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# Non-unique conical and non-conical tangents to rectifiable stationary varifolds in $\mathbb{R}^{4}$ 

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#### Abstract

We construct a rectifiable stationary 2-varifold in $\mathbb{R}^{4}$ with non-conical, and hence non-unique, tangent varifold at a point. This answers a question of L. Simon (Lectures on geometric measure theory, 1983, p. 243) and provides a new example for a related question of W. K. Allard (On the first variation of a varifold, Ann. of Math., 1972, p. 460).

There is also a (rectifiable) stationary 2 -varifold in $\mathbb{R}^{4}$ that has more than one conical tangent varifold at a point.


Keywords stationary varifold $\cdot$ varifold tangent $\cdot$ tangent cone $\cdot$ non-unique $\cdot$ non-conical $\cdot$ minimal surface $\cdot$ regularity

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## Contents

1 Introduction ..... 2
1.1 General context ..... 2
1.2 Known results ..... 4
1.3 The questions and the main result ..... 5
2 Notation and definitions ..... 6
2.1 Varifolds ..... 7
2.2 The first variation. Stationary varifolds. The mass. The curvature. ..... 7
2.3 An example ..... 9
2.4 Tangents. Conical varifolds ..... 9
3 The non-rectifiable varifold ..... 10
4 The rectifiable varifold ..... 13
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4.1 Basic surface, rings and their joins. ..... 15
4.2 Mini-layer. Details about branching ..... 22
4.3 Layers ..... 25
5 Two variants of the main result ..... 31

## 1 Introduction

### 1.1 General context

Geometrical measure theory uses various "generalized surfaces" to reach its goals, and the varifolds are among them. Most of them allow, i.a., countably many pieces of surface that are interconnected into simple or complicated networks (Figure 1). The classes of surfaces are designed to have compactness properties and to allow to obtain a generalized surface of least area among those that, say, span a given boundary. The next, equally important, step is to explore smoothness and regularity properties of the minimizer.

Uniqueness of tangents is both an important attribute encompassed in various definitions of smoothness and regularity, and an important tool. Here a tangent is (informally) defined as a limit of a sequence of blow-ups at a given point. The tangents implicitly appear already in the basic calculus of real functions: A Lipschitz function on $\mathbb{R}^{n}$ is differentiable at a point $x_{0}$ if and only if it admits a unique tangent at $x_{0}$ and the tangent is a hyperplane. For $n=1$, the existence of the two one-sided derivatives at $x_{0}$ is equivalent to uniqueness of the tangent at $x_{0}$ and the tangent is then necessarily a cone.


Fig. 1 Some networks of segments. The first two illustrate analogy and differences of linear and central (radial) configurations. The idea contained in the last one is actually used in this paper. a) The set of weak limits of sequences of vertical upward shifts of the varifold corresponding to this network is uncountable. b) With the unit density on each segment we have non-unique tangents but the varifold is not stationary. Although it can be converted to a stationary varifold by assigning suitable densities, it does not provide a stationary example with non-unique tangents. The densities necessarily converge to zero near the center, and the zero varifold is the unique tangent at the centre. c) This network is continually branching and refining in the downward direction (towards an interface line). Such a network was used by Brakke [Bra, p. $238,240,250]$ in the context of varifolds evolving by its 'mean curvature'. We use its radial variation in a more complicated arrangement.

Likewise, a junction of three smooth curves $\gamma_{i}:[0,1) \rightarrow \mathbb{R}^{2}$ at $x_{0} \in \mathbb{R}^{2}$ (Figure 2) is considered more regular if the object has a unique tangent at $x_{0}$ (the curves have a


Fig. 2 Three curves in the plane. (Think about this also as a planar section of a hypothetical joint of minimal surfaces in equilibrium.) a) Curves smooth up to the end. Unique (and conical) tangent at $x_{0}$. b) The logarithmic spirals. Non-unique tangent. The tangents are represented by the rotations of the same picture. c) "Spirals with varying speed" $r(\alpha)=e^{c\left(\alpha-\alpha_{0}\right)^{2}}, \alpha \geq \alpha_{0}$. The tangents are all $120^{\circ}$-triples of half-lines, so the tangents are conical but non-unique.
non-zero one-sided derivative at the endpoint). In this case they can also be studied as graphs of functions satisfying a differential equation, and this can be helpful if they came out of a variational problem.

The uniqueness of tangents is the regularity, or a basic degree of the regularity. It is an interpretation of what existence of the derivative would be in case we face more general objects than graphs of functions. In fact, the mathematical language is somewhat inhomogeneous in not having a single word for "the unique tangent" (of a varifold, e.g.) as a counterpart of "the derivative". ${ }^{1}$ This choice of terminology is not surprising since uniqueness of tangents in Geometrical measure theory is from the beginnings connected to open problems and later only to partial results. ${ }^{2}$

Now let us give an example of how uniqueness of tangents might be used as a tool: It is the result of Sheldon Chang that the singular set of area minimizing twodimensional integral currents consists of isolated points and that near any such point their structure is the same as that of a classical branched minimal surface [Ch].

Based on the work of B. White [W1], Chang first notes that (in the case he considers, i.e., the case of Riemannian manifolds) two dimensional area minimizing integral currents have unique tangent cones and he estimates the rate of pointwise convergence. He says that this steps are "necessary for the construction of the first center manifold." [Ch, p. 701].

[^0]The uniqueness of tangents is also used in [DL-S, Chapter 5 and Theorem 0.12] where an improvement (in Chang's spirit) of the size of the singular set of Dirminimizing Q -valued functions (on $\Omega \subset \mathbb{R}^{2}$ ) is given.

### 1.2 Known results

The structure of one-dimensional stationary varifolds with density bounded away from zero is well known [AA1].

A result about uniqueness of tangent cones of two-dimensional soap-bubble-like and soap-film-like minimal surfaces $\left((M, \xi, \delta)\right.$-minimal sets) in $\mathbb{R}^{3}$ is contained in [T].

Tangent cones to two-dimensional area-minimizing integral currents are unique by the result of B. White [W1]. As we already noted, this was generalized to Riemannian manifolds by Chang [Ch].

For more general dimensions, there are results for some special cases, with assumptions related for example to calibration. Note that the notions of ( $\omega$-)positive, (semi-)calibrated and (pseudo-)holomorphic currents are to a large extent synonymous (cf. [Be1], [Be2]). Recent results with this kind of assumptions can be found in [PR] (2-dimensional), [Be1], [Be2]. As Bellettini [Be1] notes, the integrable case $\mathbb{C}^{n}$ of his results follows already by [Siu].

Very nice result is [S2], which has a partial generalisation [S4]. Simon [S2, Corollary on page 564] does not assume calibrations. The corollary states that if $C$ is a tangent cone to a stationary varifold $V$ at a point $p, C$ has density 1 on $\operatorname{spt} C \backslash\{0\}$ (hence $C$ is integral and 0 is the only singular point of $C$ ) then $C$ is the unique tangent cone of $V$ at $p$ and we have a $\mathscr{C}^{2}$-flavour of convergence of blowups at $p$. [S2] improved earlier result [AA2] which included assumption on integrability of Jacobi fields and already covered the case of the cone over the cartesian product of two (but not more, cf. [W2]) standard spheres (of arbitrary dimensions). [S4] provides similar result where $C=C_{0} \times \mathbb{R}$ are allowed to be certain cases of cylinders with singular set $\{0\} \times \mathbb{R}$. ( $C_{0}$ is assumed to be a strictly minimizing, strictly stable codimension one cone, and to admit a nice Jacobi-field operator).

As it can be seen from the above, even the codimension one case remains open. Notably, it remains open whether the hyper-cones over $\mathbb{S}^{3} \times \mathbb{S}^{3} \times \mathbb{R}$ and $\mathbb{S}^{2} \times \mathbb{S}^{4} \times \mathbb{R}$ in $\mathbb{R}^{9}$ are always unique tangent cones when they arise at all as multiplicity one tangent cones [S4, p. 1-2]. (The question in its formulation in [S4] seemingly concerns the hyper-cones over $\mathbb{S}^{3} \times \mathbb{S}^{3}$ and $\mathbb{S}^{2} \times \mathbb{S}^{4}$ in $\mathbb{R}^{8}$ but that was already solved by [AA2, p. 215, (1) and (2)], as well as [S2].)

Kiselman's example [K] with non-unique tangent cones is mentioned in the next paragraph. There is also an example [Ko] consisting of spirals and a number of lines. It shares with the minimal surfaces an important property called the "monotonicity" - for balls centered at an arbitrary fixed point the measure ratio is non-decreassing. In [CKR-R2] and [CKR-R3] the number of lines is reduced so that the density is, everywhere in the support, between 1 and $3+\varepsilon$ (the planar example), or between 1 and $2+\varepsilon$ (the example in $\mathbb{R}^{3}$ ).
1.3 The questions and the main result

The purpose of this paper is to answer a question of L. Simon [S1, p. 243]. Simultaneously we provide a new example for a related question of W. K. Allard [A, p. 460].

Allard's question was in a different spirit already solved by [HM] because Allard's formulation allowed non-stationary varifolds. It was also answered by Kiselman $[\mathrm{K}]$, who constructed a closed positive current in $\mathbb{C}^{2}$ with non-unique tangent cones. The current is not rectifiable since its support contains separating 3-dimensional surfaces created by use of max in [K, (4.3)], at least when applied as described in Examples 4.2 and 4.3 [K]. He also uses smoothing by convolution. Kiselman's example was generalized to general bidegree $(p, p)$ in [ B 1 , Theorem 3.11]. Also this example is not rectifiable since the current $W=i \partial \bar{\partial} F$ is added on [ $\mathrm{Bl}, \mathrm{p} .528$, p. 529], where $F$ equals a power of $-\log |z|^{2}$ in a neighbourhood of 0 .

The book [S1] and the paper [A] are standard sources cited when varifolds and related regularity results are of concern. Varifolds are generalized (non-oriented) surfaces and admit compactness properties suitable to approach the problem of existence of surfaces with minimal area.

On p. 243, L. Simon recalls the definition of tangent varifolds. He proves that if $C$ is a tangent varifold (and if some natural conditions are satisfied), then $\mu_{C}$ is conical, where $\mu_{C}$ denotes the measure in $\mathbb{R}^{n}$ associated with $C$ by the direction-forgetting projection $G_{m}\left(\mathbb{R}^{n}\right) \rightarrow \mathbb{R}^{n}$. He says that it seems to be an open question whether $C$ itself has to be conical.

Likewise, W. K. Allard [A, p. 459-460] states that all $C \in \operatorname{Var}_{\operatorname{Tan}}^{a}$ $V$ are conical (under some conditions on densities of $V$ and $\delta V$ ) and then he says he knows of no varifold (with a weak condition on the densities of $V$ and $\delta V$ at $a$ ) such that $\operatorname{Var} \operatorname{Tan}_{a} V$ has more than one element. We already noted that examples of varifolds with properties specified by Allard were provided by [HM] (non-stationary, which is not natural in context of [A]) and [K] (non-rectifiable).

The result that we prove in this paper is the following (see Theorem 5.1 and Theorem 5.2).

Theorem 1.1 There exists a stationary rectifiable 2 -varifold in $\mathbb{R}^{4}$ that has a nonconical (hence non-unique) tangent at a point. There exists a stationary rectifiable 2 -varifold in $\mathbb{R}^{4}$ that has a conical but non-unique tangent at a point. (The varifolds have a positive and finite $k$-dimensional density at the point.)

Note that there is no such varifold $V$ with non-conical tangent and $\theta^{2}\left(\mu_{V}, \cdot\right)$ bounded away from zero on spt $\mu_{V}$, as the following results imply.

Lemma 1.1 Let $V$ be a stationary $m$-varifold on an open set $\Omega \subset \mathbb{R}^{n}, x_{0} \in \Omega, C \in$ $\operatorname{Var} \operatorname{Tan}_{x_{0}} V$ and $C \neq 0 .^{3}$

If $\theta^{m}(C, x)>0$ for $\mu_{C}$-almost every $x$, then $C$ is conical and rectifiable. (Stated on [S1, p. 243], proved in proof of [S1, Corollary 42.6]. Alternatively see [A, 5.2(2)(b)] for conicity of $\mu_{C}$ and then the rectifiability theorem [A,5.5(1)] for how this determines the directions $C^{(x)}$ of $C$.)

[^1]If $C$ is rectifiable, then (equivalently) $\theta^{m}(C, x)>0$ for $\mu_{C}$-almost every $x$ and hence $C$ is again conical.

If $\theta^{m}\left(\mu_{V}, \cdot\right) \geq c>0 \mu_{V}$-almost everywhere then $\theta^{m}(C, \cdot) \geq c>0 \mu_{C}$-almost everywhere and $C$ is conical (and rectifiable) [S1, proof of Corollary 42.6], [A, 6.5].

Further note that if $C$ is a tangent varifold from our example, then $\mu_{C}$ must be conical [S1, 42.2 on p. 243].

For stationary 1 -varifolds, the tangent varifolds $C$ are always conical since $\mu_{C}$ is conical and $x \in S$ (equivalently, $p_{S^{\perp}}(x)=0$ ) for all $(x, S) \in \operatorname{spt} C$ [S1, p. 243, 1. 2-3]. For stationary 1 -varifolds with density bounded away from zero, the tangent varifolds are conical and unique [AA1].

In Remark on page 449, [A] relates conicity of stationary varifolds to the constancy of its "sphere slices $B^{(r) "}$ that are implicitly defined by [A, Theorem 5.2(3)]. Namely, he writes: There is $C$ as in [A, Theorem 5.2(2), p. 446] (i.e., $C$ a stationary $k$-varifold, with the density $\left.\theta^{k}(C, 0) \geq \mu_{C}\left(B_{1}\left(\mathbb{R}^{n}\right)\right)\right)$ which is not homothetically invariant (conical) if and only if there is B as in [Theorem 5.2(3), p. 448] (i.e., for almost every $r>0$, the slice $B^{(r)}$ is a $(k-1)$-varifold in $S_{1}\left(\mathbb{R}^{n}\right)$, which is 'stationary in the manifold $S_{1}\left(\mathbb{R}^{n}\right)^{\prime}$ if $k \geq 2$ resp. balanced if $k=1$, with $r \mapsto \mu_{B^{(r)}}$ almost constant and $r \mapsto B^{(r)}$ measurable) which is not almost constant. The simpler nonrectifiable version of our examples (see Section 3) shows that both statements are (unfortunately) true. Note that the example of Kiselman [K], and our rectifiable example (Sections 4 and 5) are not applicable to these statements (the monotonicity ratio and corresponding function $r \mapsto \mu_{B^{(r)}}$ are far from constant).

## 2 Notation and definitions

For $0 \leq r<s \leq \infty$, denote by $S_{r}\left(\mathbb{R}^{n}\right)$ the sphere of radius $r$ in $\mathbb{R}^{n}$ and $A_{r}^{s}=A_{r}^{s}\left(\mathbb{R}^{n}\right)=$ $\left\{x \in \mathbb{R}^{n}: r \leq\|x\| \leq s\right\}$ the annulus (or shell) in $\mathbb{R}^{n}$. Let $\mathbb{S}^{1}=S_{1}\left(\mathbb{R}^{2}\right)$.
$X$ denotes a smooth compactly supported vector field on $\mathbb{R}^{n}$ (or on $\Omega \subset \mathbb{R}^{n}$ ).
If $v$ is a measure and $M$ is $v$-measurable then $v\llcorner M$ denotes the restriction of $v$ to $M:(v\llcorner M)(A)=v(M \cap A)$.
$\phi_{\#} \mu$ denotes the image measure [F, 2.1.2]:

$$
\begin{equation*}
\phi_{\#} \mu(A)=\mu\left(\phi^{-1}(A)\right) . \tag{1}
\end{equation*}
$$

If $V$ is a $k$-varifold in $\mathbb{R}^{n}$ (i.e., a measure on $G_{k}\left(\mathbb{R}^{n}\right)$, see Section 2.1 ), then we write $\phi_{\#} V$ for the image measure (if $\operatorname{dom} \phi \subset G_{k}\left(\mathbb{R}^{n}\right)$ ) defined by (1) and

$$
\phi_{\# \#} V
$$

for the image varifold (assuming $\operatorname{dom} \phi \subset \mathbb{R}^{n}$; see Section 2.4). The standard notation for both is the same $\left(\phi_{\#} V\right)$ which would cause difficulties when reading some expressions in this paper.

### 2.1 Varifolds

To recall basic notions we follow and extend [ O ' $\mathrm{N}, \mathrm{p} .4-5$, $\S$ Varifolds]. More details can be found in [A] and [S1]. An $m$-varifold $V$ on an open subset $\Omega \subset \mathbb{R}^{n}$ is a Radon measure on

$$
G_{m}(\Omega):=\Omega \times G(n, m) .
$$

( $G(n, m)$ denotes the Grassmann manifold consisting of $m$-dimensional linear subspaces of $\mathbb{R}^{n}$.) The space of $m$-varifolds is equipped with the weak topology given by saying that $V_{i} \rightarrow V$ if and only if $\int f \mathrm{~d} V \rightarrow \int f \mathrm{~d} V$ for all compactly supported, continuous real-valued functions on $G_{m}(\Omega)$. Varifolds can be combined using the addition which is addition of measures $\left(\left(c_{1} V_{1}+c_{2} V_{2}\right)(B)=c_{1} V_{1}(B)+c_{2} V_{2}(B)\right)$. A countable sum of varifolds is also a varifold, provided it is a Radon measure, i.e., it assign finite values to compact sets.

To a given $m$-varifold $V$, we associate a Radon measure $\mu_{V}$ on $\Omega$ by setting $\mu_{V}(A)=V\left(G_{m}(A)\right)$ for $A \subset \Omega . \mu_{V}$ is called the weight of $V$ ([S1, p. 229]). As a partial converse, to a (Radon) $m$-rectifiable measure $\mu$ (see [O'N]) we can associate an $m$-rectifiable varifold $V=V_{\mu}$ by defining

$$
\begin{equation*}
V(B)=\mu\left\{x:\left(x, T_{x}\right) \in B\right\}, \quad B \subset G_{m}(\Omega) \tag{2}
\end{equation*}
$$

where $T_{x}$ is the approximate tangent plane at $x .{ }^{4}$ If a countable sum of rectifiable varifolds is also a varifold then it is rectifiable. In this paper we need only the following particular case of rectifiable varifolds (and their countable sums): $V=V_{c} \cdot \mathscr{H}^{m}\llcorner S$ where $S=\operatorname{range}(U)$ is a smooth parameterized surface and $c \in(0, \infty)$. Then the approximate tangent plane $T_{U(x)}$ agrees ( $\mu_{V}$-almost everywhere) with the classical tangent $\operatorname{span}\left\{\partial U / \partial x^{1}, \ldots, \partial U / \partial x^{m}\right\}$ to $S$, and $V$ is exactly $c \cdot \mathbf{v}(S)$ from [A, p. 431].

The support of a measure $\mu$ is denoted by spt $\mu$. Note that if $V$ is an $m$-varifold in $\Omega \subset \mathbb{R}^{n}$ then $\operatorname{spt} V \subset G_{m}\left(\mathbb{R}^{n}\right)$ while spt $\mu_{V} \subset \mathbb{R}^{n}$. If $V$ is an $m$-varifold (hence also a measure) and we say that $V$ is supported by a set $M$ if $M \subset \mathbb{R}^{n}$ and $\mu_{V}\left(\mathbb{R}^{n} \backslash M\right)=0$ or $M \subset G_{m}\left(\mathbb{R}^{n}\right)$ and $V\left(G_{m}\left(R^{n}\right) \backslash M\right)=0$. If $V$ is an $m$-varifold on $\Omega \subset \mathbb{R}^{n}$ and $M \subset \Omega$ then $V\left\llcorner G_{m}(M)\right.$ might be called the restriction of $V$ to $M$.

The density that we use in Introduction is defined as

$$
\theta^{k}(\mu, x)=\lim _{r \rightarrow 0+} \mu\left(x+A_{0}^{r}\right) / r^{k}
$$

for a measure $\mu$ on $\mathbb{R}^{n}$, and by $\theta^{k}(V, x)=\theta^{k}\left(\mu_{V}, x\right)$ for a varifold $V$.
2.2 The first variation. Stationary varifolds. The mass. The curvature.

The first variation of an $m$-varifold $V$ is a map from the space of smooth compactly supported vector fields on $\Omega$ to $\mathbb{R}$ defined by (see [A, p. 434] and [S1, p. 234, p. 51])

$$
\begin{equation*}
\delta V(X)=\int_{\Omega} \operatorname{div}_{S} X(x) \mathrm{d} V(x, S) \tag{3}
\end{equation*}
$$

[^2]where $\operatorname{div}_{S} X(x)$ is the divergence at $x$ of the field $X$ restricted (and projected) to affine subspace $x+S$ ([S1, p. 234]). The idea is that the variation measures the rate of change in the 'size' (mass) of the varifold if it is perturbed slightly (see the alternate formula in [S1, p. 233]). The mass of the varifold (see [S1, p. 229]) is given by
$$
\mathbf{M}(V)=V\left(G_{m}(\Omega)\right)=\mu_{V}(\Omega) .
$$

If $\delta V=0$, then the varifold is said to be stationary. Varifold $V_{\mathscr{H}^{m}\llcorner S}$ associated to an $m$-dimensional affine plane $S$ in $\mathbb{R}^{n}$ is stationary.

Assume $V=V_{\mathscr{H}^{m}{ }_{L} S}$ is the rectifiable varifold associated to Hausdorff measure restricted to a smooth surface $S \subset \mathbb{R}^{n}$ such that the closure $\bar{S}$ is a $C^{2}$-smooth compact manifold with smooth ( $m-1$ )-dimensional boundary $\partial S:=\bar{M} \backslash M$. Then (3) reads

$$
\begin{equation*}
\delta V(X)=\int_{S} \operatorname{div}_{T_{x}} X(x) \mathrm{d} \mathscr{H}^{m}(x) \tag{4}
\end{equation*}
$$

and can be (see [S1, 7.6]) computed as

$$
\begin{equation*}
\delta V(X)=-\int_{S} X \cdot \mathbf{H} \mathrm{~d} \mathscr{H}^{m}-\int_{\partial S} X \cdot \eta \mathrm{~d} \mathscr{H}^{m-1} \tag{5}
\end{equation*}
$$

where $\eta$ is the inward pointing unit co-normal of $\partial S$, cf. [S1, p. 43], and $\mathbf{H}$ is the mean curvature vector ([S1, 7.4]). If $U$ is a parameterization of $S$ and $\mathscr{B}(x):=$ $\left\{\partial U / \partial x^{1}, \ldots, \partial U / \partial x^{m}\right\}$ happens to be orthonormal at $x$ then $\mathbf{H}$ can be obtained (cf. 7.4 together with the last line on p. 44 of [S1]) as

$$
\mathbf{H}(U(x))=\sum_{i=1}^{m}\left(\frac{\partial^{2} U(x)}{\left(\partial x^{i}\right)^{2}}\right)^{\perp}
$$

where $v^{\perp}$ denotes orthogonal projection of $v$ to the orthogonal complement of $T_{U(x)}=$ span $\mathscr{B}(x)$. If $\mathscr{B}(x)$ is merely orthogonal at $x$, a linear change of variables $\tilde{x}_{i}=$ $\sqrt{g^{i i}} x_{i}=\frac{1}{\left\|\partial U / \partial x^{i}\right\|} x_{i}$ reveals that

$$
\begin{equation*}
\mathbf{H}(U(x))=\sum_{i=1}^{m}\left(g^{i i} \frac{\partial^{2} U(x)}{\left(\partial x^{i}\right)^{2}}\right)^{\perp} \tag{6}
\end{equation*}
$$

We skip further derivations and note for the sake of completeness that (6) is in accordance with the general formula

$$
\begin{equation*}
\mathbf{H}(U(x))=\left(\sum_{i j} g^{i j} \frac{\partial^{2} U}{\partial x^{i} \partial x^{j}}\right)^{\perp} \tag{7}
\end{equation*}
$$

where $\left(g^{i j}\right)$ is the inverse to the metric tensor $\left(g_{i j}\right)$ of $U$ (see [O, (1.11), p. 1098]).

### 2.3 An example

Exercise 2.1 Let $H$ be a hyperplane dividing $\mathbb{R}^{3}$ into two half-spaces $H_{1}, H_{2}$. Let $S_{1}$, $S_{2}$ be 2-dimensional subspaces orthogonal to $H$. For $i=1,2$, let $V_{i}=\left(\mathscr{L}^{3}\left\llcorner H_{i}\right) \times \delta_{S_{i}}\right.$ (where $\delta_{S_{i}}$ is the Dirac measure at the point $S_{i} \in G(3,2)$ ) and $V=V_{1}+V_{2}$. Show that $V$ is a stationary varifold on $\mathbb{R}^{3}$.
Solution. From (3) and the divergence theorem we have $\delta V_{i}(X)=-\int_{H} x \cdot \eta_{i} \mathrm{~d} \mathscr{H}^{2}$ where $\eta_{i}$ is the inward pointing unit normal to $H_{i}$.

Interpretation. $V_{i}$ is the integral (or, uncountable "linear combination") of varifolds $V_{i, x}=V_{\mathscr{H}^{2}\left\llcorner\left(S_{i}+x\right)\right.}, x \in S_{i}^{\perp}$. The variations $\delta V_{i, x}$ combine in the same way, and it turns out that the result is exactly opposite for $V_{1}$ and $V_{2}$.

Remark 2.1 The varifold $V$ from Exercise 2.1 is a 2 -varifold supported by the 3 -space ( $\mu_{V}=\mathscr{L}^{3}$, spt $\mu_{V}=\mathbb{R}^{3}$ ); $V$ is non-rectifiable. $V$ can be "approximated" by a rectifiable varifold supported by many half-planes touching $H$ and parallel to $S_{1}$ (inside $H_{1}$ ) or $S_{2}$ (inside $H_{2}$ ). (The more half-planes, the better approximation and the less density on each of them.) This varifold cannot be stationary - the failure is located near $H$. There is a better "approximation" that is rectifiable and stationary, which is supported by strips of plane creating structure that branches and refines towards $H$. See Figure 1c) for a planar network of segments illustrating such a branching.
Remark 2.2 Also the 2-varifold $\tilde{V}:=\left(\mathscr{L}^{3}\left\llcorner M_{1}\right) \times \delta_{S_{1}}+\left(\mathscr{L}^{3}\left\llcorner M_{2}\right) \times \delta_{S_{2}}\right.\right.$ is stationary when $M_{1}=\bigcup_{k \in \mathbb{Z}}[2 k-1,2 k] \times \mathbb{R} \times \mathbb{R}, M_{2}=\bigcup_{k \in \mathbb{Z}}[2 k, 2 k+1] \times \mathbb{R} \times \mathbb{R}, S_{1}=\mathbb{R} \times \mathbb{R} \times$ $\{0\}, S_{2}=\mathbb{R} \times\{0\} \times \mathbb{R} . \tilde{V}$ can be again approximated by a stationary and rectifiable varifold by using the idea from Figure 1c) twice inside each $[k, k+1] \times \mathbb{R} \times \mathbb{R}$. 乞

### 2.4 Tangents. Conical varifolds

For $x \in \mathbb{R}^{n}$ and $\lambda>0$, let

$$
\begin{equation*}
\eta_{x, \lambda}(y)=\frac{y-x}{\lambda}, \quad y \in \mathbb{R}^{n} \tag{8}
\end{equation*}
$$

If $V$ and $C$ are $m$-varifolds on $\mathbb{R}^{n}$ and $x \in \mathbb{R}^{n}$, we say that $C$ is a tangent varifold to $V$ at $x, C \in \operatorname{VarTan}_{x} V$, if there exist $\lambda_{i}>0, \lambda_{i} \rightarrow 0$ such that, for every continuous function $f$ on $G_{m}\left(\mathbb{R}^{n}\right)$ with compact support,

$$
\int f(y, S) \mathrm{d} C(y, S)=\lim _{i \rightarrow \infty}\left(\lambda_{i}\right)^{-m} \int f\left(\eta_{x, \lambda_{i}}(y), S\right) \mathrm{d} V(y, S)
$$

This is equivalent to

$$
\eta_{x, \lambda_{i} \# \#} V \rightarrow C
$$

(weakly), which is the definition used in [S1, p. 242-243].
The general definition of $\#$ for varifolds is (denoted differently by \#) in [S1, p. 233] and it is slightly complicated. We need $\ldots$ only (i) with maps that are combination of translation and homothety, like (8), in which case

$$
\eta_{x, \lambda_{i} \text { \# }} V(A)=\left(\lambda_{i}\right)^{-m} V\left(\left\{(y, S):\left(\eta_{x, \lambda_{i}}(y), S\right) \in A\right\}\right) ;
$$

(ii) with orthonormal linear maps $L$, with

$$
L_{\ldots} V(A)=V(\{(y, S):(L(y), L(S)) \in A\}) .
$$

An $m$-varifold $C$ is conical if

$$
\eta_{0, \lambda \not \ldots} C=C
$$

for every $\lambda>0$.

## 3 The non-rectifiable varifold

We start with an example of a non-rectifiable varifold, which is simpler. The rectifiable varifold in later sections is in fact a suitable rectifiable approximation of this non-rectifiable example. Thus, in this section we prove the following weaker version of Theorem 1.1.

Proposition 3.1 There is a 2 -varifold in $\mathbb{R}^{4}$ that has a non-conical (hence nonunique) tangent at a point. There is a 2 -varifold in $\mathbb{R}^{4}$ that has a conical but nonunique tangent at a point.

Proof The varifold will be supported by the three-dimensional surface, ${ }^{5}$ in $\mathbb{R}^{4}$, for which we propose the name Clifford cone, ${ }^{6}$ parameterized by

$$
\begin{equation*}
F((a, b),(c, d))=a c e_{1}+b c e_{2}+a d e_{3}+b d e_{4} . \tag{9}
\end{equation*}
$$

Then, for every $t>0$,

$$
\begin{equation*}
F((t a, t b),(c, d))=t F((a, b),(c, d))=F((a, b),(t c, t d)) \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
F((a, b),(c, d))=c\left(a e_{1}+b e_{2}\right)+d\left(a e_{3}+b e_{4}\right)=a\left(c e_{1}+d e_{3}\right)+b\left(c e_{2}+d e_{4}\right) \tag{11}
\end{equation*}
$$

Now, we are ready for an informal explanation of the idea. The surface is the union of a parameterized family of two-dimensional linear subspaces. In fact there is a pair of such representations that are "orthogonal": We can fix $(a, b) \in \mathbb{S}^{1}$ as a parameter and use variables $(c, d) \in \mathbb{R}^{2}$ to create a 2-dimensional varifold $V_{1}^{(a, b)}:=$ $V_{\mathscr{H}^{2}\left\llcorner\operatorname{span}\left\{a e_{1}+b e_{2}, a e_{3}+b e_{4}\right\}\right.}$ (which is stationary because it is associated to a 2-plane).

[^3]Then we obtain a new (non-rectifiable) stationary varifold $V_{1}$ by averaging $V_{1}^{(a, b)}$ over all $(a, b) \in \mathbb{S}^{1}$. We also do the same with swapped $(a, b)$ and $(c, d)$ to obtain a different stationary varifold $V_{2}$ (yet with $\mu_{V_{1}}=\mu_{V_{2}}$ ). Suitable parts of the two varifolds can be joined together in similar way as in Exercise 2.1, with the separating hyperplane $H$ replaced by a sphere. The resulting varifold is again stationary; the quantitative aspects of the formal proof of this fact depend on the presence of "orthogonality" of the parameterizations. Moreover, we can interleave an infinite number of concentric shells containing (parts of) $V_{1}$ and $V_{2}$ to obtain the target (non-rectifiable) varifold. Now we proceed with the formal definitions, arguments and calculations.

Let $0 \leq r<s \leq \infty$,

$$
\begin{aligned}
& g_{1}((a, b),(c, d))=\operatorname{span}\left\{a e_{1}+b e_{2}, a e_{3}+b e_{4}\right\} \\
& g_{2}((a, b),(c, d))=\operatorname{span}\left\{c e_{1}+d e_{3}, c e_{2}+d e_{4}\right\}
\end{aligned}
$$

( $g_{i}$ does not depend on all its parameters),

$$
\begin{align*}
& \phi_{1, r, s}=\left(F, g_{1}\right): S_{1}\left(\mathbb{R}^{2}\right) \times A_{r}^{s}\left(\mathbb{R}^{2}\right) \rightarrow G_{2}\left(A_{r}^{s}\left(\mathbb{R}^{4}\right)\right), \\
&(a, b, c, d) \mapsto\left(F((a, b),(c, d)), g_{1}((a, b),(c, d))\right), \\
& \phi_{2, r, s}=\left(F, g_{2}\right): A_{r}^{s}\left(\mathbb{R}^{2}\right) \times S_{1}\left(\mathbb{R}^{2}\right) \rightarrow G_{2}\left(A_{r}^{s}\left(\mathbb{R}^{4}\right)\right), \\
&(a, b, c, d) \mapsto\left(F((a, b),(c, d)), g_{2}((a, b),(c, d))\right), \\
& V_{1, r, s}=\phi_{1, r, s \#}\left(\mathscr{H}^{1} \times \mathscr{L}^{2}\right),  \tag{12}\\
& V_{2, r, s}=\phi_{2, r, s \#}\left(\mathscr{L}^{2} \times \mathscr{H}^{1}\right), \tag{13}
\end{align*}
$$

where \# denotes the image of a measure (the image is a measure that happens to be a varifold), $\mathscr{H}^{1}$ is the one-dimensional Hausdorff measure in the unit sphere $S_{1}\left(\mathbb{R}^{2}\right)$ and $\mathscr{L}^{2}$ is the Lebesgue measure (on the annulus $A_{r}^{S}\left(\mathbb{R}^{2}\right) \subset \mathbb{R}^{2}$ ). From the definition of $m$-varifold we see that $V_{i, r, s}$ defined by (12), (13) are 2 -varifolds. To see that $V_{i, r, s}$ can also be obtained by "averaging" (integrating, in the weak sense) 2-rectifiable varifolds, let

$$
\begin{gather*}
\phi_{1, r, s,(a, b)}(c, d)=\phi_{1, r, s}((a, b),(c, d)), \quad(c, d) \in A_{r}^{s}\left(\mathbb{R}^{2}\right), \\
\phi_{2, r, s,(c, d)}(a, b)=\phi_{2, r, s}((a, b),(c, d)), \quad(a, b) \in A_{r}^{s}\left(\mathbb{R}^{2}\right), \\
V_{1, r, s,(a, b)}:=\phi_{1, r, s,(a, b) \#} \mathscr{L}^{2} \stackrel{*}{=} V_{\mathscr{H}^{2}\left\llcorner\left(\operatorname{span}\left\{a e_{1}+b e_{2}, a e_{3}+b e_{4}\right\} \cap A_{r}^{s}\left(\mathbb{R}^{4}\right)\right)\right.},  \tag{14}\\
V_{2, r, s,(c, d)}:=\phi_{2, r, s,(c, d) \#} \mathscr{L}^{2} \stackrel{*}{=} V_{\mathscr{H}}{ }^{2}\left\llcorner\left(\operatorname{span}\left\{c e_{1}+d e_{3}, c e_{2}+d e_{4}\right\} \cap A_{r}^{s}\left(\mathbb{R}^{4}\right)\right)\right. \tag{15}
\end{gather*},
$$

where " $\stackrel{*}{=}$ " are valid under condition $(a, b) \in \mathbb{S}^{1}$ or $(c, d) \in \mathbb{S}^{1}$, respectively. Then, by the Fubini theorem,

$$
\begin{align*}
& V_{1, r, s}=\int_{(a, b) \in \mathbb{S}^{1}} V_{1, r, s,(a, b)} \mathrm{d} \mathscr{H}^{1}  \tag{16}\\
& V_{2, r, s}=\int_{(c, d) \in \mathbb{S}^{1}} V_{2, r, s,(c, d)} \mathrm{d} \mathscr{H}^{1} \tag{17}
\end{align*}
$$

Since $V_{i, r, s,(\cdot,)}$ is just the varifold corresponding to an annulus part of a 2-plane $(\mathbf{H}=$ 0 ), its first variation corresponds to the inward pointing unit co-normal field supported
on the two circles (cf. (5)):

$$
\begin{aligned}
& \delta V_{1, r, s,(a, b)}(X)=\int_{\left\{F(a, b, c, d): c^{2}+d^{2}=s^{2}\right\}} X \cdot N \mathrm{~d} \mathscr{H}^{1}-\int_{\left\{F(a, b, c, d): c^{2}+d^{2}=r^{2}\right\}} X \cdot N \mathrm{~d} \mathscr{H}^{1}, \\
& \delta V_{2, r, s,(c, d)}(X)=\int_{\left\{F(a, b, c, d): a^{2}+b^{2}=s^{2}\right\}} X \cdot N \mathrm{~d} \mathscr{H}^{1}-\int_{\left\{F(a, b, c, d): a^{2}+b^{2}=r^{2}\right\}} X \cdot N \mathrm{~d} \mathscr{H}^{1}
\end{aligned}
$$

where $N(x)=x /\|x\|$ and where we leave out the first term if $s=\infty$. The second term is zero if $r=0$. Integrating over $(a, b) \in S_{1}\left(\mathbb{R}^{2}\right)$, respective over $(c, d) \in S_{1}\left(\mathbb{R}^{2}\right)$, and changing the variables back to the image of $F$ (where it becomes a circle of radius $s$ or $r$ ), we get
$\delta V_{1, r, s}(X)=\int_{F\left(S_{1}\left(\mathbb{R}^{2}\right) \times S_{s}\left(\mathbb{R}^{2}\right)\right)} X \cdot N \mathrm{~d} \frac{\mathscr{H}^{1}}{s} \times \mathscr{H}^{1}-\int_{F\left(S_{1}\left(\mathbb{R}^{2}\right) \times S_{r}\left(\mathbb{R}^{2}\right)\right)} X \cdot N \mathrm{~d} \frac{\mathscr{H}^{1}}{r} \times \mathscr{H}^{1}$
$\delta V_{2, r, s}(X)=\int_{F\left(S_{s}\left(\mathbb{R}^{2}\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)} X \cdot N \mathrm{~d} \mathscr{H}^{1} \times \frac{\mathscr{H}^{1}}{s}-\int_{F\left(S_{r}\left(\mathbb{R}^{2}\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)} X \cdot N \mathrm{~d} \mathscr{H}^{1} \times \frac{\mathscr{H}^{1}}{r}$
(Again, if $s=\infty$ or $r=0$, the first or second term has to be replaced by zero. In particular, $V_{i, 0, \infty}$ are stationary.) Therefore $V_{1, r, s}$ and $V_{2, r, s}$ have the same first variation, $\delta V_{1, r, s}=\delta V_{2, r, s}$, and (cf. (3))

$$
\begin{equation*}
\int_{\Omega} \operatorname{div}_{S} X(x) \mathrm{d} V_{1, r, s}(x, S)=\int_{\Omega} \operatorname{div}_{S} X(x) \mathrm{d} V_{2, r, s}(x, S) . \tag{18}
\end{equation*}
$$

We show that

$$
\begin{equation*}
V=V_{\left\{r_{i}\right\}_{i \in \mathbb{Z}}}=\sum_{i=-\infty}^{\infty}\left(V_{1, r_{2 i}, r_{2 i+1}}+V_{2, r_{2 i+1}, r_{2 i+2}}\right) \tag{19}
\end{equation*}
$$

is a stationary varifold for any increasing sequence $\left\{r_{i}\right\}_{i \in \mathbb{Z}}$ with $\lim _{i \rightarrow \infty} r_{i}=\infty$ and $\lim _{i \rightarrow-\infty} r_{i}=0$.

Indeed, V is a Radon measure on $G_{2}\left(\mathbb{R}^{4}\right)$ since, e.g., $V\left(G_{2}\left(A_{0}^{s}\right)\right)=\pi \cdot \pi s^{2}$. Using (3) and substituting from (18)

$$
\begin{aligned}
& \delta V(X)=\sum_{i=-\infty}^{\infty} \int_{\Omega} \operatorname{div}_{S} X(x) \mathrm{d} V_{1, r_{2 i}, r_{2 i+1}}+\int_{\Omega} \operatorname{div}_{S} X(x) \mathrm{d} V_{2, r_{2 i+1}, r_{2 i+2}} \\
&=\sum_{i=-\infty}^{\infty} \int_{\Omega} \operatorname{div}_{S} X(x) \mathrm{d} V_{1, r_{i}, r_{i+1}}=\int_{\Omega} \operatorname{div}_{S} X(x) \mathrm{d} V_{1,0, \infty}=0
\end{aligned}
$$

since $V_{i, 0, \infty}$ is stationary.
If $r_{i}=2^{i}$ then $C:=V$ is a non-conical tangent varifold to $V$ at $0 \in \mathbb{R}^{4}$. (Also $\eta_{0, \lambda \ldots} V \in \operatorname{VarTan}_{0} V$ for $\lambda \in(0, \infty)$.)

If $r_{i}=2^{2^{i}}$ then $V_{1,0, \infty}$ and $V_{2,0, \infty}$ are two different conical tangent varifolds to $V$ at $0 \in \mathbb{R}^{4}$. (Also $V_{1,0, r}+V_{2, r, \infty} \in \operatorname{VarTan}_{0} V$ and $V_{2,0, r}+V_{1, r, \infty} \in \operatorname{Var} \operatorname{Tan}_{0} V$ for $r \in(0, \infty)$.)

The above statements about "non-conical" tangent and about "two different" varifolds need a bit of justification and we choose to formulate them separately.

Lemma 3.1 Varifold $V=V_{\left\{r_{i}\right\}_{i \in \mathbb{Z}}}$ from (19) is not conical. Furthermore, $V_{1, r, s} \neq V_{2, r, s}$ for any $0 \leq r<s \leq \infty$.
Proof We claim that

$$
\begin{equation*}
\text { if } F(x)=F(y) \neq 0 \text { then } g_{1}(x) \neq g_{2}(y) . \tag{20}
\end{equation*}
$$

For $x=\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in \mathbb{R}^{4} \backslash\{0\}$, we have $F^{-1}(x)=\emptyset$ or

$$
F^{-1}(x) \subset\left\{\left( \pm t \sqrt{x_{1}^{2}+x_{3}^{2}}, \pm t \sqrt{x_{2}^{2}+x_{4}^{2}}, \pm \frac{\|x\|}{t} \sqrt{x_{1}^{2}+x_{2}^{2}}, \pm \frac{\|x\|}{t} \sqrt{x_{3}^{2}+x_{4}^{2}}\right): t>0\right\}
$$

First we show that

$$
\begin{equation*}
S_{1}:=g_{1}((a, b),(c, d)) \text { is different from } S_{2}:=g_{2}(( \pm a, \pm b),( \pm c, \pm d)) \tag{21}
\end{equation*}
$$

apart from singular cases $a=b=0$ or $c=d=0$ (when $F((a, b),(c, d))=0$ ): Since $g_{2}$ does not depend on $a$ and $b$, we have $S_{2}=g_{2}((a, b),( \pm c, \pm d))$. Since $g_{1}$ does not depend on $c$ and $d$, we can freely change the sign of $c$ (and $d$ ) in (21). Therefore it is enough to consider $S_{2}=g_{2}((a, b),(c, d))$. Assume that $a^{2}+b^{2} \neq 0$ and $c^{2}+d^{2} \neq 0$. $S_{1}$ and $S_{2}$ are two-dimensional subspaces and if $S_{1}=S_{2}$ then $\operatorname{span}\left(S_{1} \cup S_{2}\right)$ is twodimensional as well, i.e., the matrix

$$
\left(\begin{array}{llll}
a & b & 0 & 0 \\
0 & 0 & a & b \\
c & 0 & d & 0 \\
0 & c & 0 & d
\end{array}\right)
$$

has rank 2. Then $\left(a^{2}+b^{2}\right) c=-\left|\begin{array}{lll}a & b & 0 \\ 0 & 0 & a \\ 0 & c & 0\end{array}\right|+\left|\begin{array}{lll}a & b & 0 \\ 0 & 0 & b \\ c & 0 & 0\end{array}\right|=0+0=0,\left(a^{2}+b^{2}\right) d=\left|\begin{array}{lll}a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & d\end{array}\right|-$ $\left|\begin{array}{lll}b & 0 & 0 \\ 0 & a & b \\ 0 & d & 0\end{array}\right|=0$. Hence $c=0, d=0$, a contradiction showing that (21) is true.

Since $g_{i}((t a, t b),(u c, u d))=g_{i}((a, b),(c, d))=S_{i}$ for $i=1,2$ and $t, u \in \mathbb{R} \backslash\{0\}$, we get (20).

By (20), $V_{1, r, s}$ and $V_{2, r, s}$ are supported by disjoint subsets of $G_{2}\left(\mathbb{R}^{4}\right)$ whenever $r>0$. (For $r=0, \operatorname{spt} V_{1, r, s} \cap \operatorname{spt} V_{2, r, s} \subset\{(0,0,0,0)\} \times G(4,2)$.) Hence $V_{1, r, s} \neq V_{2, r, s}$ for any $0 \leq r<s \leq \infty$. Obviously, varifold $V=V_{\left\{r_{i}\right\}_{i \in \mathbb{Z}}}$ is not conical.

## 4 The rectifiable varifold

The rectifiable stationary varifold (let us call it $V_{\text {rect }}$ for now) will be obtained as a suitable approximation of the above non-rectifiable $V$. (The idea of approximation for the case of linear configuration was suggested in Remark 2.2, cf. Figure 1c). Since we work in central configuration, our task will be more complicated, see the caption of Figure 1b) where the simplest idea does not transfer from Figure 1a).)

Instead of planar annuli (see the support of $V_{1, r, s,\left(\cos t_{1}, \sin t_{1}\right)}$ in (14)) smoothed out by averaging in the definition of $V$, the support now consists of countably many planar annuli and countably many pieces homeomorphic to annuli (we will call them "rings") whose number will increase (through a process that we call "branching") towards the boundary of the layer.

Since now the pieces of $V_{\text {rect }}$ are not oriented radially ( $x \notin S$ for many $(x, S) \in$ $\operatorname{spt} V_{\text {rect }}$ ), the ratio $V_{\text {rect }}\left(G_{2}(B(0, r))\right) / r^{2}$ necessarily decreases as $r$ decreases. (This is a corollary to the Monotonicity formula [S1, 17.5].) Therefore we have, and do, take special care to make sure that the density $\theta^{2}\left(V_{\text {rect }}, 0\right)$ does not vanish.

The proof continues towards the end of this paper and depends on the calculations summarized in the following lemmata.

The varifold will be again supported by the three-dimensional surface parameterized by $F$, see (9) and (10).

In every point of $x \in \operatorname{range} F \backslash\{0\}$, we will frequently refer to the radial direction $N(x)=x /\|x\|$ and to a selected tangential direction. The latter is conveniently expressed by matrix multiplication.

Let $J_{13}^{24}$ be the matrix that rotates $e_{1} \rightarrow e_{3}$ and $e_{2} \rightarrow e_{4}$ given by

$$
J_{13}^{24}=J(0,1)=\left(\begin{array}{cccc}
0 & 0 & -1 & 0  \tag{22}\\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right)
$$

For $\varepsilon \geq 0$ and $x \in \mathbb{R}^{4} \backslash\{0\}$, consider the following set of 2-dimensional planes in $\mathbb{R}^{4}$

$$
G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}(x)=\{\operatorname{span}\{u, v\}:\{u, v\} \text { orthonormal, }
$$

$$
\begin{equation*}
\left.\|u-N(x)\| \leq \varepsilon,\left\|v-J_{13}^{24} N(x)\right\| \leq \varepsilon\right\} \tag{23}
\end{equation*}
$$

and related subset of $\mathbb{R}^{4} \times G(4,2)$

$$
\begin{equation*}
G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}=\left\{(x, S): x \in \mathbb{R}^{4} \backslash\{0\}, S \in G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}(x)\right\} . \tag{24}
\end{equation*}
$$

Then $\left\{G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}(x): \varepsilon>0\right\}$ is a neighbourhood base for a special point $\operatorname{span}\{N(x)$, $\left.J_{13}^{24} N(x)\right\} \in G(4,2)$, which is the span of the radial direction and the direction determined by $J_{13}^{24}$. From this comes the subscript in our notation.

Note that if we let

$$
J_{12}^{34}=\left(\begin{array}{cccc}
0 & -1 & 0 & 0  \tag{25}\\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

and define $G_{\text {rad \& } J_{12}^{34}}^{\varepsilon}(x)$ accordingly then there is $\varepsilon_{0}>0$ (independent of $x$ ) such that

$$
\begin{equation*}
G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}(x) \cap G_{\mathrm{rad} \& J_{12}^{34}}^{\varepsilon}(x)=\emptyset \tag{26}
\end{equation*}
$$

for all $\varepsilon \in\left[0, \varepsilon_{0}\right)$. To see that, we first observe that

$$
\begin{equation*}
\operatorname{span}\left\{N(x), J_{13}^{24} N(x)\right\} \neq \operatorname{span}\left\{N(x), J_{12}^{34} N(x)\right\} \tag{27}
\end{equation*}
$$

This is similar to (21) but now the proof is even easier. First consider (27) in the special case when $N(x)=e_{1}$. Then if (27) were not valid, then the matrix

$$
\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

would have range at most two. Using a rotation, we see that (27) in general is equivalent to (27) in the special case when $N(x)=e_{1}$. Moreover, we see that $\varepsilon_{0}>0$ admissible for (26) is independent of $x \in \mathbb{R}^{4} \backslash\{0\}$.
4.1 Basic surface, rings and their joins.

Let $d>0, \alpha_{0} \in \mathbb{R}$.
We will use pieces of a minimal surface that is derived by a "rotation" in $\mathbb{R}^{4}$ from planar curve (in polar coordinates)

$$
\begin{equation*}
r(\alpha)=r^{d, \alpha_{0}}(\alpha)=\sqrt{d / \cos 2\left(\alpha-\alpha_{0}\right)}, \quad \alpha-\alpha_{0} \in(-\pi / 4, \pi / 4) \tag{28}
\end{equation*}
$$

It turns out that the curve is a hyperbola. To make a geometrical picture, let us consider the special case $\alpha_{0}=\pi / 4$ (the general case is obtained by rotations in the plane); then $r(\alpha)^{2}=d / 2 \sin \alpha \cos \alpha$ and

$$
\{(r(\alpha) \cos \alpha, r(\alpha) \sin \alpha)\}=\left\{\left(x_{1}, x_{2}\right) \subset(0, \infty)^{2}: 2 x_{1} x_{2}=d\right\}
$$

The most important are the portions of the hyperbola far away from the origin, that is with $x_{1}$ (respectively, $x_{2}$ ) restricted to interval $\left(\varepsilon_{1}, \varepsilon_{2}\right)$ close to 0 . This corresponds to $\alpha-\alpha_{0} \in\left(t_{1}, t_{2}\right)$ close to either $-\pi / 4$ or $\pi / 4$.

The rotation applied is the one obtained from $F$ (see (9)):

$$
\begin{equation*}
U(\alpha, \beta)=F((r(\alpha) \cos \alpha, r(\alpha) \sin \alpha),(\cos \beta, \sin \beta)) \tag{29}
\end{equation*}
$$

where $\alpha-\alpha_{0} \in\left(t_{1}, t_{2}\right) \subset(-\pi / 4, \pi / 4)$ and $\beta \in \mathbb{R}$. The set thus obtained in (30) below is homeomorphic (and nearly isometric, for suitable pairs $t_{1}, t_{2}$ ) to a planar annulus and we call it a "ring".

For imagination of the full surface, one might notice that it is obtained by deformation of a part (a strip, since $\alpha$ is restricted to an interval) of Clifford torus $F\left(S_{1}\left(\mathbb{R}^{2}\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)$, with the middle circle $\left(\alpha=\alpha_{0}\right)$ scaled at $\sqrt{d}$ and the ends ( $\alpha \rightarrow \alpha_{0} \pm \pi / 4$ ) lifted in the radial direction to infinity. As we already indicated above, we will use only rather flat parts that are lifted high above $\sqrt{d}$ and have nearly radial directions.

The notation. The lemma below summarizes the properties of our minimal surface $S=$ range $U$. The letter $S$ with various indexes denotes parts of the surface while $V$ is used for the corresponding varifolds. The upper index is reserved for the parameters $d$ and $\alpha_{0}$; we drop them from $S_{t_{1}, t_{2}}=S_{t_{1}, t_{2}}^{d, \alpha_{0}}$ but we have to leave them in $V_{t_{1}, t_{2}}^{d, \alpha_{0}}$ (see (32) below). Lower index denotes the range for the variable $\alpha$. The range is either an interval ( $t_{1}, t_{2}$ ) (thus $S_{t_{1}, t_{2}}$ is our "ring") or a single point $t_{1}$; thus $S_{t_{1}}$ is a circle. Boundary of the ring $S_{t_{1}, t_{2}}$ consists of two circles $S_{t_{1}}, S_{t_{2}}$. Some time later the circle will be denoted by $K(\rho, \alpha)$ ( $\rho$ will be the radius) - this change in the notation will be necessary as to drop the dependence on $d$ and $\alpha_{0}$. Thus $S_{t_{1}}=K\left(r^{d, \alpha_{0}}\left(t_{1}\right), t_{1}\right)$.

Remark 4.1 $S=$ range $U$ and all other objects until Lemma 4.3 are included in the Clifford cone range $F=F\left(\mathbb{R}^{2} \times \mathbb{S}^{1}\right)=\mathbb{R} \cdot F\left(\mathbb{S}^{1} \times \mathbb{S}^{1}\right)$. Their geometry is determined in the $(r, \alpha)$ plane and all of them are invariant with respect to the parameter $\beta$ in (29) and the corresponding rotation. The rotation introduces factor $r$ into the area functional and influences thus the shape of the minimal surface. Therefore we stick with the full 4-dimensional description and do not restrict ourselves to the $(r, \alpha)$ plane where everything is determined. In Lemma 4.4, the roles of $\alpha$ and $\beta$ are interchanged.

Lemma 4.1 1. Let $d>0, \alpha_{0} \in \mathbb{R}$ and $r(\alpha)$ as in (28). Consider the parameterized surface $U(\alpha, \beta)=U^{d, \alpha_{0}}(\alpha, \beta)$,

$$
\begin{array}{r}
U(\alpha, \beta)=(r(\alpha) \cos \alpha \cos \beta, r(\alpha) \sin \alpha \cos \beta, r(\alpha) \cos \alpha \sin \beta, r(\alpha) \sin \alpha \sin \beta) \\
\alpha-\alpha_{0} \in(-\pi / 4, \pi / 4), \beta \in \mathbb{R}
\end{array}
$$

( $U$ is $2 \pi$-periodic in $\beta$, and injective on every period.) Then $U$ is a minimal surface.
2. Let

$$
\begin{align*}
S_{t_{1}, t_{2}} & :=\left\{U(\alpha, \beta): \alpha \in\left(t_{1}, t_{2}\right), \beta \in \mathbb{R}\right\}, \quad \text { (the "ring") }  \tag{30}\\
S_{t_{1}} & :=\left\{U\left(t_{1}, \beta\right): \beta \in \mathbb{R}\right\},  \tag{31}\\
S & :=\operatorname{range}(U) .
\end{align*}
$$

Then the rectifiable varifold $V_{\mathscr{H}^{2}}{ }^{\llcorner }$號 is stationary.
3. (The ring varifold and its first variation.) For every $x \in S$, find any $p$ satisfying $U(p)=x$ and let

$$
\boldsymbol{\eta}_{\alpha_{0}}(x)=N(\partial U(p) / \partial \alpha)
$$

where $N(y)=y /\|y\|$.
For $\alpha_{0}-\pi / 4<t_{1}<t_{2}<\alpha_{0}+\pi / 4$, let

$$
\begin{equation*}
V_{t_{1}, t_{2}}^{d, \alpha_{0}}=V_{\mathscr{H}^{2}\left\llcorner S_{t_{1}, t_{2}}\right.} . \tag{32}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\delta V_{t_{1}, t_{2}}^{d, \alpha_{0}}(X)=\int_{S_{t_{2}}} X(x) \cdot \boldsymbol{\eta}_{\alpha_{0}}(x) \mathrm{d} \mathscr{H}^{1}-\int_{S_{t_{1}}} X(x) \cdot \boldsymbol{\eta}_{\alpha_{0}}(x) \mathrm{d} \mathscr{H}^{1} . \tag{33}
\end{equation*}
$$

4. (Two rings at touch.) If $\alpha_{1} \leq \alpha \leq \alpha_{2}$ and $\alpha-\alpha_{1}=\alpha_{2}-\alpha \in[0, \pi / 4)$ then

$$
\begin{equation*}
U^{d, \alpha_{1}}(\alpha, \beta)=U^{d, \alpha_{2}}(\alpha, \beta) \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{\eta}_{\alpha_{1}}\left(U^{d, \alpha_{1}}(\alpha, \beta)\right)-\boldsymbol{\eta}_{\alpha_{2}}\left(U^{d, \alpha_{2}}(\alpha, \beta)\right)=2 \sin 2\left(\alpha-\alpha_{1}\right) \cdot N\left(U^{d, \alpha_{1}}(\alpha, \beta)\right) \tag{35}
\end{equation*}
$$

is a radial vector at the point.
5. The tangent plane to $U=U^{d, \alpha_{0}}$ at $x=U(\alpha, \beta)$ belongs to $G_{\operatorname{rad} \& J_{13}^{24}}^{2 \cos 2\left(\alpha-\alpha_{0}\right)}(x)$ and

$$
\operatorname{spt} V_{t_{1}, t_{2}}^{d, \alpha_{0}} \subset G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}
$$

where $\varepsilon=2 \max \cos \left(\left[2\left(t_{1}-\alpha_{0}\right), 2\left(t_{2}-\alpha_{0}\right)\right]\right)$.
6. (Mass distribution)

$$
\mathbf{M}\left(V_{t_{1}, t_{2}}^{d, \alpha_{0}}\right)=\pi d\left(\tan 2\left(t_{2}-\alpha_{0}\right)-\tan 2\left(t_{1}-\alpha_{0}\right)\right) .
$$

For every $0<\sqrt{d} \leq r_{1}<r_{2}$ there is a number $\rho=\rho\left(d, r_{1}, r_{2}\right) \in\left[r_{1}, r_{2}\right]$ such that whenever $\alpha_{0}<t_{1}<t_{2}<\alpha_{0}+\pi / 4$, and $t_{1} \leq s_{1} \leq s_{2} \leq t_{2}$ then,

$$
\begin{aligned}
& \mathbf{M}\left(V_{t_{1}, t_{2}}^{d, \alpha_{0}}\right)=\pi\left|\sqrt{r\left(t_{2}\right)^{4}-d^{2}}-\sqrt{r\left(t_{1}\right)^{4}-d^{2}}\right| \\
&=\frac{1}{\sqrt{1-d^{2} /\left(\rho\left(d, r\left(t_{1}\right), r\left(t_{2}\right)\right)\right)^{4}}} \pi\left|r\left(t_{2}\right)^{2}-r\left(t_{1}\right)^{2}\right|
\end{aligned}
$$

and

$$
\begin{aligned}
V_{t_{1}, t_{2}}^{d, \alpha_{0}}\left(G_{2}\left(A_{r\left(s_{1}\right)}^{r\left(s_{2}\right)}\right)\right)=\mathbf{M}\left(V_{s_{1}, s_{2}}^{d, \alpha_{0}}\right)=\pi \mid & \left|\sqrt{r\left(s_{2}\right)^{4}-d^{2}}-\sqrt{r\left(s_{1}\right)^{4}-d^{2}}\right| \\
& =\frac{1}{\sqrt{1-d^{2} /\left(\rho\left(d, r\left(s_{1}\right), r\left(s_{2}\right)\right)^{4}\right.}} \pi\left|r\left(s_{2}\right)^{2}-r\left(s_{1}\right)^{2}\right|
\end{aligned}
$$

If $\alpha_{0}-\pi / 4<t_{1}<t_{2}<\alpha_{0}$, the same holds with $A_{r\left(s_{1}\right)}^{r\left(s_{2}\right)}$ replaced by $A_{r\left(s_{2}\right)}^{r\left(s_{1}\right)}$ and $\rho$ extended by formula $\rho\left(d, r_{1}, r_{2}\right):=\rho\left(d, r_{2}, r_{1}\right)$ for $\sqrt{d} \leq r_{2}<r_{1}$.

Remark 4.2 For $\alpha_{0}+\pi / 4-\varepsilon<t_{1}<t_{2}<\alpha_{0}+\pi / 4$ (or analogously for $\alpha_{0}-\pi / 4<$ $\left.t_{1}<t_{2}<\alpha_{0}-\pi / 4+\varepsilon\right)$, and $r=r\left(t_{1}\right), s=r\left(t_{2}\right)$, the ring $S_{t_{1}, t_{2}}$ is intended to be a perturbation of the annulus supporting $V_{1, r, s,\left(\cos t_{1}, \sin t_{1}\right)}$ from (14).

Remark 4.3 The surface $S=\operatorname{range}(U)$ can be found in [L].
We will give two arguments for the minimality of surface $U$, the first one is easy but slightly incomplete: Let $\alpha_{0}-\pi / 4<t_{1}<t_{2}<\alpha_{0}+\pi / 4$ with $t_{2}$ close to $t_{1}$, and consider the part of the surface determined by the range $t \in\left(t_{1}, t_{2}\right)$ (cf. (30)); recall this is the surface created by a certain "rotation" from curve

$$
\gamma(t):=(r(t) \cos t, r(t) \sin t, 0,0), \quad t \in\left(t_{1}, t_{2}\right)
$$

The boundary of the selected part consists of two circles $S_{t_{1}}, S_{t_{2}}$ (see (31)). To this correspond fixed values $\gamma\left(t_{1}\right), \gamma\left(t_{2}\right)$, as boundary conditions for $\gamma$.

Our first and incomplete argument for the minimality of $U$ is based on comparing the area of the selected part of $U$ with surfaces corresponding to other possible curves $\gamma$ in $\mathbb{R}^{2} \times\{0\}^{2}$ with the same boundary condition.

The area is given by the formula

$$
A=2 \pi \int_{\left(t_{1}, t_{2}\right)}\left\|\gamma^{\prime}(t)\right\| \cdot\|\gamma(t)\| \mathrm{d} t
$$

since the length of the circle through $\gamma(t)$ is $2 \pi\|\gamma(t)\|$. We will view $\gamma$ as a curve in $\mathbb{R}^{2} \cong \mathbb{R}^{2} \times\{0\}^{2}$, and assume that $\gamma$ is the graph of a function $r$ in polar coordinates, that is $\gamma(t)=(r(t) \cos t, r(t) \sin t)$. On $\mathbb{R}^{2}=\mathbb{C}$, consider the map $z \mapsto z^{2}$ whose derivative is $2 z$. That maps curve $\gamma$ to a curve $\gamma^{2}$ (where $\gamma^{2}(t)=(\gamma(t))^{2} \in \mathbb{C}$ ) whose length

$$
L=\int_{\left(t_{1}, t_{2}\right)}\left\|\left(\gamma^{2}\right)^{\prime}(t)\right\| \mathrm{d} t=\int_{\left(t_{1}, t_{2}\right)} 2\left\|\gamma^{\prime}(t)\right\| \cdot\|\gamma(t)\| \mathrm{d} t
$$

we find to be directly proportional to $A$. It is well known that $L$ is minimal if $\gamma^{2}$ is the segment connecting its endpoints. A special case is a vertical segment given in polar coordinates by $(\tilde{r}, \tilde{\alpha})$ with $\tilde{r}=d / \cos \tilde{\alpha}$; the general case is $\tilde{r}=d / \cos \left(\tilde{\alpha}-\tilde{\alpha}_{0}\right)$. Since $z \mapsto z^{2}$ is expressed in polar coordinates as $(r, \alpha) \mapsto(\tilde{r}, \tilde{\alpha})=\left(r^{2}, 2 \alpha\right)$, we obtain the curve $\gamma(t)=(r(t) \cos t, r(t) \sin t)$ with $r(t)=\sqrt{d / \cos 2\left(t-\alpha_{0}\right)}, t \in\left[t_{1}, t_{2}\right]$. The corresponding rotation surface is our best candidate for the minimum area surface spanned between $S_{t_{1}}$ and $S_{t_{2}}$ and $U$ likely is a minimal surface.

## Proof of Lemma 4.1

1. For formal verification of the minimality of surface $U$, it is enough to verify that $\mathbf{H}(U)=0$.

For $a, b, \alpha, \beta \in \mathbb{R}$, let

$$
B=B(\beta)=J(\cos \beta, \sin \beta), \quad \text { where } \quad J(a, b)=\left(\begin{array}{cccc}
a & 0 & -b & 0 \\
0 & a & 0 & -b \\
b & 0 & a & 0 \\
0 & b & 0 & a
\end{array}\right)
$$

and (since we choose to treat the vectors, including $U$, as column vectors, we will distinguish that in notation from this moment)

$$
A=A(\alpha)=(\cos \alpha, \sin \alpha, 0,0)^{T}
$$

Then

$$
U=r B A
$$

where $r$ is a function of $\alpha$ :

$$
U(\alpha, \beta)=r(\alpha) B(\beta) A(\alpha), \quad \alpha \in(-\pi / 4, \pi / 4), \beta \in \mathbb{R}
$$

Note that obviously $\|U\|=r$, hence

$$
N(U)=B A
$$

We have

$$
\begin{align*}
& \frac{\partial U}{\partial \alpha}=r^{\prime} B A+r B A^{\prime}=B\left(r^{\prime} A+r A^{\prime}\right)  \tag{36}\\
& \frac{\partial U}{\partial \beta}=r B^{\prime} A \tag{37}
\end{align*}
$$

where

$$
A^{\prime}=(-\sin \alpha, \cos \alpha, 0,0)^{T}, \quad B^{\prime}=J(-\sin \beta, \cos \beta) .
$$

Furthermore,

$$
A^{\prime \prime}=(-\cos \alpha,-\sin \alpha, 0,0)^{T}=-A, \quad B^{\prime \prime}=J(-\cos \beta,-\sin \beta)=-B
$$

and hence

$$
\begin{align*}
& \frac{\partial^{2} U}{\partial \alpha^{2}}=B\left(r^{\prime \prime} A+2 r^{\prime} A^{\prime}+r A^{\prime \prime}\right)=B\left(\left(r^{\prime \prime}-r\right) A+2 r^{\prime} A^{\prime}\right)  \tag{38}\\
& \frac{\partial^{2} U}{\partial \beta^{2}}=r B^{\prime \prime} A=-r B A \tag{39}
\end{align*}
$$

Obviously

$$
\begin{equation*}
A^{T} A=\left(A^{\prime}\right)^{T} A^{\prime}=1 \quad A^{T} A^{\prime}=\left(A^{\prime}\right)^{T} A=0 . \tag{40}
\end{equation*}
$$

It is immediate that $J(a, b)^{T}=J(a,-b)$ and $J(a, b)^{T} J(a, b)=\left(a^{2}+b^{2}\right) I$ where $I$ is the identity matrix; in particular

$$
\begin{align*}
B^{T} B & =I,  \tag{41}\\
\left(B^{\prime}\right)^{T} B^{\prime} & =I . \tag{42}
\end{align*}
$$

Hence

$$
\begin{equation*}
B^{-1}=B^{T} \tag{43}
\end{equation*}
$$

Furthermore, $J(b, a) J(a, b)=J\left(0, a^{2}+b^{2}\right)$, in particular

$$
\begin{equation*}
B^{T} B^{\prime}=J(0,1) \tag{44}
\end{equation*}
$$

and $B^{\prime} B^{T}=J(0,1)$. Multiplying that by $B$ from the right (see (43)) we get

$$
\begin{equation*}
B^{\prime}=J(0,1) B . \tag{45}
\end{equation*}
$$

The metric tensor is

$$
\begin{align*}
& g_{11}=\frac{\partial U}{\partial \alpha} \cdot \frac{\partial U}{\partial \alpha}=\left(r^{\prime} A+r A^{\prime}\right)^{T} B^{T} B\left(r^{\prime} A+r A^{\prime}\right)  \tag{46}\\
& g_{22}=\frac{\partial U}{\partial \beta} \cdot \frac{\partial U}{\partial \beta}=r A^{T}\left(B^{\prime}\right)^{T} r B^{\prime} A \\
&\left.\stackrel{(42)}{=} r^{\prime} A+r A^{\prime}\right)^{T}\left(r^{\prime} A+r A^{\prime}\right) \stackrel{(40)}{=}\left(r^{\prime}\right)^{2}+r^{2} \\
& g_{12}=g_{21}=\frac{\partial U}{\partial \alpha} \cdot \frac{\partial U}{\partial \beta}=\left(r^{\prime} A+r A^{\prime}\right)^{T} B^{T} r B^{\prime} A \\
& \stackrel{(44)}{=} r\left(r^{\prime} A+r A^{\prime}\right)^{T} J(0,1) A=0 \tag{47}
\end{align*}
$$

since $A, A^{\prime} \in \mathbb{R}^{2} \times\{0\}^{2}$ while $J(0,1) A \in\{0\}^{2} \times \mathbb{R}^{2}$. Therefore

$$
\left(g_{i j}\right)=\left(\begin{array}{cc}
\left(r^{\prime}\right)^{2}+r^{2} & 0 \\
0 & r^{2}
\end{array}\right), \quad\left(g^{i j}\right)=\left(\begin{array}{cc}
\frac{1}{\left(r^{\prime}\right)^{2}+r^{2}} & 0 \\
0 & \frac{1}{r^{2}}
\end{array}\right)
$$

We want to verify $\mathbf{H}(U)=0$ using (6) (or, equivalently, (7)). Thus we want to verify

$$
v^{\perp}=0, \quad \text { that is, } \quad v \in \operatorname{span}\left\{\frac{\partial U}{\partial x^{i}}\right\}
$$

where

$$
v=\frac{1}{\left(r^{\prime}\right)^{2}+r^{2}} B\left(\left(r^{\prime \prime}-r\right) A+2 r^{\prime} A^{\prime}\right)+\frac{1}{r^{2}}(-r B A)
$$

That is

$$
\frac{1}{\left(r^{\prime}\right)^{2}+r^{2}} B\left(\left(r^{\prime \prime}-r\right) A+2 r^{\prime} A^{\prime}\right)+\frac{1}{r^{2}}(-r B A) \in \operatorname{span}\left\{B\left(r^{\prime} A+r A^{\prime}\right), r B^{\prime} A\right\}
$$

Multiplying by $B^{-1}$ and using (43), (44), we get equivalent relation

$$
\frac{1}{\left(r^{\prime}\right)^{2}+r^{2}}\left(\left(r^{\prime \prime}-r\right) A+2 r^{\prime} A^{\prime}\right)-\frac{1}{r} A \in \operatorname{span}\left\{r^{\prime} A+r A^{\prime}, r J(0,1) A\right\} .
$$

Since $A, A^{\prime} \in \mathbb{R}^{2} \times\{0\}^{2}$, while $r J(0,1) A \in\{0\} \times \mathbb{R}^{2}$, the latter can be removed:

$$
\frac{1}{\left(r^{\prime}\right)^{2}+r^{2}}\left(\left(r^{\prime \prime}-r\right) A+2 r^{\prime} A^{\prime}\right)-\frac{1}{r} A \in \operatorname{span}\left\{r^{\prime} A+r A^{\prime}\right\} .
$$

Now the relation reduces to $\mathbb{R}^{2} \times\{0\}^{2}$, where $A, A^{\prime}$ form an orthogonal base. We have $r^{\prime} A+r A^{\prime} \perp r A-r^{\prime} A^{\prime}$ and our relation is equivalent to

$$
\left(r A-r^{\prime} A^{\prime}\right)^{T}\left(\frac{1}{\left(r^{\prime}\right)^{2}+r^{2}}\left(\left(r^{\prime \prime}-r\right) A+2 r^{\prime} A^{\prime}\right)-\frac{1}{r} A\right)=0
$$

Using (40) this reduces to

$$
r r^{\prime \prime}-3\left(r^{\prime}\right)^{2}-2 r^{2}=0
$$

It is easy to check that our function $r(\alpha)=\sqrt{d / \cos 2\left(\alpha-\alpha_{0}\right)}$ verifies this equation.
Thus we proved that the mean curvature vector $\mathbf{H}(U)$ is identically zero and $U(\alpha, \beta)$ is a minimal surface.
2. Since $\mathbf{H}(U)=0$ and there is no boundary ( $U$ is defined on $\mathbb{R}^{2}$ and essentially injective) the associated varifold is stationary.
3. To obtain (33), it is enough to use (5); The boundary of $S_{t_{1}, t_{2}}$ is $S_{t_{1}} \cup S_{t_{2}}$, and if $U(p) \in \partial S_{t_{1}, t_{2}}$ then $\partial U(p) / \partial \beta$ is obviously tangent to $\partial S_{t_{1}, t_{2}}$ and $\eta:=\partial U(p) / \partial \alpha$ is orthogonal to it, see (47). If $p=\left(t_{1}, \boldsymbol{\beta}\right)$ then $\eta$ is an inner normal, if $p=\left(t_{2}, \boldsymbol{\beta}\right)$ then it is outer.
4. Assume now that $\alpha_{1} \leq \alpha \leq \alpha_{2}$ and

$$
\begin{equation*}
\alpha-\alpha_{1}=\alpha_{2}-\alpha \in[0, \pi / 4) \tag{48}
\end{equation*}
$$

Then $r^{d, \alpha_{1}}(\alpha)=r^{d, \alpha_{2}}(\alpha)$ and hence $U^{\alpha_{0}}(\alpha, \beta)=U^{\alpha_{1}}(\alpha, \beta)$.
At any point ( $\alpha, \beta$ ) satisfying (48) we have, by (36) and (46),

$$
\begin{align*}
\frac{\partial U}{\partial \alpha} & =r^{\prime} B A+r B A^{\prime} \\
N\left(\frac{\partial U}{\partial \alpha}\right) & =\frac{r^{\prime}}{\sqrt{r^{\prime 2}+r^{2}}} B A+\frac{r}{\sqrt{r^{\prime 2}+r^{2}}} B A^{\prime} \tag{49}
\end{align*}
$$

where $A, B$ and $r$ are the same regardless if $U^{d, \alpha_{1}}$ or $U^{d, \alpha_{2}}$ is considered. Only $r^{\prime}$ is different:

$$
\left(r^{d, \alpha_{1}}\right)^{\prime}(\alpha)=-\left(r^{d, \alpha_{2}}\right)^{\prime}(\alpha) .
$$

Letting, e.g., $\alpha_{0}:=\alpha_{1}$, we have

$$
\begin{align*}
r & =\sqrt{d} \cos ^{-1 / 2} 2\left(\alpha-\alpha_{0}\right)  \tag{50}\\
r^{\prime} & =\sqrt{d} \cos ^{-3 / 2} 2\left(\alpha-\alpha_{0}\right) \sin 2\left(\alpha-\alpha_{0}\right)  \tag{51}\\
\sqrt{r^{\prime 2}+r^{2}} & =\sqrt{d} \cos ^{-3 / 2} 2\left(\alpha-\alpha_{0}\right) . \tag{52}
\end{align*}
$$

Since (49) are the values of $\boldsymbol{\eta}_{\alpha_{1}}$ and $\boldsymbol{\eta}_{\alpha_{2}}$, we get (35), that is,

$$
\boldsymbol{\eta}_{\alpha_{1}}\left(U^{d, \alpha_{1}}(\alpha, \beta)\right)-\boldsymbol{\eta}_{\alpha_{2}}\left(U^{d, \alpha_{2}}(\alpha, \beta)\right)=c B A=c N\left(U^{d, \alpha_{1}}(\alpha, \beta)\right)
$$

where

$$
c=\frac{2 r^{\prime}}{\sqrt{r^{\prime 2}+r^{2}}}=2 \sin 2\left(\alpha-\alpha_{0}\right)
$$

To prove 5 ., it is enough to show that the tangent to $U$ at $U(\alpha, \beta)$ is the plane spanned by orthonormal base $\left\{N\left(\frac{\partial U}{\partial \alpha}(\alpha, \beta)\right), N\left(\frac{\partial U}{\partial \beta}(\alpha, \beta)\right)\right\}$ where

$$
\begin{gather*}
\left\| \pm N\left(\frac{\partial U}{\partial \alpha}(\alpha, \beta)\right)-N(U(\alpha, \beta))\right\| \leq 2 \cos 2\left(\alpha-\alpha_{0}\right)  \tag{53}\\
N\left(\frac{\partial U}{\partial \beta}(\alpha, \beta)\right)=J(0,1) U(\alpha, \beta) \tag{54}
\end{gather*}
$$

Here $\pm$ denotes the sign of $\alpha-\alpha_{0}$. The two vectors are orthogonal by (47). By (49) and (50)-(52)

$$
N\left(\frac{\partial U}{\partial \alpha}\right)=\sin 2\left(\alpha-\alpha_{0}\right) B A+\cos 2\left(\alpha-\alpha_{0}\right) B A^{\prime}
$$

Using $N(U)=B A$ and $\|B A\|=1=\left\|B A^{\prime}\right\|$ we get

$$
\left\| \pm N\left(\frac{\partial U}{\partial \alpha}\right)-N(U)\right\| \leq\left(1-\left|\sin 2\left(\alpha-\alpha_{0}\right)\right|\right)+\cos 2\left(\alpha-\alpha_{0}\right) \leq 2 \cos 2\left(\alpha-\alpha_{0}\right)
$$

which is (53). Furthermore we have

$$
N\left(\frac{\partial U}{\partial \beta}(\alpha, \beta)\right)=\frac{1}{\sqrt{g_{22}}} \frac{\partial U(\alpha, \beta)}{\partial \beta} \stackrel{(37)}{=} B^{\prime} A \stackrel{(45)}{=} J(0,1) B A=J(0,1) U(\alpha, \beta)
$$

which is (54).
6. The mass formula is directly obtained by integration. Since $g_{12}=0$, the 2volume element has a simple form.

$$
\begin{aligned}
& \mathbf{M}\left(V_{t_{1}, t_{2}}^{d, \alpha_{0}}\right)=\mathscr{H}^{2} S_{t_{1}, t_{2}}=\int_{\left[t_{1}, t_{2}\right]} \mathrm{d} \alpha \int_{[0,2 \pi]} \mathrm{d} \beta \sqrt{g_{11} g_{22}} \\
&=2 \pi \int_{\left[t_{1}, t_{2}\right]} \mathrm{d} \alpha r \sqrt{r^{\prime 2}+r^{2}} \stackrel{(52)}{=} 2 \pi \int_{\left[t_{1}, t_{2}\right]} \mathrm{d} \alpha d \cos ^{-2} 2\left(\alpha-\alpha_{0}\right) \\
&=\pi d\left(\tan 2\left(t_{2}-\alpha_{0}\right)-\tan 2\left(t_{1}-\alpha_{0}\right)\right)
\end{aligned}
$$

If $\alpha_{0} \notin\left[t_{1}, t_{2}\right]$ then $\operatorname{sgn} \tan 2\left(t_{2}-\alpha_{0}\right)=\operatorname{sgntan} 2\left(t_{1}-\alpha_{0}\right)$ and

$$
d\left|\tan 2\left(t_{i}-\alpha_{0}\right)\right|=\sqrt{\frac{d^{2}}{\cos ^{2} 2\left(t_{i}-\alpha_{0}\right)}-d^{2}}=\sqrt{r\left(t_{i}\right)^{4}-d^{2}}
$$

since $r\left(t_{i}\right)^{2}=d / \cos 2\left(t_{i}-\alpha_{0}\right)$. This gives the mass in the form

$$
\pi\left|\sqrt{r\left(t_{2}\right)^{4}-d^{2}}-\sqrt{r\left(t_{1}\right)^{4}-d^{2}}\right| .
$$

The expression that contains $\rho$ is obtained by the Mean value theorem applied to function $q \mapsto \sqrt{q^{2}-d^{2}}$ on interval $\left[r\left(t_{1}\right)^{2}, r\left(t_{2}\right)^{2}\right]$ or $\left[r\left(t_{2}\right)^{2}, r\left(t_{1}\right)^{2}\right]$. (Thus $\rho$ depends on $d, r\left(t_{1}\right)$ and $r\left(t_{2}\right)$ but, naturally, not on $\alpha_{0}$.) Since obviously $S_{t_{1}, t_{2}} \cap A_{r\left(s_{1}\right)}^{r\left(s_{2}\right)}=S_{s_{1}, s_{2}}$ we have

$$
V_{t_{1}, t_{2}}^{d, \alpha_{0}}\left(G_{2}\left(A_{r\left(s_{1}\right)}^{r\left(s_{2}\right)}\right)\right)=\mathbf{M}\left(V_{s_{1}, s_{2}}^{d, \alpha_{0}}\right)
$$

4.2 Mini-layer. Details about branching.

For $\rho>0$ and $\alpha \in \mathbb{R}$, denote

$$
\begin{equation*}
K(\rho, \alpha)=\{\rho(\cos \alpha \cos \beta, \sin \alpha \cos \beta, \cos \alpha \sin \beta, \sin \alpha \sin \beta): \beta \in \mathbb{R}\} \tag{55}
\end{equation*}
$$

which is the circle of radius $\rho$ parameterized by $\beta$ and oriented in $\mathbb{R}^{4}$ by the choice of $\alpha$.

From the ring varifolds we construct two types of (mini-layer) varifolds: $V_{1}$ branching outwards and $V_{2}$ branching inwards (Figure 3). That is, $\delta V_{1}$ is supported on a number of circles of type $K(\rho, \alpha)$ of smaller radius and twice as much circles $K(\rho, \alpha)$ of larger radius. We carefully compute the densities of $\delta V_{i}$ on the circles and record the mass distribution. The densities of $\delta V_{i}$ (see (58), (59)) determine four constants denoted by $C, c$ with decorations. $C$ is the density on larger circles $K\left(r_{2}, \cdot\right), c$ on smaller circles $K\left(r_{1}, \cdot\right)$. Tilde marks the ones related to $\delta V_{1}$ as opposed to $\delta V_{2}$. (The relation $C_{k, \gamma}=\tilde{c}_{k, \gamma}$ is best regarded as just a coincidence although it appears naturally from the manipulations with the objects and numbers.)


Fig. 3 a) Two mini-layers branching outwards. b) Three mini-layers branching inwards.

Lemma 4.2 Let $k \in \mathbb{N}, k>20$ and $\gamma \in(\pi / 8, \pi / 4)$ be fixed. Let

$$
\begin{aligned}
& \sigma=\sigma_{k, \gamma}=\sqrt{\frac{\cos 2 \gamma}{\cos 2(\gamma-\pi / k)}} \in(0,1), \\
& \varepsilon=2 \cos 2(\gamma-\pi / k)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{C}_{k, \gamma} & =4 \sin (2 \gamma) \\
\tilde{c}_{k, \gamma} & =2 \sin (2(\gamma-\pi / k))+2 \sin (2 \gamma)=4 \sin (2 \gamma-\pi / k) \cos (\pi / k) \\
C_{k, \gamma} & =\tilde{c}_{k, \gamma} \\
c_{k, \gamma} & =4 \sin (2(\gamma-\pi / k)) .
\end{aligned}
$$

Then, for every $r_{2}>0$ and for $r_{1}=\sigma r_{2}$, there are rectifiable 2 -varifolds $V_{1}=$ $V_{1}^{r_{1}, r_{2}, k, \gamma}, V_{2}=V_{2}^{r_{1}, r_{2}, k, \gamma}$ in $\mathbb{R}^{4}$ (see (70) and (71) for the definition) such that spt $\mu_{V_{i}} \subset$ $A_{r_{1}}^{r_{1}}$,

$$
\begin{equation*}
\operatorname{spt} V_{i} \subset G_{2}\left(A_{r_{1}}^{r_{2}}\right) \cap G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}, \tag{56}
\end{equation*}
$$

$$
\begin{align*}
4 \sin 2(\gamma-\pi / k) \cdot \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right) & \leq \mathbf{M}\left(V_{i}\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)=V_{i}\left(G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)\right.  \tag{57}\\
& \leq \frac{4}{\sin 2(\gamma-\pi / k)} \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right)
\end{align*}
$$

whenever $r_{1} \leq s_{1}<s_{2} \leq r_{2}$, and

$$
\begin{align*}
& \delta V_{1}(X)=\tilde{C}_{k, \gamma} \mathscr{M}_{r_{2}, 2 k}(X)-\tilde{c}_{k, \gamma} \mathscr{M}_{r_{1}, k}(X)  \tag{58}\\
& \delta V_{2}(X)=C_{k, \gamma} \mathscr{M}_{r_{2}, k}(X)-c_{k, \gamma} \mathscr{M}_{r_{1}, 2 k}(X) \tag{59}
\end{align*}
$$

where (denoting $N(x)=x /\|x\|$ and $K(\rho, \alpha)$ as in (55))

$$
\begin{equation*}
\mathscr{M}_{\rho, k}(X)=\frac{1}{k} \sum_{i=1}^{k} \int_{K(\rho, 2 i \pi / k)} X \cdot N \mathrm{~d} \mathscr{H}^{2} . \tag{60}
\end{equation*}
$$

Proof Let $d>0$ be such that

$$
\begin{align*}
& r_{2}=\sqrt{d / \cos 2 \gamma}  \tag{61}\\
& r_{1}=\sigma r_{2}=\sqrt{d / \cos 2(\gamma-\pi / k)} \tag{62}
\end{align*}
$$

Let $V_{t_{1}, t_{2}}^{d, \alpha_{0}}$ be as in Lemma 4.1, cf. (32) ( $\alpha_{0} \in \mathbb{R}$ and $\alpha_{0}-\pi / 4<t_{1}<t_{2}<\alpha_{0}+$ $\pi / 4)$.

Let

$$
\begin{align*}
& V_{01}=\sum_{i=1}^{k}\left(V_{2 i \pi / k,(2 i+1) \pi / k}^{d,(2 i+1) \pi / k-\gamma}+V_{(2 i+1) \pi / k,(2 i+2) \pi / k}^{d,(2 i+1) \pi / k+\gamma}\right)  \tag{63}\\
& V_{02}=\sum_{i=1}^{k}\left(V_{(2 i-1) \pi / k, 2 i \pi / k}^{d, 2 i \pi / k-\gamma}+V_{2 i \pi / k,(2 i+1) \pi / k}^{d, 2 i \pi / k+\gamma}\right) \tag{64}
\end{align*}
$$

(see Figure 4). The parameters of all $V_{t_{1}, t_{2}}^{d, \alpha_{0}}$ in (63), (64) are so chosen that $r\left(t_{i}\right)=$ $r^{d, \alpha_{0}}\left(t_{i}\right)$ from (28) attain exactly the values $r_{1}, r_{2}$, cf. (61), (62). Therefore all $V_{t_{1}, t_{2}}^{d, \alpha_{0}}$ are supported by $A_{r_{1}}^{r_{2}}$. The difference between $V_{01}$ and $V_{02}$ is just a rotation which allows (together with $V_{00}$ below) proper alignment with the neighbouring mini-layers as in Figure 4. From (33), (34) and (35), we have

$$
\begin{align*}
\delta V_{01}(X)= & 2 \sin (2 \gamma) \sum_{i=1}^{k} \int_{K\left(r_{2},(2 i+1) \pi / k\right)} X \cdot N \mathrm{~d} \mathscr{H}^{2}  \tag{65}\\
& -2 \sin (2(\gamma-\pi / k)) \sum_{i=1}^{k} \int_{K\left(r_{1},(2 i+2) \pi / k\right)} X \cdot N \mathrm{~d} \mathscr{H}^{2},
\end{align*}
$$



Fig. 4 The varifolds a) $V_{01}$ and b) $V_{02}$ (63), (64) on the gray background of Figure 3. (Note that a) and b) are not drawn and will not be used at the same scale.) We are patching rings by actually patching pieces of the planar curve $r(\alpha)$ from (28). The result is then rotated in $\mathbb{R}^{4}$ as indicated in by $\beta$ in (29). The radial segments (they create planar annuli by the rotation) will be added later with (a proper density), see $V_{00}$ in (67) and (70), (71).

$$
\begin{align*}
\delta V_{02}(X)= & 2 \sin (2 \gamma) \sum_{i=1}^{k} \int_{K\left(r_{2}, 2 i \pi / k\right)} X \cdot N \mathrm{~d} \mathscr{H}^{2}  \tag{66}\\
& -2 \sin (2(\gamma-\pi / k)) \sum_{i=1}^{k} \int_{K\left(r_{1},(2 i+1) \pi / k\right)} X \cdot N \mathrm{~d} \mathscr{H}^{2} .
\end{align*}
$$

Let

$$
\begin{equation*}
V_{00}=V_{00}^{r_{1}, r_{2}, k}=\frac{1}{k} \sum_{i=1}^{k} V_{1, r_{1}, r_{2},(\cos 2 i \pi / k, \sin 2 i \pi / k)} \tag{67}
\end{equation*}
$$

where $V_{1, r_{1}, r_{2},(a, b)}=V_{\mathscr{H}^{2}\left\llcorner\left(\operatorname{span}\left\{a e_{1}+b e_{2}, a e_{3}+b e_{4}\right\} \cap A_{r_{1}}^{r_{2}}\left(\mathbb{R}^{4}\right)\right)\right.}$ (see also (14), (15)). Since $\operatorname{span}\left\{a e_{1}+b e_{2}, a e_{3}+b e_{4}\right\}$ is a linear space invariant under multiplication by $J_{13}^{24}$ (see (22)), we have

$$
\begin{equation*}
\operatorname{spt} V_{00} \subset G_{2}\left(A_{r_{1}}^{r_{2}}\right) \cap G_{\mathrm{rad} \& J_{13}^{24}}^{0} \tag{68}
\end{equation*}
$$

Furthermore (cf. (5) or Section 3),

$$
\begin{align*}
\delta V_{00}(X) & =\frac{1}{k} \sum_{i=1}^{k}\left(\int_{K\left(r_{2}, 2 i \pi / k\right)} X \cdot N \mathrm{~d} \mathscr{H}^{2}-\int_{K\left(r_{1}, 2 i \pi / k\right)} X \cdot N \mathrm{~d} \mathscr{H}^{2}\right) \\
& =\mathscr{M}_{r_{2}, k}(X)-\mathscr{M}_{r_{1}, k}(X) \tag{69}
\end{align*}
$$

Let

$$
\begin{align*}
& V_{1}=\frac{1}{k} V_{01}+2 \sin (2 \gamma) V_{00}  \tag{70}\\
& V_{2}=\frac{1}{k} V_{02}+2 \sin (2(\gamma-\pi / k)) V_{00} \tag{71}
\end{align*}
$$

Then the first variation of $V_{1}$ and $V_{2}$ is exactly as stated in (58), (59). Note that

$$
\mathrm{spt}_{V_{01}} \cup \mathrm{spt}_{V_{02}} \subset G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}
$$

by Lemma 4.1, 5., and the same is true for planar varifold $V_{00}$, so also for $V_{1}$ and $V_{2}$.
Let $r_{1} \leq s_{1}<s_{2} \leq r_{2}$. We claim that

$$
\begin{equation*}
\mathbf{M}\left(V_{01}\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)=\mathbf{M}\left(V_{02}\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)=c 2 k \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right)\right.\right. \tag{72}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{1}{\sin 2 \gamma} \leq c \leq \frac{1}{\sin 2(\gamma-\pi / k)} \tag{73}
\end{equation*}
$$

Indeed, if $\rho=\rho\left(d, s_{1}, s_{2}\right) \in\left[s_{1}, s_{1}\right] \subset\left[r_{1}, r_{2}\right]$ is as in Lemma 4.1, 6., then (72) holds true with

$$
c=\frac{1}{\sqrt{1-\frac{d^{2}}{\rho^{4}}}} \leq \frac{1}{\sqrt{1-\frac{d^{2}}{\left(r_{1}\right)^{4}}}}=\frac{1}{\sqrt{1-\cos ^{2} 2(\gamma-\pi / k)}}=\frac{1}{\sin 2(\gamma-\pi / k)}
$$

On the other hand,

$$
c \geq \frac{1}{\sqrt{1-\frac{d^{2}}{\left(r_{2}\right)^{4}}}}=\frac{1}{\sqrt{1-\cos ^{2} 2 \gamma}}=\frac{1}{\sin 2 \gamma}
$$

We have exactly $\mathbf{M}\left(V_{00}\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)=\pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right)\right.$. Combining that with (73), we get (for $i=1,2$ )

$$
\begin{aligned}
4 \sin (2(\gamma-\pi / k)) \cdot \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right) & \leq\left(\frac{2}{\sin 2 \gamma}+2 \sin (2(\gamma-\pi / k))\right) \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right) \\
& \leq \mathbf{M}\left(V_{i}\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)\right. \\
& \leq\left(\frac{2}{\sin 2(\gamma-\pi / k)}+2 \sin (2 \gamma)\right) \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right) \\
& \leq \frac{4}{\sin 2(\gamma-\pi / k)} \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right)
\end{aligned}
$$

which is (57).

### 4.3 Layers.

Recall now that $F$ is defined by (9) (see also (10) and (11)).
Lemma 4.3 If $0<R_{1}<R_{2}<R_{3}<R_{4}<\infty$ and $\varepsilon>0$ then there is $c \in(1-\varepsilon, 1)$ and a rectifiable 2-varifold $V$ with $\operatorname{spt} \mu_{V} \subset A_{R_{1}}^{R_{4}}$,

$$
\begin{gather*}
\operatorname{spt} V \subset G_{2}\left(A_{R_{1}}^{R_{2}} \cup A_{R_{3}}^{R_{4}}\right) \cap G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon}  \tag{74}\\
\cup G_{2}\left(A_{R_{2}}^{R_{3}}\right) \cap G_{\mathrm{rad} \& J_{13}^{24}}^{0} \\
\subset G_{2}\left(A_{R_{1}}^{R_{4}}\right) \cap G_{\mathrm{rad} \& J_{13}^{24}}^{\varepsilon},  \tag{75}\\
(1-\varepsilon) \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right)< \tag{76}
\end{gather*} \mathbf{M}\left(V\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)<(1+\varepsilon) \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right), ~ \$\right.
$$

whenever $R_{1} \leq s_{1}<s_{2} \leq R_{4}$, and

$$
\begin{equation*}
\delta V(X)=\mathscr{M}_{R_{4}, \infty}(X)-c \mathscr{M}_{R_{1}, \infty}(X) \tag{77}
\end{equation*}
$$

where (with $N(x)=x /\|x\|)$

$$
\begin{equation*}
\mathscr{M}_{\rho, \infty}(X)=\int_{F\left(\left(\rho \cdot S_{1}\left(\mathbb{R}^{2}\right)\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)} X \cdot N \mathrm{~d} \frac{\mathscr{H}^{2}}{2 \pi \rho} \tag{78}
\end{equation*}
$$

Remark 4.4 Note that $\mathscr{M}_{\rho, \infty}$ is a vector measure on the scaled Clifford torus $F((\rho$. $\left.\left.S_{1}\left(\mathbb{R}^{2}\right)\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)$ that is uniformly diffuse in the sense that the associated total variation measure is just a constant multiple of the Hausdorff measure. This comes from the properties of the Clifford torus and from how uniformly $\mathscr{M}_{\rho, k}$ from (60) is distributed on the "parallel" circles. (For their weak convergence see also (85) below.)

This will be important later for compatibility on the interface, when $V$ from Lemma 4.3 is used together with similar but different varifold $V$ from Lemma 4.4.」

Before giving the formal proof of Lemma 4.3, we explain how the varifold $V$ (the layer) is constructed.

The space between $R_{1}$ and $r^{\left(n_{0}\right)} \in\left(R_{1}, R_{2}\right]$ is occupied by an infinite sequence of varifolds (mini-layers) from Lemma 4.2 and Figure 3b) that are branching towards the inner interface which is the Clifford torus at radius $R_{1}$. Here $n_{0}$ is a technical index (to be explained later) and $r^{\left(n_{0}\right)}$ is chosen at our convenience for using Lemma 4.2. The mini-layers are indexed by $n \geq n_{0}$ and each of them lives between suitably defined radii $r^{(n+1)}$ and $r^{(n)}$ where $r^{(n+1)}<r^{(n)}$. The connections at radii $r^{(n)}\left(n>n_{0}\right)$ (i.e., the branching) can also be seen in Figure 3b).

Likewise, the space between $R^{\left(n_{0}\right)} \in\left[R_{3}, R_{4}\right)$ and $R_{4}$ is occupied by an infinite sequence of mini-layers from Figure 3a) that are branching towards the outer interface which is the Clifford torus at radius $R_{4}$. Each of them lives between suitably defined radii $R^{(n)}$ and $R^{(n+1)}$ where $R^{(n)}<R^{(n+1)}$.

The space between $r^{\left(n_{0}\right)}$ and $R^{\left(n_{0}\right)}$ is bridged by a varifold supported on a finite number of planar annuli, which is the term $c_{1} V_{00}$ in the definition of $V$ below. (I have received a question about the purpose of $c_{1} V_{00}$. Obviously, the space between $r^{\left(n_{0}\right)}$ and $R^{\left(n_{0}\right)}$ should not be left empty if we wish to have a stationary varifold. In principle it would be possible to assume $r^{\left(n_{0}\right)}=R_{2}=R_{3}=R^{\left(n_{0}\right)}$ and avoid the term $c_{1} V_{00}$ but we have chosen an easier way. Moreover, for the proof of Theorem 5.2 it is useful to have $R_{3} / R_{2}$ very large and $c_{1} V_{00}$ is not only the most easy but also the most natural and most intuitive candidate to fill the space. Though, mini-layers have enough flexibility to replace its role.)

As it is indicated above, we use infinite sequences of mini-layers and we have to emphasize that the corresponding sequences of parameters for Lemma 4.2 need (and fortunately can) be chosen so that both 1) products of density coefficient ratios are positive and 2 ) the product of radii ratios is positive.

This then allows to choose the technical index $n_{0}$ large enough to obtain 1) estimate (76) 2) an infinite number of mini-layers that fits between $R_{1}$ and $R_{2}\left(R_{3}\right.$ and $R_{4}$, respectivelly).

The meaning of index $n_{0}$ is the following: out of a sequence of candidate minilayers (which are indexed by $n$ ) we forget the first $n_{0}$ of them and use only the tail of the sequence. After $n_{0}$ is known we decide the radii to which we scale the mini-layers as well the densities that we apply to them.

Remark 4.5 Without giving details we note that $n_{0}$ has to be chosen large if $\varepsilon$ is small. This results in the density coefficients of $V$ being bounded from above by about $1 / 2^{n_{0}}$ (and $n_{0} \rightarrow \infty$ when layers closer to $0 \in \mathbb{R}^{4}$ are considered in our application of the lemma) which is not much desired and implies that our tangent varifolds will be non-rectifiable. Actually, the presence of a large number of points of small density is unavoidable if we want to obtain non-conical tangents, see Lemma 1.1 and its references.

Furthermore, the two-dimensional density at the points of the interface (i.e., at radii $R_{1}$ and $R_{4}$ ) of each layer will be zero. Nevertheless, this is negligible in measure (as measured by the varifold) and does not prevent us from constructing a rectifiable varifold. We leave open whether points of zero two-dimensional density can be completely avoided in a varifold with non-conical tangents.

Proof (of Lemma 4.3) Choose $k^{(n)}=100 \cdot 2^{n}$ and $\gamma^{(n)}=\pi / 4-\pi / \sqrt{k^{(n)}}$.
With $C_{k, \gamma}, c_{k, \gamma} \tilde{C}_{k, \gamma}, \tilde{c}_{k, \gamma}$ and $\sigma_{k, \gamma}$ as in Lemma 4.2 we have

$$
\begin{aligned}
& 1 \geq \frac{c_{k^{(n)}, \gamma^{(n)}}}{C_{k^{(n)}, \gamma^{(n)}}} \geq \sin \left(2\left(\gamma^{(n)}-\pi / k^{(n)}\right)\right) \geq 1-8 \pi^{2} / k^{(n)}>0 \\
& 1 \geq \frac{\tilde{c}_{k^{(n)}, \gamma^{(n)}}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} \geq \sin \left(2 \gamma^{(n)}-\pi / k^{(n)}\right) \cos \left(\pi / k^{(n)}\right) \geq 1-5 \pi^{2} / k^{(n)}>0
\end{aligned}
$$

Hence

$$
\prod_{n=1}^{\infty} \frac{c_{k^{(n)}, \gamma^{(n)}}}{C_{k^{(n)}, \gamma^{(n)}}} \in(0,1), \quad \prod_{n=1}^{\infty} \frac{\tilde{c}_{k^{(n)}, \gamma^{(n)}}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} \in(0,1)
$$

Furthermore

$$
\begin{aligned}
0 & \leq 1-\sigma_{k^{(n)}, \gamma^{(n)}}^{2}=1-\frac{\sin 2 \pi / \sqrt{k^{(n)}}}{\sin \left(2 \pi / k^{(n)}+2 \pi / \sqrt{k^{(n)}}\right)} \\
& =\frac{2 \cos \left(\pi / k^{(n)}+2 \pi / \sqrt{k^{(n)}}\right) \sin \pi / k^{(n)}}{\sin \left(2 \pi / k^{(n)}+2 \pi / \sqrt{k^{(n)}}\right)} \leq \pi \frac{\pi / k^{(n)}}{2 \pi / k^{(n)}+2 \pi / \sqrt{k^{(n)}}} \leq \frac{\pi}{2} \frac{1}{\sqrt{k^{(n)}}},
\end{aligned}
$$

hence

$$
\left(\prod_{n=1}^{\infty} \sigma_{k^{(n)}, \gamma^{(n)}}\right)^{2}=\prod_{n=1}^{\infty} \sigma_{k^{(n)}, \gamma^{(n)}}^{2} \in(0,1)
$$

Choose $n_{0} \in \mathbb{N}$ so that (for $n \geq n_{0}$ )

$$
\begin{gather*}
\varepsilon_{n}:=2 \cos 2\left(\gamma^{(n)}-\pi / k^{(n)}\right)<\varepsilon, \\
\sin 2\left(\gamma^{(n)}-\pi / k^{(n)}\right)>1-\varepsilon / 3,  \tag{79}\\
M:=\frac{1}{4 \sin \left(2 \gamma^{\left(n_{0}\right)}-\pi / k^{\left(n_{0}\right)}\right) \cos \left(\pi / k^{\left(n_{0}\right)}\right)} \frac{4}{\sin 2\left(\gamma^{\left(n_{0}\right)}-\pi / k^{\left(n_{0}\right)}\right)}<1+\varepsilon, \tag{80}
\end{gather*}
$$

$$
\begin{aligned}
& c_{1}:=\prod_{n=n_{0}}^{\infty} \frac{\tilde{c}_{k^{(n)}, \gamma^{(n)}}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} \in(1-\varepsilon / 3,1), \\
& c_{2}:=\prod_{n=n_{0}}^{\infty} \frac{c_{k^{(n)}, \gamma^{(n)}}}{C_{k^{(n)}, \gamma^{(n)}}} \in(1-\varepsilon / 3,1)
\end{aligned}
$$

and

$$
\sigma:=\prod_{n=n_{0}}^{\infty} \sigma_{k^{(n)}, \gamma^{(n)}} \in\left(\max \left(R_{1} / R_{2}, R_{3} / R_{4}\right), 1\right)
$$

Let $r^{\left(n_{0}\right)}:=R_{1} / \sigma, R^{\left(n_{0}\right)}:=\sigma R_{4}$, and then inductively $r^{(n+1)}:=\sigma_{k^{(n)}, \gamma^{(n)}} r^{(n)}, R^{(n+1)}:=$ $R^{(n)} / \sigma_{k^{(n)}, \gamma^{(n)}}$. Then $\lim _{n \rightarrow \infty} r^{(n)}=R_{1}, \lim _{n \rightarrow \infty} R^{(n)}=R_{4}$,

$$
\begin{aligned}
& R_{1}<r^{\left(n_{0}\right)} \leq R_{2} \leq R_{3} \leq R^{\left(n_{0}\right)}<R_{4} \\
& R_{1}<\cdots<r^{\left(n_{0}+2\right)}<r^{\left(n_{0}+1\right)}<r^{\left(n_{0}\right)} \leq R^{\left(n_{0}\right)}<R^{\left(n_{0}+1\right)}<R^{\left(n_{0}+2\right)}<\cdots<R_{4} .
\end{aligned}
$$

Let

$$
\begin{array}{ll}
c_{1, n}:=\prod_{m=n}^{\infty} \frac{\tilde{c}_{k^{(m)}, \gamma^{(m)}}}{\tilde{C}_{k^{(m)}, \gamma^{(m)}}} \in(0,1) & \left(\text { hence } c_{1}=c_{1, n_{0}}\right) \\
c_{2, n}:=\prod_{m=n_{0}}^{n-1} \frac{c_{k^{(m)}, \gamma^{(m)}}}{C_{k^{(m)}, \gamma^{(m)}}} \in(0,1] & \left(c_{2, n_{0}}:=1 ; c_{2}=c_{2, \infty}\right),
\end{array}
$$

and let $V_{1}^{r, s, k, \gamma}, V_{2}^{r, s, k, \gamma}$ be as in Lemma 4.2 and $V_{00}=V_{00}^{r^{\left(n_{0}\right)}, R^{\left(n_{0}\right)}, k^{(n)}}$ is as in (67). Let
and

$$
V_{m}=\sum_{n=n_{0}}^{m} \frac{c_{1} c_{2, n}}{C_{k^{(n)}, \gamma^{(n)}}} V_{2}^{r^{(n+1)}, r^{(n)}, k^{(n)}, \gamma^{(n)}}+c_{1} V_{00}+\sum_{n=n_{0}}^{m} \frac{c_{1, n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} V_{1}^{R^{(n)}, R^{(n+1)}, k^{(n)}, \gamma^{(n)}} .
$$

Then (74) can be obtained from (56) and (68).
Denote also

Note that

$$
\begin{aligned}
\frac{c_{1} c_{2, n}}{C_{k^{(n)}, \gamma^{(n)}}} \frac{4}{\sin 2\left(\gamma^{(n)}-\pi / k^{(n)}\right)} \leq M, & n \geq n_{0}, \\
c_{1} \leq 1 \leq M, & \\
\frac{c_{1, n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} \frac{4}{\sin 2\left(\gamma^{(n)}-\pi / k^{(n)}\right)} \leq M, & n \geq n_{0}
\end{aligned}
$$

Hence, by (57) and (80),

$$
\begin{align*}
\mathbf{M}(V) \leq & \sum_{n=n_{0}}^{\infty} M \pi\left(\left(r^{(n)}\right)^{2}-\left(r^{(n+1)}\right)^{2}\right)  \tag{81}\\
& +M \pi\left(\left(R^{\left(n_{0}\right)}\right)^{2}-\left(r^{\left(n_{0}\right)}\right)^{2}\right)+\sum_{n=n_{0}}^{\infty} M \pi\left(\left(R^{(n+1)}\right)^{2}-\left(R^{(n)}\right)^{2}\right) \\
= & M \pi\left(\left(R_{4}\right)^{2}-\left(R_{1}\right)^{2}\right) \leq(1+\varepsilon) \pi\left(\left(R_{4}\right)^{2}-\left(R_{1}\right)^{2}\right)
\end{align*}
$$

In particular, $V$ is a Radon measure. Therefore $V$ is a varifold, obviously rectifiable. Moreover, $\mathbf{M}\left(V-V_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$.

Note also that

$$
\begin{aligned}
\frac{4 c_{1} c_{2, n}}{C_{k^{(n)}, \gamma^{(n)}}} \geq c_{1} c_{2}, & n \geq n_{0} \\
\quad c_{1} & \geq c_{1} c_{2}, \\
\frac{4 c_{1, n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}} \geq c_{1} c_{2}, & n \geq n_{0}
\end{aligned}
$$

Again by (57) (and (79)), we get

$$
\begin{align*}
\mathbf{M}(V) \geq & \sum_{n=n_{0}}^{\infty}(1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(r^{(n)}\right)^{2}-\left(r^{(n+1)}\right)^{2}\right)  \tag{82}\\
& +(1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(R^{\left(n_{0}\right)}\right)^{2}-\left(r^{\left(n_{0}\right)}\right)^{2}\right) \\
& +\sum_{n=n_{0}}^{\infty}(1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(R^{(n+1)}\right)^{2}-\left(R^{(n)}\right)^{2}\right) \\
= & (1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(R_{4}\right)^{2}-\left(R_{1}\right)^{2}\right) \geq(1-\varepsilon) \pi\left(\left(R_{4}\right)^{2}-\left(R_{1}\right)^{2}\right) .
\end{align*}
$$

From (81) and (82), (76) follows in the special case $s_{1}=R_{1}, s_{2}=R_{4}$. (Note that a special case $s_{1}=r_{1}, s_{2}=r_{2}$ of (57) was used.) Proof of the general case $R_{1} \leq s_{1}<$ $s_{2} \leq R_{4}$ of (76) is similar, with the following differences: a) some of the terms in (81), (82) might be replaced by 0 , and b) some (at most two) of the terms might be "cut" to a smaller span between radii; the general case of (57) is used in such a case. For example, (82) is to be replaced by

$$
\begin{align*}
\mathbf{M}\left(V\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right) \geq\right. & \left.\sum_{n=n_{0}}^{\infty}(1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(\widehat{\left(r^{(n)}\right.}\right)^{2}-\widehat{\left(r^{(n+1)}\right.}\right)^{2}\right)  \tag{83}\\
& +(1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(\widehat{R^{\left(n_{0}\right)}}\right)^{2}-\left(\widehat{r^{\left(n_{0}\right)}}\right)^{2}\right) \\
& +\sum_{n=n_{0}}^{\infty}(1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(\widehat{R^{(n+1)}}\right)^{2}-\left(\widehat{R^{(n)}}\right)^{2}\right) \\
= & (1-\varepsilon / 3) c_{1} c_{2} \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right) \geq(1-\varepsilon) \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right)
\end{align*}
$$

where $\widehat{\rho}=\min \left(\max \left(s_{1}, \rho\right), s_{2}\right)$.

We have $V_{n_{0}-1}=c_{1} V_{00}$ and, by (69),

$$
\delta V_{n_{0}-1}=c_{1} \mathscr{M}_{R^{\left(n_{0}\right), k\left(k^{\left(n_{0}\right)}\right.}}-c_{1} \mathscr{M}_{r^{\left(n_{0}\right)}, k^{\left(n_{0}\right)}}=c_{1, n_{0}} \mathscr{M}_{R^{\left(n_{0}\right), k} k^{\left(n_{0}\right)}}-c_{1} c_{2, n_{0}} \mathscr{M}_{r^{\left(n_{0}\right)}, k^{\left(n_{0}\right)}}
$$

where $\mathscr{M}_{\rho, k}$ is as in (60). Using (58) (59) we obtain by induction

$$
\begin{equation*}
\delta V_{n}=c_{1, n+1} \mathscr{M}_{R^{(n+1)}, k^{(n+1)}}-c_{1} c_{2, n+1} \mathscr{M}_{r^{(n+1)}, k^{(n+1)}} . \tag{84}
\end{equation*}
$$

Indeed, for $n \geq n_{0}$,

$$
\begin{aligned}
& \delta V_{n}=\frac{c_{1, n+1}}{\tilde{C}_{k^{(n)}, \gamma^{(n)}}}\left(\tilde{C}_{k^{(n)}, \gamma^{(n)}} \mathscr{M}_{R^{(n+1)}, 2 k^{(n)}}-\tilde{c}_{k^{(n)}, \gamma^{(n)}} \mathscr{M}_{R^{(n)}, k^{(n)}}\right)+ \\
& c_{1, n} \mathscr{M}_{R^{(n)}, k^{(n)}}-c_{1} c_{2, n} \mathscr{M}_{r^{(n)}, k^{(n)}}+ \\
& \frac{c_{1} c_{2, n}}{C_{k^{(n)}, \gamma^{(n)}}}\left(C_{k^{(n)}, \gamma^{(n)}} \mathscr{M}_{r^{(n)}, k^{(n)}}-c_{k^{(n)}, \gamma^{(n)}} \mathscr{M}_{r^{(n+1), 2 k^{(n)}}}\right) \\
& =c_{1, n+1} \mathscr{M}_{R^{(n+1), k^{(n+1)}}}-c_{1} c_{2, n+1} \mathscr{M}_{r^{(n+1), k^{(n+1)}}}
\end{aligned}
$$

It is easy to verify that, for every smooth vector field $X$,

$$
\begin{equation*}
\mathscr{M}_{R^{(n+1)}, k^{(n+1)}}(X) \rightarrow \mathscr{M}_{R_{4}, \infty}(X) \tag{85}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathscr{M}_{r^{(n+1)}, k^{(n+1)}}(X) \rightarrow \mathscr{M}_{R_{1}, \infty}(X) . \tag{86}
\end{equation*}
$$

Indeed, the (local) uniform continuity of $X$ can be used in the same way as when proving the simple planar exercise with Dirac masses $\frac{1}{k} \sum_{i=1}^{k} \delta_{\left(\frac{i}{k}, \frac{1}{k}\right)} \xrightarrow{w} \mathscr{H}^{1}\llcorner([0,1] \times\{0\})$. On the other hand,

$$
\left|\delta V(X)-\delta V_{n}(X)\right| \stackrel{(3)}{=}\left|\int \operatorname{div}_{S} X(x) \mathrm{d}\left(V-V_{n}\right)(x, S)\right| \leq\|X\|_{C^{1}} \cdot \mathbf{M}\left(V-V_{n}\right) \rightarrow 0
$$

as $n \rightarrow \infty$. From (84) and (85), (86) we therefore obtain the formula (77) for the first variation of $V$, with $c:=\lim c_{1} c_{2, n}=c_{1} c_{2} \in(1-\varepsilon, 1)$.

Lemma 4.4 If $0<R_{1}<R_{2}<R_{3}<R_{4}<\infty$ and $\varepsilon>0$ then there is $c \in(1-\varepsilon, 1)$ and a rectifiable 2 -varifold $V$ with $\operatorname{spt} \mu_{V} \subset A_{R_{1}}^{R_{4}}$,

$$
\begin{gather*}
\operatorname{spt} V \subset G_{2}\left(A_{R_{1}}^{R_{2}} \cup A_{R_{3}}^{R_{4}}\right) \cap G_{\mathrm{rad} \& J_{12}^{34}}^{\varepsilon}  \tag{87}\\
\cup G_{2}\left(A_{R_{2}}^{R_{3}}\right) \cap G_{\mathrm{rad} \& J_{12}^{34}}^{0} \\
\subset G_{2}\left(A_{R_{1}}^{R_{4}}\right) \cap G_{\mathrm{rad} \& J_{12}^{34}}^{\varepsilon},  \tag{88}\\
(1-\varepsilon) \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right)< \tag{89}
\end{gather*} \mathbf{M}\left(V\left\llcorner G_{2}\left(A_{s_{1}}^{s_{2}}\right)\right)<(1+\varepsilon) \pi\left(\left(s_{2}\right)^{2}-\left(s_{1}\right)^{2}\right), ~ \$\right.
$$

whenever $R_{1} \leq s_{1}<s_{2} \leq R_{4}$, and

$$
\begin{equation*}
\delta V(X)=\mathscr{M}_{R_{4}, \infty}(X)-c \mathscr{M}_{R_{1}, \infty}(X) \tag{90}
\end{equation*}
$$

where $\mathscr{M}_{\rho, \infty}$ is as in (78).

Proof The statement is the same as in Lemma 4.3, with the exception of a change of coordinates in (87) - we show that it is enough to exchange coordinates $x_{2}$ and $x_{3}$. Let $\phi\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\phi\left(x_{1}, x_{3}, x_{2}, x_{4}\right),\left(\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in \mathbb{R}^{4}\right)$, and $\Phi(x, S)=$ $(\phi(x), \phi(S))\left((x, S) \in G_{2}\left(\mathbb{R}^{4}\right)\right)$. Then $\phi\left(J_{13}^{24} x\right)=J_{12}^{34} \phi(x)$ and $\Phi\left(G_{\text {rad } \& J_{13}^{24}}^{\varepsilon}\right)=G_{\text {rad \& } J_{12}^{34}}^{\varepsilon}$ (cf. (24)). The domain of integration in (78) (which is parameterized by $F$ ) does not change under $\phi: \phi(F((\rho a, \rho b),(c, d))) \stackrel{(11)}{=} F((c, d),(\rho a, \rho b)) \stackrel{(10)}{=} F((\rho c, \rho d),(a, b))$. Since $\phi$ is an isometry, it also preserves Hausdorff measure in (78). Therefore, if $\tilde{V}$ is as in Lemma 4.3, then $V:=\phi_{\# \#} \tilde{V}=\Phi_{\#} \tilde{V}$ is a varifold with required properties.

Lemma 4.5 If $V$ is as in Lemma 4.3 or Lemma 4.4 and $r>0$ then

$$
\begin{equation*}
\mathbf{M}\left(V\left\llcorner G_{2}\left(S_{r}\left(\mathbb{R}^{4}\right)\right)\right)=0\right. \tag{91}
\end{equation*}
$$

Proof For every $0<\varepsilon_{1}<r$ we heave by (76), (89),

$$
\mathbf{M}\left(V\left\llcorner G_{2}\left(A_{r-\varepsilon_{1}}^{r+\varepsilon_{1}}\right)\right) \leq(1+\varepsilon) \pi\left(\left(r+\varepsilon_{1}\right)^{2}-\left(r-\varepsilon_{1}\right)^{2}\right) \rightarrow 0 .\right.
$$

We do the last step of our construction of a stationary rectifiable varifold in the next section.

## 5 Two variants of the main result

Theorem 5.1 There is a stationary rectifiable 2-varifold $V$ in $\mathbb{R}^{4}$ that has a nonconical (hence non-unique) tangent at 0 and $0<\theta^{2}(V, 0)<\infty$.

The proof is built around the idea of alternating layers of two types of varifolds as in our non-rectifiable example in Section 3. For each layer, the varifold of Section 3 is replaced by its rectifiable "approximation" from Lemma 4.3 and Lemma 4.4. However this introduces some excess and therefore the density coefficients must be calculated accordingly and we have to take care to get positive density at the origin, which means we have to estimate yet another infinite product.

As we emphasised above, it is important that the first variations of the layer varifolds is a measure (vector measure with the radial directions) uniformly distributed on the interfaces and therefore it is compatible for our two types of layers which differ by a rotation. This important feature is shared with Section 3.

Proof 1. The varifold $V$. For $0<R_{1}<R_{2}<R_{3}<R_{4}<\infty$ and $\varepsilon>0$ let

$$
V_{R_{1}, R_{2}, R_{3}, R_{4}, \varepsilon}^{1} \quad \text { and } \quad c_{R_{1}, R_{2}, R_{3}, R_{4}, \varepsilon}^{1} \in(1-\varepsilon, 1)
$$

denote the varifold and the number from Lemma 4.3. Let

$$
V_{R_{1}, R_{2}, R_{3}, R_{4}, \varepsilon}^{2} \quad \text { and } \quad c_{R_{1}, R_{2}, R_{3}, R_{4}, \varepsilon}^{2} \in(1-\varepsilon, 1)
$$

denote the varifold and the number from Lemma 4.4.

For $n \in \mathbb{Z}$, let

$$
\begin{aligned}
& \varepsilon^{(n)}=1 / 4\left(n^{2}+1\right) \\
& R_{1}^{(n)}=2^{-n} \\
& R_{2}^{(n)}=\left(1+\varepsilon^{(n)}\right) 2^{-n} \\
& R_{3}^{(n)}=\left(1-\varepsilon^{(n)}\right) 2^{-n+1} \\
& R_{4}^{(n)}=2^{-n+1}=R_{1}^{(n-1)} .
\end{aligned}
$$

Let

$$
V^{(n)}= \begin{cases}V_{R_{1}^{(n)}, R_{2}^{(n)}, R_{3}^{(n)}, R_{4}^{(n)}, \varepsilon^{(n)}}^{1} & \text { for } n \text { even, and } \\ V_{R_{1}^{(n)}, R_{2}^{(n)}, R_{3}^{(n)}, R_{4}^{(n)}, \mathcal{E}^{(n)}}^{2} & \text { for } n \text { odd. }\end{cases}
$$

Accordingly, let

$$
c^{(n)}= \begin{cases}c^{1} R_{1}^{(n)}, R_{2}^{(n)}, R_{3}^{(n)}, R_{4}^{(n)}, \varepsilon^{(n)} & \text { for } n \text { even, and } \\ c^{2} R_{1}^{(n)}, R_{2}^{(n)}, R_{3}^{(n)}, R_{4}^{(n)}, \varepsilon^{(n)} & \text { for } n \text { odd. }\end{cases}
$$

Let $C^{(0)}=1$ and

$$
C^{(n)}= \begin{cases}\prod_{k=0}^{n-1} c^{(k)} & \text { for } n>0, \text { and } \\ \prod_{k=n}^{-1} \frac{1}{c^{(k)}} & \text { for } n<0\end{cases}
$$

Since $c^{(k)} \geq 1-\varepsilon^{(k)}$ and $\sum_{k \geq 0} \varepsilon^{(k)}<\infty$, we have

$$
C^{(\infty)}:=\lim _{n \rightarrow \infty} C^{(n)} \in(0, \infty)
$$

Define

$$
V:=\sum_{n \in \mathbb{Z}} C^{(n)} V^{(n)} .
$$

By (76), (89),

$$
\begin{equation*}
\frac{\pi}{2}\left(\left(R_{4}^{(n)}\right)^{2}-\left(R_{1}^{(n)}\right)^{2}\right) \leq \mathbf{M}\left(V^{(n)}\right) \leq M^{(n)}:=2 \pi\left(\left(R_{4}^{(n)}\right)^{2}-\left(R_{1}^{(n)}\right)^{2}\right) \tag{92}
\end{equation*}
$$

Since $C^{(n)}$ is decreasing,

$$
\begin{equation*}
\sum_{n \geq-k} C^{(n)} \mathbf{M}\left(V^{(n)}\right) \leq \sum_{n \geq-k} C^{(-k)} M^{(n)}=C^{(-k)} 2 \pi\left(R_{4}^{(-k)}\right)^{2}<\infty . \tag{93}
\end{equation*}
$$

$V$ is a Radon measure because, for every $k$,

$$
V\left(G_{2}\left(\left\{x:\|x\|<2^{k}\right\}\right)\right) \leq \sum_{n \geq-k} C^{(n)} \mathbf{M}\left(V^{(n)}\right)<\infty .
$$

Obviously, the varifold $V$ is rectifiable.
Using (76) and (89) more wisely than in (92) we get that

$$
\begin{equation*}
C^{(\infty)}\left(1-\varepsilon^{(n)}\right) \pi R^{2} \leq V\left(G_{2}(\{x:\|x\| \leq R\})\right) \leq C^{(n)}\left(1+\varepsilon^{(n)}\right) \pi R^{2} \tag{94}
\end{equation*}
$$

whenever $R \in\left(0, R_{4}^{(n)}\right)$. Hence

$$
\theta^{2}(V, 0)=C^{(\infty)} \pi \in(0, \infty)
$$

2. The varifold $V$ is stationary. Let $X$ be a compactly supported smooth vector field on $\mathbb{R}^{4}$. Fix $k \in \mathbb{N}$ such that $\operatorname{spt} X \subset\left\{x:\|x\|<2^{k}\right\}$. We have

$$
\left|\delta V^{(n)}(X)\right|=\left|\int \operatorname{div}_{S} X(x) \mathrm{d} V^{(n)}(x, S)\right| \leq\|X\|_{C^{1}} \cdot \mathbf{M}\left(V^{(n)}\right) \leq\|X\|_{C^{1}} C^{(n)} M^{(n)}
$$

Since $\sum_{n \geq-k} C^{(n)} M^{(n)}$ converges by (93), we have

$$
\begin{align*}
& \delta V(X)=\int \operatorname{div}_{S} X(x) \mathrm{d} V(x, S) \\
&  \tag{95}\\
& =\sum_{n \geq-k} C^{(n)} \int \operatorname{div}_{S} X(x) \mathrm{d} V^{(n)}(x, S)=\sum_{n \geq-k} C^{(n)} \delta V^{(n)}(X)
\end{align*}
$$

Next we use (77) and (90) to calculate $\sum_{n=-k}^{m} C^{(n)} \delta V^{(n)}(X)$. The first term is zero since the support of $\delta V^{-k}$ is disjoint with the support of $X$, next terms mutually cancel $\left(C^{(n)} c^{(n)}=C^{(n+1)}, R_{1}^{(n)}=R_{4}^{(n+1)}\right)$ and what remains from the last one can be transformed so that we see that it converges to 0 . Formally,

$$
\begin{aligned}
& \sum_{n=-k}^{m} C^{(n)} \delta V^{(n)}(X)=\sum_{n=-k}^{m}\left(C^{(n)} \mathscr{M}_{R_{4}^{(n), \infty}}(X)-C^{(n)} c^{(n)} \mathscr{M}_{R_{1}^{(n), \infty}}(X)\right) \\
&=C^{(-k)} \mathscr{M}_{R_{4}^{(-k), \infty}}(X)-C^{(m)} c^{(m)} \mathscr{M}_{R_{1}^{(m), \infty}}(X) \\
& \stackrel{(78)}{=}-C^{(m)} c^{(m)} \int_{F\left(\left(R_{1}^{(m)} \cdot S_{1}\left(\mathbb{R}^{2}\right)\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)} X \cdot N \frac{\mathrm{~d} \mathscr{H}^{2}}{2 \pi R_{1}^{(m)}} \\
& \stackrel{x=R_{1}^{(m)} u}{=}-C^{(m+1)} \int_{F\left(S_{1}\left(\mathbb{R}^{2}\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)} X\left(R_{1}^{(m)} u\right) \cdot N(u) \frac{\mathrm{d} \mathscr{H}^{2}(u)}{2 \pi} \rightarrow 0
\end{aligned}
$$

as $m \rightarrow \infty$ since $\lim R_{1}^{(m)}=0, \lim C^{(m+1)}=C^{(\infty)}$, and mainly $X(\rho u) \rightarrow X(0)$ uniformly as $\rho \rightarrow 0$ and

$$
\int_{F\left(S_{1}\left(\mathbb{R}^{2}\right) \times S_{1}\left(\mathbb{R}^{2}\right)\right)} N(u) \frac{\mathrm{d} \mathscr{H}^{2}(u)}{2 \pi}=0
$$

Therefore the sum in (95) is zero, $\delta V(X)=0$ for arbitrary smooth compactly supported $X$, and $V$ is a stationary varifold.
3. The tangents to $V$. First we describe (without proof) the tangents to $V$ :

$$
\operatorname{Var} \operatorname{Tan}_{0} V=\underbrace{\left\{\left(C^{(\infty)} / 2 \pi\right) V_{\left\{\zeta R_{1}^{(-i)}\right\}_{i \in \mathbb{Z}}}\right.}_{V_{\zeta}}: \zeta>0\}
$$

where $R_{1}^{(i)}$ is as above and $V_{\left\{r_{i}\right\}}$ as in (19), (12), (13). Due to a "periodicity", $\zeta$ can be restricted to $\left[R_{1}^{(0)}, R_{1}^{(-2)}\right)=[1,4)$. Then $V_{\zeta}$ are mutually different and therefore not conical (cf. Lemma 3.1).
(Recall that $V_{\zeta}$ are 2-varifolds supported by a 3-dimensional cone in $\mathbb{R}^{4}$. In alternating layers, $V_{\zeta}$ assume two different directions, namely those mentioned in (27).)

To finish the formal proof of the theorem we do not need anything more than to pick out a single tangent varifold and show that it is not conical. Let $\lambda_{i}=4^{-i}$. Then $\lambda_{i} R_{1}^{(n)}=R_{1}^{(n+2 i)}$ and (see (75), (88))

$$
\operatorname{spt}\left(\eta_{0, \lambda_{i} \ldots V^{(n+2 i)}}\right) \subset G_{2}\left(A_{R_{1}^{(n)}}^{R_{4}^{(n)}}\right) \cap G_{\mathrm{rad} \& D_{n+2 i}}^{\varepsilon^{(n+2 i)}}
$$

where $D_{n}$ is either symbol $J_{13}^{24}$ ( $n$ even) or $J_{12}^{34}$ ( $n$ odd). Therefore $D_{n+2 i}=D_{n}$ and

$$
\operatorname{spt}\left(\eta_{0, \lambda_{i} \ldots} V\right) \subset \bigcup_{n \in \mathbb{Z}}\left(G_{2}\left(A_{R_{1}^{(n)}}^{R_{4}^{(n)}}\right) \cap G_{\mathrm{rad}}^{\varepsilon^{(n+2 i)}}\right)
$$

From (92),

$$
\begin{align*}
& \mathbf{M}\left(\left(\eta_{0, \lambda_{i} \#} V\right)\left\llcorner G_{2}\left(A_{R_{1}^{(0)}}^{R_{4}^{(0)}}\right)\right)=\left(\lambda_{i}\right)^{-2} \mathbf{M}\left(V\left\llcorner G_{2}\left(A_{R_{1}^{(2 i)}}^{R_{4}^{(2 i)}}\right)\right)\right.\right. \\
& \quad=\left(\lambda_{i}\right)^{-2} C^{(2 i)} \mathbf{M}\left(V^{(2 i)}\right) \geq\left(\lambda_{i}\right)^{-2} C^{(\infty)} \frac{\pi}{2}\left(\left(R_{4}^{(2 i)}\right)^{2}-\left(R_{1}^{(2 i)}\right)^{2}\right)=\frac{3 \pi}{2} C^{(\infty)} \tag{96}
\end{align*}
$$

By the compactness theorem for Radon measures ([S1, p. 242, p. 22]), there is a varifold $C$ and a subsequence of $\left\{\lambda_{i}\right\}$ (denoted by $\left\{\lambda_{i}\right\}$ again) such that $\eta_{0, \lambda_{i} \#} V \rightarrow C$. (We note without proof that in fact it is not necessary to pass to a subsequence since even the original sequence is convergent.) Hence $C \in \operatorname{VarTan}_{0} V$. From the above,

$$
\begin{equation*}
\mathbf{M}\left(C\left\llcorner G_{2}\left(A_{R_{1}^{(0)}}^{R_{4}^{(0)}}\right)\right) \geq \frac{3 \pi}{2} C^{(\infty)}>0\right. \tag{97}
\end{equation*}
$$

and

$$
\operatorname{spt} C \subset \bigcup_{n \in \mathbb{Z}}\left(G_{2}\left(A_{R_{1}^{(n)}}^{R_{4}^{(n)}}\right) \cap G_{\mathrm{rad} \& D_{n}}^{\varepsilon}\right)
$$

for every $\varepsilon>0$ and thus also for $\varepsilon=0$. In particular

$$
\begin{align*}
& \operatorname{spt} C \cap G_{2}\left(\operatorname{int} A_{R_{1}^{(0)}}^{R_{4}^{(0)}}\right) \subset G_{\mathrm{rad} \& D_{0}}^{0},  \tag{98}\\
& \operatorname{spt} C \cap G_{2}\left(\operatorname{int} A_{R_{1}^{(1)}}^{R_{4}^{(1)}}\right) \subset G_{\mathrm{rad} \& D_{1}}^{0} \tag{99}
\end{align*}
$$

where int $M$ denotes the interior of $M$. From (99),

$$
\begin{equation*}
\operatorname{spt}\left(\eta_{0,1 / 2} \not \#^{C} C\right) \cap G_{2}\left(\operatorname{int} A_{R_{1}^{(0)}}^{R_{4}^{(0)}}\right) \subset G_{\mathrm{rad} \& D_{1}}^{0} . \tag{100}
\end{equation*}
$$

Assume that $C$ is conical. Then $\operatorname{spt} C=\operatorname{spt}\left(\eta_{0,1 / 2} \# C\right)$. Since $G_{\mathrm{rad} \& D_{0}}^{0}$ and $G_{\mathrm{rad} \& D_{1}}^{0}$ are disjoint (see (26)), we see that (98) and (100) is possible only when

$$
\operatorname{spt} C \cap G_{2}\left(\operatorname{int} A_{R_{1}^{(0)}}^{R_{4}^{(0)}}\right)=\emptyset
$$

which is a contradiction with (97) and (91). Hence $C$ is not conical.
Theorem 5.2 There is a stationary rectifiable 2 -varifold $V$ in $\mathbb{R}^{4}$ that has at least two different conical tangents at 0 and $0<\theta^{2}(V, 0)<\infty$.

The proof differs from the proof of the previous theorem mainly in a different definition of the sequences of radii $R_{1}, R_{2}, R_{3}, R_{4}$ : taking $R_{3} / R_{2}$ large, the middle "conical" part becomes dominant.

Proof For $n \in \mathbb{Z}$, let

$$
\begin{aligned}
& \varepsilon^{(n)}=1 / 4\left(n^{2}+1\right) \\
& R_{1}^{(n)}=2^{-n^{3}} \\
& R_{2}^{(n)}=\left(1+\varepsilon^{(n)}\right) R_{1}^{(n)} \\
& R_{3}^{(n)}=\left(1-\varepsilon^{(n)}\right) R_{4}^{(n)} \\
& R_{4}^{(n)}=R_{1}^{(n-1)} .
\end{aligned}
$$

Note that $\left\{n^{3}\right\}$ is a strictly increasing sequence with increments at least one, hence $R_{1}^{(n)}<R_{2}^{(n)}<R_{3}^{(n)}<R_{4}^{(n)}$. Repeating the construction of Theorem 5.1 we obtain a rectifiable stationary 2 -varifold $V$, but now the varifold's tangents at 0 are different.

Without proof we claim that, with $c=C^{(\infty)} / 2 \pi, c V_{1,0, \infty}$ and $c V_{2,0, \infty}$ (see Section 3, (12), (13)) are two different (Lemma 3.1) conical tangent varifolds to $V$ at $0 \in \mathbb{R}^{4}$. There are also tangent varifolds of the form $c\left(V_{1,0, \rho}+V_{2, \rho, \infty}\right)$ and $c\left(V_{2,0, \rho}+V_{1, \rho, \infty}\right)$, $\rho>0$; they are not conical, but they are "conical near 0 "."

We will give the detailed proof for existence of two different conical tangent varifolds at 0 . Let $\lambda_{i}=i R_{1}^{(2 i)}$ and $\tilde{\lambda}_{i}=i R_{1}^{(2 i+1)}$.

Note that, for $i \rightarrow \infty, R_{1}^{(2 i)} / \lambda_{i}=1 / i \rightarrow 0$ while $R_{4}^{(2 i)} / \lambda_{i}=2^{-(2 i-1)^{3}+(2 i)^{3}} / i \rightarrow \infty$. We have

$$
\operatorname{spt}\left(\eta_{0, \lambda_{i} \ldots \#^{(2 i)}}\right) \subset G_{2}\left(A_{R_{1}^{(2 i)} / \lambda_{i}}^{R_{4}^{(2 i)} / \lambda_{i}}\right) \cap G_{\text {rad } \& D_{2 i}}^{\varepsilon^{(2 i)}}
$$

where $D_{2 i}=D_{0}$ is the symbol " $J_{13}^{24 " \text { ". Hence }}$

$$
\begin{equation*}
\operatorname{spt}\left(\eta_{0, \lambda_{i} \# \#} V\right) \subset G_{2}\left(A_{0}^{R_{1}^{(2 i)} / \lambda_{i}}\right) \cup\left(G_{2}\left(A_{R_{1}^{(2 i)} / \lambda_{i}}^{R_{4}^{(2 i)} / \lambda_{i}}\right) \cap G_{\text {rad } \& D_{0}}^{\varepsilon^{(2 i)}}\right) \cup G_{2}\left(A_{R_{4}^{(2 i)} / \lambda_{i}}^{\infty}\right) \tag{101}
\end{equation*}
$$

[^4]As in the proof of the previous theorem, we pass to a subsequence (denoted by $\left\{\lambda_{i}\right\}$ again) if necessary, so that $\eta_{0, \lambda_{i}, \ldots} V \rightarrow C \in \operatorname{Var} \operatorname{Tan}_{0} V$ and $\eta_{0, \tilde{\lambda}_{i}, \ldots} V \rightarrow \tilde{C} \in$ Var $\operatorname{Tan}_{0} V$.

By (101),

$$
\operatorname{spt} C \subset G_{2}(\{0\}) \cup \bigcap_{\varepsilon>0} G_{\mathrm{rad}}^{\varepsilon} \& D_{0}=G_{2}(\{0\}) \cup G_{\mathrm{rad} \& D_{0}}^{0}
$$

By the same argument,

$$
\operatorname{spt} \tilde{C} \subset G_{2}(\{0\}) \cup G_{\mathrm{rad}}^{0} \& D_{1}
$$

where $D_{1}=$ " $J_{12}^{34}$ ". Hence $C=\tilde{C}$ is posssible (cf. again (26)) only if $\operatorname{spt} C \cup \operatorname{spt} \tilde{C} \subset$ $G_{2}(\{0\})$. However, for sufficiently large $i \in \mathbb{N}$ we have $R_{4}^{(2 i)} / \lambda_{i}>2, R_{1}^{(2 i)} / \lambda_{i}<1$ and, by (76) and (89),

$$
\begin{aligned}
& \mathbf{M}\left(\left(\eta_{0, \lambda_{i} \#} V\right)\left\llcorner G_{2}\left(A_{1}^{2}\right)\right)=\left(\lambda_{i}\right)^{-2} \mathbf{M}\left(V\left\llcorner G_{2}\left(A_{\lambda_{i}}^{2 \lambda_{i}}\right)\right)\right.\right. \\
& =\left(\lambda_{i}\right)^{-2} C^{(2 i)} \mathbf{M}\left(V^{(2 i)}\left\llcorner G_{2}\left(A_{\lambda_{i}}^{2 \lambda_{i}}\right)\right)\right. \\
& \quad \geq\left(\lambda_{i}\right)^{-2} C^{(\infty)} \frac{\pi}{2}\left(\left(2 \lambda_{i}\right)^{2}-\left(\lambda_{i}\right)^{2}\right)=\frac{3 \pi}{2} C^{(\infty)}>0
\end{aligned}
$$

and therefore $C \neq \tilde{C}$ are two different conical tangents to $V$.

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[^0]:    ${ }^{1}$ Though, in different context the tangent cone is sometimes defined to be what we call the unique tangent cone, see for example [K, p. 159].
    ${ }^{2}$ See for example [S3, p. 591]: "... but it is far from obvious (and an open question) whether or not $\operatorname{Tan}_{X} M$ can contain more than one cone $C$ if $X \in \operatorname{sing} M$." The same paper contains a result on the uniqueness of tangents $m$-almost everywhere in the singular set [S3, p. 650, (2), (1)], where $m$ is the 'top dimension' (e.g., $m=\operatorname{dim} M-2$, depending on the context).

[^1]:    ${ }^{3}$ Since $C \neq 0$, we have $\theta^{m}\left(\mu_{V}, x_{0}\right) \in(0, \infty)$ from the Monotonicity formula for stationary varifolds, cf. [S1, 40.5]. Therefore the assumptions of Corollary 42.6. (namely 42.1.) are satisfied.

[^2]:    ${ }^{4}$ Although there are several possible definition of approximate tangent plane (see [O'N], [A, p. 428, (3) and (b)] and $[\mathrm{S} 1,11.2]$ ), they agree $\mu$-almost everywhere. The definitions of rectifiable varifolds in [A] and [ $\left.\mathrm{O}^{\prime} \mathrm{N}\right]$ essentially agree with that of [S1], cf. footnote on [S1, p. 77].

[^3]:    ${ }^{5}$ The surface is neither a linear space nor a convex set: it contains points $(1,0,0,0)(a=c=1, b=$ $d=0)$ and $(0,0,0,1)(a=c=0, b=d=1)$ but does not contain $(1 / 2,0,0,1 / 2)$. Indeed, $(t, 0,0, t)=$ $(a c, b c, a d, b d), t \neq 0$ leads to $a \neq 0, c=t / a, b \neq 0, d=t / b$, then $b t / a=0, a t / b=0$ and finally $b=0=a$, a contradiction.
    ${ }^{6}$ The surface is actually a copy of the three-dimensional cone generated by $\mathbb{S}^{1} \times \mathbb{S}^{1}$ as can be seen from the relation $(\cos \gamma, \sin \gamma, \cos \delta, \sin \delta)=(x+w, y-z, x-w, z+y)$ where $(x, y, z, w)=F((\cos \alpha, \sin \alpha)$, $(\cos \beta, \sin \beta)), \gamma=\alpha-\beta, \delta=\alpha+\beta$.
    The surface was the first known nontrivial minimal cone in $\mathbb{R}^{4},[\mathrm{O}, \mathrm{p} .1113] . \mathbb{S}^{1} \times \mathbb{S}^{1}$ is so called Clifford torus. Recently, Simon Brendle announced that (up to a congruence) it is the only embedded minimal torus in $\mathbb{S}^{3}$ [Bre].

[^4]:    7 We believe a slightly more complicated construction gives an example of a varifold whose all tangents are conical but the tangent at a point is non-unique. Basically, $\left\{J_{13}^{24}, J_{12}^{34}\right\}$ has to be replaced by a curve $\{J(t): t \in[0,1]\}$. A varifold would be used that takes directions in $G_{\mathrm{rad} \& J\left(j / 2^{k}\right)}^{1 /\left(n^{2}+1\right)}$ on $A_{R_{1}^{(n)}}^{R_{4}^{(n)}}\left(\mathbb{R}^{4}\right)$ whenever $|n|=2^{k}+j>2, k, j, \in \mathbb{N}, j \leq 2^{k}$.

