# Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

272



K. R. Parthasarathy
University of Bombay, Bombay/India
K. Schmidt
Bedford College, University of London, London/England


# Positive Definite Kernels, Continuous Tensor Products, and Central Limit Theorems of Probability Theory


Springer-Verlag
Berlin · Heidelberg · New York 1972

AMS Subject Classifications (1970): 43 A 05, 43 A 35, 43 A 65, 46 C 10, 60 B 15, 60 F 05

ISBN 3-540-05908-3 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-05908-3 Springer-Verlag New York · Heidelberg · Berlin



Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

# PREFACE

These notes are mainly based on a course of lectures given by the first named author at the Research and Training School of the Indian Statistical Institute, Calcutta during May 1971. A first and slightly shorter version of these notes has appeared under the same title as publication No. M71-1 of the Research and Training School of the Indian Statistical Institute. Some of the results were obtained when the authors were at the Statistical Laboratory, Mathematics Department, University of Manchester, in 1970.

The notion of a continuous tensor product of Hilbert spaces and group representations appears in the work of H. Araki [2], H. Araki and E.J. Woods [1] and R.F. Streater [19]. Their analysis leads to a connection between continuous tensor products and the theory of infinitely divisible distributions of Probability theory. The present work contains a systematic study of these notions in terms of positive definite kernels with invariance properties under a group action. Such analysis also leads to a unified approach to the central limit problems of Probability theory, the theory of stochastic processes with stationary increments and construction of free fields in Quantum Mechanics.

The contents of these notes are divided into three parts. In part 1 the notion of a projectively invariant positive definite kernel on an abstract G-space is introduced and obtained as expectation value of a projective unitary representation of the group G. Affine invariant conditionally positive definite kernels are investigated and a representation of such kernels in terms of unitary representations and first order cocycles is obtained. Using these ideas and the theory of multiplicative measures, continuous tensor products of Hilbert spaces and representations are constructed. The Fock-Cook construction of the Bose-Einstein field in Quantum Mechanics [4] , [18] arises as a natural consequence of this theory. Much of the inspiration for this approach was derived from the work of H. Araki [2] .

Part 2 analysis limits of products of uniformly infinitesimal families of positive definite kernels. This leads in particular to the limit laws in the theory of sums of independent random variables.

The third part contains results about first order cocycles for various classes of representations and on some special groups. In particular it is shown that first order cocycles of induced representations arise from the cocycles of the inducing representation. As corollary to this, cocycles of irreducible representations of nilpotent Lie groups are described. Cocycles of representations of semisimple Lie groups which are induced by characters of maximal solvable subgroups are also obtained.

A short list of references including most of the major contributions to these problems is included.

The first named author would like to thank the Research and Training School of the Indian Statistical Institute and the University of Bombay for providing him facilities for writing these notes. For the same reason the second author would like to express his gratitude to the Department of Mathematics, Bedford College, University of London.

K.R. Parthasarathy
K. Schmidt

CONTENTS

Part I:

Positive definite kernels and continuous tensor products . . . . 1

§ 1. Positive definite and conditionally positive definite kernels . . . . . . . . . . . . . . . . . . . . . . . . . 1

§ 2. Projectively invariant positive definite kernels. . . . . . 5

§ 3. Affine invariant conditionally positive definite kernels. . 10

§ 4. Multiplicative measures . . . . . . . . . . . . . . . . . . 13

§ 5. Factorisable families of projectively invariant positive definite kernels. . . . . . . . . . . . . . . . . . . . . 17

§ 6. Representations of current groups . . . . . . . . . . . . . 22

§ 7. Continuous tensor products of Hilbert spaces and the Araki-Woods imbedding theorem . . . . . . . . . . . . . . 26

§ 8. The Bose-Einstein field of quantum mechanics. . . . . . . . 46

Part II:

Limit theorems for uniformly infinitesimal families of positive definite kernels . . . . . . . . . . . . . . . . . . . . . . . 48

§ 9. Uniformly infinitesimal families of positive definite kernels . . . . . . . . . . . . . . . . . . . . . . . . . 48

§10. Uniformly infinitesimal families of projectively invariant positive definite kernels . . . . . . . . . . . . . . . . . 55

§11. Uniformly infinitesimal families of continuous normalised positive definite functions . . . . . . . . . . . . . . . . 59

§12. Infinitely divisible positive definite functions. . . . . . 76

Part III:
The analysis of cocycles of first order on some special groups. . 83

§13. Cocycles of direct integrals of representations. . . . . . . 83

§14. Cocycles of induced representations. . . . . . . . . . . . . 86

§15. Cocycles of compact groups . . . . . . . . . . . . . . . . . 92

§16. Cocycles of abelian groups . . . . . . . . . . . . . . . . . 93

§17. Cocycles of nilpotent Lie groups . . . . . . . . . . . . . . 100

§18. Cocycles of semi-simple Lie groups . . . . . . . . . . . . . 102

References. . . . . . . . . . . . . . . . . . . . . . . . . . . 105