Sections 2-3 are devoted to recollect some well-known material and to formulate precisely the problem of the adiabatic limit for the model of interest. Section 4 deals with the adiabatic Theorem for symplectic dynamics (Thm.4.4) which generalizes the result of [ASY] to the symplectic case. Finally Section 5 deals with the massless to massive transition, analyzing in particular the Hadamard property of the adiabatic limit as well as the adiabatic limit of vacuum and KMS states.

1.1. **Notation.** - we set $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$ for $x\in\mathbb{R}^n$.

- the domain of a closed, densely defined operator $a$ will be denoted by $\mathrm{Dom}\, a$ and equipped with the graph norm, its resolvent set by $\rho(a)$.

- if $a$ is a selfadjoint operator on a Hilbert space $\mathcal{H}$, we write $a>0$ if $a\geq 0$ and $\mathrm{Ker}\, a=\{0\}$. We set $\mathcal{S}=\{u\in\mathcal{H} : u=\mathbf{1}_{[\delta,R]}(a)u,\ \delta, R>0\}$. For $s\in\mathbb{R}$ we denote by $\langle a\rangle^s\mathcal{H}$ the completion of $\mathcal{S}$ for the norm $\|u\|_{-s}=\|\langle a\rangle^{-s}u\|$. Similarly if $a>0$ we denote by $a^s\mathcal{H}$ the completion of $\mathcal{S}$ for the norm $\|u\|=\|a^{-s}u\|$.

- functions of $a$ will be denoted by $f(a)$, in particular if $\Delta\subset\mathbb{R}$ is a Borel set, $\mathbf{1}_\Delta(a)$ denotes the spectral projection on $\Delta$ for $a$.

- if $\mathbb{R}\ni t\mapsto b(t)$ is a map with values in closed densely defined operators on $\mathcal{H}$, satisfying the conditions of Kato's theorem, see [RS, Thm. X.70] or [SG] for a recent exposition, the strongly continuous two parameter group with generator $b(t)$ will be denoted by $\mathrm{Texp}(\mathrm{i}\int_s^t b(\sigma)d\sigma)$.

- the operator of multiplication by a function $f$ will be denoted by $f$, while the operators of partial differentiation will be denoted by $\overline{\partial}_i$, so that $[\overline{\partial}_i, f]=\partial_i f$.

## 2. Free quantized Klein-Gordon fields

We now briefly recall some background material on free quantized Klein-Gordon fields, referring for example to [BGP, KM] for details. We adopt the framework of *charged fields*, corresponding to complex solutions of the Klein-Gordon equation, which we find more convenient. We refer the reader to [GW1, Sect. 1] for details.

2.1. **Charged bosonic fields.** In this framework the phase space, used to construct the CCR algebra, is a pseudo-unitary space $(\mathcal{Y}, q)$, ie $\mathcal{Y}$ is a complex vector space and $q\in L_{\mathrm{h}}(\mathcal{Y},\mathcal{Y}^*)$ a non-degenerate hermitian form, instead of a real symplectic space $(\mathcal{X},\sigma)$ as usual. Let hence $\mathcal{Y}$ a complex vector space, $\mathcal{Y}^*$ its anti-dual. Sesquilinear forms on $\mathcal{Y}$ are identified with elements of $L(\mathcal{Y},\mathcal{Y}^*)$ and the action of a sesquilinear form $\beta$ is correspondingly denoted by $\overline{y}_1\cdot\beta y_2$ for $y_1, y_2\in\mathcal{Y}$. We fix $q\in L_{\mathrm{h}}(\mathcal{Y},\mathcal{Y}^*)$ a non degenerate hermitian form on $\mathcal{Y}$.

The $^*-$algebra $\mathrm{CCR}(\mathcal{Y}, q)$ is the (complex) $^*-$algebra generated by symbols $\mathbf{1}, \psi(y), \psi^*(y), y \in \mathcal{Y}$ and the relations:

$$\begin{aligned}
&\psi(y_1 + \lambda y_2) = \psi(y_1) + \overline{\lambda}\psi(y_2),\ \ y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C},\\
&\psi^*(y_1 + \lambda y_2) = \psi^*(y_1) + \lambda\psi^*(y_2),\ \ y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C},\\
&[\psi(y_1), \psi(y_2] = [\psi^*(y_1), \psi^*(y_2)] = 0,\ \ [\psi(y_1), \psi^*(y_2)] = \overline{y}_1\!\cdot\! q y_2 \mathbf{1},\ \ y_1, y_2 \in \mathcal{Y},\\
&\psi(y)^* = \psi^*(y),\ \ y \in \mathcal{Y}.
\end{aligned}$$

A state $\omega$ on $\mathrm{CCR}(\mathcal{Y}, q)$ is *(gauge invariant) quasi-free* if

$$\omega(\prod_{i=1}^{p}\psi(y_i)\prod_{i=1}^{q}\psi^*(y_j)) = \left\{\begin{array}{l} 0 \text{ if } p \neq q,\\ \sum_{\sigma\in S_p}\prod_{i=1}^{p}\omega(\psi(y_i)\psi^*(y_{\sigma(i)})) \text{ if } p = q. \end{array}\right.$$

There is no loss of generality to restrict oneself to charged fields and gauge invariant states, see eg the discussion in [GW1, Sect. 2]. It is convenient to associate to $\omega$ its *(complex) covariances* $\lambda_\pm \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$ defined by:

$$\begin{array}{l} \omega(\psi(y_1)\psi^*(y_2)) =: \overline{y}_1\!\cdot\!\lambda_+ y_2,\\ \omega(\psi^*(y_2)\psi(y_1)) =: \overline{y}_1\!\cdot\!\lambda_- y_2, \end{array}\quad y_1, y_2 \in \mathcal{Y}.$$

It is well-known that two hermitian forms $\lambda_\pm \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$ are the covariances of a quasi-free state $\omega$ iff

$$\lambda_\pm \geq 0,\ \lambda_+ - \lambda_- = q. \tag{2.1}$$

2.2. **Free quantized Klein-Gordon fields.** Let $(M, g)$ be a globally hyperbolic spacetime, $A_a(x)dx^a$ a smooth $1-$form on $M$ and $m \in C^\infty_{\mathrm{c}}(M; \mathbb{R})$ a smooth real function. We set

$$P = -(\nabla^a - \mathrm{i}A^a(x))(\nabla_a - \mathrm{i}A_a(x)) + m(x) \tag{2.2}$$

the associated Klein-Gordon operator. Let $G^\pm$ be the advanced/retarded inverses of $P$ and $G := G^+ - G^-$ the causal propagator. Denote by $\mathrm{Sol}_{\mathrm{sc}}(KG)$ the space of smooth, complex, space-compact solutions of the Klein-Gordon equation $P\phi = 0$.

We equip $\mathrm{Sol}_{\mathrm{sc}}(KG)$ with the hermitian form

$$\overline{\phi}\!\cdot\! q\phi := \mathrm{i}\int_\Sigma \left(\overline{(\partial_a - \mathrm{i}A_a)\phi}\phi - \overline{\phi}(\partial_a - \mathrm{i}A_a)\phi\right) n^a ds_\Sigma$$

where $\Sigma$ is a spacelike Cauchy hypersurface, $n^a$ is the future directed normal to $\Sigma$ and $ds_\Sigma$ the induced density on $\Sigma$. The above expression is independent on the choice of $\Sigma$ and $(\mathrm{Sol}_{\mathrm{sc}}(KG), q)$ is a pseudo-unitary space, i.e. $q$ is non degenerate.

It is well-known that the sequence

$$0 \longrightarrow C^\infty_{\mathrm{c}}(M) \xrightarrow{P} C^\infty_{\mathrm{c}}(M) \xrightarrow{G} \mathrm{Sol}_{\mathrm{sc}}(KG) \xrightarrow{P} 0$$

is exact and

$$\overline{Gu}\!\cdot\! qGu = \mathrm{i}^{-1}(u|Gu)_M =: \overline{[u]}\!\cdot\! Q[u],\ \ [u] \in \frac{C^\infty_{\mathrm{c}}(M)}{PC^\infty_{\mathrm{c}}(M)},$$

where $(u|v)_M = \int_M \overline{u}v dVol_g$. It follows that

$$(\frac{C^\infty_{\mathrm{c}}(M)}{PC^\infty_{\mathrm{c}}(M)}, Q) \xrightarrow{G} (\mathrm{Sol}_{\mathrm{sc}}(KG), q)$$

is an isomorphism of pseudo-unitary spaces. Fixing a space-like Cauchy hypersurface $\Sigma$ and setting

$$\rho : C^\infty_{\mathrm{sc}}(M) \ni \phi \mapsto \rho\phi = \begin{pmatrix} \phi\restriction_\Sigma \\ n^a(\mathrm{i}^{-1}\partial_a\phi - A_a\phi)\restriction_\Sigma \end{pmatrix} = f \in C^\infty_{\mathrm{c}}(\Sigma) \otimes \mathbb{C}^2,$$

we obtain, since the Cauchy problem

$$\begin{cases} P\phi = 0, \\ \rho u = f \end{cases}$$

for $f \in C_{\rm c}^\infty(\Sigma)\otimes \mathbb{C}^2$ that

$$({\rm Sol}_{\rm sc}(KG), q) \xrightarrow{\rho} (C_{\rm c}^\infty(\Sigma)\oplus C_{\rm c}^\infty(\Sigma), q)$$

is pseudo-unitary, where

$$\overline{f}\cdot qf = \int_\Sigma \overline{f}_1 f_0 + \overline{f}_0 f_1 ds_\Sigma, \ f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}. \tag{2.3}$$

2.3. **Quasi-free states.** One restricts attention to quasi-free states on ${\rm CCR}(\mathcal{Y}, q)$ whose covariances are given by distributions on $M\times M$, ie such that there exists $\Lambda^\pm \in \mathscr{D}'(M\times M)$ with

$$\begin{aligned} \omega(\psi([u_1])\psi^*([u_2])) &= (u_1|\Lambda^+ u_2)_M, \\ \omega(\psi^*([u_2])\psi([u_1])) &= (u_1|\Lambda^- u_2)_M, \end{aligned} \quad u_1, u_2 \in C_{\rm c}^\infty(M). \tag{2.4}$$

In the sequel the distributions $\Lambda^\pm \in \mathscr{D}'(M\times M)$ will be called the *spacetime covariances* of the state $\omega$.

In (2.4) we identify distributions on $M$ with distributional densities using the density $dVol_g$ and use hence the notation $(u|\varphi)_M$, $u\in C_{\rm c}^\infty(M)$, $\varphi \in \mathscr{D}'(M)$ for the duality bracket. We have then

$$\begin{aligned} &P(x,\partial_x)\Lambda^\pm(x,x') = P(x',\partial_{x'})\Lambda^\pm(x,x') = 0, \\ &\Lambda^+(x,x') - \Lambda^-(x,x') = {\rm i}^{-1}G(x,x'). \end{aligned}$$

Since

$$\left(\frac{C_{\rm c}^\infty(M)}{PC_{\rm c}^\infty(M)}, Q\right) \xrightarrow{\rho\circ G} (C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2, q)$$

is an isomorphism of pseudo-unitary spaces, it follows that a quasi-free state with space-time covariances $\Lambda^\pm$ is uniquely defined by its *Cauchy surface covariances* $\lambda_\Sigma^\pm$ defined by:

$$\Lambda^\pm =: (\rho E)^*\lambda_\Sigma^\pm(\rho E). \tag{2.5}$$

Using the canonical scalar product $(f|f)_\Sigma := \int_\Sigma \overline{f}_1 f_1 + \overline{f}_0 f_0 d\sigma_\Sigma$ we identify $\lambda_\Sigma^\pm$ with operators, still denoted by $\lambda_\Sigma^\pm$, belonging to $L(C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2, \mathscr{D}'(\Sigma)\otimes\mathbb{C}^2)$.

A pair $\lambda_\Sigma^\pm$ of hermitian forms on $C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$ is the pair of Cauchy surface covariances of a quasi-free state iff

$$\lambda_\Sigma^\pm \geq 0, \ \lambda_\Sigma^+ - \lambda_\Sigma^- = q, \tag{2.6}$$

where the charge $q$ is defined in (2.3).

## 3. Adiabatic limits of quasi-free states

In this section we formulate the problem that we will consider in this paper, namely the existence of adiabatic limits for quasi-free states. The formulation relies on a $1+d$ decomposition, ie on fixing some time coordinate.

We also state Thm. 3.4 about the adiabatic limits of *vacuum states*.

3.1. $1+d$ **decompositions.** We consider simple model spacetimes $M=\mathbb{R}\times\Sigma$ equipped with the Lorentzian metric

$$g=-dt^2+h_{ij}(t,x)dx^i dx^j,$$

where $\Sigma$ is a smooth manifold and $h_{ij}(t,x)dx^i dx^j$ is a smooth, time-dependent family of complete Riemannian metrics on $\Sigma$. We also fix a smooth $1-$form $A=V(t,x)dt+\mathrm{A}_i(t,x)dx^i$ and a real function $m\in C_{\rm c}^\infty(M)$. We denote by $\tilde{P}$ the associated Klein-Gordon operator as (2.2):

$$\tilde{P}=(\overline{\partial}_t-\mathrm{i}V(t))^2+r(t)(\overline{\partial}_t-\mathrm{i}V(t))+a(t,x,\overline{\partial}_x), \tag{3.1}$$

where $V(t), r(t)$ are the operators of multiplication by $V(t,x), |h_t|^{-\frac12}\partial_t|h_t|^{\frac12}(x)$ and

$$\begin{aligned}&\tilde{a}(t,x,\overline{\partial}_x)\\ =\;& -|h_t|^{-\frac12}(x)(\overline{\partial}_j-\mathrm{i}\mathrm{A}_j(t,x))|h_t|^{\frac12}(x)h_t(x)^{jk}(\overline{\partial}_k-\mathrm{i}\mathrm{A}_k(t,x))+m(t,x),\end{aligned}$$

is formally selfadjoint on $\mathcal{H}_t=L^2(\Sigma,|h_t|dx)$.

It is convenient to equip $\Sigma$ with the time-independent density $|h_0|^{\frac12}dx$ and to set

$$c^2(t,x):=|h_t|^{-\frac12}|h_0|^{\frac12}(x).$$

Using the unitary transformation

$$U:L^2(M,|h_0|^{\frac12}dxdt)\ni\phi\mapsto\psi=c\phi\in L^2(M,|h_t|^{\frac12}dxdt)$$

for $c^2=|h_t|^{-\frac12}|h_0|^{\frac12}$, we transform $\tilde{P}$ into $P=U\tilde{P}U^{-1}=c^{-1}\tilde{P}c$.

Using that $(\overline{\partial}_t-\mathrm{i}V)c=c(\overline{\partial}_t-\mathrm{i}V)+\partial_t c$ and $r=-2c^{-1}\partial_t c$, we obtain after an easy computation that:

$$\begin{aligned}&P=(\overline{\partial}_t-\mathrm{i}V)^2+a(t,x,\overline{\partial}_x),\\ &a(t,x,\overline{\partial}_x)=a(t)=c^{-1}\tilde{a}(t,x,\overline{\partial}_x)c+c^{-1}\partial_t^2c-2(c^{-1}\partial_t c)^2,\end{aligned} \tag{3.2}$$

which is formally selfadjoint on $\mathcal{H}=L^2(\Sigma,|h_0|^{\frac12}dx)$. The conserved charge for the solutions of $P\phi=0$ is:

$$\overline{\phi}q\phi:=\int_\Sigma\left((\overline{\mathrm{i}^{-1}\partial_t\phi(t)-V(t)\phi(t)})\phi(t)+\overline{\phi(t)}(\mathrm{i}^{-1}\partial_t\phi(t)-V(t)\phi(t))\right)|h_0|^{\frac12}dx.$$

The corresponding identities for causal propagators and spacetime two-point functions of a quasi-free state are:

$$G=c^{-1}\tilde{G}c,\;\Lambda^\pm=c^{-1}\tilde{\Lambda}^\pm c,$$

and in the sequel we will consider quantized Klein-Gordon fields for $P$, instead of the original operator $\tilde{P}$, since both are equivalent.

3.2. **Assumptions.** We will assume that for any interval $I\Subset\mathbb{R}$ there exist constants $C_{I,n}>0$, $n\in\mathbb{N}$ such that for $t\in I, x\in\Sigma$:

$$\begin{aligned}(Hi)&\quad C_{I,0}^{-1}h_0(x)\leq h_t(x)\leq C_{I,0}h_0(x),\\ (Hii)&\quad |\partial_t^n h_t(x)|\leq C_{I,n}h_0(x),\\ (Hiii)&\quad |\partial_t^n V(t,x)|+|\partial_t^n\mathrm{A}_i(t,x)h_0^{ij}(x)\partial_t^n\mathrm{A}_j(t,x)|+|\partial_t^n m(t,x)|\leq C_{I,n},\\ (Hiv)&\quad \partial_i\mathrm{A}_i(t,x)h_0^{ij}(x)\partial_j\mathrm{A}_j(t,x)\leq C_{I,0}.\end{aligned}$$

Let us set

$$a_0=a_0(x,\overline{\partial}_x)=-|h_0|^{-\frac12}\overline{\partial}_j|h_0|^{\frac12}h_0^{jk}\overline{\partial}_k,$$

which by Chernoff's theorem [C] (recall that $h_t(x)dx^2$ is assumed to be complete), is essentially selfadjoint on $C_{\rm c}^\infty(\Sigma)$. We set $H^1(\Sigma):=\mathrm{Dom}\,a_0^{\frac12}$ and $H^{-1}(\Sigma):=H^1(\Sigma)^*$

its anti-dual. We have continuous and dense embeddings $H^1(\Sigma) \hookrightarrow L^2(\Sigma) \hookrightarrow H^{-1}(\Sigma)$.

Similarly $a(t)$ is essentially selfadjoint on $C_c^\infty(\Sigma)$ and using $(H)$ we easily see that $H^1(\Sigma) = \mathrm{Dom}\,|a(t)|^{\frac{1}{2}}$.

We will need later stronger conditions than $(H)$. Setting $f^{(k)} = \partial_t^k f$, we require that for $k = 1, 2$:

$$(Di) \quad a^{(k)}(t)a^{-1}(t) \text{ is bounded on } \mathcal{H} \text{ locally uniformly in } t,$$

$$(Dii) \quad a(t)^{\frac{1}{2}}a^{(k)}(t)a(t)^{-3/2} \text{ is bounded on } \mathcal{H} \text{ locally uniformly in } t,$$

$$(Diii) \quad a(t)^{-\frac{1}{2}}[a^{(k)}(t), a(t)]a(t)^{-1} \text{ is bounded on } \mathcal{H} \text{ locally uniformly in } t.$$

These conditions will be used in Lemma 4.1 to estimate time derivatives of $a(t)^{\frac{1}{2}}$. They are a substitute for the lack of knowledge of $\mathrm{Dom}\,a(t)$ in our abstract setting.

**Remark 3.1.** *A convenient setup where conditions $(D)$ are satisfied is the following: we assume that $(\Sigma, h_0)$ is of bounded geometry, see* [CG, Ro] *or* [GOW] *for a self contained exposition. One can then define the spaces $\mathrm{BT}_q^p(\Sigma, h_0)$ of smooth bounded $(q, p)$ tensors. If we assume that $h \in C^\infty(\mathbb{R}; \mathrm{BT}_2^0(\Sigma, h_0))$, $h^{-1} \in C^\infty(\mathbb{R}; \mathrm{BT}_0^2(\Sigma, h_0))$, $V, m \in C^\infty(\mathbb{R}; \mathrm{BT}_0^0(\Sigma, h_0))$, $\mathrm{A} \in C^\infty(\mathbb{R}, \mathrm{BT}_1^0(\Sigma, h_0))$, then these assumptions are satisfied. We refer the interested reader to* [GOW, Sects. 2, 5] *for details.*

3.3. **Cauchy evolution.** Denoting by

$$\rho_s\phi(x) = \begin{pmatrix} \phi(s,x) \\ \mathrm{i}^{-1}\partial_t\phi(s,x) - V(s,x)\phi(s,x) \end{pmatrix}, \ s \in I,$$

the trace operator on $\Sigma_s = \{s\} \times \Sigma$, we know that the Cauchy problem

$$\left\{ \begin{array}{l} P\phi = 0, \\ \rho_s\phi = f \in C_c^\infty(\Sigma) \otimes \mathbb{C}^2 \end{array} \right. \tag{3.3}$$

is globally well posed. If $\phi = U_s f$ for $f \in C_c^\infty(\Sigma) \otimes \mathbb{C}^2$ is the solution of (3.3), we denote by $U(t,s)f = \rho_t U_s f$ the Cauchy evolution for (3.3).

If $\omega$ is a quasi-free state for $P$, the Cauchy surface covariances of $\omega$ for the Cauchy surface $\Sigma_s$ will be denoted by $\lambda_s^\pm$. Clearly we have

$$\lambda_t^\pm = U(s,t)^*\lambda_s^\pm U(s,t).$$

3.4. **Energy spaces.** Let $I \Subset \mathbb{R}$ a compact interval. Let us introduce the following positivity condition:

$$(P) \qquad a(t, x, \overline{\partial}_x) - V^2(t,x) \geq C_I\mathbf{1} \text{ on } \mathcal{H} \text{ for } t \in I.$$

In practice $(P)$ is satisfied if we choose $m(t,x) = m^2$ large enough. If $(P)$ holds we introduce the energy norm:

$$E_t(f,f) := (f_1 + V(t)f_0|f_1 + V(t)f_0) + (f_0|p(t)f_0), \tag{3.4}$$

where $p(t) = a(t) - V^2(t)$ and $(u|v)$ denotes the scalar product in $\mathcal{H} = L^2(\Sigma, |h_0|^{\frac{1}{2}}dx)$. By $(P)$ $E_t(\cdot,\cdot)$ is positive definite and using $(H)$ and $(P)$ we see that the norm $E_t(f,f)^{\frac{1}{2}}$ is equivalent to $\|f_0\|_{H^1(\Sigma)} + \|f_1\|_{L^2(\Sigma)}$, uniformly for $t \in I$.

**Definition 3.2.** *The space $H^1(\Sigma) \oplus L^2(\Sigma)$ with norm $\|f_0\|_{H^1(\Sigma)} \oplus \|f_1\|_{L^2(\Sigma)}$, resp. $E_t(f,f)^{\frac{1}{2}}$ will be denoted by $\mathcal{E}$, resp. $\mathcal{E}_t$.*

The norms $\|f\|_{\mathcal{E}}$ and $\|f\|_{\mathcal{E}_t}$ are uniformly equivalent for $t \in I$, using $(H)$, and $C_c^\infty(\Sigma) \otimes \mathbb{C}^2$ is dense in $\mathcal{E} = \mathcal{E}_t$.

We will prove later on in Sect. 4 the following proposition.

**Proposition 3.3.** *The two parameter group* $\{U(t,s)\}_{t,s\in I}$ *acting on* $C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$ *extends uniquely to a strongly continuous two parameter group* $\{U(t,s)\}_{t,s\in I}$ *such that* $U(t,s):\mathcal{E}_s\to\mathcal{E}_t$ *is unitary and* $I^2\ni(t,s)\mapsto U(t,s)$ *is strongly continuous (for the common topology of all the* $\mathcal{E}_t$*).*

*Denoting by* $H(t)$ *its infinitesimal generator we have:*

$$\begin{array}{ll} i) & Dom H(t) = \mathrm{Dom}\, a(t)\oplus H^1(\Sigma), \\ ii) & H(t) \text{ is selfadjoint on } \mathcal{E}_t \text{ and } 0\in\rho(H(t)). \end{array}$$

3.5. **Vacuum states.** If $Q(t,\overline{\partial}_t,x,\overline{\partial}_x)$ is a differential operator and $t_0\in I$ we denote

$$Q_{t_0}=Q(t_0,\overline{\partial}_t,x,\overline{\partial}_x) \tag{3.5}$$

the operator $Q$ with coefficients frozen at $t=t_0$. In particular $P_{t_0}=P(t_0,\overline{\partial}_t,x,\overline{\partial}_x)$ is the Klein-Gordon operator $P$ with coefficients frozen at $t=t_0$. The associated Cauchy evolution is $\mathrm{e}^{\mathrm{i}(t-s)H(t_0)}$. Since $P_{t_0}$ is invariant under time translations, and because of condition $(P)$, the quantized Klein-Gordon field for $P_{t_0}$ admits a *vacuum state* $\omega_{t_0}^{\rm vac}$. Its covariances $\lambda_{t_0}^{\pm,\rm vac}$ are given by:

$$\lambda_{t_0}^{\pm,\rm vac}=\pm q\circ\mathbf{1}_{\mathbb{R}^\pm}(H(t_0)), \tag{3.6}$$

where $\mathbf{1}_{\mathbb{R}^\pm}(H(t_0))$ are the spectral projections on $\mathbb{R}^\pm$ for $H(t_0)$, which are well defined by Prop. 3.3.

3.6. **Adiabatic limits.** We fix a compact interval $I\Subset\mathbb{R}$ (for definiteness $I=[-1,1]$) and consider for $T\gg 1$ the Klein-Gordon operator

$$P^T(t,\overline{\partial}_t,x,\overline{\partial}_x):=P(T^{-1}t,\overline{\partial}_t,x,\overline{\partial}_x), \tag{3.7}$$

ie $h_{ij}(T^{-1}t,x)dx^idx^j$, $\mathrm{A}_i(T^{-1}t,x)dx^i$, $V(T^{-1}t,x)$ and $m(T^{-1}t,x)$ are slowly varied from $t=-T$ to $t=T$. Recalling the notation in (3.5) we have

$$P^T_{\pm T}=P_{\pm 1}.$$

The associated Cauchy evolution $U_T(t,s)$ has generator $H(T^{-1}t)$.

If $\lambda_{-1}^\pm$ are the covariances at time $t=-1$ of a state $\omega$ for the time-independent Klein-Gordon operator $P_{-1}$, we can investigate the existence of the adiabatic limit

$$\lambda_1^{\rm ad}=:\mathrm{w-}\lim_{T\to+\infty}U_T(-T,T)^*\lambda_{-1}^\pm U_T(-T,T)\text{ on }C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2. \tag{3.8}$$

If the limits (3.8) exist, then they are the time $t=1$ covariances of a quasi-free state $\omega^{\rm ad}$ for the time-independent Klein-Gordon operator $P_1$.

We now state the main result of this paper.

**Theorem 3.4.** *Let* $\lambda_{-1}^{\pm,\rm vac}$ *be the Cauchy surface covariances of the vacuum state for the time-independent Klein-Gordon operator* $P_{-1}$*. Then the adiabatic limits*

$$\mathrm{w-}\lim_{T\to+\infty}U_T(-T,T)^*\lambda_{-1}^{\pm,\rm vac}U_T(-T,T)\ \text{exist on } C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$$

*and are the Cauchy surface covariances* $\lambda_1^{\pm,\rm vac}$ *of the vacuum state for the time-independent Klein-Gordon operator* $P_1$*.*

The proof, which follows directly from the adiabatic theorem Thm. 4.4, is given in Subsect. 4.4.

## 4. An adiabatic theorem for symplectic dynamics

In this section we prove a version of the adiabatic theorem of [ASY] for a symplectic (instead of a unitary) dynamics. We will use the setup of Subsects. 3.1, 3.2 although it is likely that the adiabatic theorem proved in Thm. 4.4 extends to a more general framework. A natural situation would be a two parameter (linear) symplectic flow generated by a time-dependent quadratic Hamiltonian which is positive definite, corresponding to our condition $(P)$. Of course this positivity condition has to be supplemented by abstract versions of $(H)$, $(D)$, implying for example that the energy norms are locally uniformly equivalent to some reference Hilbert norm. We assume hence hypotheses $(H)$, $(P)$, $(D)$. We start by proving Prop. 3.3.

### 4.1. Proof of Prop. 3.3.

On $C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$ we have:

$$\partial_t U(t,s) = {\rm i}H(t)U(t,s),\ \partial_s U(t,s) = -{\rm i}U(t,s)H(s),$$

for

$$H(t) = \begin{pmatrix} V(t) & \mathbf{1} \\ a(t) & V(t)\end{pmatrix}. \tag{4.1}$$

It is convenient to set:

$$\hat\rho_s\phi(x) = \begin{pmatrix} \phi(s,x) \\ {\rm i}^{-1}\partial_t\phi(s,x)\end{pmatrix} =: g$$

so that $\hat\rho_s\phi = S(s)\rho_s\phi$, $S(s) = \begin{pmatrix} \mathbf{1} & 0 \\ V(s) & \mathbf{1}\end{pmatrix}$. The associated evolution is

$$W(t,s) = S(t)U(t,s)S^{-1}(s), \tag{4.2}$$

with generator

$$\begin{aligned} K(t) = &\ \ S(t)H(t)S^{-1}(t) - {\rm i}\partial_t S(t)S^{-1}(t) \\ = &\ \ \begin{pmatrix} 0 & \mathbf{1} \\ p(t) - {\rm i}\partial_t V(t) & 2V(t)\end{pmatrix}, \end{aligned}$$

where we recall that $p(t) = a(t) - V^2(t)$. We set

$$F_t(g,g) = (g_1|g_1) + (g_0|p(t)g_0).$$

Again the completion of $C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$ for $F_t^{\frac12}$ equals $\mathcal{E}_t$. We obtain that if $g(t) = W(t,s)g$, $g\in C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$:

$$\begin{aligned} \partial_t F_t(g(t),g(t)) = &\ (g_1(t)|\partial_t V(t)g_0(t)) \\ + &\ \ (g_0(t)|\partial_t V(t)g_1(t)) + (g_0(t)|\partial_t p(t)g_0(t)). \end{aligned} \tag{4.3}$$

Using $(H)$ and $(P)$ we obtain that for $t\in I$ one has:

$$|(g_0(t)|\partial_t p(t)g_0(t))| \leq C_I F_t(g_t,g_t),$$

which using also $(H)$ for the other terms in the rhs of (4.3) yields

$$|\partial_t F_t(g_t,g_t)| \leq C_I F_t(g_t,g_t),\ t\in I.$$

By Gronwall's inequality this implies that for any $I\Subset\mathbb{R}$ we have:

$$\sup_{t,s\in I}\|W(t,s)\|_{B(\mathcal{E}_t)} \leq C_I.$$

Since $W(t,s)$ is strongly continuous on the dense subspace $C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$ it is strongly continuous on $\mathcal{E}_t$. By (4.2) the same is true for $U(t,s)$.

The operator $K(t)$ preserves $C_{\rm c}^\infty(\Sigma)\otimes\mathbb{C}^2$ and is bounded from $\mathcal{E}_t$ to $\mathcal{E}_t^* = L^2(\Sigma)\oplus H^{-1}(\Sigma)$. Its domain as the infinitesimal generator of $W(t,s)$ is

$${\rm Dom}\, K(t) = \{g\in\mathcal{E}_t\ :\ K(t)g\in\mathcal{E}_t\} = {\rm Dom}\, a(t)\oplus H^1(\Sigma),$$

by direct inspection, using that $\mathrm{Dom}\, a(t) = \{u \in H^1(\Sigma) \ : \ a(t)u \in L^2(\Sigma)\}$.

Note that using $(Hiv)$ we obtain that $V(t) : H^1(\Sigma) \to H^1(\Sigma)$ hence $S(t)$ is an isomorphism of both $\mathcal{E}_t$ and of $\mathrm{Dom}\, a(t) \oplus H^1(\Sigma)$. Therefore the domain of $H(t)$ as infinitesimal generator of $U(t,s)$ equals $S(t)^{-1}\, \mathrm{Dom}\, K(t) = \mathrm{Dom}\, a(t) \oplus H^1(\Sigma)$.

We now study the operator $H(t)$. Let us set

$$L(t) = S(t)H(t)S(t)^{-1} = \begin{pmatrix} 0 & \mathbf{1} \\ p(t) & 2V(t) \end{pmatrix}.$$

From $(P)$ we know that $0 \in \rho(p(t))$ hence $0 \in \rho(L(t))$ by [GGH, Prop. 5.3]. Using then [GGH, Thm. 5.4] we obtain that $L(t)$ is selfadjoint on $\mathcal{E}_t$. This implies that $H(t)$ is selfadjoint on $\mathcal{E}_t$ with $0 \in \rho(H(t))$. $\square$

4.2. **Smoothness of spectral projections.** Since by Prop. 3.3 $H(t)$ is selfadjoint on $\mathcal{E}_t$ we can define the spectral projection

$$P(t) = \mathbf{1}_{\mathbb{R}^+}(H(t)) \in B(\mathcal{E}_t).$$

Moreover since $0 \in \rho(H(t))$, for each $I \Subset \mathbb{R}$ there exist $\chi \in C^\infty(\mathbb{R})$, $\chi \equiv 1$ near $+\infty$ such that $P(t) = \chi(H(t))$ for $t \in I$. In this subsection we examine the smoothness of $P(t)$ w.r.t. $t$.

4.2.1. *Almost analytic extensions and functional calculus.* Let us set

$$S^\rho(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \partial_\lambda^n f(\lambda) \in O(\langle\lambda\rangle^{\rho-n}), n \in \mathbb{N}\}, \ \rho \in \mathbb{R}.$$

We equip $S^\rho(\mathbb{R})$ with the semi-norms $\|f\|_{\rho,n} = \sup_{\lambda\in\mathbb{R}} |\langle\lambda\rangle^{\rho-n}\partial_\lambda^n f(\lambda)|$.

For $f \in S^\rho(\mathbb{R})$ we denote by $\tilde{f} \in C^\infty(\mathbb{C})$ an *almost analytic extension* of $f$ satisfying:

$$(4.4)\qquad \begin{array}{rl} i) & \tilde{f}\restriction_{\mathbb{R}} = f, \\ ii) & \mathrm{supp}\tilde{f} \subset \{|\mathrm{Im}z| \leq C|\mathrm{Re}z|\}, \\ iii) & |\partial_{\overline{z}}\tilde{f}(z)| \in O(\langle z\rangle)^{\rho-1-k}|\mathrm{Im}z|^k, \ \forall k \in \mathbb{N}, \end{array}$$

see for example [DG1, Prop. C.2.2] for a construction of $\tilde{f}$. If $H$ is selfadjoint on a Hilbert space $\mathcal{H}$ we have the bounds:

$$(4.5)\qquad \begin{array}{rl} i) & \|(H-z)^{-1}\| \leq |\mathrm{Im}z|^{-1}, \\ ii) & \|(H+\mathrm{i})(H-z)^{-1}\| \leq c\langle z\rangle|\mathrm{Im}z|^{-1} \text{ for } |\mathrm{Im}z| \leq C|\mathrm{Re}z|. \end{array}$$

and $f \in S^\rho(\mathbb{R})$, for $\rho < 0$, then one has

$$(4.6)\qquad f(H) = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-H)^{-1}dz \wedge d\overline{z},$$

the integral being norm convergent, using (4.5) *i)*.

Let us now explain how to extend (4.6) to the case $\rho \geq 0$. Let us fix $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ with $\chi = 1$ near 0 and $\tilde{\chi} \in C_{\mathrm{c}}^\infty(\mathbb{C})$ an almost analytic extension of $\chi$. We set $\chi_R(x) = \chi(R^{-1}x)$ and $\tilde{\chi}_R(z) = \tilde{\chi}(R^{-1}z)$, which is an almost analytic extension of $\chi_R$.

For $f \in S^\rho(\mathbb{R})$ and $\rho \geq 0$, we set $f_R(\lambda) = f(\lambda)\chi_R(\lambda)$. We have:

$$(4.7)\qquad \begin{array}{l} \{f_R\}_{R\geq 1} \text{ is bounded in } S^\rho(\mathbb{R}), \\ R^{\rho'-\rho}\{(f_R - f)\}_{R\geq 1} \text{ is bounded in } S^{\rho'}(\mathbb{R}), \ \forall \rho' > \rho. \end{array}$$

Let us set $\tilde{f}_R = \tilde{f}\tilde{\chi}_R$, which is an almost analytic extension of $f_R$. The following properties of $\tilde{f}_R$ follow from (4.7) and the construction of $\tilde{f}, \tilde{\chi}$ in [DG1]:

$$
(4.8)\quad \begin{array}{rl}
i) & \tilde{f}_R\lceil_{\mathbb{R}} = f_R,\\
ii) & \mathrm{supp}\tilde{f}_R \subset \{|\mathrm{Im}z| \leq C|\mathrm{Re}z|\} \cap \{|z| \leq CR\},\\
iii) & |\partial_{\overline{z}}\tilde{f}_R(z)| \in O(\langle z\rangle)^{\rho-1-k}|\mathrm{Im}z|^k,\ \forall k \in \mathbb{N},\ \text{uniformly for } R \geq 1,\\
iv) & |\partial_{\overline{z}}(\tilde{f}(z) - \tilde{f}_R(z))| \in O(\langle z\rangle)^{\rho'-1-k}R^{\rho-\rho'}|\mathrm{Im}z|^k,\ \forall k \in \mathbb{N},\ \rho' > \rho.
\end{array}
$$

Since $f(H) = \mathrm{s-}\lim_{R\to+\infty} f_R(H)$ in $B(\mathrm{Dom}\,|H|^\rho, \mathcal{H})$ we have:

$$
(4.9)\quad f(H) = \mathrm{s-}\lim_{R\to+\infty} \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}}(\tilde{f}_R)(z)(z-H)^{-1}dz \wedge d\overline{z},\ \text{in } B(\mathrm{Dom}\,|H|^\rho, \mathcal{H}).
$$

**Lemma 4.1.** *Assume $(H)$, $(P)$ and $(D)$. Let $\epsilon(t) = a(t)^{\frac{1}{2}}$. Then for $k = 1, 2$:*

$$
\begin{array}{rl}
i) & \epsilon(t)(\epsilon^{-1})^{(k)}(t) \text{ is bounded on } \mathcal{H},\ \text{locally uniformly in } t,\\
ii) & \epsilon^{-1}(t)\epsilon^{(k)}(t) \text{ is bounded on } \mathcal{H},\ \text{locally uniformly in } t.
\end{array}
$$

**Proof.** Note that by duality and interpolation $(Di)$ implies that $\epsilon^{-1}(t)a'(t)\epsilon^{-1}(t)$ is bounded on $\mathcal{H}$ locally uniformly in $t$. Let us first prove $i)$. We have $\epsilon^{-1}(t) = \pi^{-1}\int_0^{+\infty}\lambda^{-\frac{1}{2}}(a(t)+\lambda)^{-1}d\lambda$, hence:

$$
\begin{array}{rl}
\epsilon(t)(\epsilon^{-1})'(t) = & -\pi^{-1}\int_0^{+\infty}\lambda^{-\frac{1}{2}}a(t)^{\frac{1}{2}}(a(t)+\lambda)^{-1}a'(t)(a(t)+\lambda)^{-1}d\lambda\\
= & -\pi^{-1}\int_0^{+\infty}\lambda^{-\frac{1}{2}}a(t)^{\frac{1}{2}}a'(t)(a(t)+\lambda)^{-2}d\lambda\\
& +\pi^{-1}\int_0^{+\infty}\lambda^{-\frac{1}{2}}a(t)^{\frac{1}{2}}(a(t)+\lambda)^{-1}[a(t), a'(t)](a(t)+\lambda)^{-2}d\lambda.
\end{array}
$$

The first term equals $a(t)^{\frac{1}{2}}a'(t)a(t)^{-3/2}$ which is bounded by $(Dii)$. We write the second term as

$$
\pi^{-1}\int_0^{+\infty}\lambda^{-\frac{1}{2}}a(t)(a(t)+\lambda)^{-1}a(t)^{-\frac{1}{2}}[a(t), a'(t)]a(t)^{-1}(a(t)+\lambda)^{-2}a(t)d\lambda.
$$

The integral is norm convergent using $(Diii)$ since $a(t)(a(t)+\lambda)^{-1} \in O(1)$. Therefore $\epsilon(t)(\epsilon^{-1})'(t)$ is bounded on $L^2(\Sigma)$ which proves $i)$. To prove $ii)$ we write $\epsilon(t) = \epsilon^{-1}(t)a(t)$ hence

$$
\epsilon'(t)\epsilon^{-1}(t) = \epsilon^{-1}(t)a'(t)\epsilon^{-1}(t) + (\epsilon^{-1})'(t)a(t)\epsilon^{-1}(t)
$$

which is bounded on $L^2(\Sigma)$ by $i)$ and $(Di)$. Using the same argument we prove the estimates for second derivatives. $\Box$

**Proposition 4.2.** *Assume $(H)$, $(P)$ and $(D)$. Let $P(t) := \mathbf{1}_{\mathbb{R}^+}(H(t))$. Then $\mathbb{R} \ni t \mapsto P(t) \in B(\mathcal{E}_t)$ is strongly $C^2$ and $P^{(k)}(t)$ is bounded on $\mathcal{E}_t$ locally uniformly in $t$ for $k = 1, 2$.*

**Proof.** In the sequel we write $A(t) \in O(1)$ if $\|A(t)\|_{B(\mathcal{E}_t)} \in O(1)$ for $t \in I$. We set $H_0(t) = \begin{pmatrix} 0 & \mathbf{1} \\ a(t) & 0 \end{pmatrix}$, $W(t) := H(t) - H_0(t) \in O(1)$, using $(Hiii)$). Since $0 \in \rho(H_{(0)}(t))$ we have $\mathbf{1}_{\mathbb{R}^+}(H_{(0)}(t)) = f(H_{(0)}(t))$, for $t \in I$, for some $f \in S^0(\mathbb{R})$, $f = 1$ near $+\infty$, $f = 0$ near $-\infty$. We have by (4.9):

$$
f(H(t)) = f(H_0(t)) + R(t),
$$

for

$$
R(t) = \mathrm{s-}\lim_{R\to+\infty}\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{f}_R(z)(z-H(t))^{-1}W(t)(z-H_0(t))^{-1}dz \wedge d\overline{z},\ \text{in } \mathcal{E}_t.
$$

Using (4.8) *iv*) we then obtain

$$R(t) = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-H(t))^{-1}W(t)(z-H_0(t))^{-1}dz\wedge d\overline{z},$$

the integral being norm convergent on $B(\mathcal{E}_t)$.

We have by an easy computation $f(H_0(t)) = \mathbf{1}_{\mathbb{R}^+}(H_0(t)) = \frac{1}{2}\begin{pmatrix}\mathbf{1} & \epsilon(t)^{-1}\\ \epsilon(t) & \mathbf{1}\end{pmatrix}$, which using Lemma 4.1 implies that $f(H_0(t))'$ is uniformly bounded on $\mathcal{E}_t$ for $t\in I$. Next we compute:

$$\begin{aligned}
&R'(t) = \tfrac{\mathrm{i}}{2\pi}\textstyle\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-H(t))^{-1}W'(t)(z-H_0(t))^{-1}dz\wedge d\overline{z}\\
&+\tfrac{\mathrm{i}}{2\pi}\textstyle\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-H(t))^{-1}H'(t)(z-H(t))^{-1}W(t)(z-H_0(t))^{-1}dz\wedge d\overline{z}\\
&+\tfrac{\mathrm{i}}{2\pi}\textstyle\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-H(t))^{-1}W(t)(z-H_0(t))^{-1}H_0'(t)(z-H_0(t))^{-1}dz\wedge d\overline{z}.
\end{aligned}$$

From $(H)$ we see easily that $W'(t)\in O(1)$ and from $(Di)$ that $H_0'(t)(H_0(t)+\mathrm{i})^{-1}$, $H'(t)(H(t)+\mathrm{i})^{-1}\in O(1)$. Using also (4.5) we obtain that the integrands in the rhs above are bounded by either $|\mathrm{Im}z|^{-2}$ or by $\langle z\rangle|\mathrm{Im}z|^{-3}$, uniformly for $t\in I$. Since $\partial_{\overline{z}}\tilde{f}\in O(\langle z\rangle^{-1-k})|\mathrm{Im}z|^{k}$ we obtain that $R'(t)\in O(1)$. We use the same argument to estimate $P''(t)$. □

□

The following lemma is a version of [ASY, Lemma 2.5], where the case when $P(t)$ is the spectral projection on a bounded interval was considered.

**Lemma 4.3.** *Let $I\ni t\mapsto H(t)$ be a map with values in selfadjoint operators on a Hilbert space $\mathcal{H}$ and $I\ni t\mapsto X(t)\in B(\mathcal{H})$ be strongly $C^1$. Assume that $\mathscr{D}=\mathrm{Dom}\,H(t)$ is independent on $t$ and that:*

$$H(t):\mathscr{D}\to\mathcal{H} \text{ is strongly differentiable, } [-\alpha,\alpha]\subset\rho(H(t)) \text{ for } t\in I.$$

*Let $P(t)=\mathbf{1}_{\mathbb{R}^+}(H(t))$ and let us fix $f\in S^0(\mathbb{R})$ such that $f(\lambda)=\mathbf{1}_{\mathbb{R}^+}(\lambda)$ $\mathbb{R}\backslash[-\alpha,\alpha]$ and $\tilde{f}$ an almost analytic extension of $f$ satisfying* (4.4)*. Then the integral*

$$\tilde{X}(t) := -\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-H(t))^{-1}X(t)(z-H(t))^{-1}dz\wedge d\overline{z}, \tag{4.10}$$

*is norm convergent in $B(\mathcal{H})$.*

*The map $I\ni t\mapsto\tilde{X}(t)\in B(\mathcal{H})$ is strongly $C^1$ and*

$$[P(t),X(t)] = [\tilde{X}(t),H(t)], \text{ as quadratic forms on } \mathscr{D}.$$

**Proof.** We first fix $t\in I$ and omit the parameter $t$ for simplicity of notation. Using (4.8) we obtain

$$P = \mathrm{s-}\lim_{R\to+\infty}\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}_R(z)(z-H)^{-1}dz\wedge d\overline{z}, \text{ in } B(\mathcal{H}),$$

hence:

$$[P,X] = \mathrm{s-}\lim_{R\to+\infty}\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \frac{\partial\tilde{f}_R}{\partial\overline{z}}(z)[(z-H)^{-1},X]dz\wedge d\overline{z}, \text{ in } B(\mathcal{H}).$$

We recall that $\mathscr{D}=\mathrm{Dom}\,H(t)$ is independent on $t$ and denote by $\mathscr{D}'$ its topological dual. Since $[(z-H)^{-1},X]=[H,(z-H)^{-1}X(z-H)^{-1}]$ on $B(\mathscr{D},\mathscr{D}')$, we obtain

$$[P,X] = \mathrm{s-}\lim_{R\to+\infty}\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}_R(z)[H,(z-H)^{-1}X(z-H)^{-1}]dz\wedge d\overline{z} \text{ in } B(\mathscr{D},\mathscr{D}'). \tag{4.11}$$

Using (4.8) *iv)* we can compute the strong limit in the rhs of (4.11) and obtain $[P, X] = [H, \tilde{X}]$ for

$$\tilde{X} = -\frac{1}{2\mathrm{i}\pi}\int \partial_{\overline{z}}\tilde{f}(z)(z-H)^{-1}X(z-H)^{-1}dz\wedge d\overline{z}.$$

It remains to check that $t \mapsto \tilde{X}(t)$ is strongly $C^1$. This follows from differentiating in $t$ the rhs of (4.10), using $\frac{d}{dt}(z-H(t))^{-1} = (z-H(t))^{-1}H'(t)(z-H(t))^{-1}$. The details are left to the reader. $\Box$

4.3. **Adiabatic evolution.** We recall from Subsect. 3.6 that $U_T(t,s)$ is the Cauchy evolution associated to the Klein-Gordon operator $P(T^{-1}t, \overline{\partial}_t, x, \overline{\partial}_x)$. Repeating the computations in the proof of Prop. 3.3 we obtain for $f(t) = \tilde{U}_T(t,s)f$ that

$$\partial_t E_t(f(t), f(t)) \leq CT^{-1}E_t(f(t), f(t)), t\in[-T,T],$$

hence by Gronwall's inequality

$$\|U_T(t,s)f)\|_{\mathcal{E}_t} \leq C\|f\|_{\mathcal{E}_s},\ t,s\in[-T,T],$$

where we recall that the norm $\|\cdot\|_{\mathcal{E}_t}$ is defined in Def. 3.2.

$$\mathcal{E}_t = H^1(\Sigma)\oplus L^2(\Sigma),$$

equipped with the norm $E_t(f,f)^{\frac{1}{2}}$ introduced in (3.4). This implies that

$$\|U_T(t,s)\|_{B(\mathcal{E})} \leq C,\ t,s\in[-T,T].$$

We set

$$\hat{U}_T(t,s) = U_T(Tt, Ts),\ t,s\in[-1,1].$$

whose generator is $TH(t)$. We obtain that:

$$\|\hat{U}_T(t,s)\|_{B(\mathcal{E})} \leq C,\ t,s\in[-1,1],\ T\geq 1. \tag{4.12}$$

We set

$$H_T^{\mathrm{ad}}(t) := H(t) + \mathrm{i}T^{-1}[P(t), P'(t)],$$

where $P(t) = \mathbf{1}_{\mathbb{R}^+}(H(t))$. Since $[-1,1]\ni t\mapsto P(t), P'(t)\in B(\mathcal{E}_t)$ are strongly continuous by Prop. 4.2, the evolution group $\hat{U}_T^{\mathrm{ad}}(t,s)$ with generator $TH_T^{\mathrm{ad}}(t)$ can be constructed by setting:

$$\hat{U}_T^{\mathrm{ad}}(t,s) =: \hat{U}_T(t,0)Z_T(t,s)\hat{U}_T(0,s),$$

where

$$\begin{aligned}&\partial_t Z_T(t,s) = K_T(t)Z_T(t,s),\ Z_T(s,s) = \mathbf{1},\\ &K_T(t) = \hat{U}_T(0,t)[P'(t), P(t)]\hat{U}_T(t,0)\in B(\mathcal{E}_t).\end{aligned}$$

By a standard argument (see eg [ASY, Lemma 2.3]) one obtains that:

$$P(t)\hat{U}_T^{\mathrm{ad}}(t,s) = \hat{U}_T^{\mathrm{ad}}(t,s)P(s),\ t,s\in[-1,1]. \tag{4.13}$$

In fact it suffices to differentiate both terms in $t$, after acting on a vector in $\mathrm{Dom}\, H(t)$. Moreover from (4.12) we obtain

$$\|\hat{U}_T^{\mathrm{ad}}(t,s)\|_{B(\mathcal{E})} \leq C,\ t,s\in[-1,1],\ T\geq 1. \tag{4.14}$$

4.4. **Adiabatic theorem.** We now state a version of the adiabatic theorem which is sufficient for our purposes.

**Theorem 4.4.** *Assume (H), (P) and (D). Then there exists $C > 0$ such that:*

$$\|\hat{U}_T(t,s) - \hat{U}_T^{\rm ad}(t,s)\|_{B(\mathcal{E})} \leq CT^{-1},\ \ t,s \in [-1,1].$$

The theorem can be proved by repeating the arguments in the proof of [ASY, Thm. 2.4]. For the reader's convenience we will sketch its main steps.

We often remove the time variable for simplicity of notation. We set $\overline{P} = 1 - P$, and denote by $\tilde{X}$ the operator constructed in Lemma 4.3 for some strongly $C^1$ map $t \mapsto X(t)$. From $P^2 = P$ we obtain $PP' + P'P = P'$ hence

$$[P, P'] = 2PP' - P' = 2P'\overline{P} - P'. \tag{4.15}$$

It follows that:

$$\begin{aligned} &\overline{P}XP = \overline{P}[X,P]P = \overline{P}[H,\tilde{X}]P = \overline{P}[H^{\rm ad},\tilde{X}]P) - {\rm i}T^{-1}\overline{P}[[P,P'],\tilde{X}]P \\ = \ &\overline{P}[H^{\rm ad},\tilde{X}]P + {\rm i}T^{-1}\overline{P}[P',\tilde{X}]P. \end{aligned} \tag{4.16}$$

**Lemma 4.5.** *Assume that $[-1,1] \ni t \mapsto X(t), Y(t) \in B(\mathcal{E})$ are strongly $C^1$. Then for $t, s \in [-1,1]$:*

$$\begin{aligned} &\textstyle\int_s^t \overline{P}(s)\hat{U}_T^{\rm ad}(s,t_1)X(t_1)\hat{U}_T^{\rm ad}(t_1,s)P(s)Y(t_1)dt_1 \\ = \ &-{\rm i}T^{-1}\left[\overline{P}(s)\hat{U}_T^{\rm ad}(s,t_1)\tilde{X}(t_1)\hat{U}_T^{\rm ad}(t_1,s)P(s)Y(t_1)\right]_s^t \\ &+{\rm i}T^{-1}\textstyle\int_s^t \overline{P}(s)\hat{U}_T^{\rm ad}(s,t_1)\tilde{X}'(t_1)\hat{U}_T^{\rm ad}(t_1,s)P(s)Y(t_1)dt_1 \\ &+{\rm i}T^{-1}\textstyle\int_s^t \overline{P}(s)\hat{U}_T^{\rm ad}(s,t_1)\tilde{X}(t_1)\hat{U}_T^{\rm ad}(t_1,s)P(s)Y'(t_1)dt_1 \\ &+{\rm i}T^{-1}\textstyle\int_s^t \overline{P}(s)\hat{U}_T^{\rm ad}(s,t_1)[P'(t_1),\tilde{X}(t_1)]\hat{U}_T^{\rm ad}(t_1,s)P(s)Y(t_1)dt_1. \end{aligned}$$

*In particular we have:*

$$\|\int_s^t \overline{P}(s)\hat{U}_T^{\rm ad}(s,t_1)X(t_1)\hat{U}_T^{\rm ad}(t_1,s)P(s)Y(t_1)dt_1\|_{B(\mathcal{E})} \leq CT^{-1}, \tag{4.17}$$

*where the constant $C$ depends only on* $\sup_{t\in[-1,1]}\|X^{(k)}(t)\| + \|Y^{(k)}(t)\| + \|P^{(k)}(t)\|$, $k = 0, 1$.

**Proof.** From (4.16) we obtain:

$$\begin{aligned} &\overline{P}(s)\hat{U}_T^{\rm ad}(s,t)X(t)\hat{U}_T^{\rm ad}(t,s)P(s) = \hat{U}_T^{\rm ad}(s,t)\overline{P}(t)X(t)P(t)\hat{U}_T^{\rm ad}(t,s) \\ &= \hat{U}_T^{\rm ad}(s,t)\overline{P}(t)[H^{\rm ad}(t),\tilde{X}(t)]P(t)\hat{U}_T^{\rm ad}(t,s) + {\rm i}T^{-1}\hat{U}_T^{\rm ad}(s,t)\overline{P}(t)[P'(t),\tilde{X}(t)]P(t)\hat{U}_T^{\rm ad}(t,s) \\ &= {\rm i}T^{-1}\overline{P}(s)\left(-\partial_t(\hat{U}_T^{\rm ad}(s,t)\tilde{X}(t)\hat{U}_T^{\rm ad}(t,s)) + \hat{U}_T^{\rm ad}(s,t)(\tilde{X}'(t) + [P'(t),\tilde{X}(t)])\hat{U}_T^{\rm ad}(t,s)\right)P(s). \end{aligned}$$

The lemma follows by integration by parts. □

**Proof of Thm. 4.4.**

We set for fixed $s \in [-1,1]$ $\Omega_T(t,s) := \hat{U}_T^{\rm ad}(s,t)\hat{U}_T(t,s)$, so that

$$\begin{aligned} \Omega_T(t,s) &= \mathbf{1} + \int_s^t R_T(t_1)\Omega_T(t_1,s)dt_1, \\ R_T(t) &= \hat{U}_T^{\rm ad}(s,t)[P(t),P'(t)]\hat{U}_T^{\rm ad}(t,s). \end{aligned} \tag{4.18}$$

From (4.13), (4.15) we have:

$$\begin{aligned} &\overline{P}(s)R_T(t) = \hat{U}_T^{\rm ad}(s,t)\overline{P}(t)[P(t),P'(t)]\hat{U}_T^{\rm ad}(t,s) \\ &= \hat{U}_T^{\rm ad}(s,t)\overline{P}(t)[P(t),P'(t)]P(t)\hat{U}_T^{\rm ad}(t,s) \\ &= -\overline{P}(s)\hat{U}_T^{\rm ad}(s,t)P'(t)\hat{U}_T^{\rm ad}(t,s)P(s). \end{aligned}$$

Applying Lemma 4.5 to $X(t) = P'(t)$, $Y(t) = \Omega_T(t,s)$ we obtain from (4.17)

$$\| \int_s^t \overline{P}(s) R_T(t_1)\Omega_T(t_1,s)dt_1 \| \leq CT^{-1}.$$

Exchanging the role of $P$ and $\overline{P}$ we also obtain

$$\| \int_s^t P(s) R_T(t_1)\Omega_T(t_1,s)dt_1 \| \leq CT^{-1},$$

hence

$$\|\Omega_T(t,s) - \mathbf{1}\|_{B(\mathcal{E})} \leq CT^{-1}, \ t,s \in [-1,1].$$

This implies that

$$\|\hat{U}_T(t,s) - \hat{U}_T^{\rm ad}(t,s)\|_{B(\mathcal{E})} \leq CT^{-1}, \ t,s \in [-1,1]$$

and completes the proof. $\Box$

**Proof of Thm. 3.4.** Since $U_T(t,s)$ is symplectic, we have $U_T(-T,T)^* q U_T(-T,T) = q$, hence using (3.6):

$$\begin{aligned}
& U_T(-T,T)^*\lambda_{-1}^{\pm,\rm vac}U_T(-T,T) = qU_T(T,-T)\mathbf{1}_{\mathbb{R}^\pm}(H(-1))U_T(-T,T) \\
= \ & q\hat{U}_T(1,-1)\mathbf{1}_{\mathbb{R}^\pm}(H(-1))\hat{U}_T(-1,1) = q\hat{U}_T^{\rm ad}(1,-1)\mathbf{1}_{\mathbb{R}^\pm}(H(-1))\hat{U}_T^{\rm ad}(-1,1) + O(T^{-1}) \\
= \ & q\mathbf{1}_{\mathbb{R}^\pm}(H(1)) + O(T^{-1}) = \lambda_1^{\pm,\rm vac} + O(T^{-1}),
\end{aligned}$$

using Thm. 4.4 and (4.13) (remember that $P(t) = \mathbf{1}_{\mathbb{R}^+}(H(t))$). $\Box$

## 5. Further results in the separable case

We consider now a simpler version of the setup in Subsects. 3.1, 3.2 where $A = 0$, $h_t = h$ is independent on $t$ and $m(t,x) = m^2(t)$. We assume

$(HC)$ $\Sigma$ is non compact, $\sigma(-\Delta_h) = [0, +\infty[$ is purely absolutely continuous.

The Klein-Gordon operator takes the form:

$$\tilde{P} = P = \overline{\partial}_t^2 - \Delta_h + \chi(t)m^2 =: \overline{\partial}_t^2 + a(t),$$

where $a(t)$ commutes with $-\Delta_h$. It follows that the Klein-Gordon equation $P\phi = 0$ can be reduced to a family of $1-d$ Schrödinger equations:

$$\partial_t^2\phi + m^2\chi(t)\phi + \epsilon^2\phi = 0, \qquad (5.1)$$

where $\epsilon = (-\Delta_h)^{\frac{1}{2}}$, if one introduces a spectral decomposition of $\epsilon$. This is known in the physics literature as the *mode decomposition* method.

We show in Thm. 5.7 that the conclusion of Thm. 3.4 still holds when the initial or final mass $m(\mp 1)$ vanishes, ie when the stability condition $(P)$ is violated.

We next consider the adiabatic limit of an initial *thermal state* at temperature $\beta^{-1}$, and show in Thm. 5.3 that its adiabatic limit is *not* a thermal state (unless the initial and final masses are the same).

Finally we consider the adiabatic limits of (infrared regular) Hadamard states and show in Thm. 5.8 that their adiabatic limits are again Hadamard states.

5.1. **Energy estimates.** We will set:

$$\epsilon := (-\Delta_h)^{\frac{1}{2}},\ \epsilon_t = \epsilon(t) := a(t)^{\frac{1}{2}},\ m_t = m(t).$$

We set as in Subsect. 4.3:

$$\hat{U}_T(t,s) = \mathrm{Texp}(\mathrm{i}T\int_s^t H(\sigma)d\sigma),\ t,s \in [-1,1],$$

where now:

$$H(t) = \begin{pmatrix} 0 & \mathbf{1} \\ a(t) & 0 \end{pmatrix}.$$

We will consider the following three cases:

$$\begin{aligned} A:&\quad m(t) > 0,\ t \in [-1,1], \\ B:&\quad m_{-1} = 0,\ m(t) \text{ strictly increasing}, \\ C:&\quad m_1 = 0,\ m(t) \text{ strictly decreasing}. \end{aligned}$$

Conditions $(H)$ and $(D)$ are always satisfied but condition $(P)$ is not satisfied in cases $B$ and $C$.

5.1.1. *Modified energy spaces.* We set (recall that $\mathcal{H} = L^2(\Sigma, dVol_h)$):

$$\begin{aligned} \mathscr{A} &:= \langle\epsilon\rangle^{-\frac{1}{2}}\mathcal{H} \oplus \langle\epsilon\rangle^{\frac{1}{2}}\mathcal{H}, \\ \mathscr{B}_t &:= \epsilon_t^{-1}\epsilon^{\frac{1}{2}}\mathcal{H} \oplus \epsilon^{\frac{1}{2}}\mathcal{H}, \\ \mathscr{C}_t &:= \langle\epsilon\rangle^{-\frac{1}{2}}\mathcal{H} \oplus \epsilon_t\langle\epsilon\rangle^{-\frac{1}{2}}\mathcal{H}. \end{aligned} \tag{5.2}$$

which are well defined since $\mathrm{Ker}\,\epsilon = \{0\}$. We recall from Subsect. 1.1 that if

$$\mathcal{S} = \{f \in \mathcal{H}\otimes\mathbb{C}^2 : f = \mathbf{1}_{[\delta,R]}(\epsilon)f,\ R,\delta > 0\}, \tag{5.3}$$

$\mathscr{A}$, $\mathscr{B}_t$ $\mathscr{C}_t$ are the completion of $\mathcal{S}$ for the norms:

$$\begin{aligned} \|f\|^2_{\mathscr{A}} &:= \|\langle\epsilon\rangle^{\frac{1}{2}}f_0\|^2_{\mathcal{H}} \oplus \|\langle\epsilon\rangle^{-\frac{1}{2}}f_1\|^2_{\mathcal{H}}, \\ \|f\|^2_{\mathscr{B}_t} &:= \|\epsilon^{-\frac{1}{2}}\epsilon_t f_0\|^2_{\mathcal{H}} + \|\epsilon^{-\frac{1}{2}}f_1\|^2_{\mathcal{H}}, \\ \|f\|^2_{\mathscr{C}_t} &:= \|\langle\epsilon\rangle^{\frac{1}{2}}f_0\|^2_{\mathcal{H}} + \|\langle\epsilon\rangle^{\frac{1}{2}}\epsilon_t^{-1}f_1\|^2_{\mathcal{H}}. \end{aligned} \tag{5.4}$$

5.1.2. *Energy estimates.*

**Lemma 5.1.** *The following estimates hold for $t \leq s$, $t,s \in [-1,1]$:*

$$\begin{aligned} \textit{case } A:&\ \|\hat{U}_T(t,s)\|_{B(\mathscr{A})} \leq C, \\ \textit{case } B:&\ \|\hat{U}_T(t,s)\|_{B(\mathscr{B}_s,\mathscr{B}_t)} \leq C, \\ \textit{case } C:&\ \|\hat{U}_T(t,s)\|_{B(\mathscr{C}_s,\mathscr{C}_t)} \leq C. \end{aligned} \tag{5.5}$$

**Proof.** The estimate for case $A$ follows from (4.12), using that $\hat{U}_T(t,s)$ commutes with $\langle\epsilon\rangle^{\frac{1}{2}}$. In case $B$, if $f(t) = \hat{U}_T(t,s)f$ since $a'(t) = 2m(t)m'(t) \geq 0$, we obtain that $\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathscr{B}_t} \geq 0$ which implies the desired estimate. The same argument can be used for case $C$. $\Box$

5.2. **Adiabatic limit of sesquilinear forms.** We identify sesquilinear forms on $\mathscr{A}$, $\mathscr{B}_t$ or $\mathscr{C}_t$ with linear operators. In fact the canonical scalar product on $\mathcal{H}\otimes\mathbb{C}^2$ allows to identify $\mathscr{A}^*$ with $\langle\epsilon\rangle^{\frac12}\mathcal{H}\oplus\langle\epsilon\rangle^{-\frac12}\mathcal{H}$, $\mathscr{B}_t^*$ with $\epsilon^{\frac12}\epsilon_t\mathcal{H}\oplus\epsilon^{-\frac12}\mathcal{H}$, $\mathscr{C}_t$ with $\langle\epsilon\rangle^{\frac12}\mathcal{H}\oplus\langle\epsilon\rangle^{\frac12}\epsilon_t^{-1}\mathcal{H}$. In this way we will identify a sesquilinear form $\lambda$ with a linear operator, still denoted by $\lambda$, by

$$\overline{f}\cdot\lambda f =: (f|\lambda f)_{\mathcal{H}\otimes\mathbb{C}^2}.$$

We denote such an operator by $\lambda(\epsilon)$ if all its entries are functions of the selfadjoint operator $\epsilon$.

If $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ we set $A^{\mathrm{diag}}=\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}$. We set also

$$\mathscr{T}(t):=2^{-\frac12}\begin{pmatrix}\epsilon_t^{-\frac12} & -\epsilon_t^{-\frac12}\\ \epsilon_t^{\frac12} & \epsilon_t^{\frac12}\end{pmatrix},\ \mathscr{T}^{-1}(t)=2^{-\frac12}\begin{pmatrix}\epsilon_t^{\frac12} & \epsilon_t^{-\frac12}\\ -\epsilon_t^{\frac12} & \epsilon_t^{-\frac12}\end{pmatrix}. \tag{5.6}$$

**Proposition 5.2.** *Let $\lambda_{-1}=\lambda_{-1}(\epsilon)$ be a bounded sesquilinear form on $\mathscr{A}$, $\mathscr{B}_{-1}$, $\mathscr{C}_{-1}$ in cases (A), (B), (C) respectively. Then*

$$\lambda_1^{\mathrm{ad}} := \mathrm{w}-\lim_{T\to+\infty}\hat U_T(-1,1)^*\lambda_{-1}\hat U_T(-1,1)$$

*exists on $\mathscr{A}$, $\mathscr{B}_1$, $\mathscr{C}_1$ and*

$$\lambda_1^{\mathrm{ad}}=\mathscr{T}^{-1}(1)^*\left(\mathscr{T}(-1)^*\lambda_{-1}\mathscr{T}(-1)\right)^{\mathrm{diag}}\mathscr{T}^{-1}(1). \tag{5.7}$$

**Proof.** We first derive an asymptotic expansion in powers of $T^{-1}$ for $\hat U_T(t,s)=\mathrm{Texp}(\mathrm{i}T\int_s^t H(\sigma)d\sigma)$ valid for $t,s\in[-1,1]$. Setting $h=T^{-1}$ this essentially amounts to the construction of WKB solutions of a Schröedinger equation.

We will find this expansion by following the construction of a parametrix for the Cauchy problem for Klein-Gordon equations done in [GW1, GOW], taking advantage of the fact that the equation

$$(T^{-1}\overline{\partial}_t)^2\phi+a(t)\phi=0 \tag{5.8}$$

is separable. We first look for solutions of (5.8) of the form $\phi=\mathrm{Texp}(\mathrm{i}T\int_s^t b_T(\sigma)d\sigma)u$ and obtain that $\phi$ solves (5.8) iff $b_T(t)$ solves the following Riccati equation:

$$\mathrm{i}T^{-1}\partial_t b_T(t)-b_T^2(t)+a(t)=0. \tag{5.9}$$

We can solve (5.9) modulo errors of size $O(T^{-2})$ by

$$b_T(t)=\epsilon(t)+\frac{\mathrm{i}}{2}T^{-1}\partial_t\ln\epsilon(t). \tag{5.10}$$

We have then

$$\mathrm{i}\partial_t b_T(t)-b_T^2(t)+a(t)=T^{-2}\Big(\frac14(\partial_t\ln\epsilon)^2-\frac12\partial_t^2\ln\epsilon\Big)(t). \tag{5.11}$$

We set $b_T^+(t)=b_T(t)$, $b_T^-(t)=-b_T^*(t)$,

$$\begin{aligned}\mathscr{T}_T(t)&:=\begin{pmatrix}1 & -1\\ b_T^+ & -b_T^-\end{pmatrix}(t)(b_T^+-b_T^-)^{-\frac12}(t),\\ \mathscr{T}_T^{-1}(t)&=\begin{pmatrix}-b_T^- & 1\\ -b_T^+ & 1\end{pmatrix}(t)(b_T^+-b_T^-)^{-\frac12}(t),\end{aligned} \tag{5.12}$$

and

$$\hat U_T(t,s)=:\mathscr{T}_T(t)V_T(t,s)\mathscr{T}_T^{-1}(s).$$

Mimicking the computations in [GOW, Subsect. 6.4], we easily obtain that

$$V_T(t,s)=\mathrm{Texp}(\mathrm{i}T\int_s^t\hat H_T(\sigma)d\sigma),$$

for $\hat{H}_T(t) = H^{\rm diag}(t) + T^{-2}R(t)$, and:

$$
\begin{aligned}
&H^{\rm diag}(t) = \begin{pmatrix} \epsilon(t) & 0 \\ 0 & -\epsilon(t)\end{pmatrix},\\
&R_2(t) = (2\epsilon)^{-1}(t)(\tfrac{1}{4}(\partial_t \ln\epsilon)^2 - \tfrac{1}{2}\partial_t^2\ln\epsilon)(t)\begin{pmatrix} -1 & 1 \\ -1 & 1\end{pmatrix}.
\end{aligned}
\tag{5.13}
$$

Let us set

$$
V_T^{\rm diag}(t,s) := {\rm Texp}({\rm i}T\int_s^t H^{\rm diag}(\sigma)d\sigma).
$$

In case $A$, $\epsilon_t$ is bounded from below by a strictly positive constant, uniformly for $t\in[-1,1]$ and we immediately deduce from (5.13) that

$$
\|\hat{U}_T(t,s) - \mathscr{T}_T(t)V_T^{\rm diag}(t,s)\mathscr{T}_T^{-1}(s)\|_{B(\mathscr{A})} \in O(T^{-1})
$$

uniformly for $t\leq s$, $t,s\in[-1,1]$. We cannot use this argument in cases $B, C$ since $0\in\sigma(\epsilon_t)$, either for $t=-1$ or $t=1$. Instead we use a density argument, that we will explain for case $B$, case $C$ being similar.

From Lemma 5.1 we see that the family of sesquilinear forms

$$
\hat{U}_T(-1,1)^*\lambda_{-1}\hat{U}_T(-1,1)
$$

is bounded on $\mathscr{B}_1$ uniformly for $T\geq 1$. Therefore it suffices to prove (5.7) on the dense subspace $\mathcal{S}$ defined in (5.3).

We have to compute the limit of $(\hat{U}_T(-1,1)f|\lambda_{-1}\hat{U}_T(-1,1)f)_{\mathcal{H}\otimes\mathbb{C}^2}$ for $f\in\mathcal{S}$. Since $\hat{U}_T(-1,1)$ and $\lambda_{-1}$ commute with $\epsilon$ we see that if $f = \mathbf{1}_{[\delta,R]}(\epsilon)f$ we can replace $\epsilon$ by some function $F(\epsilon)$ such that $\frac{1}{2}\delta\leq F\leq 2R$, $F(\lambda)=\lambda$ on $[\delta,R]$. Equivalently we can assume that $\epsilon$ is boundedly invertible on $\mathcal{H}$.

In this way we deduce from Lemma 5.1 and (5.13) that

$$
\lim_{T\to+\infty}\hat{U}_T(-1,1)f - \mathscr{T}_T(-1)V_T^{\rm diag}(-1,1)\mathscr{T}_T^{-1}(1)f = 0,\ \forall f\in\mathcal{S}.
$$

Therefore we have as sesquilinear forms on $\mathcal{S}$:

$$
\begin{aligned}
&\hat{U}_T(-1,1)^*\lambda_{-1}\hat{U}_T(-1,1)\\
=\ &\mathscr{T}_T^{-1}(1)^*V_T^{\rm diag}(-1,1)^*\hat{\lambda}_{-1,T}V_T^{\rm diag}(-1,1)\mathscr{T}_T^{-1}(1) + o(T^0),
\end{aligned}
$$

for $\hat{\lambda}_{-1,T} = \mathscr{T}_T(-1)^*\lambda_{-1}\mathscr{T}_T(-1)$. We have

$$
V_T^{\rm diag}(t,s) = \begin{pmatrix} u_T^+(t,s) & 0 \\ 0 & u_T^-(t,s)\end{pmatrix}
$$

for

$$
u_T^{\pm}(t,s) = {\rm e}^{\pm{\rm i}T\int_s^t\epsilon(\sigma)d\sigma}.
$$

Since $u_T^{\pm}(-1,1)$ is unitary on $\mathcal{H}$ we can replace $\hat{\lambda}_{-1,T}$ by

$$
\hat{\lambda}_{-1} := \mathscr{T}(-1)^*\lambda_{-1}\mathscr{T}(-1),
\tag{5.14}
$$

where $\mathscr{T}(t)$ is defined in (5.6). The error terms will again be $o(T^0)$, by (5.10). We write then $\hat{\lambda}_{-1}$ as

$$
\hat{\lambda}_{-1} = \begin{pmatrix} \hat{\lambda}_{-1}^{++} & \hat{\lambda}_{-1}^{+-} \\ \hat{\lambda}_{-1}^{-+} & \hat{\lambda}_{-1}^{--}\end{pmatrix}.
$$

Using that $\hat{\lambda}_{-1}^{\alpha\beta}$ for $\alpha,\beta\in\{+,-\}$ are functions of $\epsilon$, we obtain that:

$$
\begin{aligned}
&V_T^{\rm diag}(-1,1)^*\hat{\lambda}_{-1}V_T^{\rm diag}(-1,1)\\
=\ &\begin{pmatrix} u_T^+(-1,1)^*\hat{\lambda}_{-1}^{++}u_T^+(-1,1) & u_T^+(-1,1)^*\hat{\lambda}_{-1}^{+-}u_T^-(-1,1) \\ u_T^-(-1,1)^*\hat{\lambda}_{-1}^{-+}u_T^+(-1,1) & u_T^-(-1,1)^*\hat{\lambda}_{-1}^{--}u_T^-(-1,1)\end{pmatrix}
\end{aligned}
$$

Now $u_T^+(t,s) = u_T^-(t,s)^*$ and $\mathrm{w}-\lim_{T\to+\infty} u_T^\pm(t,s) = 0$ in $\mathcal{H}$, since the spectrum of $-\Delta_h$ is purely absolutely continuous. This implies that

$$\mathrm{w}-\lim_{T\to+\infty} V_T^{\mathrm{diag}}(-1,1)^*\hat{\lambda}_{-1}V_T^{\mathrm{diag}}(-1,1) = \begin{pmatrix}\hat{\lambda}_{-1}^{++} & 0\\ 0 & \hat{\lambda}_{-1}^{--}\end{pmatrix} = \hat{\lambda}_{-1}^{\mathrm{diag}}$$

in $\mathcal{S}$. This completes the proof of the proposition in case $B$, the other cases being similar. □

5.3. **Adiabatic limit of vacuum, thermal states and Hadamard states.** In the sequel instead of the pair $\lambda^\pm$ of Cauchy surface covariances of some quasi-free state, we will consider only $\lambda^+$, (since $\lambda^- = \lambda^+ - q$) and denote it simply by $\lambda$. The necessary and sufficient condition (2.6) becomes

$$\lambda \geq 0,\ \ \lambda - q \geq 0. \tag{5.15}$$

Let $\omega_{-1}$ be a quasi-free state for the Klein-Gordon operator at time $t=-1$, ie $P_{-1} = \overline{\partial}_t^2 - \Delta_h + m_{-1}^2$ and let $\lambda_{-1}$ its covariance at time $t=-1$. In order to be able to apply Prop. 5.2 to study the adiabatic limit $\lambda_1^{\mathrm{ad}}$ of $\lambda_{-1}$ we need that the following properties are satisfied:

(1) $\lambda_{-1}$ is bounded on $\mathscr{A}$, resp. $\mathscr{B}_{-1}$, $\mathscr{C}_{-1}$;
(2) $C_\mathrm{c}^\infty(\Sigma)\otimes\mathbb{C}^2 \subset \mathscr{A}$, resp. $\mathscr{B}_{-1}$, $\mathscr{C}_{-1}$ continuously;
(3) $C_\mathrm{c}^\infty(\Sigma)\otimes\mathbb{C}^2 \subset \mathscr{A}$, resp. $\mathscr{B}_{1}$, $\mathscr{C}_{1}$ continuously.

In fact (1) is needed to obtain the existence of the adiabatic limit $\lambda_1^{\mathrm{ad}}$ on $\mathscr{A}$, resp. $\mathscr{B}_1$, $\mathscr{C}_1$, while (2) and (3) imply that the initial covariance $\lambda_{-1}$ and final covariance $\lambda_1^{\mathrm{ad}}$ are well defined on $C_\mathrm{c}^\infty(\Sigma)\otimes\mathbb{C}^2$.

In particular since (5.15) is automatically satisfied by $\lambda_1^{\mathrm{ad}}$, $\lambda_1^{\mathrm{ad}}$ is the covariance at time $t=1$ of a quasi-free state $\omega_1^{\mathrm{ad}}$ for the Klein-Gordon operator at time $t=1$, ie $P_1 = \overline{\partial}_t^2 - \Delta_h + m_1^2$.

5.3.1. *Adiabatic limit of thermal states (case A).* We assume we are in case $A$ and take as initial state the $\beta-$*KMS state* at time $t=-1$, given by the covariance:

$$\lambda_{-1}^\beta = \frac{1}{2}\begin{pmatrix}\epsilon_{-1}\coth(\beta\epsilon_{-1}/2) & \mathbf{1}\\ \mathbf{1} & \epsilon_{-1}^{-1}\coth(\beta\epsilon_{-1}/2)\end{pmatrix}.$$

**Theorem 5.3.** *The adiabatic limit*

$$\lambda_1^{\beta,\mathrm{ad}} = \mathrm{w}-\lim_{T\to+\infty}\hat{U}_T(-1,1)^*\lambda_{-1}^\beta\hat{U}_T(-1,1)$$

*exists on* $C_\mathrm{c}^\infty(\Sigma)\otimes\mathbb{C}^2$*. The adiabatic limit state* $\omega_1^{\beta,\mathrm{ad}}$ *is* not *the* $\beta-$*KMS state at time* $t=1$*, unless* $m_1 = m_{-1}$*.*

**Proof.** Properties (1), (2), (3) are immediate for the space $\mathscr{A}$, using that the mass of the field is strictly positive. A routine computation shows that the limit covariance $\lambda_1^{\beta,\mathrm{ad}}$ in Prop. 5.2 equals:

$$\lambda_1^{\beta,\mathrm{ad}} = \frac{1}{2}\begin{pmatrix}\epsilon_{1}\coth(\beta\epsilon_{-1}/2) & \mathbf{1}\\ \mathbf{1} & \epsilon_{1}^{-1}\coth(\beta\epsilon_{-1}/2)\end{pmatrix}. \tag{5.16}$$

This is not the covariance of the $\beta-$KMS state at time $t=1$, unless $m_1 = m_{-1}$. □

**Remark 5.4.** *The instability of KMS states under adiabatic limits can be related to the failure of the return to equilibrium property analyzed in* [DFP]*. In this paper the authors consider a couple of KMS states* $\omega^\beta$*,* $\omega_V^\beta$ *with respect to different dynamics* $\tau$*,*$\tau^V$*. Here,* $\tau^V$ *is the one-parameter group of* $*$*-automorphism obtained by perturbing the dynamics* $\tau$ *with a self-adjoint element* $V$*. The state* $\omega^\beta$ *is said to satisfy the return to equilibrium property if* $w\text{-}\lim_{t\to\infty}\omega^\beta\circ\tau_t^V = \omega_V^\beta$*. In* [DFP] *it has been shown that, for quantum fields, such a property is linked to the support*

*properties of $V$. Actually, if the spatial support of $V$ is compact, then $\omega^\beta$ satisfies the return to equilibrium property, while if $V$ has non-compact spatial support this is not the case.*
*In our case the adiabatic limit* $\mathrm{w}-\lim_{T\to+\infty}\hat{U}_T(-1,1)^*\lambda^\beta_{-1}\hat{U}_T(-1,1)$ *can be related with* $\lim_{t\to\infty}\omega^\beta\circ\tau_t^V$*, by identifying the perturbation $V$ with the quadratic perturbation* $\int m^2\chi(t)\phi^2(x)dVol_g$*, which is not of compact spatial support.*

5.3.2. *The infrared problem.* To verify properties (2) (3), in particular the inclusions $C^\infty_{\rm c}(\Sigma)\otimes\mathbb{C}^2\subset\mathscr{B}_{-1},\mathscr{C}_1$, one is faced with a version of the *infrared problem*, ie the fact that $0\in\sigma(\epsilon)$. In the lemma below we give a sufficient condition for (2), (3) which is easy to verify in applications.

**Lemma 5.5.** *Assume that:*

$(IR)$ *there exists a continuous function* $c:\Sigma\to\mathbb{R}$, $c(x)>0$ *such that* $-\Delta_h\geq c^{-2}(x)$.

*Then (2), (3) are satisfied.*

**Remark 5.6.** *If* $\Sigma=\mathbb{R}^d$ *and the metric $h$ satisfies*

$$h_{ij}(x)\geq C\delta_{ij},\ \partial^\alpha_x h_{ij}(x)\text{ bounded for all }\alpha\in\mathbb{N}^d,\tag{5.17}$$

*then $(IR)$ holds for* $c(x)=C\langle x\rangle$, *see* [GGH, Prop. A2].

**Proof.** We immediately see that if

$$C^\infty_{\rm c}(\Sigma)\subset\mathrm{Dom}\,\epsilon^{-\frac12}\cap\mathrm{Dom}\langle\epsilon\rangle^{\frac12}\epsilon^{-1}\cap\mathrm{Dom}\langle\epsilon\rangle\epsilon^{-\frac12}\tag{5.18}$$

then properties (2) and (3) are satisfied. From functional calculus (5.18) holds if $C^\infty_{\rm c}(\Sigma)\subset\mathrm{Dom}\,\epsilon^{-1}$. Setting $A=c^{-2}(x)$, $B=-\Delta_h=\epsilon^2$ we have $0<A\leq B$, which by definition means that $\mathrm{Ker}\,A=\{0\}$, $\mathrm{Dom}\,B^{\frac12}\subset\mathrm{Dom}\,A^{\frac12}$ and $(u|Au)\leq(u|Bu)$ for $u\in\mathrm{Dom}\,B^{\frac12}$. By [K, Thm. V.2.21] this implies that $0<(B+\delta)^{-1}\leq(A+\delta)^{-1}$ for any $\delta>0$. Letting $\delta\to0^+$ we obtain $0<B^{-1}\leq A^{-1}$ ie $\mathrm{Dom}\,c\subset\mathrm{Dom}\,\epsilon^{-1}$, which completes the proof since $C^\infty_{\rm c}(\Sigma)\subset\mathrm{Dom}\,c$ □

5.3.3. *Adiabatic limit of vacuum state (cases B, C).* We assume that we are in case $B$ or $C$ and take as initial state the *vacuum state* at time $t=-1$ given by the covariance

$$\lambda^{\rm vac}_{-1}=\frac12\begin{pmatrix}\epsilon_{-1} & \mathbf{1}\\ \mathbf{1} & \epsilon^{-1}_{-1}\end{pmatrix}.$$

**Theorem 5.7.** *Assume that $(IR)$ holds. Then the adiabatic limit*

$$\lambda^{\rm vac,ad}_1=\mathrm{w}-\lim_{T\to+\infty}\hat{U}_T(-1,1)^*\lambda^{\rm vac}_{-1}\hat{U}_T(-1,1)$$

*exists on* $C^\infty_{\rm c}(\Sigma)\otimes\mathbb{C}^2$. *The adiabatic limit state* $\omega^{\rm vac,ad}_1$ *is the vacuum state at time* $t=1$.

**Proof.** Property (1) holds by direct computation and (2), (3) hold by Lemma 5.5. We apply then Prop. 5.2. The same computation as Thm. 5.3, which amounts to set $\beta=+\infty$ in (5.16), shows that $\lambda^{\rm vac,ad}_1$ is the covariance of the vacuum state at time $t=1$. □

5.3.4. *Adiabatic limit for a class of Hadamard states (cases A, B, C).* We now take as initial state a *Hadamard state* at time $t=-1$, whose covariance $\lambda_{-1}$ is a function of $\epsilon$, as in Prop. 5.2. This corresponds exactly to a Hadamard state obtained by mode decomposition arguments.

Let us first discuss the form of the covariance $\lambda_{-1}$.

Recall that we have set $\mathscr{T}(-1)^*\lambda_{-1}\mathscr{T}(-1) =: \hat{\lambda}_{-1}$. Using that

$$\mathscr{T}(-1)^* q \mathscr{T}(-1) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -1 \end{pmatrix} =: \hat{q},$$

the positivity condition (2.1) becomes

$$\hat{\lambda}_{-1} \geq 0, \hat{\lambda}_{-1} \geq \hat{q}.$$

This is satisfied if

$$\hat{\lambda}_{-1} = \begin{pmatrix} \mathbf{1} + b^*b & b^*dc \\ c^*db & c^*c \end{pmatrix},$$

for $b, c, d \in L(\mathcal{H})$ and $\|d\|_{B(\mathcal{H})} \leq 1$, see eg [GW1, Prop. 7.4]. The operators $b, c, d$ should be functions of $\epsilon$, ie $b = b(\epsilon), c = c(\epsilon), d = d(\epsilon)$ for Borel measurable functions $b, c, d : \mathbb{R}^+ \to \mathbb{R}$, the requirement $\|d(\epsilon)\| \leq 1$ being insured if $|d(s)| \leq 1$ for $s \in \mathbb{R}^+$.

Finally $\lambda_{-1}$ should be a Hadamard state, which is ensured if $\lambda_{-1} - \lambda_{-1}^{\mathrm{vac}}$ is infinitely smoothing. Using the ellipticity of $-\Delta_h$, this is the case if

$$b(s), c(s) \in O(\langle s\rangle^{-\infty}).$$

We now discuss the conditions (1), (2), (3) in the beginning of Subsect. 5.3. We saw in Lemma 5.5 that (2), (3) are satisfied if condition $(IR)$ holds, so it remains to discuss condition (1), ie the fact that $\lambda_{-1}$ is bounded on $\mathscr{A}$, $\mathscr{B}_{-1}$ or $\mathscr{C}_{-1}$. Equivalently if $\hat{\mathscr{A}}, \hat{\mathscr{B}}_{-1}, \hat{\mathscr{C}}_{-1}$ are the images of $\mathscr{A}, \mathscr{B}_{-1}, \mathscr{C}_{-1}$ under $\mathscr{T}(-1)^{-1}$, $\hat{\lambda}_{-1}$ should be bounded on $\hat{\mathscr{A}}, \hat{\mathscr{B}}_{-1}, \hat{\mathscr{C}}_{-1}$, in cases $(A), (B), (C)$.

An easy computation yields that:

$$\hat{\mathscr{A}} = \hat{\mathscr{B}}_{-1} = \hat{\mathscr{C}}_{-1} = \mathscr{H} \oplus \mathscr{H},$$

hence condition (1) is satisfied if $b, c, d$ are bounded functions. Summarizing we impose the following condition on the initial covariance:

(5.19)

$$\hat{\lambda}_{-1} = \begin{pmatrix} \mathbf{1} + b^*b(\epsilon) & b^*dc(\epsilon) \\ c^*db(\epsilon) & c^*c(\epsilon) \end{pmatrix}, \text{for } b, c, d : \mathbb{R}^+ \to \mathbb{R}, b(s), c(s) \in O(\langle s\rangle^{-\infty}), |d(s)| \leq 1.$$

**Theorem 5.8.** *Let $\omega_{-1}$ be a Hadamard state at time $t=-1$, whose covariance $\lambda_{-1}$ is such that $\hat{\lambda}_{-1}$ satisfies* (5.19)*. In cases (B), (C) we assume moreover condition (IR). Then the adiabatic limit*

$$\lambda_1^{\mathrm{ad}} = \mathrm{w-}\lim_{T\to+\infty} \hat{U}_T(-1,1)^* \lambda_{-1}^{\mathrm{Had}} \hat{U}_T(-1,1)$$

*exists on $C_{\mathrm{c}}^\infty(\Sigma) \otimes \mathbb{C}^2$. The adiabatic limit state $\omega_1^{\mathrm{ad}}$ is a Hadamard state at time $t=1$.*

**Proof.** the existence of $\lambda_1^{\mathrm{ad}}$ follows from Prop. 5.2. We obtain that

$$(\mathscr{T}(-1)^*\lambda_{-1}\mathscr{T}(-1))^{\mathrm{diag}} = \hat{\lambda}_{-1}^{\mathrm{diag}} = \begin{pmatrix} 1 + |b|^2(\epsilon) & 0 \\ 0 & |c|^2(\epsilon) \end{pmatrix}.$$

It follows that $\lambda_1^{\mathrm{ad}} = \lambda_1^{\mathrm{vac}} + r$, where

$$r = \begin{pmatrix} \epsilon_1(|b|^2(\epsilon) + |c|^2(\epsilon) & |b|^2(\epsilon) - |c|^2(\epsilon) \\ |b|^2(\epsilon) - |c|^2(\epsilon) & \epsilon_1^{-1}(|b|^2(\epsilon) + |c|^2(\epsilon)) \end{pmatrix}.$$

Using that by $(IR)$ $C_c^\infty(\Sigma) \subset \mathrm{Dom}\,\epsilon_1^{-1}$ and the fact that $b(s), c(s) \in O(\langle s\rangle^{-\infty})$ we obtain that $r$ is smoothing, hence $\omega_1^{\rm ad}$ is Hadamard. $\Box$

## References


[ASY] Avron, J., Seiler, R., Yaffe, L.: *Adiabatic theorem and applications to the quantum Hall effect*, Comm. Math. Phys. **110**, 33-49 (1987).

[BDM] Benini, M., Dappiaggi C., Murro S.: *Radiative observables for linearized gravity on asymptotically at spacetimes and their boundary induced states*, Journal of Math. Phys. **55**, 082301 (2014).

[BFDY] Brunetti R. , Dappiaggi C., Fredenhagen K., Yngvason Y.: , *Advances in Algebraic Quantum Field Theory*, Mathematical Physics Studies Springer International Publishing (2015).

[BFV] Brunetti R., Fredenhagen K., Verch R.: *The Generally covariant locality principle: A New paradigm for local quantum field theory*, Comm. Math. Phys. **237**, 31-68 (2003).

[BGP] Bär, C., Ginoux, N., Pfäffle, F.: *Wave equation on Lorentzian manifolds and quantization*, ESI Lectures in Mathematics and Physics, EMS Publishing House, 2007.

[CG] Cheeger, J., Gromov, M.: *Bounds on the von Neumann dimension of $L^2$-cohomology and the Gauss-Bonnet theorem for open manifolds*, J. Diff. Geom. **21**, 1-34 (1985).

[C] Chernoff, P.: Essential self-adjointness of powers of generators of hyperbolic equations, J. Funct. Anal. **12**, 401-414 (1973).

[DD] Dappiaggi, C., Drago N.: *Constructing Hadamard states via an extended Möller operator*, arXiv:1506.09122 (2015) to appear in Lett. Math. Phys. **106** (2016).

[DG1] Derezinski J., Gérard, C.: *Scattering Theory of Classical and Quantum N-particle Systems*, Texts and Monographs in Physics, Springer Verlag Berlin Heidelberg (1997).

[DG2] Dereziński, J., Gérard: C. *Mathematics of Quantization and Quantum Fields*, Cambridge Monographs n Mathematical Physics Cambridge University Press, Cambridge (2013).

[DFP] Drago N., Faldino F., Pinamonti N.: *On the stability of KMS states in perturbative algebraic quantum field theories*, arXiv:1609.01124.

[DHP] Drago N., Hack,T.-P., Pinamonti N.: *The generalized principle of perturbative agreement and the thermal mass*, http://arxiv.org/abs/1502.02705.

[DMP] Dappiaggi, C., Moretti V., Pinamonti N.: , *Rigorous steps towards holography in asymptotically at spacetimes*, Reviews in Mathematical Physics **18**, 349-416 (2006).

[FMR] Finster F., Murro S., Röken C.: *The Fermionic Projector in a Time-Dependent External Potential: Mass Oscillation Property and Hadamard States* preprint ArXiv math-ph 1501.05522, (2015).

[F] Fulling, S. A., *Aspects of Quantum Field Theory in Curved Space-time*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge (1989).

[FNW1] Fulling, S. A., Narcowich F. J., Wald R. M.: *Singularity structure of the two-point function in quantum field theory in curved spacetime, II*, Annals of Physics **136**, 243-272 (1981).

[FNW2] Fulling, S. A., Narcowich F. J., Wald R. M.:*Singularity structure of the two-point function in quantum field theory in curved spacetime*, Comm. Math. Phys. **3**, 257-264 (1978).

[FV] Fewster, C. J., Verch R.: *The necessity of the Hadamard condition*, Class. Quant. Grav. **30**, 235027 (2013).

[GK] Gonnella G., Kay B.S.: *Can Locally Hadamard Quantum States Have Nonlocal Singularities?*, Class. Quant. Grav. **6** 1445 (1989).

[GW1] Gérard, C., Wrochna, M.: *Construction of Hadamard states by pseudodifferential calculus*, Comm. Math. Phys. **325**, 713-755 (2013).

[GW2] Gérard, C., Wrochna, M.: *Construction of Hadamard states by characteristic Cauchy problem*, Anal. PDE **9**, 111-149 (2016).

[GOW] Gérard, C., Oulghazi, O., Wrochna, M.: *Hadamard states for the Klein-Gordon equation on Lorentzian manifolds of bounded geometry,* preprint ArXiv math-ph 1602.00930, (2016).

[GGH] Georgescu V., Gérard C. , Häfner D.: *Resolvent and propagation estimates for Klein-Gordon equations with non-positive energy*, Journal of Spectral Theory **5**, 113-192 (2015).

[H] Haag R.: *Local quantum physics: Fields, particles, algebras*, Texts and Monographs in Physics, Springer-Verlag Berlin Heidelberg (2012).

[HK] Haag, R., Kastler, D.: *An algebraic approach to quantum field theory*, J. Math. Phys. **5**, 848-861 (1964).

[HW] Hollands S., Wald R.M: *Existence of local covariant time ordered products of quantum fields in curved space-time*, Comm. Math. Phys. **231**, 309-345 (2002).

[K] Kato, T.: *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer (1995).

[KM] Khavkine, I., Moretti V.: *Algebraic QFT in Curved Spacetime and quasi-free Hadamard states: an introduction.*, arXiv:1412.5945 (2014).

[R1] Radzikowski, M. J.: *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Comm. Math. Phys. **179**, 529-553 (1996).

[R2] Radzikowski, M. J.: *A Local to global singularity theorem for quantum field theory on curved space-time*, Comm. Math. Phys. **180**, 1-22 (1996).

[RS] Reed M., Simon, B.: *Methods of Modern Mathematical Physics II: Fourier analysis, self-adjointness*, Academic Press, 1975.

[Ro] Roe, J., *An index theorem on open manifolds I*, J. Diff. Geom. **27**, 87-113 (1988).

[S] Sanders, K.: *Thermal equilibrium states of a linear scalar quantum field in stationary spacetimes*, Internat. J. Modern Phys. A **28**, 1330010 (2013).

[SG] Schmid, J., Griesemer M.: *Kato's theorem on the integration of non-autonomous linear evolution equations*, Math. Phys. Anal. Geom. **17**, 265-271 (2014).

[W] Wald, R. M.: *Quantum field theory in curved spacetime and black hole thermodynamics*, Chicago Lectures in Physics, University of Chicago Press, Chicago (1994).

[WZ] Wrochna M., Zahn J.: , *Classical phase space and Hadamard states in the BRST formalism for gauge field theories on curved spacetime*, arXiv:1407.8079 [math-ph].



Dipartimento di Matematica, Università di Genova - Via Dodecaneso 35, I-16146 Genova, Italy

*E-mail address*: drago@dima.unige.it

Université Paris-Sud XI, Département de Mathématiques, 91405 Orsay Cedex, France

*E-mail address*: christian.gerard@math.u-psud.fr