Generalized Uncertainty Principle, Modified Dispersion Relations and the Early Universe Thermodynamics

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Abstract

In this paper, we study the effects of Generalized Uncertainty Principle(GUP) and Modified Dispersion Relations(MDRs) on the thermodynamics of ultra-relativistic particles in early universe. We show that limitations imposed by GUP and particle horizon on the measurement processes, lead to certain modifications of early universe thermodynamics.

PACS: 04.60.-m, 05.70.-a

Key Words: Quantum Gravity, Generalized Uncertainty Principle, Modified Dispersion Relations, Thermodynamics of Early Universe

1 Introduction

Generalized Uncertainty Principle is a common feature of all promising candidates of quantum gravity. String theory, loop quantum gravity and noncommutative geometry (with deeper insight to the nature of spacetime at Planck scale), all indicate the modification of standard Heisenberg principle [1-10]. Recently it has been indicated that within quantum gravity scenarios, a modification of dispersion relation (relation between energy and momentum of a given particle) is unavoidable[11-13]. There are some conceptual relations between GUP and MDRs. These possible relations have been studied recently[14,15].

These quantum gravity effects, in spite of being small, are important since they can modify experimental results. There are several efforts to provide experimental evidence of these small effects. For example, Amelino-Camelia *et al*, by investigation of potential sensitivity of Gamma-Ray Burster observations to wave dispersion in vacuo, have outlined aspects of an observational programme that could address possible detection of these quantum gravity effects [16]. Amelino-Camelia and Piran have argued that Planck-scale deformation of Lorentz symmetry can be a solution to the Ultra High Energy Cosmic Rays(UHECR) with energies above the GZK threshold and the TeV- γ paradoxes[17]. Gambini and Pullin have studied light propagation in the picture of semi-classical spacetime that emerges in canonical quantum gravity in the loop representation [18]. They have argued that in such a picture, where space-time exhibits a polymer-like structure at microscales, it is natural to expect departures from the perfect non-dispersiveness of ordinary vacuum. They have evaluated these departures by computing the modifications to Maxwell's equations due to quantum gravity, and showing that under certain circumstances, non-vanishing corrections appear that depend on the helicity of propagating waves. These effects could lead to observable cosmological predictions of the discrete nature of quantum spacetime. Then, they have addressed to observations of non-dispersiveness in the spectra of gamma-ray bursts at various energies to constrain the type of semi-classical state that describes the universe. Jacobson et al have shown that threshold effects and Planck scale Lorentz violation are combined constraints from high energy astrophysics [19]. These literatures provide possible experimental schemes for detection of small quantum gravity effects. However, there are two extreme domains: black hole structure and early stages of the universe evolution where these quantum gravity effects are dominant. Corrections to black hole thermodynamics due to quantum gravitational effects of minimal length and GUP have been studied extensively (see [20] and references therein). On the other hand, part of the thermodynamical implications of GUP and MDR have been studied by Amelino-Camelia et al[21] and Nozari et al[22]. Thermodynamics of early universe within standard Heisenberg principle has been studied by Rahvar et al[23]. Since quantum gravitational effects are very important in early stages of the universe evolution, it is natural to investigate early universe thermodynamics within GUP and MDRs frameworks. Here we are going to formulate thermodynamics of ultra-relativistic particles in early universe within GUP and MDRs frameworks. In the first step, using GUP as our primary input, we calculate thermodynamical properties of ultra-relativistic particles in early universe. In formulation of the early universe thermodynamics within GUP framework, due to limitations

imposed on the measurement processes, two main points should be considered: first due to casual structure of spacetime, maximum distance for causal relation is particle horizon radius and secondly, there is a minimum momentum imposed by GUP which restricts the minimum value of energy. In the next step, for a general gaseous system composed of ultra-relativistic particles, we find density of states using MDRs with Bose-Einstein or Fermi-Dirac statistics and then thermodynamics of the system will be followed. In each step we discuss ordinary limits of our equations and we compare consequences of two approaches.

2 Preliminaries

Emergence of the generalized uncertainty principle can be motivated and finds support in the direct analysis of any quantum gravity scenario. This means that GUP itself is a model independent concept. Generally, GUP can be written as[24]

$$\delta x \delta p \ge \frac{\hbar}{2} \Big(1 + \kappa (\delta x)^2 + \eta (\delta p)^2 + \gamma \Big), \tag{1}$$

where κ , η and γ are positive and independent of δx and δp (but may in general depend on the expectation values of x and p). This GUP leads to a nonzero minimal uncertainty in both position and momentum for positive κ and η [24]. If we set $\kappa = 0$ we find

$$\delta x \delta p \ge \frac{\hbar}{2} \left(1 + \eta (\delta p)^2 + \gamma \right). \tag{2}$$

Since we are going to deal with absolutely smallest uncertainties, we set $\gamma = 0$ from now on. So we find

$$\delta x \delta p \ge \frac{\hbar}{2} (1 + \eta (\delta p)^2). \tag{3}$$

This relation leads to a nonzero minimal observable length of the order of Planck length, $(\delta x)_{min} = \hbar \sqrt{\eta}$. Any position measurement in quantum gravity has at least $(\delta x)_{min}$ as its lower limit of position uncertainty. This relation has an immediate consequence for the rest of statistical mechanics: it modifies the *fundamental volume* ω_0 of accessible phase space for representative points. In ordinary statistical mechanics, it is impossible to define the position of a representative point in the phase space of the given system more accurately than the situation which is given by $(\delta q \, \delta p)_{min} \geq \hbar$. In another words, around any point (q, p) of the (two dimensional) phase space, there exists an area of the order \hbar which the position of the representative point cannot be pin-pointed. In ordinary statistical mechanics we have the following definition of fundamental volume

$$\omega_0 = (\delta q \, \delta p)^{3N}.\tag{4}$$

Since in quantum gravity era $\delta p \sim p$, we can interpret equation (3) as a generalization of \hbar ,

$$\hbar_{eff} = \hbar \left(1 + \eta p^2 \right).$$

Therefore, we find the following generalization of the fundamental volume

$$(\omega_0)_{eff} = [\hbar (1 + \eta p^2)]^{3N} \equiv (\hbar_{eff})^{3N}.$$
 (5)

Since the total number of microstates is given by $\Omega = \frac{\omega}{(\omega_0)_{eff}}$ (here ω is the volume of the accessible phase space), we see that GUP leads to a reduction of accessible microstates and therefore a reduction of entropy. In other words, when we approaches Planck scale regime with high energy and momentum particles, the volume of the fundamental cell increases in such away that eventually the number of microstates tends to unity and therefore entropy vanishes. This is a novel prediction of quantum gravity. Recently we have calculated microcanonical entropy of an ideal gaseous system and we have observed an unusual thermodynamics of systems in very short distances or equivalently very high energy regime[22].

Another consequence of GUP in the form of relation (3), has been formulated by Kempf $et \ al[24]$. They have shown that within the momentum representation, the generalization of the scalar products reads

$$\langle \psi | \phi \rangle = \int \frac{dp}{1 + \eta p^2} \psi^*(p) \phi(p), \tag{6}$$

where ϕ and ψ are momentum space state functions. For ultra-relativistic particles with E = pc, we should consider the following generalization

$$dE \longrightarrow \frac{dE}{1+\eta E^2},$$
 (7)

where we have set c = 1.

On the other hand, if we set $\eta = 0$ in (1), we find

$$\delta x \delta p \ge \frac{\hbar}{2} \Big(1 + \kappa (\delta x)^2 + \gamma \Big), \tag{8}$$

where for positive κ leads to nonzero minimal uncertainty in momentum. This statement leads to a space-dependent generalization of \hbar . This type of generalization has nothing to do with dynamics and there is no explicit physical interpretation of it at least up to now. From another perspective, in scenarios which consider spacetime foam intuition in the study of quantum gravity phenomena, emergence of modified dispersion relations takes place naturally [25]. As a consequence, wave dispersion in the spacetime foam might resemble wave dispersion in other media. Since Planck length fundamentally set the minimum allowed value for wavelengths, a modified dispersion relation can also be favored. Recently it has been shown that a modified energy-momentum dispersion relation can also be introduced as an observer-independent law [26]. In this case, the Planckian minimum-wavelength hypothesis can be introduced as a physical law valid in every frame. Therefore, the analysis of some quantum-gravity scenarios has shown some explicit mechanisms for the emergence of modified dispersion relations. For example, in the framework of noncommutative geometry and loop quantum gravity approaches this modified dispersion relations have been motivated (see for example [21] and references therein). In most cases one is led to consider a dispersion relation of the type (note that from now on we set $c = \hbar = k_B = 1)$

$$(\vec{p})^2 = f(E,m;l_p) \simeq E^2 - \mu^2 + \alpha_1 l_p E^3 + \alpha_2 l_p^2 E^4 + O\left(l_p^3 E^5\right)$$
(9)

where f is the function that gives the exact dispersion relation, and on the right-hand side we have assumed the applicability of a Taylor-series expansion for $E \ll 1/l_p$. The coefficients α_i can take different values in different quantum-gravity proposals. Note that m is the rest energy of the particle and the mass parameter μ on the right-hand side is directly related to the rest energy, but $\mu \neq m$ if the α_i do not all vanish. Since we are working in Planck regime where the rest mass is much smaller than the particle kinetic energy, there is no risk of confusing between m and μ . While in the parametrization of (3) we have included a possible correction term suppressed only by one power of the Planck length, in GUP such a linear-in- l_p term is assumed not to be present. For the MDR a large number of alternative formulations, including some with the linear-in- l_p term, are being considered, as they find support in different approaches to the quantum-gravity problem, whereas all the discussions of a GUP assume that the leading-order correction should be proportional to the square of l_p (as has been indicated by Amelino-Camelia *et al*[21], linear-in- l_p term in MDR has no support in string theory analysis of black holes entropyarea relation and therefore it seems that this term should not be present in MDR. Recently we have shown that coefficients of all odd power of E in MDR should be zero[15]).

Within quantum field theory, the relation between particle localization and its energy is given by $E \geq \frac{1}{\delta x}$, where δx is particle position uncertainty. It is obvious that due to both GUP and MDR this relation should be modified. In a simple analysis based on the familiar derivation of the relation $E \geq \frac{1}{\delta x}[27]$, one can obtain the corresponding generalized relation. Since we need this generalization in forthcoming arguments, we give a brief outline of its derivation here. We focus on the case of a particle of mass M at rest, whose position is being measured by a procedure involving a collision with a photon of energy E_{γ} and momentum p_{γ} . According to Heisenberg's uncertainty principle, in order to measure the particle position with precision δx , one should use a photon with momentum uncertainty $\delta p_{\gamma} \geq \frac{1}{\delta x}$. Following the standard argument[28], one takes this $\delta p_{\gamma} \geq \frac{1}{\delta x}$ relation and converts it into the relation $\delta E_{\gamma} \geq \frac{1}{\delta x}$ using the special relativistic dispersion relation. Finally $\delta E_{\gamma} \geq \frac{1}{\delta x}$ is converted into the relation $M \geq \frac{1}{\delta x}$ because the measurement procedure requires $\delta E \leq M$, in order to ensure that the relevant energy uncertainties are not large enough to allow the production of additional copies of the particle whose position is being measured. If indeed our quantum-gravity scenario hosts a Planck-scale modification of the dispersion relation of the form (9) then clearly the relation between δp_{γ} and δE_{γ} should be re-written as follows

$$\delta p_{\gamma} \simeq \left[1 + \alpha_1 l_p E + 3\left(\frac{\alpha_2}{2} - \frac{\alpha_1^2}{8}\right) l_p^2 E^2\right] \delta E_{\gamma}.$$
(10)

This relation will modify density of states for statistical systems. Note that one can use GUP to find such relation between δp_{γ} and $\delta E_{\gamma}[15]$.

3 GUP and Early Universe Thermodynamics

Now we are going to calculate thermodynamical properties of ultra-relativistic particles in early universe, using the generalized uncertainty principle. We consider the following GUP as our primary input,

$$\delta x \delta p \ge \pi \left(1 + \xi^2 \frac{(\delta x)^2}{l_p^2} \right),\tag{11}$$

where ξ is a dimensionless constant. Consider the early stages of the universe evolution. Analogue to a particle inside a box, in the case of the early universe one can consider a causal box (i.e. particle horizon) which any observer in the universe has to do measurements within this scale[29]. In the language of wave mechanics, if Ψ denotes the wave function of a given particle, the probability of finding this particle by an observer outside its horizon is zero, i. e. $|\Psi(x > horizon)|^2 = 0$. From the theory of relativity, measurement of a stick length can be done by sending simultaneous signals to the observer from the two endpoints, where for the scales larger than the causal size, those signals need more than the age of the universe to be received. Looking back to the history of the universe, the particle horizon after the Planck era grows as H^{-1} , but inflates to a huge size by the beginning of inflationary epoch. Here H is Hubble parameter. In the pre-inflationary epoch, the maximum uncertainty in the location of a particle, $\delta x = H^{-1}$ results in an uncertainty in the momentum of the particle which is given by

$$\delta p \ge \frac{\pi \xi^2}{l_p^2 H} + \pi H. \tag{12}$$

This leads to a minimum uncertainty in momentum as

$$(\delta p)_{min} = \frac{\pi \xi^2}{l_p^2 H} + \pi H. \tag{13}$$

Therefore, we can conclude that (assuming that $p \sim \delta p$)

$$p_{min} = \frac{\pi\xi^2}{l_p^2 H} + \pi H,\tag{14}$$

which leads to

$$E_{min} = \sqrt{3} \Big(\frac{\pi \xi^2}{l_p^2 H} + \pi H \Big), \tag{15}$$

for ultra-relativistic particles in three space dimensions. Now, suppose that

$$E_n = n\vartheta,\tag{16}$$

where ϑ is given by

$$\vartheta = \frac{\pi\xi^2}{l_p^2 H} + \pi H. \tag{17}$$

To obtain complete thermodynamics of the system, we calculate partition function of the system and then we use standard thermodynamical relations. In classical statistical mechanics, partition function for a system composed of ultra-relativistic noninteracting monatomic particles (Fermions or Bosons) is given by

$$\ln Z = \pm g \int_0^\infty \frac{4\pi n^2}{8} \ln(1 \pm e^{-\beta E_n}) dn.$$
 (18)

In our case, due to limitation imposed by GUP and particle horizon, we should consider the following generalization

$$\ln Z = \pm \frac{g\pi}{2\vartheta^3} \int_{E_{min}}^{\infty} \frac{E^2}{1+\eta E^2} \ln(1\pm e^{-\beta E}) dE,$$
(19)

where we have used relations (7), (15) and (16) respectively. By definition, the entropy of the system is given by

$$S = -\frac{1}{V}\frac{\partial F}{\partial T},\tag{20}$$

where F is the free energy of the system defined as

$$F = -\frac{1}{\beta} \ln Z. \tag{21}$$

So the entropy of the system can be written as

$$S = \frac{1}{V} \bigg[\pm \frac{g\pi}{2\vartheta^3} \int_{E_{min}}^{\infty} \frac{E^2}{1+\eta E^2} \ln(1\pm e^{-\beta E}) dE + \frac{g\pi\beta}{2\vartheta^3} \int_{E_{min}}^{\infty} \frac{E^3}{1+\eta E^2} \frac{dE}{e^{\beta E}\pm 1} \bigg].$$
(22)

For ultra-relativistic fermions this relation leads to the following expression

$$S_{f} = \frac{g}{2\pi^{2}(1+D)} \left[\frac{7}{90} \frac{\pi^{4}}{\beta^{3}} - \frac{31}{210} \frac{\eta\pi^{6}}{\beta^{5}} + \frac{1016}{1680} \frac{\eta^{2}\pi^{8}}{\beta^{7}} - E_{min}^{3} \left(\frac{1}{3} - \frac{\eta E_{min}^{2}}{5} + \frac{\eta^{2} E_{min}^{4}}{7} \right) \ln(1 + e^{-\beta E_{min}}) - \frac{4}{3}\beta I_{3} + \frac{6}{5}\eta\beta I_{5} - \frac{8}{7}\eta^{2}\beta I_{7} + \dots \right],$$

$$(23)$$

where for simplicity we have defined

$$I_j = \int_0^{E_{min}} \frac{E^j dE}{e^{\beta E} + 1}.$$

While for bosons we find

$$S_{b} = \frac{g}{2\pi^{2}(1+D)} \left[\frac{4}{45} \frac{\pi^{4}}{\beta^{3}} - \frac{48}{315} \frac{\eta\pi^{6}}{\beta^{5}} + \frac{64}{105} \frac{\eta^{2}\pi^{8}}{\beta^{7}} + E_{min}^{3} \left(\frac{1}{3} - \frac{\eta E_{min}^{2}}{5} + \frac{\eta^{2} E_{min}^{4}}{7} \right) \ln(1 - e^{-\beta E_{min}}) - \frac{4}{3}\beta J_{3} + \frac{6}{5}\eta\beta J_{5} - \frac{8}{7}\eta^{2}\beta J_{7} + \dots \right],$$

$$(24)$$

where

$$J_j = \int_0^{E_{min}} \frac{E^j dE}{e^{\beta E} - 1}.$$

In these equations D is defined as

$$D = \left(\frac{A^3}{B^3} + 3\frac{A^2}{B^2} + 3\frac{A}{B}\right) \qquad and \quad \vartheta = A + B$$

with $A = \frac{\pi \xi^2}{l_p^2 H}$ and $B = \pi H$. Note that both of the equations (23) and (24) are well behavior in high and low temperature limits. In the standard situation, we have $\xi = 0$, $\eta = 0$ and $E_{min} = 0$. So we find the well-known and standard results for entropy of the corresponding ultra-relativistic fermionic or bosonic systems. From (22) we find the following expression for standard entropy

$$S = \frac{4}{3} \frac{\beta g}{2\pi^2} \int_0^\infty \frac{E^3 dE}{e^{\beta E} \pm 1},$$
 (25)

which leads to

$$S_f = \frac{g}{2\pi^2} \frac{7}{90} \frac{\pi^4}{\beta^3},$$
(26)

and

$$S_b = \frac{g}{2\pi^2} \frac{4}{45} \frac{\pi^4}{\beta^3},\tag{27}$$

for fermions and bosons respectively.

Now the pressure of the ultra-relativistic gas is given by $P = \frac{1}{\beta V} \ln Z$. For fermions and bosons we find respectively

$$P_{f} = \frac{g}{2\pi^{2}(1+D)\beta} \left[\frac{7}{360} \frac{\pi^{4}}{\beta^{3}} - \frac{31}{1260} \frac{\eta\pi^{6}}{\beta^{5}} + \frac{127}{1680} \frac{\eta^{2}\pi^{8}}{\beta^{7}} - E_{min}^{3} \left(\frac{1}{3} - \eta \frac{E_{min}^{2}}{5} + \eta^{2} \frac{E_{min}^{4}}{7} \right) \ln(1 + e^{-\beta E_{min}}) - \frac{\beta}{3} I_{3} + \frac{\beta}{5} \eta I_{5} - \frac{\beta}{7} \eta^{2} I_{7} + \dots \right],$$

$$(28)$$

and

$$P_{b} = \frac{g}{2\pi^{2}(1+D)\beta} \left[\frac{1}{45} \frac{\pi^{4}}{\beta^{3}} - \frac{8}{315} \frac{\eta\pi^{6}}{\beta^{5}} + \frac{8}{105} \frac{\eta^{2}\pi^{8}}{\beta^{7}} + E_{min}^{3} \left(\frac{1}{3} - \eta \frac{E_{min}^{2}}{5} + \eta^{2} \frac{E_{min}^{4}}{7} \right) \ln(1 - e^{-\beta E_{min}}) - \frac{\beta}{3} J_{3} + \frac{\beta}{5} \eta J_{5} - \frac{\beta}{7} \eta^{2} J_{7} + \dots \right].$$

$$(29)$$

In the standard situation, we find the following well-known result

$$P = \frac{g}{6\pi^2} \int_0^\infty \frac{E^3 dE}{e^{\beta E} \pm 1},$$
 (30)

which leads to

$$P_f = \frac{g}{2\pi^2} \frac{7}{360} \frac{\pi^4}{\beta^4},\tag{31}$$

and

$$P_b = \frac{g}{2\pi^2} \frac{1}{45} \frac{\pi^4}{\beta^4},\tag{32}$$

for fermions and bosons respectively.

The specific heat of the system which is defined as

$$C_V = T \left(\frac{\partial S}{\partial T}\right)_V,\tag{33}$$

can be written in the following closed form

$$C_V = \frac{g\beta^2}{2\pi^2(1+D)} \int_{E_{min}}^{\infty} \frac{E^4}{1+\eta E^2} \frac{dE}{e^{-\beta E}(e^{\beta E}\pm 1)^2}.$$
 (34)

One can obtain explicit form of C_V for fermions and bosons using relation (33), (23) and (24). A simple calculation gives

$$C_{Vf} = \frac{g}{2\pi^2(1+D)} \left[\frac{21}{90} \frac{\pi^4}{\beta^3} - \frac{155}{210} \frac{\eta\pi^6}{\beta^5} + \frac{7112}{1680} \frac{\eta^2\pi^8}{\beta^7} - \frac{\beta E_{min}^4}{e^{\beta E_{min}} + 1} \left(\frac{1}{3} - \frac{\eta E_{min}^2}{5} + \frac{\eta^2 E_{min}^4}{7} \right) + \frac{4}{3}\beta I_3 - \frac{6}{5}\eta\beta I_5 + \frac{8}{7}\eta^2\beta I_7 - \frac{4}{3}\frac{dI_3}{dT} + \frac{6}{5}\eta\frac{dI_5}{dT} - \frac{8}{7}\eta^2\frac{dI_7}{dT} + \dots \right],$$
(35)

and

$$C_{Vb} = \frac{g}{2\pi^2(1+D)} \left[\frac{12}{45} \frac{\pi^4}{\beta^3} - \frac{240}{315} \frac{\eta\pi^6}{\beta^5} + \frac{448}{105} \frac{\eta^2\pi^8}{\beta^7} + \frac{\beta E_{min}^4}{e^{\beta E_{min}} - 1} \left(\frac{1}{3} - \frac{\eta E_{min}^2}{5} + \frac{\eta^2 E_{min}^4}{7} \right) + \frac{4}{3}\beta J_3 - \frac{6}{5}\eta\beta J_5 + \frac{8}{7}\eta^2\beta J_7 - \frac{4}{3} \frac{dJ_3}{dT} + \frac{6}{5}\eta \frac{dJ_5}{dT} - \frac{8}{7}\eta^2 \frac{dJ_7}{dT} + \dots \right],$$
(36)

for specific heat of fermions and bosons respectively. In the standard case we find

$$C_{Vf} = 3 \times \frac{g}{2\pi^2} \frac{7}{90} \frac{\pi^4}{\beta^3} = 3S_f, \tag{37}$$

and

$$C_{Vb} = 3 \times \frac{g}{2\pi^2} \frac{4}{45} \frac{\pi^4}{\beta^3} = 3S_b.$$
(38)

Figure 1 shows the values of entropy in different situations. In standard thermodynamics of ultra-relativistic fermionic or bosonic gas, the entropy of the system tends to zero in $T_0 = 0$. This situation is shown in Figure 1, (a) and (b). Within GUP framework, entropy tends to zero in a nonzero temperature, that is, for $T > T_0$. This is a result of quantum fluctuation of spacetime itself. Figure 2 shows the corresponding behavior of pressure as a function of temperature. Note that these figures are plotted in arbitrary units and they show only general behaviors of the functions. Figure 3 shows the behavior of specific heat of the system in various conditions. In GUP framework, the general behavior of C_V has considerable departure from its standard counterpart in high temperature regime.

4 MDR and Early Universe Thermodynamics

Now we are going to formulate early universe thermodynamics within MDR framework. We consider a gaseous system composed of ultra-relativistic monatomic, non-interacting particles. First we derive the density of states. Consider a cubical box with edges of length L (and volume $V = L^3$) consisting black body radiation(photons). The wavelengths of the photons are subject to the boundary condition $\frac{1}{\lambda} = \frac{n}{2L}$, where n is a positive integer. This condition implies, assuming that the de Broglie relation is left unchanged, that the photons have (space-)momenta that take values $p = \frac{n}{2L}$. Thus momentum space is divided into cells of volume $V_p = \left(\frac{1}{2L}\right)^3 = \frac{1}{8V}$. From this point, it follows that the number of modes with momentum in the interval [p, p + dp] is given by

$$g(p)dp = 8\pi V p^2 dp \tag{39}$$

Assuming a MDR of the type parameterized in (9) one then finds that (m = 0 for photons)

$$p \simeq E \left[1 + \frac{\alpha_1}{2} l_p E + \left(\frac{\alpha_2}{2} - \frac{\alpha_1^2}{8} \right) l_p^2 E^2 \right]$$
(40)

and

$$dp \simeq \left[1 + \alpha_1 l_p E + (\frac{3}{2}\alpha_2 - \frac{3}{8}\alpha_1^2) l_p^2 E^2\right] dE$$
(41)

Using this relation in (39), one obtains

$$g(E)dE = 8\pi V \left[1 + 2\alpha_1 l_p E + 5\left(\frac{1}{2}\alpha_2 + \frac{1}{8}\alpha_1^2\right) l_p^2 E^2 \right] E^2 dE.$$
(42)

This is density of states which we use in our calculations. Note that we have not set $\alpha_1 = 0$ to ensure generality of our discussions, but we will discuss corresponding situation at the end of our calculations.

To obtain thermodynamics of the system under consideration, we start with the partition function of fermions and bosons,

$$\ln Z = \pm \int_{E_{min}}^{\infty} g(E) \ln(1 \pm e^{-\beta E}) dE, \qquad (43)$$

where + and - stand for fermions and bosons respectively and $\beta = 1/T$ since $k_B = 1$. Using equation(42) in the following form

$$g(E)dE = 8\pi V(1 + aE + bE^2)E^2dE,$$
(44)

where for simplicity we have defined $a = 2\alpha_1 l_p$ and $b = 5\left(\frac{1}{2}\alpha_2 + \frac{1}{8}\alpha_1^2\right)l_p^2$, one can compute the integral of equation(43) to find the following expression for entropy of fermions and bosons

$$S = \pm \frac{1}{V} \int_{E_{min}}^{\infty} g(E) \ln(1 \pm e^{-\beta E}) dE + \frac{\beta}{V} \int_{E_{min}}^{\infty} \frac{g(E)EdE}{e^{\beta E} \pm 1}.$$
 (45)

This relation can be written as follows

$$S = \pm \frac{1}{V} \int_{0}^{\infty} g(E) \ln(1 \pm e^{-\beta E}) dE + \frac{\beta}{V} \int_{0}^{\infty} \frac{g(E)EdE}{e^{\beta E} \pm 1}$$

$$\mp \frac{1}{V} \int_{0}^{E_{min}} g(E) \ln(1 \pm e^{-\beta E}) dE - \frac{\beta}{V} \int_{0}^{E_{min}} \frac{g(E)EdE}{e^{\beta E} \pm 1}.$$
 (46)

By calculating this integral, we find for fermions and bosons respectively

$$S_{f} = 8\pi \left[\frac{7}{90} \frac{\pi^{4}}{\beta^{3}} + \frac{225}{8} \frac{a\zeta(5)}{\beta^{4}} + \frac{31}{210} \frac{b\pi^{6}}{\beta^{5}} - E_{min}^{3} \left(\frac{1}{3} + \frac{aE_{min}}{4} + \frac{bE_{min}^{2}}{5} \right) \ln(1 + e^{-\beta E_{min}}) - \frac{4}{3}\beta I_{3} - \frac{5}{4}\beta a I_{4} - \frac{6}{5}\beta b I_{5} + \dots \right]$$

$$(47)$$

and

$$S_{b} = 8\pi \left[\frac{4}{45} \frac{\pi^{4}}{\beta^{3}} + \frac{30a\zeta(5)}{\beta^{4}} + \frac{48}{315} \frac{b\pi^{6}}{\beta^{5}} + E_{min}^{3} \left(\frac{1}{3} + \frac{aE_{min}}{4} + \frac{bE_{min}^{2}}{5} \right) \ln(1 - e^{-\beta E_{min}}) - \frac{4}{3}\beta J_{3} - \frac{5}{4}\beta a J_{4} - \frac{6}{5}\beta b J_{5} + \dots \right].$$
(48)

One may ask about the relation between these two results and corresponding results of GUP, that is, relations (23) and (24). Although these results seem to be different in their β dependence, but note that if we set $\alpha_1 = 0$ (which is reasonable regarding the argument presented in page 5), we find a = 0 and then β dependence of our findings will

coincide with each other. The only difference which remains is the differences between numerical factors. This argument shows that essentially the results of GUP and MDRs for thermodynamics of the early universe do not differ with each other in their temperature dependence and overall behaviors.

In the standard situation, we have a = b = 0 and $E_{min} = 0$, so we find

$$S = \frac{4}{3} (8\pi\beta) \int_0^\infty \frac{E^3 dE}{e^{\beta E} \pm 1}.$$
 (49)

For entropy of fermions and bosons we find respectively

$$S_f = 8\pi \frac{7}{90} \frac{\pi^4}{\beta^3},\tag{50}$$

and

$$S_b = 8\pi \frac{4}{45} \frac{\pi^4}{\beta^3}.$$
 (51)

In the presence of MDR, the pressure of corresponding systems are

$$P_{f} = \frac{8\pi}{\beta} \left[\frac{7}{360} \frac{\pi^{4}}{\beta^{3}} + \frac{45}{8} \frac{a\zeta(5)}{\beta^{4}} + \frac{31}{1260} \frac{b\pi^{6}}{\beta^{5}} - E_{min}^{3} \left(\frac{1}{3} + a \frac{E_{min}}{4} + b \frac{E_{min}^{5}}{5} \right) \ln(1 + e^{-\beta E_{min}}) - \frac{\beta}{3} I_{3} - \frac{\beta}{4} a I_{4} - \frac{\beta}{5} b I_{5} + \dots \right],$$

$$P_{b} = \frac{8\pi}{\beta} \left[\frac{1}{45} \frac{\pi^{4}}{\beta^{3}} + 6 \frac{a\zeta(5)}{\beta^{4}} + \frac{8}{315} \frac{b\pi^{6}}{\beta^{5}} + E_{min}^{3} \left(\frac{1}{3} + a \frac{E_{min}}{4} + b \frac{E_{min}^{5}}{5} \right) \ln(1 - e^{-\beta E_{min}}) - \frac{\beta}{3} J_{3} - \frac{\beta}{4} a J_{4} - \frac{\beta}{5} b J_{5} + \dots \right],$$

$$(52)$$

for fermions and bosons respectively. In the standard situation we find the following well-known relation

$$P = \frac{8\pi}{3} \int_0^\infty \frac{E^3 dE}{e^{\beta E} \pm 1},$$
 (54)

which for fermions and bosons leads to

$$P_f = 8\pi \frac{7}{360} \frac{\pi^4}{\beta^4},\tag{55}$$

and

$$P_b = 8\pi \frac{1}{45} \frac{\pi^4}{\beta^4}$$
(56)

respectively.

The specific heat of the system can be written in the following closed form

$$C_V = \frac{\beta^2}{V} \int_{E_{min}}^{\infty} \frac{g(E)E^2 dE}{e^{-\beta E}(e^{\beta E} \pm 1)^2}.$$
 (57)

One can use relations (33), (47) and (48) to find the following explicit results for fermions and bosons respectively

$$C_{Vf} = 8\pi \left[\frac{21}{90} \frac{\pi^4}{\beta^3} + \frac{900}{8} \frac{a\zeta(5)}{\beta^4} + \frac{155}{210} \frac{b\pi^6}{\beta^5} - \frac{\beta E_{min}^4}{e^{\beta E_{min}} + 1} \left(\frac{1}{3} + \frac{aE_{min}}{4} + \frac{bE_{min}^2}{5} \right) + \frac{4}{3}\beta I_3 + \frac{5}{4}\beta a I_4 + \frac{6}{5}\beta b I_5 - \frac{4}{3} \frac{dI_3}{dT} - \frac{5}{4}a \frac{dI_4}{dT} - \frac{6}{5}b \frac{dI_5}{dT} + \dots \right],$$
(58)

and

$$C_{Vb} = 8\pi \left[\frac{12}{45} \frac{\pi^4}{\beta^3} + \frac{120a\zeta(5)}{\beta^4} + \frac{240}{315} \frac{b\pi^6}{\beta^5} + \frac{\beta E_{min}^4}{e^{\beta E_{min}} - 1} \left(\frac{1}{3} + \frac{aE_{min}}{4} + \frac{bE_{min}^2}{5} \right) + \frac{4}{3}\beta J_3 + \frac{5}{4}\beta a J_4 + \frac{6}{5}\beta b J_5 - \frac{4}{3} \frac{dJ_3}{dT} - \frac{5}{4}a \frac{dJ_4}{dT} - \frac{6}{5}b \frac{dJ_5}{dT} + \dots \right].$$
(59)

In the standard case we find

$$C_{Vf} = 3 \times 8\pi \frac{7}{90} \frac{\pi^4}{\beta^3} = 3S_f,$$
(60)

and

$$C_{Vb} = 3 \times 8\pi \frac{4}{45} \frac{\pi^4}{\beta^3} = 3S_b \tag{61}$$

respectively.

As has been indicated, there are severe constraints on the functional form of MDR which these constraint are motivated when one compares black hole entropy-area relation in different points of view[15,21]. In this case we should set $\alpha_1 = 0$ which leads to a = 0. We find from (47) and (48) the following expressions for entropy of fermions and bosons respectively

$$S_{f} = 8\pi \left[\frac{7}{90} \frac{\pi^{4}}{\beta^{3}} + \frac{31}{210} \frac{b' \pi^{6}}{\beta^{5}} - E_{min}^{3} \left(\frac{1}{3} + \frac{b' E_{min}^{2}}{5} \right) \ln(1 + e^{-\beta E_{min}}) - \frac{4}{3}\beta I_{3} - \frac{6}{5}\beta b' I_{5} + \dots \right],$$

$$(62)$$

and

$$S_{b} = 8\pi \left[\frac{4}{45} \frac{\pi^{4}}{\beta^{3}} + \frac{48}{315} \frac{b'\pi^{6}}{\beta^{5}} + E_{min}^{3} \left(\frac{1}{3} + \frac{b'E_{min}^{2}}{5} \right) \ln(1 - e^{-\beta E_{min}}) - \frac{4}{3}\beta J_{3} - \frac{6}{5}\beta b'J_{5} + \dots \right],$$
(63)

where $b' = \frac{5}{2} \alpha_2 l_p^2$. These statements for partition function are more realistic since black hole thermodynamics within MDRs when is compared with exact solution of string theory, suggest the vanishing of α_1 .

It is important to note that the formalism presented in this section is not restricted to early universe. Actually, it can be applied to any statistical system composed of ultrarelativistic monatomic noninteracting particles which has a minimum accessible energy. The Possible relation between GUP and MDRs itself is under investigation[14,15]. Generally these two features of quantum gravity scenarios are not equivalent, but as Hossenfelder has shown, they can be related to each other[14](see also [15]). As a result, it is natural to expect that under special circumstances, our results for early universe thermodynamics within GUP and MDRs should transform to each other. This is a transformation between coefficients of our equations and overall behaviors of thermodynamical quantities, specially their temperature dependence are similar.

5 Summary

GUP and MDRs have found strong supports from string theory, noncommutative geometry and loop quantum gravity. There are many implications, originated from GUP and MDRs, for the rest of the physics. From a statistical mechanics point of view, GUP changes the volume of the fundamental cell of the phase space in a momentum dependent manner. On the other hand, MDR leads to a modification of density of states. These quantum gravity features have novel implications for statistical properties of thermodynamical systems. Here we have studied thermodynamics of early universe within both GUP and MDRs. We have considered early universe as a statistical system composed of ultra-relativistic particles. Since both particle horizon distance and GUP impose severe constraint on measurement processes, the statistical mechanics of the system should be modified to contain these constraint. Since GUP and MDRs are quantum gravitation effects, the modified thermodynamics within GUP and MDRs tends to standard thermodynamics in classical limits. There are severe constraints on the functional form of MDRs from string theory considerations. When we consider these constraints, the results of MDRs and GUP for thermodynamics of early universe tends to each other in their general temperature dependence and they differs only in their numerical factors. This fact may be interpreted so that GUP and MDRs essentially are not different concepts of quantum gravity proposal. Although the exact relation between GUP and MDRs is not known yet, our formalism of early universe shows the very close relation between these two aspects of quantum gravity.

In standard statistical mechanics of bosonic and fermionic gases, the entropy of the system tends to zero in $T_0 = 0$. As our equations and corresponding numerical result show, within GUP framework entropy of the system tends to zero in a temperature larger than zero($T > T_0$). This is a consequence of the relation (5). The volume of the fundamental cell of phase space increases due to GUP. Note that MDRs give entropy-temperature relation which has no difference with GUP result in its general behavior. Figure 2 shows the pressure of the system versus temperature. Pressure tends to zero in a temperature larger than $T_0 = 0$. The same behavior is repeated by specific heat of the system. So, our analysis shows an unusual thermodynamics for statistical systems in quantum gravity eras. This unusual behaviors have been seen in other context such as black hole thermodynamics[30,31].

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Figure 1: Entropy of ultra-relativistic monoatomic gaseous system for (a)standard bosonic gas (b) standard fermionic gas (c) bosonic gas within GUP and (d) fermionic gas within GUP.



Figure 2: Pressure of ultra-relativistic monoatomic gaseous system for (a)standard bosonic gas (b) standard fermionic gas (c) bosonic and fermionic gas within GUP. The difference between bosonic and fermionic gasses in this case is not considerable.



Figure 3: Heat Capacity of ultra-relativistic monoatomic gaseous system for (a)standard bosonic gas (b) standard fermionic gas (c) bosonic gas within GUP and (d) fermionic gas within GUP.