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The Relation between KMS States for Different Temperatures

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Abstract. Given a thermal field theory for some temperature β^{-1}, we construct the theory at an arbitrary temperature $1/\beta'$. Our work is based on a construction invented by Buchholz and Junglas, which we adapt to thermal field theories. In a first step we construct states which closely resemble KMS states for the new temperature in a local region $\mathcal{O}_\circ \subset \mathbb{R}^4$, but coincide with the given KMS state in the space-like complement of a slightly larger region $\hat{\mathcal{O}}$. By a weak*-compactness argument there always exists a convergent subnet of states as the size of $\mathcal{O}_\circ$ and $\hat{\mathcal{O}}$ tends towards $\mathbb{R}^4$. Whether or not such a limit state is a global KMS state for the new temperature, depends on the surface energy contained in the layer in between the boundaries of $\mathcal{O}_\circ$ and $\hat{\mathcal{O}}$. We show that this surface energy can be controlled by a generalized cluster condition.

1 Introduction

A quantum field theory can be specified by a C^*-algebra $\mathcal{A}$ together with a net

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}), \qquad \mathcal{O} \subset \mathbb{R}^4,$$

of subalgebras associated with open, bounded space-time regions $\mathcal{O}$ in Minkowski space (as described in the monograph by Haag [H]; see also [HK]). The Hermitian elements of $\mathcal{A}(\mathcal{O})$ are interpreted as the observables which can be measured at times and locations in $\mathcal{O}$. Technically the algebra $\mathcal{A}(\mathcal{O})$ may be thought of as being generated by bounded functions of the underlying smeared quantum fields (see, e.g., [BoY]). For instance, if $\phi(x)$ is a hermitian quantum field and if $f(x)$ is a real test function with support in a bounded region $\mathcal{O}$ of space-time, then the unitary operator $a := \exp\bigl(i \int \mathrm{d}x\, f(x)\phi(x)\bigr)$ is a typical element of $\mathcal{A}(\mathcal{O})$. In this way the quantum fields provide a "coordinate system" for the algebra $\mathcal{A}$. However, as emphasized by Haag and Kastler, only the algebraic relations between the elements of $\mathcal{A}$ are of physical significance.

If the time evolution is given by a strongly continuous one-parameter group of automorphisms $\{\tau_t\}_{t\in\mathbb{R}}$ of $\mathcal{A}$, then the pair $(\mathcal{A}, \tau)$ forms a C^*-dynamical system. Such a description of a QFT fits nicely into the structure of algebraic quantum statistical mechanics (see, e.g., [BR], [E], [R], [Se], [Th]) and we can therefore rely on this well-developed framework.

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Up till now non-relativistic quantum field theories and spin systems were favored in the framework of algebraic quantum statistical mechanics. In low dimensions the latter have been worked out in great detail (see, e.g., [BR]). Only recently the benefits of formulating thermal field theory in the algebraic framework were emphasized in a series of papers [BJu 86], [BJu 89], [BB 94], [N], [Jä 98], [Jä 99], [Jä 04].

Equilibrium states can be characterized by first principles in the algebraic framework: equilibrium states are invariant under the time-evolution and stable against small dynamical (or adiabatic [NT]) perturbations of the time-evolution [HKTP]. Adding a few technical assumptions such a heuristical characterization of an equilibrium state leads to a sharp mathematical criterion [HHW], named for Kubo [K], Martin and Schwinger [MS]:

Definition. A state ω_β over $\mathcal{A}$ is called a (τ, β)-KMS state for some $\beta \in \mathbb{R} \cup \{\pm\infty\}$, if

$$\omega_\beta\big(a\tau_{i\beta}(b)\big) = \omega_\beta(ba) \tag{1}$$

for all a, b in a norm dense, τ-invariant $*$-subalgebra of $\mathcal{A}_\tau$. Here $\mathcal{A}_\tau \subset \mathcal{A}$ denotes the set of analytic elements for τ.

We note that there are C^*-dynamical systems $(\mathcal{A}, \tau)$, for which a KMS state exists at one and only one value $\beta \in \mathbb{R}$ (see [BR, 5.3.27]). But for a QFT one can specify conditions on the phase-space properties in the vacuum representation, such that KMS states exist for all temperatures $\beta^{-1} > 0$ [BJu 89]. These conditions exclude (see [BJu 86]) the class of models with a countable number of free scalar particles proposed by Hagedorn [Ha]. These models obey all the Wightman and Haag-Kastler axioms but they do not allow equilibrium states above a certain critical temperature.

For a generic model one expects that for high temperatures and low densities the set of KMS states contains a unique element[1], whereas at low temperature it should contain many disjoint extremal KMS states and their convex combinations corresponding to various thermodynamic phases and their possible mixtures. The symmetry, or lack of symmetry of the extremal KMS states is automatically determined by this decomposition. Consequently, spontaneous symmetry breaking may occur, when we change the temperature in the sequel.

Given a KMS state ω_β over $\mathcal{A}$ the GNS-representation $(\pi_\beta, \mathcal{H}_\beta, \Omega_\beta)$ provides a net of von Neumann algebras:

$$\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O}) := \pi_\beta\big(\mathcal{A}(\mathcal{O})\big)'', \qquad \mathcal{O} \in \mathbb{R}^4.$$

Under fairly general circumstances KMS states for different values of the temperature β^{-1} lead to unitarily inequivalent GNS-representations (see [T], [BR, 5.3.35]). Hence thermal field theories for different temperatures are frequently treated as completely disjoint objects even if they refer to the same vacuum theory, i.e., even

[1] For non-relativistic fermions with pair-interaction see [Jä 95].

if they show identical interactions on the microscopic level. To understand the relations between these 'disjoint thermal field theories' seems to be highly desirable.

One simple case is well known ([Pe, 8.12.10]): Assume that the time-evolution τ can be approximated by a net of inner automorphisms such that, for $a \in \mathcal{A}$ fixed,

$$\lim_{\Lambda \to \infty} \| \tau_z(a) - \mathrm{e}^{izh_\Lambda} a \mathrm{e}^{-izh_\Lambda} \| = 0, \qquad h_\Lambda = h_\Lambda^* \in \mathcal{A},$$

uniformly in z on compact subsets of $\mathbb{C}$. If $(\mathcal{A}, \tau)$ has a KMS state ω_β at some $\beta \neq 0$, then the net of states $\Lambda \mapsto \omega_\Lambda$,

$$\omega_\Lambda(a) = \frac{\omega_\beta\big(\mathrm{e}^{\frac{1}{2}(\beta-\beta')h_\Lambda} a \mathrm{e}^{\frac{1}{2}(\beta-\beta')h_\Lambda}\big)}{\omega_\beta\big(\mathrm{e}^{(\beta-\beta')h_\Lambda}\big)}, \qquad a \in \mathcal{A},$$

has convergent subnets and the limit points $\omega_{\beta'} := \lim_\Lambda \omega_\Lambda$ are (τ, β')-KMS states for the new temperature $1/\beta'$ $(0 < \beta' < \infty)$.

But in general, phase transitions may occur while we change the temperature. Consequently "... there is no simple prescription for connecting the (τ, β)-KMS states for different β's" (c.f. [BR, p. 78]). Nevertheless, we will provide a prescription which covers, as far as relativistic systems are concerned, the physically relevant cases.

We start form a thermal field theory $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$, whose number of local degrees of freedom is restricted in a physically sensible manner. Using a method, which is essentially due to Buchholz and Junglas [BJu 89], we construct a KMS state $\omega_{\beta'}$ and a thermal field theory

$$\mathcal{O} \to \mathcal{R}_{\beta'}(\mathcal{O}), \qquad \beta' \in \mathbb{R}^+,$$

for a new temperature $1/\beta' > 0$. Although we almost repeat their line of arguments, there are some nontrivial deviations due to the mathematical structure we encounter in thermal field theory.

In a first step we construct product states ω_Λ, $\Lambda = (\mathcal{O}_\circ, \hat{\mathcal{O}})$, which – up to boundary effects – resemble KMS states for the new temperature $1/\beta'$ in a local region $\mathcal{O}_\circ \subset \mathbb{R}^4$, but coincide with the given KMS state ω_β in the space-like complement of a slightly larger region $\hat{\mathcal{O}}$:

$$\omega_\Lambda(ab) = \omega_\Lambda(a) \cdot \omega_\beta(b) \qquad \forall a \in \mathcal{A}(\mathcal{O}_\circ), \quad \forall b \in \mathcal{A}(\hat{\mathcal{O}}').$$

At this point our method is semi-constructive; the product states ω_Λ is not uniquely fixed. Intuitively the choice of a particular product state ω_Λ corresponds to a choice of the boundary conditions which decouple the local region $\mathcal{O}_\circ$, where the state already resembles an equilibrium state for the new temperature, from the space-like complement of $\hat{\mathcal{O}}$. Different choices ω_Λ, ${\omega_\Lambda}'$ should manifest themselves in different expectation values for observables localized in between the two regions $\mathcal{O}_\circ$ and $\hat{\mathcal{O}}'$. I.e., we expect

$$\omega_\Lambda \neq {\omega_\Lambda}' \quad \Rightarrow \quad \exists a \in \mathcal{A}(\mathcal{O}_\circ' \cap \hat{\mathcal{O}}) \text{ such that } \omega_\Lambda(a) \neq {\omega_\Lambda}'(a).$$

It follows from standard compactness arguments that the net of states $\Lambda \to \omega_\Lambda$ has convergent subnets. Whether or not these subnets converge to a global KMS states for the new temperature depends on the surface energy contained in between the two regions $\mathcal{O}_\circ$ and $\hat{\mathcal{O}}'$ as their size increases. Introducing an auxiliary structure, which can be understood as a local purification, and assuming a cluster condition, we will control these surface energies in all thermal theories which satisfy a certain "nuclearity condition" (see, e.g., [BW], [BD'AL 90a], [BD'AL 90b], [BY] for related work). Consequently, we can single out (generalized) sequences $\Lambda_i = \big(\mathcal{O}_\circ^{(i)}, \hat{\mathcal{O}}^{(i)}\big)$ such that the limit points[2]

$$\omega_{\beta'}(a) := \lim_{i\to\infty} \omega_{\Lambda_i}(a), \qquad a \in \mathcal{A},$$

are KMS states for the new temperature $1/\beta'$ $(0 \le \beta' \le \infty)$. We emphasize that phase transitions are not excluded by our method: by choosing different "boundary conditions" we may encounter disjoint KMS states for the new temperature in the thermodynamic limit.

Loosely speaking, we provide a method to heat up or cool down a quantum field theory.

2 Definitions and preliminary results

For the Lagrangian formulation of a thermal field theory we refer the reader to the books by Kapusta [Ka], Le Bellac [L] and Umezawa [U], and the excellent review article by Landsman and van Weert [LvW]. Recent work in the Wightman framework can be found in [BB 92], [BB 95], [BB 96], [St]. In this section we will outline the basic structure of a thermal field theory in the algebraic framework.

2.1 List of assumptions

Although it would be more natural – from the viewpoint of algebraic quantum statistical mechanics – to start from a C^*-dynamical system $(\mathcal{A}, \tau)$ and then characterize equilibrium states ω_β and thermal representations π_β with respect to the dynamics, we will assume here that we are given a thermal field theory $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ acting on some Hilbert space $\mathcal{H}_\beta$. How we can reconstruct a C^*-dynamical system $(\mathcal{A}, \tau)$ from the W^*-dynamical system $(\mathcal{R}_\beta, \hat{\tau})$ is well known and will be indicated in the next subsection ($\hat{\tau}$ will be defined in (2.1)).

We now provide a list of assumptions:

i) (Thermal QFT). A thermal QFT is specified by a von Neumann algebra $\mathcal{R}_\beta$, acting on a separable Hilbert space $\mathcal{H}_\beta$, together with a net

(Net structure) $\qquad \mathcal{O} \to \mathcal{R}_\beta(\mathcal{O}), \qquad \mathcal{O} \subset \mathbb{R}^4,$

[2] We have simplified the notation here. In fact, we will have to adjust the relative sizes of a triple $\Lambda_i = (\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}, \hat{\mathcal{O}}^{(i)})$ of space-time regions.

of subalgebras associated with open bounded space-time regions $\mathcal{O}$ in Minkowski space. The net $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ satisfies

(Isotony) $\quad \mathcal{R}_\beta(\mathcal{O}_1) \subset \mathcal{R}_\beta(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2$

and

(Locality) $\quad \mathcal{R}_\beta(\mathcal{O}_1) \subset \mathcal{R}_\beta(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'$.

As before, $\mathcal{O}'$ denotes the space-like complement of $\mathcal{O}$.

ii) (Dynamical law). The time-evolution $\hat{\tau}\colon t \mapsto \hat{\tau}_t$,

$$\hat{\tau}_t(\,.\,) = \mathrm{e}^{iH_\beta t} \;.\; \mathrm{e}^{-iH_\beta t},$$

is induced by a strongly continuous one-parameter group of unitaries $\{\mathrm{e}^{iH_\beta t}\}_{t\in\mathbb{R}}$. It acts geometrically, i.e., $\hat{\tau}_t\big(\mathcal{R}_\beta(\mathcal{O})\big) \subset \mathcal{R}_\beta(\mathcal{O} + te)$ for all $t \in \mathbb{R}$. Here e is the unit-vector in the time-direction in the Lorentz-frame distinguished by the KMS state.

iii) (Unique KMS vector). There exists a distinguished vector Ω_β, cyclic and separating for $\mathcal{R}_\beta$, such that the associated vector state $\omega_\beta(\,.\,) := (\Omega_\beta\,,\,.\,\Omega_\beta)$ satisfies the KMS condition (1) w.r.t. the time-evolution τ. Restricting attention to pure phases we assume that Ω_β is the unique – up to a phase – normalized eigenvector with eigenvalue $\{0\}$ of H_β.

iv) (Reeh-Schlieder property). The KMS vector Ω_β is cyclic and separating for the local algebra $\mathcal{R}_\beta(\mathcal{O})$, if the space-like complement of $\mathcal{O} \subset \mathbb{R}^4$ is not empty.

v) (Nuclearity condition). The thermal field theory $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ has the following phase-space properties: for $\mathcal{O}$ bounded the maps $\Theta_{\alpha,\mathcal{O}}\colon \mathcal{R}_\beta(\mathcal{O}) \to \mathcal{H}_\beta$ given by

$$\Theta_{\alpha,\mathcal{O}}(A) = \mathrm{e}^{-\alpha\beta H_\beta} A\Omega_\beta, \qquad 0 \le \alpha \le 1/2,$$

are nuclear for $0 < \alpha < 1/2$ and the nuclear norm (for $\alpha \searrow 0$ or $\alpha \nearrow 1/2$ and large diameters r of $\mathcal{O}$) satisfies

$$\|\Theta_{\alpha,\mathcal{O}}\| \le \mathrm{e}^{cr^d\left(\alpha^{-m} + (1/2-\alpha)^{-m}\right)}, \tag{2}$$

where c, m, d are positive constants. (We expect that the constant d in this bound can be put equal to the dimension of space in realistic theories, but we do not make such an assumption here. The constant $m > 0$ may depend on the interaction and the KMS state.)

vi) (Regularity from the outside). The net $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ is regular from the outside, i.e.,

$$\bigcap_{\hat{\mathcal{O}}^{(i)} \supset \mathcal{O}} \mathcal{R}_\beta\big(\hat{\mathcal{O}}^{(i)}\big) = \mathcal{R}_\beta(\mathcal{O}), \qquad \hat{\mathcal{O}}^{(i)} \searrow \mathcal{O}.$$

(This property can usually be achieved by defining the local algebras in an appropriate way.)

vii) (Cluster assumption). Let $\mathcal{O}_\circ$ and $\mathcal{O}$ be two space-time regions such that $\mathcal{O}_\circ + te \subset \mathcal{O}$ for $|t| < \delta_\circ$. Let J denote the modular conjugation (see Subsection 2.3) for the pair $(\mathcal{R}_\beta, \Omega_\beta)$. Let $M_j \in \mathcal{R}_\beta(\mathcal{O}_\circ) \vee J\mathcal{R}_\beta(\mathcal{O}_\circ)J$ and $N_j \in \big(\mathcal{R}_\beta(\mathcal{O}) \vee J\mathcal{R}_\beta(\mathcal{O})J\big)'$. Then, for $\delta_\circ$ large compared to the thermal wave-length β,

$$\Big|\sum_{j=1}^{N}(\Omega_\beta\,,\, M_j\Omega_\beta)(\Omega_\beta\,,\, N_j\Omega_\beta) - (\Omega_\beta\,,\, M_jN_j\Omega_\beta)\Big| \le c'\, r_\circ^{d'}\delta_\circ^{-\gamma}\cdot\Big\|\sum_{j=1}^{N} M_jN_j\Big\|, \tag{3}$$

where c', d' and γ are positive constants which do not depend on $\mathcal{O}_\circ$ or $\mathcal{O}$. Here $r_\circ$ denotes the diameter of $\mathcal{O}_\circ$.

Remarks

i) The Reeh-Schlieder property is a consequence [Jä 00] of additivity[3] and the relativistic KMS condition proposed by Bros and Buchholz [BB 94]. If the KMS state is locally normal w.r.t. the vacuum representation, then the standard KMS condition (together with additivity of the net in the vacuum representation) is sufficient to ensure the Reeh-Schlieder property of the KMS vector Ω_β [J].

ii) If the KMS state is locally normal w.r.t. the vacuum representation, then it is sufficient to assume that

$$\bigcap_{\hat{\mathcal{O}}^{(i)}\supset\mathcal{O}} \mathcal{R}\big(\hat{\mathcal{O}}^{(i)}\big) = \mathcal{R}(\mathcal{O}), \qquad \hat{\mathcal{O}}^{(i)} \searrow \mathcal{O},$$

holds true in the vacuum representation. For the free scalar field this property was shown by Araki [A 64].

iii) One might try to establish the cluster condition starting from a sharper nuclearity condition. For instance, we might assume that the map $\Theta^\sharp_{\alpha,\mathcal{O}}\colon \mathcal{R}_\beta(\mathcal{O}) \to \mathcal{H}_\beta$ given by

$$\Theta^\sharp_{\alpha,\mathcal{O}}(A) = \mathrm{e}^{-\alpha|H_\beta|}\big(A - (\Omega_\beta\,,\, A\Omega_\beta)\big)\Omega_\beta, \qquad \alpha > 0,$$

is nuclear too and satisfies (for α^m large in comparison with r^d) the following bound on its nuclear norm

$$\|\Theta^\sharp_{\alpha,\mathcal{O}}\| \le c'\cdot r^d\alpha^{-m}.$$

Formally the bound on the nuclear norm $\|\Theta^\sharp_{\alpha,\mathcal{O}}\|$ follows from taking the limit α^m large in comparison with r^d in the expression $\exp(cr^d\alpha^{-m}) - 1$, where the one is due to the subtraction of the thermal expectation value. (The expression $\exp(cr^d\alpha^{-m})$ should provide an upper bound for the nuclear norm of the map $A \mapsto \exp(-\alpha|H_\beta|)A\Omega_\beta$, where $\alpha > 0$.)

[3] The net $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ is called additive if $\cup_i\mathcal{O}_i = \mathcal{O} \Rightarrow \vee_i\mathcal{R}_\beta(\mathcal{O}_i) = \mathcal{R}_\beta(\mathcal{O})$. Here $\vee_i\mathcal{R}_\beta(\mathcal{O}_i)$ denotes the von Neumann algebra generated by the algebras $\mathcal{R}_\beta(\mathcal{O}_i)$.

iv) The product state appearing in (3) is induced by a product vectors χ_i, which satisfies

$$(\chi_i\,,\,MN\,\chi_i) = (\Omega_\beta\,,\,M\Omega_\beta)(\Omega_\beta\,,\,N\Omega_\beta)$$

for $M \in \mathcal{M}_\beta\big(\mathcal{O}_\circ^{(i)}\big)$ and $N \in \mathcal{M}_\beta\big(\mathcal{O}^{(i)}\big)'$. The convergence of the product vector $\chi \to \Omega_\beta$ follows from $\big(\cup_{\mathcal{O}}\,\mathcal{M}_\beta(\mathcal{O})\big)' = \mathbb{1}$ (see [D'ADFL]).

2.2 The restricted C^*-dynamical system

If the weakly continuous one-parameter group $\hat\tau\colon t \mapsto \hat\tau_t$ fails to be strongly continuous, then we can reconstruct the underlying C^*-dynamical system by a suitable smoothening procedure (once again we refer to [S, 1.18]): given a thermal field theory $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ there exists

(i) a C^*-algebra $\mathcal{A}$ and a representation $\pi_\beta\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\beta)$ such that $\pi_\beta(\mathcal{A})$ is a σ-weakly dense C^*-subalgebra of $\mathcal{R}_\beta$;

(ii) a net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ of C^*-subalgebras of $\mathcal{A}$ such that $\pi_\beta\big(\mathcal{A}(\mathcal{O})\big)$ is a σ-weakly dense C^*-subalgebra of $\mathcal{R}_\beta(\mathcal{O})$ for all $\mathcal{O} \subset \mathbb{R}^4$;

(iii) a strongly continuous automorphism group $t \mapsto \tau_t$ of $\mathcal{A}$ such that $\pi_\beta\big(\tau_t(a)\big) = \hat\tau_t\big(\pi_\beta(a)\big)$ for all $a \in \mathcal{A}$.

Moreover, the net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ satisfies isotony and locality and τ respects the local structure of the net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$, i.e., $\tau_t\big(\mathcal{A}(\mathcal{O})\big) = \mathcal{A}(\mathcal{O} + te)$ for $t \in \mathbb{R}$.

We can now introduce subalgebras $\mathcal{A}_p$ of almost local elements in $\mathcal{A}$ which are analytic with respect to time-translations [BJu 89]. For the existence of these subalgebras it is crucial that the time-evolution $t \mapsto \tau_t$ is strongly continuous, i.e., if we fix some $a \in \mathcal{A}$, then $\lim_{t\to 0} \|\tau_t(a) - a\| = 0$.

Lemma 2.1. (Buchholz and Junglas). *Let $p \in \mathbb{N}$ be fixed and let $\mathcal{A}_p \subset \mathcal{A}$ be the $*$-algebra generated by all finite sums and products of operators of the form*

$$a(f) = \int \mathrm{d}t\, f(t)\tau_t(a),$$

where f is any one of the functions

$$f(t) = const.\, \mathrm{e}^{-\kappa(t+w)^{2p}}$$

(with $\kappa > 0$, $w \in \mathbb{C}$) and $a \in \cup_{\mathcal{O}}\mathcal{A}(\mathcal{O})$ is any strictly local operator. It follows that

(i) *each $b \in \mathcal{A}_p$ is an analytic element with respect to τ, i.e., the operator-valued function $t \mapsto \tau_t(b)$ can be extended to a holomorphic function on $\mathbb{C}$;*

(ii) *each $b \in \mathcal{A}_p$ is almost local in the sense that for any $r^{(i)} > 0$ there is a local operator $b^{(i)} \in \mathcal{A}(\mathcal{O}^{(i)})$ such that*

$$\|b^{(i)} - b\| \le C\,\mathrm{e}^{-\kappa(r^{(i)}/2)^{2p}}, \qquad \kappa > 0,$$

where the constant $C > 0$ does not depend on $r^{(i)}$;

(iii) *the algebra $\mathcal{A}_p$ is invariant under τ_z, $z \in \mathbb{C}$, and norm dense in $\mathcal{A}$.*

The new state $\omega_{\beta'}$, which we will construct in the sequel, will be a (τ, β')-KMS state for the pair $(\mathcal{A}, \tau)$. More precisely, it will satisfy the KMS condition (1) for $a, b \in \mathcal{A}_p$ for some p (p will be specified in Subsection 4.3). As we have just seen, $\mathcal{A}_p$ is a norm dense, τ-invariant subalgebra of $\mathcal{A}_\tau$.

Remark. If the new state $\omega_{\beta'}$ is locally normal w.r.t. π_β, then one might expect that the KMS condition extends to

$$\mathcal{F} := \overline{\bigcup_{\mathcal{O} \in \mathbb{R}^4} \mathcal{R}_\beta(\mathcal{O})}^{C^*} .$$

However, the representations $\pi_{\beta'}$ and π_β of $\mathcal{F}$ will be inequivalent for $\beta' \neq \beta$, and therefore the weak closures $\pi_{\beta'}(\mathcal{F})''$ and $\pi_\beta(\mathcal{F})''$ will in general be non-isomorphic.

2.3 The opposite net of local algebras

By assumption the KMS vector Ω_β is cyclic and separating for $\mathcal{R}_\beta$. Thus Tomita-Takesaki theory applies: the polar decomposition $S = J\Delta^{1/2}$ of the closeable operator $S_\circ\colon A\Omega_\beta \mapsto A^*\Omega_\beta$, $A \in \mathcal{R}_\beta$, provides a conjugate-linear isometric mapping J from $\mathcal{H}_\beta$ onto $\mathcal{H}_\beta$ and a positive self-adjoint (in general, unbounded, but densely defined and invertible) operator Δ acting on $\mathcal{H}_\beta$. The modular conjugation J satisfies $J^2 = \mathbb{1}$ and

$$J\Delta^{1/2} A\Omega_\beta = A^*\Omega_\beta \qquad \forall A \in \mathcal{R}_\beta .$$

Δ is called the modular operator. J induces a $*$-anti-isomorphism $j\colon A \mapsto JA^*J$ between the algebra of quasi-local observables $\mathcal{R}_\beta$ and its commutant (Tomita's theorem). The opposite net

$$\mathcal{O} \to j\big(\mathcal{R}_\beta(\mathcal{O})\big), \qquad \mathcal{O} \subset \mathbb{R}^4,$$

provides a perfect mirror image of the net of local observables. The unitary operators Δ^{is}, $s \in \mathbb{R}$, induce a one-parameter group of $*$-automorphism $\sigma\colon s \mapsto \sigma_s$ of $\mathcal{R}_\beta$,

$$\sigma_s(A) = \Delta^{is} A \Delta^{-is}, \qquad s \in \mathbb{R}, \quad A \in \mathcal{R}_\beta .$$

σ is called the modular automorphism. Takesaki has shown that ω_β is a $(\sigma, -1)$-KMS state. Moreover, σ is uniquely determined by this condition and consequently $\Delta^{is} = \exp\big(-is\beta H_\beta\big)$.

We conclude that in a thermal field theory the modular automorphism σ coincides – up to a scaling factor – with the time-evolution $\hat{\tau}$. Consequently, the modular automorphism respects the net structure too, i.e.,

$$\sigma_s\big(\mathcal{R}_\beta(\mathcal{O})\big) = \mathcal{R}_\beta(\mathcal{O} + s\beta \cdot e) \qquad \forall s \in \mathbb{R}. \tag{4}$$

The real parameter $\beta \in \mathbb{R}^+$ appearing (until now β was just a dummy index) in (4) distinguishes a length scale, which is called the thermal wave-length. In fact, we can turn the argument up side down: given a thermal field theory $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$, it is not necessary to provide an explicit expression for the effective Hamiltonian H_β. It is already uniquely specified by the pair $(\mathcal{R}_\beta, \Omega_\beta)$: by Stone's theorem there exists a unique self-adjoint generator H_β such that $\Delta = \exp(-\beta H_\beta)$. Modular theory implies that for $0 \leq \beta < \infty$ the operator H_β is not semi-bounded but its spectrum is symmetric and consists typically of the whole real line [A 72], [tBW].

2.4 Doubling the degrees of freedom

We now present the first step of our construction, which can be understood as a local purification. Consider some $\delta > 0$ and two space-time regions $\mathcal{O}$ and $\hat{\mathcal{O}}$ such that $\mathcal{O} + te \subset \hat{\mathcal{O}}$ for $|t| < \delta$. In a forthcoming paper [Jä 04] we will show that the so-called split property for the net of von Neumann algebras $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ follows from the nuclearity condition (2). It asserts that there exists a type I factor $\mathcal{N}$ such that

$$\mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{N} \subset \mathcal{R}_\beta(\hat{\mathcal{O}}). \tag{5}$$

Remark. If the KMS state is locally normal w.r.t. the vacuum representation, then the split property for the vacuum representation automatically implies the split property for the thermal representation.

The following result is a consequence of the split inclusion (5).

Lemma 2.2. *Let $\mathcal{O}$ be an open and bounded space-time region. Then the von Neumann algebra*

$$\mathcal{M}_\beta(\mathcal{O}) := \mathcal{R}_\beta(\mathcal{O}) \vee j\big(\mathcal{R}_\beta(\mathcal{O})\big)$$

is naturally isomorphic to the tensor product of $\mathcal{R}_\beta(\mathcal{O})$ and $j\big(\mathcal{R}_\beta(\mathcal{O})\big)$. I.e., there exists a unitary operator $V\colon \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ such that

$$V \mathcal{M}_\beta(\mathcal{O}) V^* = \mathcal{R}_\beta(\mathcal{O}) \otimes j\big(\mathcal{R}_\beta(\mathcal{O})\big). \tag{6}$$

Proof. The split property (5) implies that there exists a product vector $\Omega_p \in \mathcal{H}_\beta$, cyclic and separating for $\mathcal{R}_\beta(\mathcal{O}) \vee \mathcal{R}_\beta(\hat{\mathcal{O}})'$, such that

$$(\Omega_p\,, AB\Omega_p) = (\Omega_\beta\,, A\Omega_\beta)(\Omega_\beta\,, B\Omega_\beta)$$

for all $A \in \mathcal{R}_\beta(\mathcal{O})$ and $B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$ [Jä 04]. The product vector Ω_p can be utilized to define a linear operator $V\colon \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ by linear extension of

$$VAB\Omega_p = A\Omega_\beta \otimes B\Omega_\beta, \tag{7}$$

where $A \in \mathcal{R}_\beta(\mathcal{O})$ and $B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$. The operator V is unitary. Inspecting (7) we find

$$V\mathcal{R}_\beta(\mathcal{O})V^* = \mathcal{R}_\beta(\mathcal{O}) \otimes \mathbb{1} \quad \text{and} \quad V\mathcal{R}_\beta(\hat{\mathcal{O}})'V^* = \mathbb{1} \otimes \mathcal{R}_\beta(\hat{\mathcal{O}})'. \tag{8}$$

The inclusion $j\big(\mathcal{R}_\beta(\mathcal{O})\big) \subset \mathcal{R}_\beta(\hat{\mathcal{O}})'$ implies that the von Neumann algebra $\mathcal{M}_\beta(\mathcal{O})$ is naturally isomorphic to the tensor product of $\mathcal{R}_\beta(\mathcal{O})$ and $j\big(\mathcal{R}_\beta(\mathcal{O})\big)$ and the relation (6) is a consequence of (8). □

Remark. The algebras $\mathcal{R}_\beta(\mathcal{O})$ and $j\big(\mathcal{R}_\beta(\mathcal{O})\big)$ are weakly statistically independent, i.e., $0 \neq A \in \mathcal{R}_\beta(\mathcal{O})$ and $0 \neq B \in j\big(\mathcal{R}_\beta(\mathcal{O})\big)$ implies $AB \neq 0$ (Schlieder property) [Jä 04]. In this sense one can speak of a doubling of degrees of freedom.

The elements of $\mathcal{M}_\beta(\mathcal{O})$ will in general not show analyticity properties with respect to Ω_β. Thus it seems that the essence of a thermal field theory gets lost, when we 'double the degrees of freedom' and consider the net $\mathcal{O} \to \mathcal{M}_\beta(\mathcal{O})$ instead of the net of observables $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$. But, due to the natural tensor product structure of $\mathcal{M}_\beta(\mathcal{O})$, we can recover certain analyticity properties w.r.t. Ω_p and a new auxiliary one-parameter group of unitary operators:

Definition. A one-parameter group of unitary operators $s \mapsto \Delta_p^{-is} \colon \mathcal{H}_\beta \to \mathcal{H}_\beta$, $s \in \mathbb{R}$, and an anti-unitary operator $J_p \colon \mathcal{H}_\beta \to \mathcal{H}_\beta$ are given by linear extension of

$$\Delta_p^{-is} AB\Omega_p := V^*\big(\Delta^{-is} A\Omega_\beta \otimes \Delta^{is} B\Omega_\beta\big), \qquad s \in \mathbb{R}, \tag{9}$$

and, respectively,

$$J_p AB\Omega_p := V^*\big(JA\Omega_\beta \otimes JB\Omega_\beta\big),$$

where $A \in \mathcal{R}_\beta(\mathcal{O})$ and $B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$.

By Stone's theorem there exists a unique self-adjoint operator H_p such that

$$\Delta_p = \mathrm{e}^{-\beta H_p} \qquad \text{and} \qquad H_p\Omega_p = 0.$$

The vector $\Omega_p \in \mathcal{H}_\beta$ is cyclic and separating for $\mathcal{R}_\beta(\mathcal{O}) \vee \mathcal{R}_\beta(\hat{\mathcal{O}})'$. It follows from the definition (7) of V and the Reeh-Schlieder property of Ω_β that the product vector Ω_p is cyclic (and of course separating) for $\mathcal{M}_\beta(\mathcal{O})$ too.

Theorem 2.3. *Let $\mathcal{O}_\circ$ and $\mathcal{O}$ be two space-time regions such that $\mathcal{O}_\circ + te \subset \mathcal{O}$ for $|t| < \delta_\circ$. Then*

(i) Δ_p^{-is} *respects the local structure of $\mathcal{M}_\beta(\mathcal{O})$ for $|s|$ sufficiently small, i.e.,*

$$\Delta_p^{-is}\mathcal{M}_\beta(\mathcal{O}_\circ)\Delta_p^{is} \subset \mathcal{M}_\beta(\mathcal{O}_\circ + s\beta\cdot e) \qquad \forall |s\beta| < \delta_\circ. \tag{10}$$

(ii) *the group of unitaries $s \mapsto \Delta_p^{-is}$ coincides for $a \in \mathcal{A}(\mathcal{O}_\circ)$ and $|s\beta| < \delta_\circ$ – up to rescaling – with the time-evolution, i.e.,*

$$\Delta_p^{-is}\pi_\beta(a)\Delta_p^{is} = \pi_\beta\big(\tau_{s\beta}(a)\big)$$

for $|s\beta| < \delta_\circ$.

Proof. The inclusion (10) follows from the definition (9) of Δ_p and the inclusions

$$\hat{\tau}_t\big(\mathcal{R}_\beta(\mathcal{O}_\circ)\big) \subset \mathcal{R}_\beta(\mathcal{O}_\circ + te) \quad \text{and} \quad \hat{\tau}_t\big(j(\mathcal{R}_\beta(\mathcal{O}_\circ))\big) \subset j\big(\mathcal{R}_\beta(\mathcal{O}_\circ + te)\big),$$

which hold for $|t| < \delta_\circ$. □

Lemma 2.4. *Consider some $\delta > 0$ and two space-time regions $\mathcal{O}$ and $\hat{\mathcal{O}}$ such that $\mathcal{O} + te \subset \hat{\mathcal{O}}$ for $|t| < \delta$. Let Ω_p and Δ_p be the product vector specified in (7) and the operator defined in (9). Then $\mathcal{M}_\beta(\mathcal{O})\Omega_p$ is in the domain $\mathcal{D}(\Delta_p^\alpha)$ of Δ_p^α for $0 \leq \alpha \leq 1/2$. Moreover, the identity $J_p\Delta_p^{1/2}M\Omega_p = M^*\Omega_p$ holds true for all $M \in \mathcal{M}_\beta(\mathcal{O})$.*

Proof. By definition, $J_p^2 = \mathbb{1}$, $J_p\Omega_p = \Omega_p$ and

$$J_p\Delta_p^{1/2}AB\Omega_p = V^*\big(A^*\Omega_\beta \otimes B^*\Omega_\beta\big) = A^*B^*\Omega_p = (AB)^*\Omega_p$$

for all $A \in \mathcal{R}_\beta(\mathcal{O})$ and $B \in j\big(\mathcal{R}_\beta(\mathcal{O})\big)$. Since $p^\alpha \leq \max(1,p) < 1+p$ for $0 \leq \alpha \leq 1$ and $p > 0$, the spectral resolution of the positive operator $\Delta_p^{1/2}$ implies that $\mathcal{M}_\beta(\mathcal{O})\Omega_p \subset \mathcal{D}(\Delta_p^\alpha)$ for $0 \leq \alpha \leq 1/2$. □

Nevertheless, J_p and Δ_p are *not* the modular objects associated to $\big(\mathcal{M}_\beta(\mathcal{O}), \Omega_p\big)$.

Theorem 2.5. *Let $\mathcal{O}_\circ$ and $\mathcal{O}$ be two space-time regions such that $\mathcal{O}_\circ + te \subset \mathcal{O}$ for $|t| < \delta_\circ$. Then the inclusion of von Neumann algebras $\mathcal{M}_\beta(\mathcal{O}_\circ) \subset \mathcal{M}_\beta(\mathcal{O})$ is a standard split inclusion and there exists a unitary operator $W\colon \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ such that*

$$W\mathcal{M}_\beta(\mathcal{O}_\circ)W^* = \mathcal{M}_\beta(\mathcal{O}_\circ) \otimes \mathbb{1} \quad \text{and} \quad W\mathcal{M}_\beta(\mathcal{O})'W^* = \mathbb{1} \otimes \mathcal{M}_\beta(\mathcal{O})'.$$

(A split inclusion $\mathcal{A} \subset \mathcal{B}$ is called standard (see [DL]), if there exists a vector Ω which is cyclic for $\mathcal{A}' \wedge \mathcal{B}$ as well as for $\mathcal{A}$ and $\mathcal{B}$.)

Proof. From the split inclusions

$$\mathcal{R}_\beta(\mathcal{O}_\circ) \subset \mathcal{N}_\circ \subset \mathcal{R}_\beta(\mathcal{O}) \quad \text{and} \quad j\big(\mathcal{R}_\beta(\mathcal{O}_\circ)\big) \subset j\big(\mathcal{N}_\circ\big) \subset j\big(\mathcal{R}_\beta(\mathcal{O})\big)$$

we infer that there exists a type I factor, namely $\mathcal{N}_\circ \vee j\big(\mathcal{N}_\circ\big)$, such that

$$\mathcal{M}_\beta(\mathcal{O}_\circ) \subset \mathcal{N}_\circ \vee j\big(\mathcal{N}_\circ\big) \subset \mathcal{M}_\beta(\mathcal{O}).$$

All infinite type I factors with infinite commutant on the separable Hilbert space $\mathcal{H}_\beta$ are unitarily equivalent to $\mathcal{B}(\mathcal{H}_\beta) \otimes \mathbb{1}$ ([KR], Chapter 9.3). Thus there exists a unitary operator $W\colon \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ such that $\mathcal{N}_\circ \vee j\big(\mathcal{N}_\circ\big) = W^*\big(\mathcal{B}(\mathcal{H}_\beta) \otimes \mathbb{1}\big)W$. Now consider $\omega_\beta(\,.\,) := (\Omega_\beta\,,\,.\,\Omega_\beta)$ and $\omega_p(\,.\,) := (\Omega_p\,,\,.\,\Omega_p)$ as two normal states over $\mathcal{M}_\beta(\mathcal{O}_\circ)$ and $\mathcal{M}_\beta(\mathcal{O})'$, respectively. Set

$$\phi_p(C) := (\omega_\beta \otimes \omega_p)(WCW^*) \qquad \forall C \in \mathcal{M}_\beta(\mathcal{O}_\circ) \vee \mathcal{M}_\beta(\mathcal{O})'.$$

Then ϕ_p is a normal state over $\mathcal{M}_\beta(\mathcal{O}_\circ) \vee \mathcal{M}_\beta(\mathcal{O})'$, which satisfies $\phi_p(MN) = \omega_\beta(M) \cdot \omega_p(N)$ for all $M \in \mathcal{M}_\beta(\mathcal{O}_\circ)$ and $N \in \mathcal{M}_\beta(\mathcal{O})'$. In the presence of a separating vector each normal state is a vector state ([KR, 7.2.3]). In fact, there exists a unique vector η in the natural positive cone $\mathcal{P}^\natural\big(\mathcal{M}_\beta(\mathcal{O}_\circ) \vee \mathcal{M}_\beta(\mathcal{O})', \Omega_\beta\big)$ such that

$$(\eta\,,\,MN\eta) = \phi_p(MN) = (\Omega_\beta\,,\,M\Omega_\beta)(\Omega_p\,,\,N\Omega_p)$$

for all $M \in \mathcal{M}_\beta(\mathcal{O}_\circ)$ and $N \in \mathcal{M}_\beta(\mathcal{O})'$ ([BR, 2.5.31]). Thus the operator $W\colon \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ can now be specified by linear extension of

$$WMN\eta = M\Omega_\beta \otimes N\Omega_p, \tag{11}$$

where $M \in \mathcal{M}_\beta(\mathcal{O}_\circ)$ and $N \in \mathcal{M}_\beta(\mathcal{O})'$. Consequently,

$$W\mathcal{M}_\beta(\mathcal{O}_\circ)W^* = \mathcal{M}_\beta(\mathcal{O}_\circ) \otimes \mathbb{1} \quad \text{and} \quad W\mathcal{M}_\beta(\mathcal{O})'W^* = \mathbb{1} \otimes \mathcal{M}_\beta(\mathcal{O})'.$$

The vector Ω_β is cyclic and separating for $\mathcal{M}_\beta(\mathcal{O}_\circ)$ and the vector Ω_p is cyclic and separating for $\mathcal{M}_\beta(\mathcal{O})'$. Thus the vector $\Omega_\beta \otimes \Omega_p$ is cyclic and separating for $\mathcal{M}_\beta(\mathcal{O}_\circ) \otimes \mathcal{M}_\beta(\mathcal{O})'$ and the split inclusion $\mathcal{M}_\beta(\mathcal{O}_\circ) \subset \mathcal{M}_\beta(\mathcal{O})$ is standard. □

3 Localized excitations of a KMS state

Taking the auxiliary structure developed in the previous section into account, we can now adapt the method of Buchholz and Junglas to thermal representations.

3.1 Consequences of the nuclearity condition

Imposing strict localization on an excitation (see Proposition 3.3 (iii) below) does not lead to a convenient notion. The split property provides the key to a more convenient definition of a localized excitation. However, it leaves a lot of freedom, for instance one could request additional properties for some subregion in $\mathcal{O}_\circ \cap \hat{\mathcal{O}}$. In this sense the following definition only provides one possible choice, fixed by choosing a specific product vector η.

Definition. Let $\mathcal{O}_\circ, \mathcal{O}$ and $\hat{\mathcal{O}}$ denote three space-time regions such that for some $\delta_\circ$, $\delta > 0$

$$\mathcal{O}_\circ + te \subset \mathcal{O} \quad \forall |t| < \delta_\circ \quad \text{and} \quad \mathcal{O} + te \subset \hat{\mathcal{O}} \quad \forall |t| < \delta. \tag{12}$$

The Hilbert space $\mathcal{H}_\Lambda \subset \mathcal{H}_\beta$, $\Lambda := (\mathcal{O}_\circ, \mathcal{O}, \hat{\mathcal{O}})$, of localized excitations of the KMS state ω_β is given by

$$\mathcal{H}_\Lambda := \overline{\mathcal{M}_\beta(\mathcal{O}_\circ)\eta}. \tag{13}$$

The projection onto $\mathcal{H}_\Lambda$ is denoted by E_Λ.

Notation. Here $\mathcal{M}_\beta(\mathcal{O}_\circ)$ denotes the von Neumann algebra generated by $\mathcal{R}_\beta(\mathcal{O}_\circ)$ and $j\big(\mathcal{R}_\beta(\mathcal{O}_\circ)\big)$ and $\eta \in \mathcal{H}_\beta$ denotes the unique[4] product vector in the natural positive cone $\mathcal{P}^\natural\big(\mathcal{M}_\beta(\mathcal{O}_\circ) \vee \mathcal{M}_\beta(\mathcal{O})', \Omega_\beta\big)$ satisfying

$$(\eta\,,\, MN\eta) = (\Omega_\beta\,,\, M\Omega_\beta)(\Omega_p\,,\, N\Omega_p) \tag{14}$$

[4]Fixing the product vector with respect to some natural positive cone is mathematically convenient, but not necessary. In fact, we expect that different ‘boundary conditions’ are realized by different choices of η. In the thermodynamic limit different choices of the boundary conditions might lead to different phases.

for all $M \in \mathcal{M}_\beta(\mathcal{O}_\circ)$ and $N \in \mathcal{M}_\beta(\mathcal{O})'$. As before, Ω_p denotes the unique product vector in the natural positive cone $\mathcal{P}^\natural\big(\mathcal{R}_\beta(\mathcal{O}) \vee \mathcal{R}_\beta(\hat{\mathcal{O}})', \Omega_\beta\big)$ satisfying

$$(\Omega_p \,,\, AB\Omega_p) = (\Omega_\beta \,,\, A\Omega_\beta)(\Omega_\beta \,,\, B\Omega_\beta) \tag{15}$$

for all $A \in \mathcal{R}_\beta(\mathcal{O})$ and $B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$.

Note that W – as specified in (11) – is unitary and $WMNW^* = M \otimes N$ for $M \in \mathcal{M}_\beta(\mathcal{O}_\circ)$ and $N \in \mathcal{M}_\beta(\mathcal{O})'$. Using the isometry W we can write

$$\mathcal{H}_\Lambda = W^* \overline{\mathcal{M}_\beta(\mathcal{O}_\circ)\Omega_\beta \otimes \Omega_p} = W^*(\mathcal{H}_\beta \otimes \Omega_p)$$

and $E_\Lambda = W^*(\mathbb{1} \otimes P_{\Omega_p})W$. Here $P_{\Omega_p} \in \mathcal{B}(\mathcal{H}_\beta)$ denotes the projection onto $\mathbb{C} \cdot \Omega_p$.

The following proposition summarizes the properties of the Hilbert space $\mathcal{H}_\Lambda$. It justifies the claim stated at the beginning of this subsection.

Proposition 3.1. *Given a triple $\Lambda := (\mathcal{O}_\circ, \mathcal{O}, \hat{\mathcal{O}})$ of space-time regions as specified in* (12) *we find:*

(i) *The Hilbert space $\mathcal{H}_\Lambda$ is invariant under the action of elements of $\mathcal{M}_\beta(\mathcal{O}_\circ)$, i.e., $\mathcal{M}_\beta(\mathcal{O}_\circ)\mathcal{H}_\Lambda = \mathcal{H}_\Lambda$.*

(ii) *Vectors in $\mathcal{H}_\Lambda$ induce product states for the pair $\big(\mathcal{M}_\beta(\mathcal{O}_\circ), \mathcal{M}_\beta(\mathcal{O})'\big)$: if $\Psi \in \mathcal{H}_\Lambda$, then*

$$(\Psi \,,\, MN\Psi) = (\Psi \,,\, M\Psi)(\Omega_p \,,\, N\Omega_p)$$

for all $M \in \mathcal{M}_\beta(\mathcal{O}_\circ)$ and $N \in \mathcal{M}_\beta(\mathcal{O})'$.

(iii) *The vector states associated with $\mathcal{H}_\Lambda$ represent strictly localized excitations of the KMS state, i.e., they coincide with the original KMS state ω_β in the space-like complement of $\hat{\mathcal{O}}$: if $\Psi \in \mathcal{H}_\Lambda$, then*

$$(\Psi \,,\, \pi_\beta(a)\Psi) = \omega_\beta(a) \qquad \forall a \in \mathcal{A}^c(\hat{\mathcal{O}}).$$

Here $\mathcal{A}^c(\hat{\mathcal{O}})$ denotes the C^*-algebra generated by $\{a \in \mathcal{A} : [a,b] = 0 \; \forall b \in \mathcal{A}(\hat{\mathcal{O}})\}$ and not *the commutant of $\pi_\beta\big(\mathcal{A}(\hat{\mathcal{O}})\big)$ in $\mathcal{B}(\mathcal{H}_\beta)$.*

(iv) *$\mathcal{H}_\Lambda$ is complete in the following sense: to every normal state ϕ on $\mathcal{M}_\beta(\mathcal{O}_\circ)$ there exists a $\Phi \in \mathcal{H}_\Lambda$ such that $(\Phi \,,\, M\Phi) = \phi(M)$ for all $M \in \mathcal{M}_\beta(\mathcal{O}_\circ)$.*

Proof. We simply adapt the proof of the corresponding result by Buchholz and Junglas to our situation:

(i) follows from the definition;

(ii) follows from (13) and (14);

(iii) follows from (13), (14) and (15).

(iv) Since $\mathcal{M}_\beta(\mathcal{O}_\circ)$ has a cyclic and separating vector, there exists a vector $\tilde{\Phi} \in \mathcal{H}_\beta$ which induces the given normal state ϕ on $\mathcal{M}_\beta(\mathcal{O}_\circ)$. It follows that the vector $\Phi := W^*(\tilde{\Phi} \otimes \Omega_p) \in \mathcal{H}_\Lambda$ satisfies (iv). □

We need one more lemma, in order to show that the restriction of the operator Δ_p^α to the subspace $\mathcal{H}_\Lambda$ is trace class for $0 < \alpha < 1/2$.

Lemma 3.2. *Assume that the nuclearity condition* (2) *holds true. It follows that*

(i) *the maps* $\vartheta_{\alpha,\mathcal{O}}\colon \mathcal{M}_\beta(\mathcal{O}) \to \mathcal{H}_\beta$,

$$M \mapsto \Delta_p^\alpha M \Omega_p, \qquad 0 \le \alpha \le 1/2,$$

are nuclear for $0 < \alpha < 1/2$*;*

(ii) *the nuclear norm of* $\vartheta_{\alpha,\mathcal{O}}$ *is bounded by*

$$\|\vartheta_{\alpha,\mathcal{O}}\| \le \mathrm{e}^{2cr^d\left(\alpha^{-m}+(1/2-\alpha)^{-m}\right)}, \qquad c,m,d>0,$$

where r *denotes the diameter of* $\mathcal{O}$ *and* c,m,d *are the constants appearing in the bound (2) on the nuclear norm of the map* $\Theta_{\alpha,\mathcal{O}}$.

Proof. Let $A \in \mathcal{R}_\beta(\mathcal{O})$ and $B \in j\big(\mathcal{R}_\beta(\mathcal{O})\big)$. By definition,

$$\vartheta_{\alpha,\mathcal{O}}(AB) = V^*\big(\Delta^\alpha A\Omega_\beta \otimes \Delta^{-\alpha} B\Omega_\beta\big).$$

The maps $A \mapsto \Delta^\alpha A\Omega_\beta$ and $B \mapsto \Delta^{-\alpha}B\Omega_\beta$ are nuclear for $0 < \alpha < 1/2$. The tensor product of two nuclear maps itself is a nuclear map and the norm is bounded by the product of the nuclear norms [P].

Proposition 3.3. *Let* $\Lambda(\mathcal{O}_\circ, \mathcal{O}, \hat{\mathcal{O}})$ *be a triple of space-time regions as specified in* (12)*. Assume the nuclearity condition* (2) *holds true. It follows that the operator* $\Delta_p^\alpha E_\Lambda$*, acting on the Hilbert space* $\mathcal{H}_\beta$*, is of trace-class for* $0<\alpha<1/2$*, and*

$$\mathrm{Tr}\ |\Delta_p^\alpha E_\Lambda| \le \mathrm{e}^{2cr^d\left(\alpha^{-m}+(1/2-\alpha)^{-m}\right)}, \qquad c,m,d>0,$$

where r *denotes the diameter of* $\mathcal{O}$ *and* c,m,d *are the constants appearing in the bound* (2) *on the nuclear norm of the map* $\Theta_{\alpha,\mathcal{O}}$.

Proof. The proof of this proposition is more or less identical to the one given by Buchholz and Junglas [BJu 89] for the vacuum case. We present it for completeness only.

i) The first step is to construct a convenient orthonormal basis of $\mathcal{H}_\Lambda$. Let $\{\Psi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_\beta$ with $\Psi_1 = \Omega_\beta$. Set

$$U_{i,j} := W^*(M_{i,j} \otimes \mathbb{1})W, \tag{16}$$

where $M_{i,j} \in \mathcal{B}(\mathcal{H}_\beta)$ are matrix units given by

$$M_{i,j}\Psi := (\Psi_j, \Psi)\Psi_i \qquad \forall \Psi \in \mathcal{H}_\beta.$$

Since $W^*\big(\mathcal{B}(\mathcal{H}_\beta)\otimes\mathbb{1}\big)W = \mathcal{N}_\circ \vee j\big(\mathcal{N}_\circ\big)$, we infer from (16) that $U_{i,j} \in \mathcal{N}_\circ \vee j\big(\mathcal{N}_\circ\big)$. Furthermore,

$$U_{i,j}^* = U_{j,i}, \quad U_{i,j}U_{k,l} = \delta_{j,k}U_{i,l}, \quad \text{and} \quad \mathrm{s}-\lim_{N\to\infty}\sum_{i=1}^N U_{i,i} = \mathbb{1}\,.$$

Combining (11) and (16) we find $U_{i,1}\eta = W^*(\Psi_i \otimes \Omega_p)$. Thus $\{U_{i,1}\eta\}_{i\in\mathbb{N}}$ is the desired orthonormal basis of $\mathcal{H}_\Lambda$. Note that (11) holds true for all $N \in \big(\mathcal{N}_\circ \vee j(\mathcal{N}_\circ)\big)'$. Therefore $\|N\eta\| = \|N\Omega_p\|$. Consequently, we can introduce an isometry $I \in \mathcal{N}_\circ \vee j\big(\mathcal{N}_\circ\big)$ by setting

$$N\eta = IN\Omega_p \qquad \forall N \in \big(\mathcal{N}_\circ \vee j(\mathcal{N}_\circ)\big)'.$$

We can now represent the orthonormal basis $\{U_{i,1}\eta\}_{i\in\mathbb{N}}$ by vectors $\Gamma_i := U_{i,1}\eta = U_{i,1}I\Omega_p$, where $U_{i,1}I \in \mathcal{N}_\circ \vee j\big(\mathcal{N}_\circ\big) \subset \mathcal{M}_\beta(\mathcal{O})$. It follows that $\Gamma_i \in \mathcal{D}\big(\Delta_p^\alpha\big)$ for $0 < \alpha < 1/2$ and $i \in \mathbb{N}$. Especially, $\eta =: \Gamma_1 \in \mathcal{D}\big(\Delta_p^\alpha\big)$ for $0 < \alpha < 1/2$.

ii) Polar decomposition of the closeable operator $\Delta_p^\alpha E_\Lambda$ yields $\Delta_p^\alpha E_\Lambda = F \cdot |\Delta_p^\alpha E_\Lambda|$, where F is a partial isometry with range in $\mathcal{H}_\Lambda$. Introducing a set of linear functionals ϕ_i (which can be chosen to be continuous with respect to the ultra-weakly topology induced by $\mathcal{M}_\beta(\mathcal{O})$ [BD'AL 90b]) and vectors $\Phi_i \in \mathcal{H}_\beta$ corresponding to the nuclear map $\vartheta_{\alpha,\mathcal{O}}$ we obtain

$$\begin{aligned}
\mathrm{Tr}\ |\Delta_p^\alpha E_\Lambda| &= \sum_i \big(U_{i,1}I\Omega_p\,,\ F^*\Delta_p^\alpha U_{i,1}I\Omega_p\big)\\
&= \sum_i \big(U_{i,1}I\Omega_p\,,\ F^*\vartheta_{\alpha,\mathcal{O}}(U_{i,1}I)\big)\\
&= \sum_i\sum_n \phi_n(U_{i,1}I)\cdot\big(U_{i,1}I\Omega_p\,,\ F^*\Phi_n\big)\\
&\le \sum_i\sum_n |\phi_n(U_{i,1}I)|\cdot\|U_{1,i}F^*\Phi_n\|.
\end{aligned}$$

Buchholz and Junglas have shown the following inequality [BJu 89]:

$$\sum_i |\psi(U_{i,1})|\cdot\|U_{1,i}\Psi\| \le \|\psi\|\,\|\Psi\|,$$

for $\Psi \in \mathcal{H}_\beta$ and ψ an ultra-weakly continuous linear functional on $\mathcal{M}_\beta(\mathcal{O})$. Consequently, $\mathrm{Tr}\ |\Delta_p^\alpha E_\Lambda| \le \sum_n \|\phi_n\|\,\|\Phi_n\|$. Taking the infimum with respect to all decompositions of the respective nuclear maps we find $\mathrm{Tr}\ |\Delta_p^\alpha E_\Lambda| \le \|\vartheta_{\alpha,\mathcal{O}}\|$. □

3.2 Local KMS states for a new temperature

Proposition 3.3 allow us to define "local quasi-Gibbs" states, which are *local* (τ,β')-KMS states for the new temperature $1/\beta'$ in the interior of $\mathcal{O}_\circ$ and (τ,β)-KMS states for the original temperature $1/\beta$ outside of $\hat{\mathcal{O}}$. Before we do so, we give a precise meaning to the statement that a local excitation ω_Λ of a KMS state ω_β satisfies a *local KMS condition* for the new temperature $1/\beta'$ in a bounded region $\mathcal{O}_\circ$. Note that any β' $(0 < \beta' < \infty)$ can be decomposed into some α $(0 < \alpha < 1/2)$ and some (minimal) $n \in \mathbb{N}$ such that $\beta' = \alpha n\beta$.

Definition. Let $\beta' > 0$ and let $n \in \mathbb{N}$ be the smallest natural number such that $n\alpha\beta = \beta'$ for some α $(0 \leq \alpha \leq 1/2)$. A state ω_Λ satisfies the *local KMS condition* at temperature $1/\beta'$ in some bounded space-time region $\mathcal{O}_\circ \subset \mathbb{R}^4$ if for any subregion $\mathcal{O}_{\circ\circ} \subset \mathcal{O}_\circ$ whose closure is contained in the interior of $\mathcal{O}_\circ$ there exists some $\delta_{\circ\circ} > 0$ and a function $F_{a,b}$ for every pair of operators a, $b \in \mathcal{A}(\mathcal{O}_{\circ\circ})$ such that

(i) $F_{a,b}$ is defined on

$$\mathcal{G}_{n,\alpha} := \{z \in \mathbb{C} \mid 0 < \Im z < n\alpha\beta\} \setminus \{z \in \mathbb{C} \mid |\Re z| \geq \delta_{\circ\circ}, \\ \Im z = k\alpha\beta,\ k = 1, \ldots, n-1\};$$

(ii) $F_{a,b}$ is bounded and analytic in its domain of definition;

(iii) $F_{a,b}$ is continuous for $\Im z \searrow k\alpha\beta$ and $\Im z \nearrow k\alpha\beta$, $k = 1, \ldots, n-1$;

(iv) $F_{a,b}$ is continuous at the boundary for $\Im z \searrow 0$ and $\Im z \nearrow n\alpha\beta$;

(v) The respective boundary values are

$$F_{a,b}(t) = \omega_\Lambda\big(a\tau_t(b)\big) \text{ and } F_{a,b}(t + in\alpha\beta) = \omega_\Lambda\big(\tau_t(b)a\big) \text{ for } |t| < \delta_{\circ\circ}. \quad (17)$$

Remark. To heat up the system locally is quite simple: For $\beta' < \beta/2$ we find $n = 1$, i.e., no cuts appear in $\mathcal{G}_{1,\alpha} = \{z \in \mathbb{C} \mid 0 < \Im z < \alpha\beta\}$. To cool down the system locally is more delicate. One needs at least n cuts, where n is the minimal natural number such that $\beta' = n\alpha\beta$ $(0 < \alpha < 1/2)$. Whether or not it is useful to operate with more cuts then necessary is unknown to us.

Proposition 3.4. *Let $\Lambda := (\mathcal{O}_\circ, \mathcal{O}, \hat{\mathcal{O}})$ be a triple of space-time regions, as specified in* (12)*. Let $n \in \mathbb{N}$ be the minimal natural number such that $\beta' = n\alpha\beta$, $0 < \alpha < 1/2$. Set, for n and α fixed,*

$$\rho_\Lambda := \frac{\big(E_\Lambda \Delta_p^\alpha E_\Lambda\big)^n}{\mathrm{Tr}\,\big(\Delta_p^\alpha E_\Lambda\big)^n} \quad \textit{and} \quad \omega_\Lambda(a) := \mathrm{Tr}\ \rho_\Lambda \pi_\beta(a) \quad \forall a \in \mathcal{A}. \quad (18)$$

Then ρ_Λ is a density matrix, i.e., $\rho_\Lambda > 0$ and $\mathrm{Tr}\ \rho_\Lambda = 1$, and the following statements hold true:

(i) *The states ω_Λ are product states, which coincide with the given KMS state ω_β in the space-like complement of $\hat{\mathcal{O}}$; i.e.,*

$$\omega_\Lambda(ab') = \omega_\Lambda(a)\,\omega_\beta(b')$$

for all $a \in \mathcal{A}(\mathcal{O}_\circ)$ and $b' \in \mathcal{A}^c(\hat{\mathcal{O}})$. As before, $\mathcal{A}^c(\hat{\mathcal{O}})$ denotes the C^*-algebra generated by $\{a \in \mathcal{A} \mid [a, b] = 0\ \forall b \in \mathcal{A}(\hat{\mathcal{O}})\}$.

(ii) *The states ω_Λ are local $(\tau, n\alpha\beta)$-KMS states for the space-time region $\mathcal{O}_\circ$.*

Remark. For $\mathcal{O}_\circ, \mathcal{O}, \hat{\mathcal{O}} \to \mathbb{R}^4$ the denominator in (18) might go to ∞ or 0. In any case we will leave the representation: we will have no operator convergence, neither in the weak nor in the strong sense and therefore we can only rely on expectation values. After performing the thermodynamic limit, we will use these expectation values to construct a new representation and check whether the new state satisfies the KMS condition [Na]. We will see that it will do so, under the assumptions we have imposed on the phase-space properties of our thermal field theory.

Proof. (i) Let $a \in \mathcal{A}(\mathcal{O}_\circ)$ and $b' \in \mathcal{A}^c(\hat{\mathcal{O}})$ and let P_{Ω_p} denote the projection onto $\mathbb{C} \cdot \Omega_p$. Since $E_\Lambda \in \mathcal{M}_\beta(\mathcal{O}_\circ)' \subset \pi_\beta\big(\mathcal{A}(\mathcal{O}_\circ)\big)'$, it follows that $[E_\Lambda, \pi_\beta(a)] = 0$. Moreover, $E_\Lambda = W^*(\mathbb{1} \otimes P_{\Omega_p})W$ implies

$$E_\Lambda \pi_\beta(b') E_\Lambda = \omega_\beta(b') E_\Lambda \qquad \forall b' \in \mathcal{A}^c(\hat{\mathcal{O}}).$$

Using the cyclicity of the trace we find

$$\begin{aligned}\omega_\Lambda(ab') &= \frac{\mathrm{Tr}\big(E_\Lambda \Delta_p^\alpha E_\Lambda\big)^n \pi_\beta(a) E_\Lambda \pi_\beta(b') E_\Lambda}{\mathrm{Tr}\big(\Delta_p^\alpha E_\Lambda\big)^n} \\ &= \omega_\Lambda(a)\omega_\beta(b').\end{aligned}$$

(ii) Consider the case $n = 2$. Let $\delta_{\circ\circ} > 0$ and $\mathcal{O}_{\circ\circ}$ be an open space-time region such that $\mathcal{O}_{\circ\circ} + te \subset \mathcal{O}_\circ$ for $|t| < \delta_{\circ\circ}$. Let $a, b \in \mathcal{A}(\mathcal{O}_{\circ\circ})$. By assumption, $a\tau_t(b) \in \mathcal{A}(\mathcal{O}_\circ)$ for $|t| < \delta_{\circ\circ}$. Set

$$F^{(1)}_{a,b}(z) := \frac{\mathrm{Tr}\ \pi_\beta(a) E_\Lambda \Delta_p^{-iz/\beta} \pi_\beta(b) \Delta_p^{\alpha + iz/\beta} E_\Lambda \Delta_p^\alpha E_\Lambda}{\mathrm{Tr}\ \big(\Delta_p^\alpha E_\Lambda\big)^2}$$

for $0 < \Im z < \alpha\beta$. The function $F^{(1)}_{a,b}(z)$ is analytic in its domain and continuous at the boundary. We recall that $\Delta_p^{-it/\beta} \pi_\beta(b) \Delta_p^{it/\beta} = \pi_\beta\big(\tau_t(b)\big) \in \pi_\beta\big(\mathcal{A}(\mathcal{O}_\circ)\big)$ for $|t| < \delta_{\circ\circ}$. Using once again the cyclicity of the trace and $E_\Lambda \in \pi_\beta\big(\mathcal{A}(\mathcal{O}_\circ)\big)'$, we conclude that

$$\begin{aligned}\lim_{\Im z \searrow 0} F^{(1)}_{a,b}(z) &= \frac{\mathrm{Tr}\ \pi_\beta(a) E_\Lambda \pi_\beta\big(\tau_{\Re z}(b)\big) \Delta_p^\alpha E_\Lambda \Delta_p^\alpha E_\Lambda}{\mathrm{Tr}\ \big(\Delta_p^\alpha E_\Lambda\big)^2} \\ &= \frac{\mathrm{Tr}\ \pi_\beta\big(a\tau_{\Re z}(b)\big)\big(E_\Lambda \Delta_p^\alpha E_\Lambda\big)^2}{\mathrm{Tr}\ \big(\Delta_p^\alpha E_\Lambda\big)^2} \qquad \forall |\Re z| < \delta_{\circ\circ}.\end{aligned}$$

Thus

$$\lim_{\Im z \searrow 0} F^{(1)}_{a,b}(z) = \omega_\Lambda\big(a\tau_{\Re z}(b)\big) \qquad \forall |\Re z| < \delta_{\circ\circ}. \tag{19}$$

On the other hand, for $|\Re z| < \delta_{\circ\circ}$,

$$\begin{aligned}\lim_{\Im z \nearrow \alpha\beta} F^{(1)}_{a,b}(z) &= \frac{\mathrm{Tr}\ \pi_\beta(a) E_\Lambda \Delta_p^\alpha \pi_\beta\big(\tau_{\Re z}(b)\big) E_\Lambda \Delta_p^\alpha E_\Lambda}{\mathrm{Tr}\ \big(\Delta_p^\alpha E_\Lambda\big)^2} \\ &= \frac{\mathrm{Tr}\ \pi_\beta(a) E_\Lambda \Delta_p^\alpha E_\Lambda \pi_\beta\big(\tau_{\Re z}(b)\big) \Delta_p^\alpha E_\Lambda}{\mathrm{Tr}\ \big(\Delta_p^\alpha E_\Lambda\big)^2}.\end{aligned}$$

For $\alpha\beta < \Im z < 2\alpha\beta$ we set

$$F^{(2)}_{a,b}(z) := \frac{\mathrm{Tr}\ \pi_\beta(a)E_\Lambda\Delta_p^\alpha E_\Lambda\Delta_p^{-\alpha-iz/\beta}\pi_\beta(b)\Delta_p^{2\alpha+iz/\beta}E_\Lambda}{\mathrm{Tr}\left(\Delta_p^\alpha E_\Lambda\right)^2}.$$

The function $F^{(2)}_{a,b}(z)$ is analytic in its domain and continuous at the boundary. By definition,

$$\begin{aligned}\lim_{\Im z\searrow\alpha\beta} F^{(2)}_{a,b}(z) &= \frac{\mathrm{Tr}\ \pi_\beta(a)E_\Lambda\Delta_p^\alpha E_\Lambda\pi_\beta\big(\tau_{\Re z}(b)\big)\Delta_p^\alpha E_\Lambda}{\mathrm{Tr}\left(\Delta_p^\alpha E_\Lambda\right)^2}\\ &= \lim_{\Im z\nearrow\alpha\beta} F^{(1)}_{a,b}(z) \quad \forall|\Re z| < \delta_{\circ\circ}.\end{aligned}$$

Furthermore, $F^{(2)}_{a,b}$ satisfies

$$\begin{aligned}\lim_{\Im z\nearrow 2\alpha\beta} F^{(2)}_{a,b}(z) &= \frac{\mathrm{Tr}\ \pi_\beta(a)E_\Lambda\Delta_p^\alpha E_\Lambda\Delta_p^\alpha\pi_\beta\big(\tau_{\Re z}(b)\big)E_\Lambda}{\mathrm{Tr}\left(\Delta_p^\alpha E_\Lambda\right)^2}\\ &= \frac{\mathrm{Tr}\ \pi_\beta(a)\big(E_\Lambda\Delta_p^\alpha E_\Lambda\big)^2\pi_\beta\big(\tau_{\Re z}(b)\big)}{\mathrm{Tr}\left(\Delta_p^\alpha E_\Lambda\right)^2} \qquad \forall|\Re z| < \delta_{\circ\circ}.\end{aligned}$$

Thus

$$\lim_{\Im z\nearrow 2\alpha\beta} F^{(2)}_{a,b}(z) = \omega_\Lambda\big(\tau_{\Re z}(b)a\big) \qquad \forall|\Re z| < \delta_{\circ\circ}. \tag{20}$$

Using the Edge-of-the-Wedge theorem [SW] we conclude that $F^{(1)}_{a,b}$ and $F^{(2)}_{a,b}$ are the restrictions to the upper (resp. lower) half of the double cut strip

$$\mathcal{G}_{2,\alpha} = \{z\in\mathbb{C} \mid 0 < \Im z < 2\alpha\beta\} \setminus \{z\in\mathbb{C} \mid |\Re z| \geq \delta_{\circ\circ}, \Im z = \alpha\beta\}$$

of a function

$$F_{a,b}(z) := \begin{Bmatrix} F^{(2)}_{a,b}(z) \\ F^{(1)}_{a,b}(z)\end{Bmatrix} \text{ for } \begin{Bmatrix} \alpha\beta < \Im z < 2\alpha\beta, \\ 0 < \Im z < \alpha\beta, \end{Bmatrix}$$

defined and continuous on the closure of $\mathcal{G}_{2,\alpha}$ and analytic for $z \in \mathcal{G}_{2,\alpha}$. From (19) and (20) we infer $F_{a,b}(t) = \omega_\Lambda\big(a\tau_t(b)\big)$ and $F_{a,b}(t + i2\alpha\beta) = \omega_\Lambda\big(\tau_t(b)a\big)$ for $|t| < \delta_{\circ\circ}$. Analogous results for arbitrary $n \in \mathbb{N}$ can be established by the same line of arguments but with considerable more effort. $\square$

4 The thermodynamic limit

We will now control the surface energies in the limit $\mathcal{O}_\circ, \mathcal{O}, \hat{\mathcal{O}} \to \mathbb{R}^4$. Since we do not have explicit expressions for the surface energies, our approach is quite involved. The first step is to control the convergence of product vectors.

4.1 Consequences of the cluster condition

Let us introduce some notation: Let $\Lambda_i = \big(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}, \hat{\mathcal{O}}^{(i)}\big)$ be a sequence of triples of double cones with diameters $\big(r_\circ^{(i)}, r^{(i)}, \hat{r}^{(i)}\big)$. We consider the product vectors $\Omega_p^{(i)}$ and $\eta_i, \chi_i \in \mathcal{P}^\natural\big(\mathcal{M}_\beta(\mathcal{O}_\circ^{(i)}) \vee \mathcal{M}_\beta(\mathcal{O}^{(i)})', \Omega_\beta\big)$, which satisfy

$$\big(\Omega_p^{(i)}\,,\, AB\,\Omega_p^{(i)}\big) = (\Omega_\beta\,,\, A\Omega_\beta)(\Omega_\beta\,,\, B\Omega_\beta)$$

for $A \in \mathcal{R}_\beta\big(\mathcal{O}^{(i)}\big)$ and $B \in \mathcal{R}_\beta\big(\hat{\mathcal{O}}^{(i)}\big)'$, and

$$\begin{aligned}(\eta_i\,,\, MN\,\eta_i) &= \big(\Omega_\beta\,,\, M\Omega_\beta\big)\big(\Omega_p^{(i)}\,,\, N\Omega_p^{(i)}\big)\\ (\chi_i\,,\, MN\,\chi_i) &= (\Omega_\beta\,,\, M\Omega_\beta)(\Omega_\beta\,,\, N\Omega_\beta)\end{aligned}$$

for $M \in \mathcal{M}_\beta\big(\mathcal{O}_\circ^{(i)}\big)$ and $N \in \mathcal{M}_\beta\big(\mathcal{O}^{(i)}\big)'$.

So far there was no restriction on the relative size of the regions $\mathcal{O}^{(i)}$ and $\hat{\mathcal{O}}^{(i)}$. We will now exploit this freedom: If the net of local observables $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ is regular from the outside, then $\mathcal{M}_\beta(\mathcal{O})' \cap \mathcal{M}_\beta(\hat{\mathcal{O}}) \to \mathbb{C}\cdot\mathbb{1}$ as $\hat{\mathcal{O}} \searrow \mathcal{O}$. Our aim is to control $\|\eta_i - \Omega_\beta\|$. The following lemma shows that in order to do so it is sufficient to control $\|\chi_i - \Omega_\beta\|$.

Lemma 4.1. *Let $\big\{(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)})\big\}_{i\in\mathbb{N}}$ be a sequence of pairs of double cones. Then one can find a sequence of double cones $\{\hat{\mathcal{O}}^{(i)}\}_{i\in\mathbb{N}}$ such that* (12) *holds true and* $\lim_{i\to\infty}\|\chi_i - \eta_i\| = 0$.

Proof. Consider a sequences of pairs of double cones $\big\{\big(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}\big)\big\}_{i\in\mathbb{N}}$ eventually exhausting all of $\mathbb{R}^4$. For each $i \in \mathbb{N}$ fixed we consider a sequence of double cones $\big\{\hat{\mathcal{O}}^{(i,k)}\big\}_{k\in\mathbb{N}}$ such that $\hat{\mathcal{O}}^{(i,k)} \searrow \mathcal{O}^{(i)}$ for $k \to \infty$. In order to ease the notation we set

$$\mathcal{A}_i := \mathcal{M}_\beta\big(\mathcal{O}_\circ^{(i)}\big), \qquad \mathcal{B}_i := \mathcal{M}_\beta\big(\mathcal{O}^{(i)}\big), \qquad \mathcal{C}_{i,k} := \mathcal{M}_\beta\big(\hat{\mathcal{O}}^{(i,k)}\big),$$

$\mathcal{D}_i := \mathcal{A}_i \vee \mathcal{B}_i'$, and $\mathcal{E}_{i,k} := \mathcal{A}_i \vee \mathcal{C}_{i,k}'$. For each $i \in \mathbb{N}$ fixed, the sequence $\{\mathcal{E}_{i,k}\}_{k\in\mathbb{N}}$ of algebras satisfies $\mathcal{E}_{i,k+1} \subset \mathcal{E}_{i,k}$ (this follows from $\mathcal{C}_{i,k+1} \subset \mathcal{C}_{i,k}$) and $\cap_k \mathcal{E}_{i,k} = \mathcal{D}_i$. Now let $\Omega_p^{(i,k)}$ denote the unique product vector in the natural positive cone $\mathcal{P}^\natural\big(\mathcal{R}_\beta(\mathcal{O}^{(i)}) \vee \mathcal{R}_\beta(\hat{\mathcal{O}}^{(i,k)})', \Omega_\beta\big)$ satisfying

$$\big(\Omega_p^{(i,k)}\,,\, AB\Omega_p^{(i,k)}\big) = (\Omega_\beta\,,\, A\Omega_\beta)(\Omega_\beta\,,\, B\Omega_\beta)$$

for all $A \in \mathcal{R}_\beta\big(\mathcal{O}^{(i)}\big)$ and $B \in \mathcal{R}_\beta\big(\hat{\mathcal{O}}^{(i,k)}\big)'$. Note that for $C_{i,k} \in \mathcal{C}_{i,k}'$

$$\big(\Omega_p^{(i,k)}\,,\, C_{i,k}\Omega_p^{(i,k)}\big) = (\Omega_\beta\,,\, C_{i,k}\Omega_\beta).$$

If we choose product vectors $\eta_{i,k}$ and χ_i in the natural cone $\mathcal{P}^\natural(\mathcal{D}_i, \Omega_\beta)$ such that

$$(\eta_{i,k}\,,\, MN\eta_{i,k}) = \big(\Omega_\beta\,,\, M\Omega_\beta\big)\big(\Omega_p^{(i,k)}\,,\, N\Omega_p^{(i,k)}\big)$$

and $(\chi_i\,,\,MN\chi_i) = (\Omega_\beta\,,\,M\Omega_\beta)(\Omega_\beta\,,\,N\Omega_\beta)$ for all $M \in \mathcal{A}_i$ and $N \in \mathcal{B}_i'$, then by a result of Araki [A 74]

$$\|\eta_{i,k} - \chi_i\|^2 \le \sup_{D_i \in \mathcal{D}_i, \|D_i\|=1} \big|(\eta_{i,k}, D_i\eta_{i,k}) - (\chi_i, D_i\chi_i)\big|.$$

Now assume that for each $i \in \mathbb{N}$ fixed there exist a sequence $\{E_{i,k} \in \mathcal{E}_{i,k} \mid \|E_{i,k}\| = 1\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \big|(\eta_{i,k}\,,\,E_{i,k}\eta_{i,k}) - (\chi_i\,,\,E_{i,k}\chi_i)\big| \ge \epsilon_i. \tag{21}$$

We demonstrate that this leads to a contradiction. The linear functional $(\eta_{i,k}\,,\,.\,\eta_{i,k}) - (\chi_i\,,\,.\,\chi_i)$ is ultra-weakly continuous on the von Neumann algebra $\mathcal{D}_i$. Therefore the sequence $\{E_{i,k} \in \mathcal{E}_{i,k} \mid \|E_{i,k}\| = 1\}_{k\in\mathbb{N}}$ has a weak limit point $w - \lim_{k\to\infty} E_{i,k} =: D_i \in \mathcal{D}_i = \cap_k \mathcal{E}_{i,k}$ such that

$$\big|(\eta_{i,k}, D_i\eta_{i,k}) - (\chi_i, D_i\chi_i)\big| > \frac{1}{2}\epsilon_i \qquad \forall k > k_i$$

and some $k_i \in \mathbb{N}$, in contradiction to

$$\big|(\eta_{i,k}, E_{i,k}\eta_{i,k}) - (\chi_i, E_{i,k}\chi_i)\big| = 0 \qquad \forall E_{i,k} \in \mathcal{E}_{i,k}, \quad \forall k \in \mathbb{N}.$$

Therefore, the assumption (21) can not hold true. It follows that there exists some $k_i \in \mathbb{N}$ such that

$$\sup_{D\in\mathcal{D}_i, \|D\|=1} \big|(\eta_{i,k}\,,\,D\eta_{i,k}) - (\chi_i\,,\,D\chi_i)\big| < \epsilon_i \qquad \forall k \ge k_i.$$

If we set $\hat{\mathcal{O}}^{(i)} := \hat{\mathcal{O}}^{(i,k_i)}$, then we can choose ϵ_i such that $\lim_{i\to\infty} \|\chi_i - \eta_i\| = 0$. $\Box$

We will now show that the product vector χ converges to Ω_β if $\mathcal{O}_\circ$ and $\mathcal{O}$ tend to $\mathbb{R}^4$ and the relative size of $\mathcal{O}_\circ$ and $\mathcal{O}$ obeys the restrictions imposed by the cluster condition.

Lemma 4.2. *Let* $\big\{\big(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}\big)\big\}_{i\in\mathbb{N}}$ *denote a sequence of pairs of double cones with diameters* $\big(r_\circ^{(i)}, r^{(i)}\big)$, $i \in \mathbb{N}$. *Assume that* $\lim_{i\to\infty}\big(r_\circ^{(i)}\big)^{d'}\big(\delta_\circ^{(i)}\big)^{-\gamma} = 0$. *It follows that* $\|\chi_i - \Omega_\beta\| \to 0$ *as* $i \to \infty$.

Proof. Since $\chi_i \in \mathcal{P}^\natural\big(\mathcal{M}_\beta(\mathcal{O}_\circ^{(i)}) \vee \mathcal{M}_\beta(\mathcal{O}^{(i)})', \Omega_\beta\big)$, we can again rely on the result of Araki [A 74] concerning the distance of two vectors which belong to the natural positive cone $\mathcal{P}^\natural\big(\mathcal{M}_\beta(\mathcal{O}_\circ^{(i)}) \vee \mathcal{M}_\beta(\mathcal{O}^{(i)})', \Omega_\beta\big)$:

$$\|\chi_i - \Omega_\beta\|^2 \le \sup_{\|D_i\|=1} \big|(\chi_i, D_i\chi_i) - (\Omega_\beta, D_i\Omega_\beta)\big|;$$

where the supremum has to be evaluated over all elements $D_i \in \mathcal{M}_\beta\big(\mathcal{O}_\circ^{(i)}\big) \vee \mathcal{M}_\beta\big(\mathcal{O}^{(i)}\big)'$. Thus $\lim_{i\to\infty} \|\chi_i - \Omega_\beta\| = 0$ follows from the cluster condition (3) and the assumptions concerning the relative size of $\mathcal{O}_\circ$ and $\mathcal{O}$ stated in the lemma. $\Box$

Combining Lemma 4.1 and Lemma 4.2 we conclude that $\lim_{i\to\infty} \|\eta_i - \Omega_\beta\| = 0$ for an appropriate choice of the relative size of $\mathcal{O}_\circ^{(i)}$, $\mathcal{O}^{(i)}$ and $\hat{\mathcal{O}}^{(i)}$.

4.2 Bounds on the quasi-partition function

Let us consider a sequence $\{\Lambda_i\} = \{(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}, \hat{\mathcal{O}}^{(i)})\}$ of triples of double cones with diameters $(r_\circ^{(i)}, r^{(i)}, \hat{r}^{(i)})$. In order to ensure that (for $0 < \alpha < 1/2$ and $n \in \mathbb{N}$ fixed) the 'quasi-partition function'

$$Z_{\Lambda_i}(\alpha, n) := \mathrm{Tr}\,\big(E_{\Lambda_i} \Delta_{p,i}^{\alpha} E_{\Lambda_i}\big)^n, \qquad \Lambda_i = \big(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}, \hat{\mathcal{O}}^{(i)}\big),$$

is bounded from below as $i \to \infty$, it is necessary that $\mathcal{O}^{(i)}$ grows rapidely with $\mathcal{O}_\circ^{(i)}$. Otherwise the energy contained in the boundary, which is necessary to decouple the local region from the outside, lessens the eigenvalues of $E_{\Lambda_i} \Delta_{p,i}^{\alpha} E_{\Lambda_i}$ so drastically that it outruns the increase in the number of states contributing to the trace by enlarging $\mathcal{O}_\circ^{(i)}$. Following once again [BJu 89] we will now demonstrate that the condition on the relative size of $r_\circ^{(i)}$ and $r^{(i)}$ which we imposed in order to show that χ_i converges to Ω_β is already sufficient to exclude this possibility.

Lemma 4.3. *(Buchholz and Junglas). Let $\{\Lambda_i\}_{i\in\mathbb{N}}$ be a sequence of triples of increasing space-time regions such that $\|\eta_i - \Omega_\beta\| \to 0$ for $i \to \infty$. It follows that $\mathcal{H}_{\Lambda_i}$ tends to the whole Hilbert space $\mathcal{H}_\beta$, i.e., $s - \lim_{i\to\infty} E_{\Lambda_i} = \mathbb{1}$.*

Proof. By assumption $\eta_i = W_i^*\big(\Omega_\beta \otimes \Omega_p^{(i)}\big)$ converges to Ω_β. Therefore the unitary operators W_i specified in (11) fulfill $W_i^*\big(\Phi \otimes \Omega_p^{(i)}\big) \to \Phi$ for $\Phi \in \mathcal{H}_\beta$ as $i \to \infty$. Recall that $E_{\Lambda_i} = W_i^*\big(\mathbb{1} \otimes P_{\Omega_p^{(i)}}\big)W_i$, where $P_{\Omega_p^{(i)}}$ denotes the projection onto $\mathbb{C} \cdot \Omega_p^{(i)}$. Hence

$$E_{\Lambda_i}\Phi = W_i^*\big(\mathbb{1} \otimes P_{\Omega_p^{(i)}}\big)W_i\big(\Phi - W_i^*(\Phi \otimes \Omega_p^{(i)})\big) + W_i^*\big(\Phi \otimes \Omega_p^{(i)}\big) \to \Phi \quad \forall \Phi \in \mathcal{H}_\beta,$$

as $i \to \infty$. I.e., $s - \lim_{i\to\infty} E_{\Lambda_i} = \mathbb{1}$. □

Lemma 4.4. *Let $\{(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)})\}_{i\in\mathbb{N}}$ denote a sequence of pairs of double cones with diameters $(r_\circ^{(i)}, r^{(i)})$, $i \in \mathbb{N}$. Assume that $\lim_{i\to\infty}\big(r_\circ^{(i)}\big)^{d'}\big(\delta_\circ^{(i)}\big)^{-\gamma} = 0$. It follows that*

$$\liminf_i \mathrm{Tr}\,\big(E_{\Lambda_i}\Delta_{p,i}^{\alpha}E_{\Lambda_i}\big)^n > 0 \qquad \forall n \in \mathbb{N}.$$

Proof. By definition, $\Delta_{p,i}^{1/2}$ is a positive operator. The vector $\Omega_p^{(i)}$ is the unique eigenvector of H_p for the simple eigenvalue $\{0\}$. Let $\{\Omega_\beta, \Psi_1, \Psi_2, \ldots\}$ be an orthonormal basis in $\mathcal{H}_\beta$ and set $\Gamma_j^{(i)} = W_i^*\big(\Psi_j^{(i)} \otimes \Omega_p\big) \in \mathcal{H}_\beta$. For $0 < \alpha < 1/2$ and $j \in \mathbb{N}$ this implies that $\big(\Gamma_j^{(i)}\,,\, \Delta_{p,i}^{2\alpha}\Gamma_j^{(i)}\big) = \big(\Psi_j\,,\, \Delta^{2\alpha}\Psi_j\big)(\Omega_p\,,\,\Omega_p) > 0$. Since $s - \lim_{i\to\infty} E_{\Lambda_i} = \mathbb{1}$, it follows that

$$\liminf_i \mathrm{Tr}\,\big(E_{\Lambda_i}\Delta_{p,i}^{\alpha}E_{\Lambda_i}\big)^2 \geq \liminf_i \sum_{j=1}^{\infty}\Big(E_{\Lambda_i}\Gamma_j^{(i)}\,,\,\Delta_{p,i}^{\alpha}E_{\Lambda_i}\Delta_{p,i}^{\alpha}E_{\Lambda_i}\Gamma_j^{(i)}\Big)$$

$$= \liminf_i \sum_{j=1}^{\infty}\Big(\Delta_{p,i}^{\alpha}\Gamma_j^{(i)}\,,\,E_{\Lambda_i}\Delta_{p,i}^{\alpha}\Gamma_j^{(i)}\Big) = \liminf_i \sum_{j=1}^{\infty}(\Psi_j\,,\,\Delta^{2\alpha}\Psi_j) > 0. \qquad \square$$

4.3 Commutator estimates

The unit ball in $\mathcal{A}^*$ is weak*-compact. Thus for every net of states $\Lambda(\mathcal{O}_\circ, \mathcal{O}, \hat{\mathcal{O}}) \to \omega_\Lambda$ there exists a subnet of $\{\omega_{\Lambda_i}\}_{i\in I}$ converging to some state ω. Whether or not this state is a $(\tau, n\alpha\beta)$-KMS state depends on the energy contained in the boundary, i.e., the choice of the relative size of $\mathcal{O}_\circ^{(i)}$, $\mathcal{O}^{(i)}$ and $\hat{\mathcal{O}}^{(i)}$. We show that the necessary quantitative information restricting the surface energy can be drawn from the bounds on the nuclear norm of the map $\Theta_{\alpha,\mathcal{O}}$ introduced in (2) and the cluster condition (3).

Let $\{\Lambda_i = (\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}, \hat{\mathcal{O}}^{(i)})\}_{i\in\mathbb{N}}$ be a sequence[5] of triples of double cones with diameters $(r_\circ^{(i)}, r^{(i)}, \hat{r}^{(i)})$, $i \in \mathbb{N}$. We will now exploit the fact that the elements of $\mathcal{A}_p$, $p \in \mathbb{N}$, introduced at the end of Subsection 2.2, have good localization properties in space-time: we will show that there exists some $p \in \mathbb{N}$ such that

$$\left|\omega_{\Lambda_i}\big(a\tau_{in\alpha\beta}(b)\big) - \omega_{\Lambda_i}(ba)\right| < \epsilon_i \qquad \forall a, b \in \mathcal{A}_p, \tag{22}$$

where $\epsilon_i \searrow 0$ as $i \to \infty$. Thus the surface energy can be controlled by adjusting the relative size of $r_\circ^{(i)}$, $r^{(i)}$ and $\hat{r}^{(i)}$.

Inspecting the definition (18) of ω_{Λ_i} we recognize that in order to prove (22) it is sufficient to control

$$\mathrm{Tr}\ \rho_{\Lambda_i}\ \pi_\beta(a)\big[\pi_\beta\big(\tau_{ik\alpha\beta}(b)\big), E_{\Lambda_i}\big], \qquad k = 1, \ldots, n.$$

Let us consider the case $n = 2$. Let $a, b \in \mathcal{A}_p$, $p \in \mathbb{N}$ fixed. It follows that $\tau_{ik\alpha\beta}(b) \in \mathcal{A}_p$ for $k = 1, 2$. Since a and b as well as $c := \tau_{i\alpha\beta}(b)$ and $d := \tau_{2i\alpha\beta}(b)$ are almost localized in $\mathcal{O}_\circ^{(i)}$ for i sufficiently large, they almost commute with E_{Λ_i}. For example,

$$\begin{aligned}
&\left|\mathrm{Tr}\ \rho_{\Lambda_i}\ \pi_\beta(a)\big[\pi_\beta\big(\tau_{i2\alpha\beta}(b)\big), E_{\Lambda_i}\big]\right| \\
&\quad = \frac{\left|\mathrm{Tr}\,[\pi_\beta\big(\tau_{i2\alpha\beta}(b)\big), E_{\Lambda_i}] \cdot \big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2 \pi_\beta(a)\right|}{\mathrm{Tr}\,\big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2} \\
&\quad \leq \frac{\|\,[\pi_\beta\big(\tau_{i2\alpha\beta}(b) - d_i\big), E_{\Lambda_i}]\,\| \cdot \mathrm{Tr}\,|\big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2| \cdot \|a\|}{\mathrm{Tr}\,\big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2} \\
&\quad \leq \frac{2\|a\|}{\mathrm{Tr}\,\big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}\, \|\tau_{i2\alpha\beta}(b) - d_i\| \cdot \big(\mathrm{Tr}\,|\Delta^\alpha_{p,i}E_{\Lambda_i}|\big)^2.
\end{aligned}$$

Here $d_i \in \mathcal{A}(\mathcal{O}_\circ^{(i)})$ denotes a local approximation of $d := \tau_{i2\alpha\beta}(b) \in \mathcal{A}_p$ which satisfies $[E_{\Lambda_i}, d_i] = 0$. Thus

$$\left|\mathrm{Tr}\ \rho_{\Lambda_i}\ \pi_\beta(a)\big[\pi_\beta\big(\tau_{i2\alpha\beta}(b)\big), E_{\Lambda_i}\big]\right| \leq \frac{c_1}{\mathrm{Tr}\,\big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2} \cdot \mathrm{e}^{-c_2 (r_\circ^{(i)})^{2p}} \cdot \mathrm{e}^{c_3 (r^{(i)})^d} \tag{23}$$

[5]Note that it is sufficient to work with sequences if the operators a and b appearing in (22) are fixed.

for certain positive constants $c_1 = 2C\,\|a\|$, $c_2 = \kappa/(2^{2p})$ and $c_3 = 2c\big(\alpha^{-m} + (1/2 - \alpha)^{-m}\big)$, where $m > 0$. In the last inequality we made use of Proposition 3.3 and the second part of Lemma 2.1. Inspecting the r.h.s. of (23) closely, we find that the numerator vanishes in the limit $i \to \infty$, if $\exp\big(-c_2(r_\circ^{(i)})^{2p}\big) \cdot \exp\big(c_3(r^{(i)})^d\big)$ goes to zero as $i \to \infty$. As has been shown in the previous section, the denominator does not vanish as $i \to \infty$, but is bounded from below by some positive constant, if $\lim_{i\to\infty}(r_\circ^{(i)})^{d'}(\delta_\circ^{(i)})^{-\gamma} = 0$.

In other words, the distance $\delta_\circ^{(i)}$ has to grow sufficiently fast such that $\eta_i \to \Omega_\beta$, and p has to be chosen sufficiently large such that the elements in $\mathcal{A}_p$ are sufficiently well localized to fulfill the boundary condition (17) up to some small error term.

We will now establish the KMS property for all weak limit points of $\{\omega_\Lambda\}$, provided the regions $\Lambda_i = \big(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}, \hat{\mathcal{O}}^{(i)}\big)$ tend to the whole space-time in agreement with the restrictions imposed on the relative size of $r_\circ^{(i)}$, $r^{(i)}$ and $\hat{r}^{(i)}$.

Theorem 4.5. *Assume that both the nuclearity condition* (2) *and the cluster condition* (3) *hold. Then there exists a choice of triples of space-time regions Λ_i such that every weak limit point of the (generalized) sequence $\{\omega_{\Lambda_i}\}_{i\in I}$ is a τ-KMS state at temperature $1/\beta' > 0$.*

Proof. Let $n \in \mathbb{N}$ and $0 < \alpha < 1/2$ be fixed such that $\beta' = n\alpha\beta$. Moreover, let $\Lambda_i = \big(\mathcal{O}_\circ^{(i)}, \mathcal{O}^{(i)}, \hat{\mathcal{O}}^{(i)}\big)$ be a sequence of triples of double cones with diameters $r_\circ^{(i)}$, $r^{(i)}$ and $\hat{r}^{(i)}$ such that $\lim_{i\to\infty}\big(r_\circ^{(i)}\big)^{d'}\big(\delta_\circ^{(i)}\big)^{-\gamma} = 0$ and $\hat{\mathcal{O}}^{(i)} \searrow \mathcal{O}^{(i)}$ sufficiently fast as $i \to \infty$ such that $\lim_{i\to\infty}\|\eta_i - \Omega_\beta\| = 0$.

Let us recall: the nuclearity condition fixes the constants d and m and the cluster condition fixes the constants d' and γ. We will now fix $p \in \mathbb{N}$. Taking into account the restrictions on the relative size of $r_\circ^{(i)}$ and $r^{(i)} = r_\circ^{(i)} + 2\delta_\circ^{(i)}$ imposed by the cluster condition (3) – it is sufficient that $\big(r_\circ^{(i)}\big)^{d'}\big(\delta_\circ^{(i)}\big)^{-\gamma}$ goes to zero as i goes to infinity – we can now chose p such that $\exp\big(-c_2(r_\circ^{(i)})^{2p}\big) \cdot \exp\big(c_3(r^{(i)})^d\big)$ goes to zero as $i \to \infty$.

Let $a, b \in \mathcal{A}_p$ and consider the case $n = 2$.

i) Let $\omega_{2\alpha\beta}$ denote the limit state of a convergent subnet $\{\omega_{\Lambda_i}\}_{i\in I}$. For every $\epsilon > 0$ we can find an index $i \in I$ such that

$$\big|\omega_{2\alpha\beta}\big(a\tau_{i2\alpha\beta}(b) - ba\big)\big| \le \big|\omega_{\Lambda_i}\big(a\tau_{i2\alpha\beta}(b) - ba\big)\big| + \epsilon.$$

ii) We now approximate $\tau_{i2\alpha\beta}(b), \tau_{i\alpha\beta}(b)$ and b by local elements in $\mathcal{A}\big(\mathcal{O}_\circ^{(i)}\big)$ and apply the commutator estimate (23) several times: for suitable (large) $i \in \mathbb{N}$

we find

$$
\begin{aligned}
&\big|\omega_{2\alpha\beta}\big(a\tau_{i2\alpha\beta}(b)-ba\big)\big| \\
&\qquad\le \Big|\frac{\mathrm{Tr}\ \pi_\beta\big(a\tau_{i2\alpha\beta}(b)\big)\big(E_{\Lambda_i}\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}{\mathrm{Tr}\ \big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}-\omega_{\Lambda_i}(ba)\Big|+\epsilon \\
&\qquad\le \Big|\frac{\mathrm{Tr}\ \pi_\beta(a)E_{\Lambda_i}\pi_\beta\big(\tau_{i2\alpha\beta}(b)\big)\big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}{\mathrm{Tr}\ \big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}-\omega_{\Lambda_i}(ba)\Big|+2\epsilon \\
&\qquad= \Big|\frac{\mathrm{Tr}\ \pi_\beta(a)E_{\Lambda_i}\Delta^\alpha_{p,i}\pi_\beta\big(\tau_{i\alpha\beta}(b)\big)E_{\Lambda_i}\Delta^\alpha_{p,i}E_{\Lambda_i}}{\mathrm{Tr}\ \big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}-\omega_{\Lambda_i}(ba)\Big|+2\epsilon \\
&\qquad\le \Big|\frac{\mathrm{Tr}\ \pi_\beta(a)\big(E_{\Lambda_i}\Delta^\alpha_{p,i}\big)^2\pi_\beta(b)E_{\Lambda_i}}{\mathrm{Tr}\ \big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}-\omega_{\Lambda_i}(ba)\Big|+3\epsilon \\
&\qquad\le \Big|\frac{\mathrm{Tr}\ \pi_\beta(a)\big(E_{\Lambda_i}\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2\pi_\beta(b)}{\mathrm{Tr}\ \big(\Delta^\alpha_{p,i}E_{\Lambda_i}\big)^2}-\omega_{\Lambda_i}(ba)\Big|+4\epsilon \\
&\qquad= 4\epsilon.
\end{aligned}
$$

Thus $\omega_{2\alpha\beta}\big(a\tau_{2i\alpha\beta}(b)\big)=\omega_{2\alpha\beta}(ba)$ for all $a,b\in\mathcal{A}_p$. Now recall that $\mathcal{A}_p$ (for each $p\in\mathbb{N}$) is a τ-invariant $*$-subalgebra of the set $\mathcal{A}_\tau$ of analytic elements of $\mathcal{A}$ with respect to τ. Consequently, $\omega_{2\alpha\beta}$ is a $(\tau,2\alpha\beta)$-KMS state.

Similar results for arbitrary $n\in\mathbb{N}$ can be established by the same line of arguments but with considerable more effort. □

Once we have constructed a (τ,β')-KMS state $\omega_{\beta'}$, the GNS-representation $\pi_{\beta'}$ leads to a new thermal field theory

$$\mathcal{O}\to\mathcal{R}_{\beta'}(\mathcal{O}):=\pi_{\beta'}\big(\mathcal{A}(\mathcal{O})\big)'',\qquad \mathcal{O}\in\mathbb{R}^4,$$

acting on a new Hilbert space $\mathcal{H}_{\beta'}$ with GNS-vector $\Omega_{\beta'}$. If $\beta\neq\beta'$, then the new thermal field theory will not be unitarily equivalent to the old one [T]. In fact, there might even be several extremal $(\tau,n\alpha\beta)$-KMS states, which induce unitarily inequivalent representations, i.e., "disjoint thermal field theories", at the same temperature $1/\beta'=(n\alpha\beta)^{-1}$.

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