Advances in Computational Mathematics (2019) 45:1863–1879
https://doi.org/10.1007/s10444-019-09663-3

An arriving decision problem in a discrete-time queueing system

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Received: 25 September 2018 / Accepted: 4 January 2019 /
Published online: 21 January 2019

Abstract

This paper discusses a discrete-time queueing system in which an arriving customer may adopt four different strategies; two of them correspond to a LCFS discipline where displacements or expulsions occur, and in the other two, the arriving customer decides to follow a FCFS discipline or to become a negative customer eliminating the customer in the server, if any. The different choices of the involved parameters make this model to enjoy a great versatility, having several special cases of interest. We carry out a thorough analysis of the system, and using a generating function approach, we derive analytical results for the stationary distributions obtaining performance measures for the number of customers in the queue and in the system. Also, recursive formulae for calculating the steady-state distributions of the queue and system size has been developed. Making use of the busy period of an auxiliary system, the sojourn times of a customer in the queue and in the system have also been obtained. Finally, some numerical examples are given.

Keywords Discrete-time system · Decision problem · Trigger customers · Negative customers

1 Introduction

A feature that is usually found when a message is being processed in computers, in communications switching queues, etc. is that sometimes the information incom-

Communicated by: Pavel Solin

This work was partially supported by Grant No. TIN2015-70266-C2-1-P of the Science and Innovation Ministry of Spain.

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ing to the server is more actual than the one on service. In that case, the message is moved to another place if the contained information can be used later on, or if the information is not any more valuable, it is deleted; in both cases, the server is upgraded. The mechanism of moving messages by the arrival of one of them is called synchronized or triggered motion. There are several mechanisms on how and where the messages are moved, for a survey on them refer to [1] and [2, 4, 9]. This mechanism leads to service interruptions were first studied in [17] where authors studied an $M/M/1$ pre-emptive two-priority queueing model with exponentially distributed service interruption. An extensive study on such models can be consulted, for example, in [3, 8, 11, 12, 16] and for a detailed review on queues with service interruptions, we refer to [13].

A certain type of movement can be also considered when customers are deleted. The process of deleting customers or killing them is defined in queueing theory as negative customers. This type of movement can be related to the arrival of viruses at the system. Pioneering work on discrete-time considering negative arrivals without retrials was done by [5, 6] who considered several killing strategies for negative customers. For applications in engineering, we refer to [7] and for application in communication networks and packet transmission systems, refer to [10, 14] and [15].

The strategy used in this paper for moving customers is the one that displaces them from the server to the first place of the queue, and it seems a realistic one because the displaced customer can begin its service after the service completion of the customer that has caused its displacement. The arrival of negative customers has the effect, in this model, of eliminating the job that is currently being served, and has no influence on the system if the server is idle.

The rest of this paper is organised as follows. The next section gives a description of the queueing model. In Section 3, the Markov chain is studied. The queue and system size distributions are obtained together with several performance measures of the model. In Section 4, the busy period is obtained. In Section 5, the generating functions of the sojourn time of a customer in the server, the queue and the system, as well as some associated performance measures are provided. Finally, numerical results and a section of conclusions where the main results of the paper are discussed.

2 The mathematical model

We regard a discrete-time queueing system in which the time axis is segmented into a sequence of equal intervals, called slots. It should be pointed out that the discrete-time model differs from the corresponding continuous-time model in the following sense: the probability of simultaneous arrivals and departures is zero in continuous-time and positive in discrete-time. That is why we must detail the order in which the arrivals and departures occur in case of simultaneity in a discrete-time system. Basically, there are two rules: (i) if an arrival takes precedence over a departure, it is identified with Late Arrival System (LAS) (see Fig. 1a); (ii) if a departure takes precedence over an arrival, it is recognised by Early Arrival System (EAS) (see Fig. 1b). The former case is also known as Arrival First (AF) policy and the latter as Departure

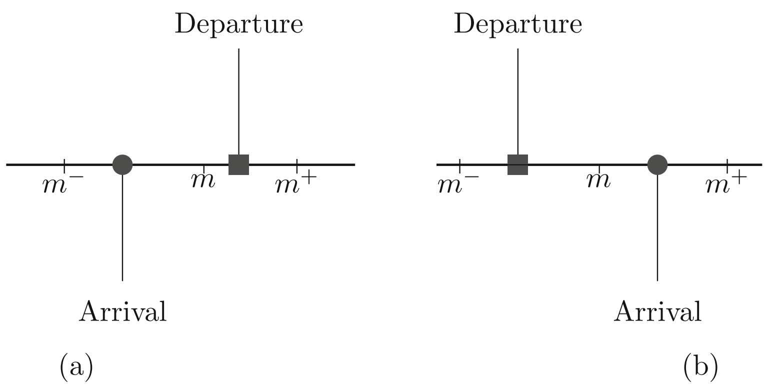

Fig. 1 Options of the arrival models

First (DF) policy (for more details on these and related concepts, see Hunter (1983)). In the present paper, we will follow the second policy.

Let the time axis be marked by $0, 1, \ldots, m, \ldots$. Consider the epoch m and suppose that the departures occur in (m^-, m) and external arrivals, and the arrivals to the server from the queue, occur, in this order in (m, m^+).

It is assumed that customers arrive according to a geometrical process with rate a, that is, a is the probability that an arrival occurs in a slot. If an arriving customer finds the server idle, he commences his service immediately; otherwise, with probability θ_0, it goes directly to the last place in the queue, with probability θ_1 displaces the customer in the server to the first place in the queue and begins its service, with probability θ_2 it expels out of the system the customer in the server and begins its service and with probability θ_3 it becomes a negative customer. The probabilities θ_i, $i = 0, \ldots, 3$ satisfy the condition $\sum_{i=0}^{3} \theta_i = 1$. The service times are independent and distributed with arbitrary distribution $\{s_i\}_{i=1}^{\infty}$, and generating function (GF) $S(x) = \sum_{i=1}^{\infty} s_i x^i$, $0 \leq x \leq 1$. Hence, s_i is the probability that a service lasts i slots. We will denote by $S_k = \sum_{i=k}^{\infty} s_i$, the probability that the service lasts not less than k slots.

3 The Markov chain

At time m^+, the system can be described by the Markov process $\{X_m, \ m \in \mathbb{N}\}$ with $X_m = (C_m, \xi_m, N_m)$ where C_m denotes the state of the server 0, or 1 according to whether the server is free or busy, and N_m is the number of customers in the queue.

If $C_m = 1$, then ξ_m represents the remaining service time of the customer currently being served.

It can be shown that $\{X_m\ ,\ m \in \mathbb{N}\}$ is the Markov chain of the queueing system under consideration, whose states space is

$$\{(0); (1, i, k) : i \geq 1, k \geq 0\}.$$

Our first task is to find the stationary distribution:

$$\begin{aligned}
\pi_0 &= \lim_{m\to\infty} P[C_m = 0],\\
\pi_{1,i,k} &= \lim_{m\to\infty} P[C_m = 1, \xi_m = i, N_m = k],\ \ i \geq 1,\ k \geq 0.
\end{aligned}$$

of the Markov chain $\{X_m\ ,\ m \in \mathbb{N}\}$.

The Kolmogorov equations for the stationary distribution are

$$\pi_0 = (\bar{a} + a\theta_3)\pi_0 + (\bar{a} + a\theta_3)\pi_{1,1,0} + a\theta_3 \sum_{i=2}^{\infty} \pi_{1,i,0} \Leftrightarrow$$

$$a(1-\theta_3)\pi_0 = \bar{a}\pi_{1,1,0} + a\theta_3 \sum_{i=1}^{\infty} \pi_{1,i,0}, \tag{1}$$

$$\begin{aligned}
\pi_{1,i,k} = {} & \delta_{0,k} a(1-\theta_3)s_i\pi_0 + a(1-\theta_3)s_i\pi_{1,1,k} + (\bar{a} + a\theta_3)s_i\pi_{1,1,k+1} + \\
& + (1-\delta_{0,k})a\theta_0\pi_{1,i+1,k-1} + \bar{a}\pi_{1,i+1,k} + \\
& + (1-\delta_{0,k})a\theta_1 s_i \sum_{j=2}^{\infty} \pi_{1,j,k-1} + a\theta_2 s_i \sum_{j=2}^{\infty} \pi_{1,j,k} + \\
& + a\theta_3 s_i \sum_{j=2}^{\infty} \pi_{1,j,k+1},\ \ i \geq 1,\ \ k \geq 0,
\end{aligned} \tag{2}$$

where $\bar{a} = 1 - a$ and $\delta_{i,j}$ denotes the Kronecker's delta.

The normalization condition is

$$\pi_0 + \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \pi_{1,i,k} = 1.$$

With the aim of solving Eqs. 1 and 2, the following generating function is introduced

$$\varphi_1(x, z) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \pi_{1,i,k}\, x^i z^k,\ 0 \leq x, z \leq 1,$$

and the auxiliary generating functions

$$\varphi_{1,i}(z) = \sum_{k=0}^{\infty} \pi_{1,i,k}\, z^k,\ 0 \leq z \leq 1,\ \ i \geq 1.$$

Multiplying Eq. 2 by z^k, summing over k and taking into account Eq. 1 yields

$$\varphi_{1,i}(z) = (\bar{a}+a\theta_0 z)\varphi_{1,i+1(z)} + \frac{\bar{a}+a\theta_0 z + a\theta_1 z(1-z)}{z} s_i\varphi_{1,1}(z) + \\ + \frac{a(\theta_1 z^2+\theta_2 z+\theta_3)}{z} s_i\varphi_1(1,z) - \frac{1-z}{z} a(1-\theta_3)s_i\pi_0, \ i \geq 1. \quad (3)$$

Next, multiplying the above equation by x^i and summing over i gives

$$z\frac{x-(\bar{a}+a\theta_0 z)}{x}\varphi_1(x,z) = [(\bar{a}+a\theta_0 z+a\theta_1 z(1-z))S(x) - z(\bar{a}+a\theta_0 z)]\,\varphi_{1,1}(z) \\ +a(\theta_1 z^2+\theta_2 z+\theta_3)S(x)\varphi_1(1,z) \\ -(1-z)a(1-\theta_3)S(x)\pi_0. \quad (4)$$

By letting $x=1$ in Eq. 4 it is obtained

$$a[(\theta_0+\theta_1)z-\theta_3]\varphi_1(1,z) = [\bar{a}+az(\theta_0+\theta_1)]\varphi_{1,1}(z) - a(1-\theta_3)\pi_0. \quad (5)$$

Inserting the above equation into (4) gives

$$[(\theta_0+\theta_1)z-\theta_3]\frac{x-(\bar{a}+a\theta_0 z)}{x}\varphi_1(x,z) = \\ = [((\bar{a}a\theta_0 z)(1-\theta_3)+\theta_1 z)\,S(x) - [(\theta_0+\theta_1)z-\theta_3](\bar{a}+a\theta_0 z)]\varphi_{1,1}(z) - \\ -a(1-\theta_3)(1-\theta_0 z)S(x)\pi_0. \quad (6)$$

Choosing $x=(\bar{a}+a\theta_0 z)$ in Eq. 6 yields

$$\varphi_{1,1}(z) = \frac{a(1-\theta_3)(1-\theta_0 z)S(\bar{a}+a\theta_0 z)}{D(z)}\pi_0, \quad (7)$$

where

$$D(z) = ((\bar{a}+a\theta_0 z)(1-\theta_3)+\theta_1 z)S(\bar{a}+a\theta_0 z) - [(\theta_0+\theta_1)z-\theta_3](\bar{a}+a\theta_0 z).$$

Finally, by substituting (7) into (6), it is obtained

$$\varphi_1(x,z) = \frac{S(x)-S(\bar{a}+a\theta_0 z)}{x-(\bar{a}+a\theta_0 z)} \cdot \frac{xa(1-\theta_3)(1-\theta_0 z)(\bar{a}+a\theta_0 z)}{D(z)}\pi_0. \quad (8)$$

Using the normalization condition that can be written as $\pi_0+\varphi_1(1,1)=1$, we find the value of the unknown constant π_0:

$$\pi_0 = \frac{D(1)}{\theta_1 S(\bar{a}+a\theta_0)+(\theta_2+\theta_3)(\bar{a}+a\theta_0)}, \quad (9)$$

where

$$D(1) = [(\bar{a}+a\theta_0)(1-\theta_3)+\theta_1]S(\bar{a}+a\theta_0) - [\theta_0+\theta_1-\theta_3](\bar{a}+a\theta_0).$$

Therefore, the condition

$$D(1) > 0 \quad (10)$$

is a necessary condition for the system's stability. Applying Foster's theorem, it can be shown that this condition is also sufficient for the stability of the system. The above results are summarized in the following theorem

Theorem 1 *If $D(1) > 0$, the stationary distribution of the Markov chain $\{X_m\ ,\ m \in \mathbb{N}\}$ has the following generating function*

$$\varphi_1(x,z) = \frac{S(x) - S[(\bar{a} + a\theta_0 z)]}{x - (\bar{a} + a\theta_0 z)} \frac{xa(1-\theta_0 z)(\bar{a} + a\theta_0 z)}{D(z)} \pi_0,$$

where

$$\pi_0 = \frac{D(1)}{\theta_1 S(\bar{a} + a\theta_0) + \theta_2(\bar{a} + a\theta_0)}.$$

Corollary 1 1. *The probability generating function of the queue size (i.e., of the variable N) is given by*

$$\begin{aligned}\Psi(z) &= \pi_0 + \varphi_1(1,z)\\ &= \frac{\theta_1 z S(\bar{a} + a\theta_0 z) + (\bar{a} + a\theta_0 z)[1 - z(\theta_0 + \theta_1)]}{D(z)} \pi_0,\ 0 \le z \le 1.\end{aligned}$$

2. *The probability generating function of the system size (i.e., of the variable L) is given by*

$$\begin{aligned}\Phi(z) &= \pi_0 + z\varphi_1(1,z)\\ &= \frac{[(\bar{a} + a\theta_0 z)(1-z)(1-\theta_3) + \theta_1 z]S(\bar{a} + a\theta_0 z) + (\bar{a} + a\theta_0 z)(\theta_2 z + \theta_3)}{D(z)} \pi_0,\\ &\quad 0 \le z \le 1.\end{aligned}$$

Corollary 2 1. *The mean queue size is given by*

$$\begin{aligned}E[N] &= \frac{1}{\Big[\theta_1 S(\bar{a} + a\theta_0) + (\theta_2 + \theta_3)(\bar{a} + a\theta_0)\Big] D(1)} \times\\ &\quad \times \Big\{ \Big[\theta_1 \Big(S(\bar{a} + a\theta_0) + a\theta_0 S'(\bar{a} + a\theta_0)\Big) + a\theta_0(\theta_2 + \theta_3) -\\ &\quad -(\bar{a} + a\theta_0)(\theta_0 + \theta_1)\Big] D(1) - \Big[\theta_1 S(\bar{a} + a\theta_0) + (\theta_2 + \theta_3)(\bar{a} + a\theta_0)\Big] D'(1)\Big\},\end{aligned}$$

where

$$\begin{aligned}D'(1) &= [a\theta_0(1-\theta_3) + \theta_1]S(\bar{a} + a\theta_0) + [(\bar{a} + a\theta_0)(1-\theta_3) + \theta_1]a\theta_0 S'(\bar{a} + a\theta_0)\\ &\quad - (\theta_0 + \theta_1)(\bar{a} + 2a\theta_0) + a\theta_0\theta_3.\end{aligned}$$

2. *The mean system size is given by*

$$E[L] = \Phi'(1) = E[N] + 1 - \pi_0.$$

4 Calculation of the steady-state probabilities

This section is devoted to develop some recursive formulae for calculating the more characteristics stationary distributions associated with our system.

Theorem 2 *The steady-state distribution of the queue size is given by the following recursive formulae*

$$\psi_0 = P[N=0] = \frac{\pi_0}{(1-\theta_3)S(\bar{a})+\theta_3}, \tag{11}$$

$$\psi_k = P[N=k] = \frac{-\sum_{n=0}^{k-1}\psi_n g_{k-n} + l_k\pi_0}{(1-\theta_3)s(\bar{a})+\theta_3},\ k\geq 1, \tag{12}$$

where

$$\begin{aligned}
g_1 &= \frac{\bar{a}^2 c_1 + [a\theta_0(1-\theta_3)+\theta_1]S(\bar{a}) - \bar{a}(\theta_0+\theta_1)}{\bar{a}}, \\
g_n &= \bar{a}^2 c_n + [a\theta_0(1-\theta_3)+\theta_1]c_{n-1},\ n\geq 2, \\
l_0 &= \pi_0, \\
l_1 &= \frac{\theta_1 S(\bar{a}) - \bar{a}(\theta_0+\theta_1)}{\bar{a}}\pi_0, \\
l_n &= \theta_1 c_{n-1}\pi_0,\ n\geq 2,
\end{aligned}$$

and

$$c_n = \sum_{i=n}^{\infty}\binom{i}{n} s_{i+1}\bar{a}^{\,i-n}(a\theta_0)^n,\ n\geq 1.$$

Proof Let us note that the GF $\Psi(z)$ of the number of customers in the queue satisfies the relation

$$\Psi(z)G(z) = \left(1 + z\left[\theta_1\frac{S(\bar{a}+a\theta_0 z)}{\bar{a}+a\theta_0 z} - (\theta_0+\theta_1)\right]\right)\pi_0, \tag{13}$$

where

$$\begin{aligned}
G(z) &= [\bar{a}(1-\theta_3) + (a\theta_0(1-\theta_3)+\theta_1)z]\frac{S(\bar{a}+a\theta_0 z)}{\bar{a}+a\theta_0 z} + \theta_3 - (\theta_0+\theta_1)z \\
&= \sum_{n=0}^{\infty} g_n z^n = \theta_3 + (1-\theta_3)S(\bar{a}) + \sum_{n=1}^{\infty} g_n z^n,\ 0\leq z\leq 1.
\end{aligned}$$

Using the properties of the generating functions and the Newton's binomial, we get

$$\frac{S(\bar{a}+a\theta_0 z)}{\bar{a}+a\theta_0 z} = \frac{S(\bar{a})}{\bar{a}} + \sum_{n=1}^{\infty} c_n z^n,$$

and the expression of the coefficients $g_n, n\geq 1$ are given by

$$\begin{aligned}
g_1 &= \frac{\bar{a}^2(1-\theta_3)c_1 + [a\theta_0(1-\theta_3)+\theta_1]S(\bar{a}) - \bar{a}(\theta_0+\theta_1)}{\bar{a}}, \\
g_n &= \bar{a}(1-\theta_3)c_n + [a\theta_0(1-\theta_3)+\theta_1]c_{n-1},\ n\geq 2.
\end{aligned}$$

In a similar way, we obtain the expression in power series of the right-hand side of (11): $\sum_{n=0}^{\infty} l_n z^n$, where $l_0 = \pi_0, l_1 = \dfrac{\theta_1 S(\bar{a}) - \bar{a}(\theta_0 + \theta_1)}{\bar{a}}\pi_0$, and $l_n = \theta_1 c_{n-1}\pi_0,\ n \geq 2$.

After comparing the coefficients of z^k on both sides in equation (13), we have

$$\psi_0 g_0 = l_0,$$
$$\sum_{n=0}^{k} \psi_n g_{k-n} = l_k,\ k \geq 1,$$

relations that directly lead to formulae (11) and (12). □

With respect to the steady-state probabilities of the system size, let us note that

$$\begin{aligned}
\phi_0 &= P[L=0] = \pi_0,\\
\phi_1 &= P[L=1] = \psi_0 - \phi_0,\\
\phi_k &= P[L=k] = \psi_{k-1},\ k \geq 2.
\end{aligned} \tag{14}$$

5 Busy period

A busy period (BP) is defined to begin with the arrival of a customer to an empty system and to end when the system next becomes empty and no external arrival takes place.

In this paragraph, we will consider the busy period of an auxiliary system that will be useful to study the customers delay in our initial model. This auxiliary system differs from the original one by the fact that the probability of an arrival is equal to $a\theta$, $\theta = \theta_1 + \theta_2 + \theta_3 = 1 - \theta_0$, and that a customer who enters in the system goes directly to the server interrupting the service of the customer that is currently being served, if any. The parameter a is the probability. The interrupted customer may be displaced to the head of the queue with probability $a\theta_1$, or expelled out of the system with probability $a\theta_2$. The arriving customer with probability $a\theta_3$ becomes a negative customer killing the customer that is in the server.

We will denote by h_k, $k \geq 0$, the probability that the BP lasts exactly k slots. Then, we have

$$\begin{aligned}
h_0 &= 0,\\
h_k &= (1-a\theta)^{k-1} s_k [1 - a(\theta_1+\theta_2)] + (1-a\theta)^{k-1} S_{k+1} a\theta_3\\
&\quad + \sum_{i=1}^{k} (1-a\theta)^{i-1} s_i a(\theta_1+\theta_2) h_{k-i}\\
&\quad + \sum_{i=1}^{k} (1-a\theta)^{i-1} S_{i+1} a\theta_1 \sum_{j=0}^{k-i} h_j h_{k-i-j}\\
&\quad + \sum_{i=1}^{k} (1-a\theta)^{i-1} S_{i+1} a\theta_2 h_{k-i},\ k \geq 1.
\end{aligned} \tag{15}$$

The generating function $h(x) = \sum_{k=0}^{\infty} h_k x^k$, $0 \leq x \leq 1$, of the BP satisfies the following relation:

$$h(x) = \frac{1-a(\theta_1+\theta_2)}{1-a\theta} S[(1-a\theta)x] + \frac{a\theta_3}{1-a\theta} \cdot \frac{(1-a\theta)x - S[(1-a\theta)x]}{1-(1-a\theta)x}$$
$$+ \frac{a(\theta_1+\theta_2)}{1-a\theta} \cdot S[(1-a\theta)x]h(x)$$
$$+ \frac{a\theta_1}{1-a\theta} \cdot \frac{(1-a\theta)x - S[(1-a\theta)x]}{1-(1-a\theta)x} h^2(x)$$
$$+ \frac{a\theta_2}{1-a\theta} \cdot \frac{(1-a\theta)x - S[(1-a\theta)x]}{1-(1-a\theta)x} h(x). \tag{16}$$

The above formula can be written as

$$\begin{aligned} & a\theta_1[(1-a\theta)x - s[(1-a\theta)x]]h^2(x) + \\ & + [a(\theta_1+\theta_2)[1-(1-a\theta)x]S[(1-a\theta)x] \\ & \quad + a\theta_2[(1-a\theta)x - S[(1-a\theta)x]] - (1-a\theta)[1-(1-a\theta)x]]h(x) \\ & + [1-a(\theta_1+\theta_2)][1-(1-a\theta)x]S[[(1-a\theta)x]] \\ & + a\theta_3[(1-a\theta)x - S[(1-a\theta)x]] = 0. \end{aligned} \tag{17}$$

Therefore, the generating function $h = h(x)$ satisfies the quadratic equation

$$f(h) = 0 \tag{18}$$

where

$$\begin{aligned} f(h) = & \; a\theta_1[(1-a\theta)x - s[(1-a\theta)x]]\, h^2 \\ & + [a(\theta_1+\theta_2)[1-(1-a\theta)x]S[(1-a\theta)x] \\ & \quad + a\theta_2[(1-a\theta)x - S[(1-a\theta)x]] - (1-a\theta)[1-(1-a\theta)x]]\, h \\ & + [1-a(\theta_1+\theta_2)][1-(1-a\theta)x]S[[(1-a\theta)x]] \\ & + a\theta_3[(1-a\theta)x - S[(1-a\theta)x]]. \end{aligned} \tag{19}$$

Let us note that for any $x \in (0, 1)$,

$$\begin{aligned} & a\theta_1[(1-a\theta)x - S[(1-a\theta)x]] > 0 \\ f(0) = & \; [1-a(\theta_1+\theta_2)][1-(1-a\theta)x]S[(1-a\theta)x] \\ & + a\theta_3[(1-a\theta)x - S[(1-a\theta)x]] > 0, \end{aligned} \tag{20}$$

and

$$f(1) = (1-a\theta)(x-1)[1-S[(1-a\theta)x]] < 0. \tag{21}$$

The above inequalities show that for any $x \in (0, 1)$, the equation (19) has two solutions, $h(x)$ and $h^*(x)$ satisfying the inequalities $0 < h(x) < 1 < h^*(x)$, and given by

$$h(x) = \frac{-b(x) - [b^2(x) - 4a(x)c(x)]^{1/2}}{2a(x)},$$
$$h^*(x) = \frac{-b(x) + [b^2(x) - 4a(x)c(x)]^{1/2}}{2a(x)},$$

where

$$\begin{aligned}
a(x) &= a\theta_1[(1-a\theta)x - S[(1-a\theta)x]],\\
b(x) &= [a(\theta_1+\theta_2)[1-(1-a\theta)x]S[(1-a\theta)x],\\
&\quad +a\theta_2[(1-a\theta)x - S[(1-a\theta)x]] - (1-a\theta)[1-(1-a\theta)x]],\\
c(x) &= [1-a(\theta_1+\theta_2)][1-(1-a\theta)x]S[[(1-a\theta)x]]\\
&\quad + a\theta_3[(1-a\theta)x - S[(1-a\theta)x]]. \qquad (22)
\end{aligned}$$

For $x = 1$ is $f(1) = 0$, which means that at least one of the two solutions $h(x)$ or $h^*(x)$ takes the value 1 for $x = 1$. Let us observe that the inequality $h^*(x) > 1$ holds if and only if

$$\begin{aligned}
&[b^2(1) - 4a(1)c(1)]^{1/2} > 2a(1) + b(1)\\
&= (1-a\theta)(\theta_1-\theta_3) - [\theta_1 + (\theta_1+\theta_2)(1-a\theta)]S(1-a\theta).
\end{aligned}$$

If the stability condition is fulfilled, the right-hand side of the above inequality is negative, and then $h^*(1) > 1$. Consequently $h(1) = 1$, and the generating function of the BP is $h(x)$.

The mean length of the BP is given by

$$\bar{h} = h'(1) = \frac{(1-a\theta)[1-S(1-a\theta)]}{a[[\theta_1+(\theta_1+\theta_2)(1-a\theta)]S(1-a\theta) - (1-a\theta)(\theta_1-\theta_3)]}. \qquad (23)$$

In order to find the generating function of the time that a customer spends in the queue, we need to consider the the GF $h(x;m)$ of the BP that starts with a customer to which remains m slots to finish its service. This GF $h(x;m)$ is given by

$$\begin{aligned}
h(x;m) &= \frac{[(1-a\theta)x]^m}{1-a\theta}[1-a(\theta_1+\theta_2)+a(\theta_1+\theta_2)h(x)]\\
&\quad + x\frac{1-[(1-a\theta)x]^{m-1}}{1-(1-a\theta)x}[a\theta_1 h^2(x) + a\theta_2 h(x) + a\theta_3]. \qquad (24)
\end{aligned}$$

Let us explain the above formula:

If after the first $m-1$ slots no customers have arrived to the system (with probability $(1-a\theta)^{m-1}$), then the BP ends with probability $1-a(\theta_1+\theta_2)$, or if in the slot m a non-negative new customer arrives (with probability $a(\theta_1+\theta_2)$, a new BP is opened with GF $h(x)$.

If after $i-1$ slots, $i = 1, \ldots, m-1$ no customers have arrived to the system (with probability $(1-a\theta)^{i-1}$) and in the slot i: a new customer arrives, then

- With probability $a\theta_1$, two BP area added with GF $h(x)\cdot h(x)$, one is opened by the new customer and the other is opened by the displaced customer to the head of the queue,
- With probability $a\theta_2$, a BP is opened with GF $h(x)$.
- With probability $a\theta_3$, the BP ends,

summing over i from 1 to $m-1$ the given formula of $h(x;m)$ is obtained.

Let us note that the above expression of $h(x;m)$ can be written as

$$\begin{aligned} h(x;m) \; = \; & \frac{1}{(1-a\theta)[1-(1-a\theta)x]} \\ & \Big[[(1-a\theta)]^m([1-(1-a\theta)x][1-a(\theta_1+\theta_2)+a(\theta_1+\theta_2)h(x)] \\ & -[a\theta_1 h^2(x)+a\theta_2 h(x)+a\theta_3]) \\ & +(1-a\theta)x[a\theta_1]h^2(x)+a\theta_2 h(x)+a\theta_3\Big]. \end{aligned} \tag{25}$$

6 Sojourn times

6.1 Sojourn time of a customer in the server

In this section, we will find the distribution of the time that a customer spends in the server. We will denote by b_k the probability that the sojourn time of a customer in the server (taking into account possible interruptions) last exactly k slots.

The distribution $\{b_k, k \geq 0\}$ is governed by the following recursive formulae:

$$\begin{aligned} b_0 &= 0, \\ b_k &= (\bar{a}+a\theta_0)^{k-1}s_k + (\bar{a}+a\theta_0)^{k-1}S_{k+1}a(\theta_2+\theta_3) + \\ &\quad + \sum_{i=1}^{k}(\bar{a}+a\theta_0)^{i-1}S_{i+1}a\theta_1 b_{k-i}, \; k \geq 1. \end{aligned}$$

The GF $b(x) = \sum_{k=0}^{\infty} b_k x^k$ is given by

$$\begin{aligned} b(x) = \; & \frac{1}{\bar{a}+a\theta_0}S[(\bar{a}+a\theta_0)x] + \frac{a(\theta_2+\theta_3)}{\bar{a}+a\theta_0} \cdot \\ & \frac{(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]}{1-(\bar{a}+a\theta_0)x} + \\ & \frac{a\theta_1}{\bar{a}+a\theta_0}\frac{(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]}{1-(\bar{a}+a\theta_0)x}b(x), \end{aligned} \tag{26}$$

that is

$$b(x) = \frac{[1-(\bar{a}+a\theta_0)x]S[(\bar{a}+a\theta_0)x] + a(\theta_2+\theta_3)[(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]]}{(\bar{a}+a\theta_0)[1-(\bar{a}+a\theta_0)x] - a\theta_1[(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]]},$$

and the corresponding time is given by

$$\bar{b} = b'(1) = \frac{(\bar{a}+a\theta_0)[1-S(\bar{a}+a\theta_0)]}{a[\theta_1 S(\bar{a}+a\theta_0)+(\bar{a}+a\theta_0)(\theta_2+\theta_3)]}.$$

Let us note that the stability condition (10) can be written as

$$\rho = a(1-\theta_3)\bar{b} < 1.$$

6.2 Sojourn time of a customer in the system

Firstly, we will find the distribution of the period of time that a customer spends in the system from the beginning of its service until the moment of its departure.
Let g_k be the probability that this period of time lasts exactly k slots. Then, we have

$$\begin{aligned} g_0 &= 0, \\ g_k &= (\bar{a}+a\theta_0)^{k-1}s_k + (\bar{a}+a\theta_0)^{k-1}a(\theta_2+\theta_3)S_{k+1} \\ &\quad + \sum_{i=1}^{k}(\bar{a}+a\theta_0)^{i-1}S_{i+1}a\theta_1\sum_{j=0}^{k-i}h_j g_{k-i-j},\ k \geq 1. \end{aligned} \tag{27}$$

The GF $g(x) = \sum_{k=0}^{\infty} g_k x^k,\ 0 \leq x \leq 1$, is given by

$$\begin{aligned} g(x) &= \frac{1}{\bar{a}+a\theta_0}S[(\bar{a}+a\theta_0)x] + \frac{a(\theta_2+\theta_3)}{\bar{a}+a\theta_0}\cdot\frac{(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]}{1-(\bar{a}+a\theta_0)x} \\ &\quad + \frac{a\theta_1}{\bar{a}+a\theta_0}\cdot\frac{(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]}{1-(\bar{a}+a\theta_0)x}h(x)g(x), \end{aligned}$$

that is

$$g(x) = \frac{[1-(\bar{a}+a\theta_0)x]S[(\bar{a}+a\theta_0)x] + a(\theta_2+\theta_3)[(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]]}{(\bar{a}+a\theta_0)[1-(\bar{a}+a\theta_0)x] - a\theta_1 h(x)[(\bar{a}+a\theta_0)x - S[(\bar{a}+a\theta_0)x]]},$$

and the corresponding mean time is given by

$$\bar{g} = g'(1) = \frac{(\bar{a}+a\theta_0)[1-S(\bar{a}+a\theta_0)] + a\theta_1\bar{h}[(\bar{a}+a\theta_0) - S(\bar{a}+a\theta_0)]}{a[\theta_1 S(\bar{a}+a\theta_0) + (\bar{a}+a\theta_0)(\theta_2+\theta_3)]}.$$

Let us observe that if $\theta_1 = 0$, then $g(x) = b(x)$.

6.2.1 Sojourn time of a customer in the queue

The stationary distribution of the sojourn time that a customer spends in the queue until the beginning of its service has the following GF:

$$\begin{aligned} w(x) &= \pi_0 + \varphi_{1,1}(1) + (1-\theta_0)\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i+1,k} + \theta_0\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i+1,k}h(x;i)h(x)^k, \\ &\quad 0 \leq x \leq 1. \end{aligned}$$

From (25) and using the GF's introduced in Section 3, the above formula becomes

$$\begin{aligned} w(x) &= \pi_0 + \theta_0\varphi_{1,1} + (1-\theta_0)\varphi_1(1,1) + \theta_0\Big[F_1(x)\frac{\varphi_1[(1-a\theta)x, h(x)]}{(1-a\theta)x} \\ &\quad + F_2(x)\varphi_1(1,h(x)) - [F_1(x)+F_2(x)]\varphi_{1,1}(h(x))\Big], \end{aligned}$$

where

$$F_1(x) = \frac{1}{(1-a\theta)[1-(1-a\theta)x]}\Big[[1-(1-a\theta)x[1-a(\theta_+\theta_2)] + a(\theta_1+\theta_2)h(x)] \\ -[a\theta_1 h^2(x) + a\theta_2 h(x) + a\theta_3]\Big],$$

$$F_2(x) = \frac{x}{1-(1-a\theta)x}\Big[a\theta_1 h^2(x) + a\theta_2 h(x) + a\theta_3\Big].$$

The corresponding mean time is given by

$$\bar{w} = w'(1) = \theta_0\Big[F_1'(1)\frac{\varphi_1[1-a\theta, 1]}{1-a\theta} + F_2'(1)\varphi_1(1,1) + \varphi_1'(1, h(x))|_{x=1} \\ -(F_1'(1) + F_2'(1))\varphi_{1,1}(1) - \varphi_{1,1}'(h(x))|_{x=1}\Big],$$

where

$$F_1'(1) = -\frac{1}{a\theta(1-a\theta)}\Big[1 - a\theta + a\bar{h}[a(\theta_1+\theta_2)(1-a\theta) + \theta_1]\Big],$$

$$F_2'(1) = \frac{1 + a\bar{h}[2\theta_1 + \theta_2]}{a\theta},$$

$$\varphi_1[1-a\theta, 1] = S'(1-a\theta)\frac{a(1-a\theta)(1-\theta_0)(1-\theta_3)}{\theta_1 S(1-a\theta) + (\theta_2+\theta_3)(1-a\theta)},$$

$$\varphi_1'(1, h(x))|_{x=1} = \frac{1 - S(1-a\theta) - (1-a\theta)S'(1-a\theta)}{D(1)\Big[\theta_1 S(1-a\theta) + (\theta_2+\theta_3)(1-a\theta)\Big]}a^2\theta_0(1-\theta_3)\bar{h},$$

$$\varphi_{1,1}'(h(x))|_{x=1} = \frac{\theta_0\Big[a(1-\theta_0)S'(1-a\theta) - S(1-a\theta)\Big]D(1) - (1-\theta_0)S(1-a\theta)D'(1)}{D(1)\Big[\theta_1 S(1-a\theta) + (\theta_2+\theta_3)(1-a\theta)\Big]} \\ \cdot\ a(1-\theta_3)\bar{h}.$$

The total time that a customer spends in the queue is given by

$$\bar{W} = \bar{w} + \bar{g} - \bar{b}.$$

Finally, the GF $v(x),\ 0 \le x \le 1$, of the stationary distribution of the sojourn time of a customer in the system is given by

$$v(x) = w(x)\, g(x),$$

and the mean sojourn time of a customer in the system is given by

$$\bar{v} = v'(1) = \bar{w} + \bar{g}.$$

7 Numerical results

In this section, we present some numerical results to illustrate the effect of varying parameters on the main performance measures of our system.

In the following figures and tables, we suppose that the service times take exactly two slots.

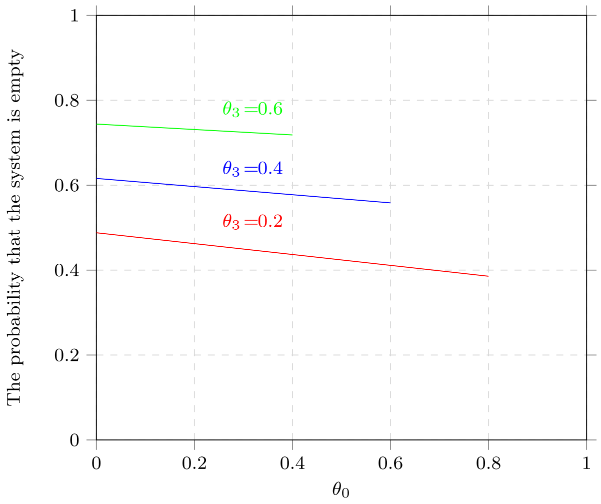

Fig. 2 Probability of an empty system against θ_0 for $a = 0.4,\ \theta_1 = 0$

In Fig. 2, the probability that the system is empty is plotted versus the parameter θ_0. We have presented three curves which correspond to $\theta_3 = 0.2\,0.4\,0.6$ respectively. As we expect, the probability that the system is empty decreases with increasing values of the parameter θ_0 and increases with increasing values of the θ_3.

In Fig. 3, we illustrate the behavior of $E[N]$ as a function of θ_0. As intuition tells us, $E[N]$ increases when θ_0 increases and θ_3 decreases.

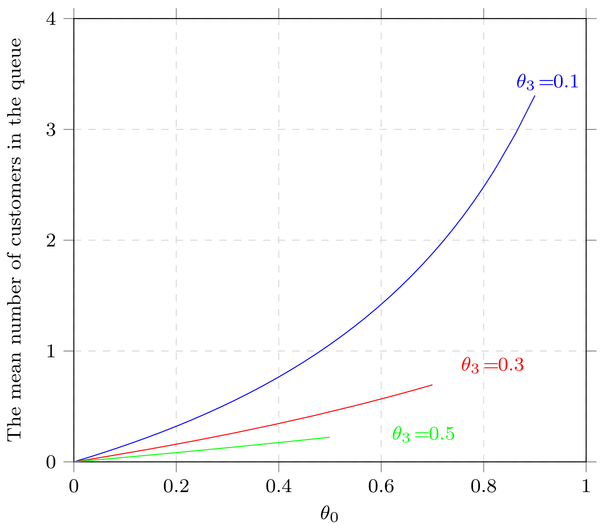

Fig. 3 The mean number of customers in the queue against θ_0 for $a = 0.5,\ \theta_1 = 0$

Table 1 for $a = 0.4,\ \theta_1 = 0$

	$\theta_0 = 0$	$\theta_0 = 0.3$	$\theta_0 = 0.6$	$\theta_0 = 1$
ψ_0	1	0.866666	0.733333	0.555555
ψ_1	0	0.115555	0.195555	0.246913
ψ_2	0	0.015407	0.052148	0.109739
ψ_3	0	0.002054	0.013906	0.048773
ψ_4	0	0.000274	0.003708	0.021677
ψ_5	0	0.000037	0.000989	0.009634
ψ_6	0	0.000005	0.000264	0.004282
ψ_7	0	0.000001	0.000070	0.001903
ψ_8	0	0	0.000019	0.000846

Table 2 For $a = 0.4,\ \theta_1 = 0$

	$\theta_0 = 0$	$\theta_0 = 0.3$	$\theta_0 = 0.6$	$\theta_0 = 1$
ϕ_0	0.36	0.312	0.264	0.2
ϕ_1	0.64	0.554666	0.469333	0.355555
ϕ_2	0	0.115555	0.195555	0.246913
ϕ_3	0	0.015407	0.052148	0.109739
ϕ_4	0	0.002054	0.013906	0.048773
ϕ_5	0	0.000274	0.003708	0.021677
ϕ_6	0	0.000037	0.000989	0.009634
ϕ_7	0	0.000005	0.000264	0.004282
ϕ_8	0	0.000001	0.000070	0.001903

Table 3 For $a = 0.4,\ \theta_0 = 0$

	$\theta_1 = 0$	$\theta_1 = 0.3$	$\theta_1 = 0.6$	$\theta_1 = 0.8$
ψ_0	1	0.757575	0.438596	0.163398
ψ_1	0	0.161616	0.187134	0.092955
ψ_2	0	0.053872	0.124754	0.082626
ψ_3	0	0.017957	0.083169	0.073445
ψ_4	0	0.005985	0.055446	0.065284
ψ_5	0	0.001995	0.036964	0.058030
ψ_6	0	0.000665	0.024642	0.051582
ψ_7	0	0.000221	0.016428	0.045851
ψ_8	0	0.000073	0.010952	0.040756

Table 4 For $a = 0.4,\ \theta_0 = 0$

	$\theta_1 = 0$	$\theta_1 = 0.3$	$\theta_1 = 0.6$	$\theta_1 = 0.8$
ϕ_0	0.36	0.272727	0.157894	0.058823
ϕ_1	0.64	0.484853	0.280701	0.104575
ϕ_2	0	0.161616	0.187134	0.092955
ϕ_3	0	0.053872	0.124754	0.082626
ϕ_4	0	0.017957	0.083169	0.073445
ϕ_5	0	0.005985	0.055446	0.065284
ϕ_6	0	0.001995	0.036964	0.058030
ϕ_7	0	0.000665	0.024642	0.051582
ϕ_8	0	0.000221	0.016428	0.045851

An important feature of this work is in the recursion scheme provided by Theorem 1 and 2. The formulae (11), (12) and (14) have been implemented in Tables 1, 2, 3 and 4 for:

8 Conclusions

In this paper, a discrete-time queueing system in which the arriving customers may opt to choose several strategies is studied. These different strategies give to the considered model a wide versatility which allows to include special cases of interest.

A thorough analysis of the system has been carried out obtaining the generating functions of the number of customers in the queue and in the system and its expected values.

The recursive algorithm provided by theorems 2 and 3 for calculating the steady-state probabilities of the number of customers in the queue and in the system is an important feature of this paper. The busy period of an auxiliary system useful to study the customer's delay has been studied.

The analysis carried out to obtain the distribution of the sojourn time that a customer spends in the queue and in the system constitutes an important research contribution of the paper. Finally, tables and graphics that illustrate the effect of the parameters on several performance characteristics are provided.

Acknowledgments We would like to thank the anonymous referees for their helpful comments.

References

1. Artalejo, J.: G-networks: A versatile approach for work removal in queueing networks. Eur. J. Oper. Res. **126**, 233–249 (2000)

2. Atencia, I.: A discrete-time system with service control and repairs. Int. J. Appl. Math. Comput. Sci. **24**(3), 471–484 (2014)
3. Atencia, I.: A discrete-time system queueing system with server breakdowns and changes in the repair times. Annals of Operations Research. https://doi.org/10.1007/s10479-015-1940-3 (2015)
4. Atencia, I., Fortes, I., Sánchez, S.: Discrete-time queueing system with expulsions. Communications in Computer and Information Science **356**, 20–25 (2013)
5. Atencia, I., Moreno, P.: The discrete-time Geo/Geo/1 queue with negative customers and disasters. Comput. Oper. Res. **31**(9), 1537–1548 (2004)
6. Atencia, I., Moreno, P.: A single-server G-queue in discrete-time with geometrical arrival and service process. Perform. Eval. **59**, 85–97 (2005)
7. Chao, X., Miyazawa, M., Pinedo, M.: Queueing Networks: Customers, Signals and Product Form Solutions. Wiley, New York (1999)
8. Fiems, D., Steyaert, B., Bruneel, H.: Randomly interrupted $GI/G/1$ queues: service strategies and stability issues. Ann. Oper. Res. **112**, 171–183 (2002)
9. Gelenbe, E., Label, A.: G-networks with multiple classes of signals and positive customers. Eur. J. Oper. Res. **108**, 293–305 (1998)
10. Harrison, P.G., Patel, N.M., Pitel, E.: Reliability modelling using G-queues. Eur. J. Oper. Res. **126**, 273–287 (2000)
11. Krishnamoorthy, A., Gopakumar, B., Viswanath Narayanan, V.: A retrial queue with server interruptions, resumption and restart of service. Operations Research International Journal **12**, 133–149 (2012)
12. Krishnamoorthy, A., Pramod, P., Chakravarthy, S.: A note on characterizing service interruptions with phase-type distribution. Journal of Stochastic Analysis and Applications **31**(4), 671–683 (2013)
13. Krishnamoorthy, A., Pramod, P., Chakravarthy, S.: A survey on queues with interruptions. TOP **22**, 290–320 (2014)
14. Park, H.M., Yang, W.S., Chae, K.C.: The $Geo/G/1$ queue with negative customers and disasters. Stoch. Model. **25**, 673–688 (2009)
15. Piórkowski, A., Werewka, J.: Minimization of the total completion time for asinchronous transmission in a packet data-transmission system. Int. J. Appl. Math. Comput. Sci. **20**, 391–400 (2010)
16. Walraevens, J., Steyaert, B., Bruneel, H.: A preemptive repeat priority queue with resampling: performance analysis. Ann. Oper. Res. **146**, 189–202 (2006)
17. White, H., Christie, L.: Queuing with preemptive priorities or with breakdown. Oper. Res. **6**(1), 79–95 (1958)