INTERIOR POINT METHODS FOR LINEAR OPTIMIZATION

Revised Edition

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By

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Dedicated to our wives

Gerda, Gabriella and Marie

and our children

Jacoline, Geranda, Marijn

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Preface

Linear Optimization[1] (LO) is one of the most widely taught and applied mathematical techniques. Due to revolutionary developments both in computer technology and algorithms for linear optimization, 'the last ten years have seen an estimated six orders of magnitude speed improvement'.[2] This means that problems that could not be solved 10 years ago, due to a required computational time of one year, say, can now be solved within some minutes. For example, linear models of airline crew scheduling problems with as many as 13 million variables have recently been solved within three minutes on a four-processor Silicon Graphics Power Challenge workstation. The achieved acceleration is due partly to advances in computer technology and for a significant part also to the developments in the field of so-called *interior-point methods* for linear optimization.

Until very recently, the method of choice for solving linear optimization problems was the Simplex Method of Dantzig [59]. Since the initial formulation in 1947, this method has been constantly improved. It is generally recognized to be very robust and efficient and it is routinely used to solve problems in Operations Research, Business, Economics and Engineering. In an effort to explain the remarkable efficiency of the Simplex Method, people strived to prove, using the theory of complexity, that the computational effort to solve a linear optimization problem via the Simplex Method is polynomially bounded with the size of the problem instance. This question is still unsettled today, but it stimulated two important proposals of new algorithms for LO. The first one is due to Khachiyan in 1979 [167]: it is based on the ellipsoid technique for nonlinear optimization of Shor [255]. With this technique, Khachiyan proved that LO belongs to the class of polynomially solvable problems. Although this result has had a great theoretical impact, the new algorithm failed to deliver its promises in actual computational efficiency. The second proposal was made in 1984 by Karmarkar [165]. Karmarkar's algorithm is also polynomial, with a better complexity bound

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1 The field of Linear Optimization has been given the name *Linear Programming* in the past. The origin of this name goes back to the Dutch Nobel prize winner Koopmans. See Dantzig [60]. Nowadays the word 'programming' usually refers to the activity of writing computer programs, and as a consequence its use instead of the more natural word 'optimization' gives rise to confusion. Following others, like Padberg [230], we prefer to use the name *Linear Optimization* in the book. It may be noted that in the nonlinear branches of the field of Mathematical Programming (like *Combinatorial Optimization*, *Discrete Optimization*, *Semidefinite Optimization*, etc.) this terminology has already become generally accepted.

2 This claim is due to R.E. Bixby, professor of Computational and Applied Mathematics at Rice University, and director of CPLEX Optimization, Inc., a company that markets algorithms for linear and mixed-integer optimization. See the news bulletin of the Center For Research on Parallel Computation, Volume 4, Issue 1, Winter 1996. Bixby adds that parallelization may lead to 'at least eight orders of magnitude improvement—the difference between a year and a fraction of a second!'

than Khachiyan, but it has the further advantage of being highly efficient in practice. After an initial controversy it has been established that for very large, sparse problems, subsequent variants of Karmarkar's method often outperform the Simplex Method.

Though the field of LO was considered more or less mature some ten years ago, after Karmarkar's paper it suddenly surfaced as one of the most active areas of research in optimization. In the period 1984–1989 more than 1300 papers were published on the subject, which became known as Interior Point Methods (IPMs) for LO.[3] Originally the aim of the research was to get a better understanding of the so-called Projective Method of Karmarkar. Soon it became apparent that this method was related to classical methods like the Affine Scaling Method of Dikin [63, 64, 65], the Logarithmic Barrier Method of Frisch [86, 87, 88] and the Center Method of Huard [148, 149], and that the last two methods could also be proved to be polynomial. Moreover, it turned out that the IPM approach to LO has a natural generalization to the related field of convex nonlinear optimization, which resulted in a new stream of research and an excellent monograph of Nesterov and Nemirovski [226]. Promising numerical performances of IPMs for convex optimization were recently reported by Breitfeld and Shanno [50] and Jarre, Kocvara and Zowe [162]. The monograph of Nesterov and Nemirovski opened the way into another new subfield of optimization, called Semidefinite Optimization, with important applications in System Theory, Discrete Optimization, and many other areas. For a survey of these developments the reader may consult Vandenberghe and Boyd [48].

As a consequence of the above developments, there are now profound reasons why people may want to learn about IPMs. We hope that this book answers the need of professors who want to teach their students the principles of IPMs, of colleagues who need a unified presentation of a desperately burgeoning field, of users of LO who want to understand what is behind the new IPM solvers in commercial codes (CPLEX, OSL, ...) and how to interpret results from those codes, and of other users who want to exploit the new algorithms as part of a more general software toolbox in optimization.

Let us briefly indicate here what the book offers, and what does it not. Part I contains a small but complete and self-contained introduction to LO. We deal with the duality theory for LO and we present a first polynomial method for solving an LO problem. We also present an elegant method for the initialization of the method, using the so-called self-dual embedding technique. Then in Part II we present a comprehensive treatment of Logarithmic Barrier Methods. These methods are applied to the LO problem in standard format, the format that has become most popular in the field because the Simplex Method was originally devised for that format. This part contains the basic elements for the design of efficient algorithms for LO. Several types of algorithm are considered and analyzed. Very often the analysis improves the existing analysis and leads to sharper complexity bounds than known in the literature. In Part III we deal with the so-called Target-following Approach to IPMs. This is a unifying framework that enables us to treat many other IPMs, like the Center Method, in an easy way. Part IV covers some additional topics. It starts with the description and analysis of the Projective Method of Karmarkar. Then we discuss some more

[3] We refer the reader to the extensive bibliography of Kranich [179, 180] for a survey of the literature on the subject until 1989. A more recent (annotated) bibliography was given by Roos and Terlaky [242]. A valuable source of information is the World Wide Web interior point archive: `http://www.mcs.anl.gov/home/otc/InteriorPoint.archive.html`.

interesting theoretical properties of the central path. We also discuss two interesting methods to enhance the efficiency of IPMs, namely Partial Updating, and so-called Higher-Order Methods. This part also contains chapters on parametric and sensitivity analysis and on computational aspects of IPMs.

It may be clear from this description that we restrict ourselves to Linear Optimization in this book. We do not dwell on such interesting subjects as Convex Optimization and Semidefinite Optimization, but we consider the book as a preparation for the study of IPMs for these types of optimization problem, and refer the reader to the existing literature.[4]

Some popular topics in IPMs for LO are not covered by the book. For example, we do not treat the (Primal) Affine Scaling Method of Dikin.[5] The reason for this is that we restrict ourselves in this book to polynomial methods and until now the polynomiality question for the (Primal) Affine Scaling Method is unsettled. Instead we describe in Appendix E a primal-dual version of Dikin's affine-scaling method that is polynomial. Chapter 18 describes a higher-order version of this primal-dual affine-scaling method that has the best possible complexity bound known until now for interior-point methods.

Another topic not touched in the book is (Primal-Dual) Infeasible Start Methods. These methods, which have drawn a lot of attention in the last years, deal with the situation when no feasible starting point is available.[6] In fact, Part I of the book provides a much more elegant solution to this problem; there we show that any given LO problem can be embedded in a self-dual problem for which a feasible interior starting point is known. Further, the approach in Part I is theoretically more efficient than using an Infeasible Start Method, and from a computational point of view is not more involved, as we show in Chapter 20.

We hope that the book will be useful to students, users and researchers, inside and outside the field, in offering them, under a single cover, a presentation of the most successful ideas in interior-point methods.

Kees Roos
Tamás Terlaky
Jean-Philippe Vial

Preface to the 2005 edition

Twenty years after Karmarkar's [165] epoch making paper *interior point methods (IPMs)* made their way to all areas of optimization theory and practice. The theory of IPMs matured, their professional software implementations significantly pushed the boundary of efficiently solvable problems. Eight years passed since the first edition of this book was published. In these years the theory of IPMs further crystallized. One of the notable developments is that the significance of the self-dual embedding

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[4] For Convex Optimization the reader may consult den Hertog [140], Nesterov and Nemirovski [226] and Jarre [161]. For Semidefinite Optimization we refer to Nesterov and Nemirovski [226], Vandenberghe and Boyd [48] and Ramana and Pardalos [236]. We also mention Shanno and Breitfeld and Simantiraki [252] for the related topic of barrier methods for nonlinear programming.

[5] A recent survey on affine scaling methods was given by Tsuchiya [272].

[6] We refer the reader to, e.g., Potra [235], Bonnans and Potra [45], Wright [295, 297], Wright and Ralph [296] and the recent book of Wright [298].

model –that is a distinctive feature of this book– got fully recognized. Leading linear and conic-linear optimization software packages, such as MOSEK[7] and SeDuMi[8] are developed on the bedrock of the self-dual model, and the leading commercial linear optimization package CPLEX[9] includes the embedding model as a proposed option to solve difficult practical problems.

This new edition of this book features a completely rewritten first part. While keeping the simplicity of the presentation and accessibility of complexity analysis, the featured IPM in Part I is now a standard, primal-dual path-following Newton algorithm. This choice allows us to reach the so-far best known complexity result in an elementary way, immediately in the first part of the book.

As always, the authors had to make choices when and how to cut the expansion of the material of the book, and which new results to include in this edition. We cannot resist mentioning two developments after the publication of the first edition.

The first development can be considered as a direct consequence of the approach taken in the book. In our approach properties of the univariate function $\psi(t)$, as defined in Section 5.5 (page 92), play a key role. The book makes clear that the primal-, dual- and primal-dual logarithmic barrier function can be defined in terms of $\psi(t)$, and as such $\psi(t)$ is at the heart of all logarithmic barrier functions; we call it now the kernel function of the logarithmic barrier function. After the completion of the book it became clear that more efficient large-update IPMs than those considered in this book, which are all based on the logarithmic barrier function, can be obtained simply by replacing $\psi(t)$ by other kernel functions. A large class of such kernel functions, that allowed to improve the worst case complexity of large-update IPMs, is the family of self-regular functions, which is the subject of the monograph [233]; more kernel functions were considered in [32].

A second, more recent development, deals with the complexity of IPMs. Until now, the best iteration bound for IPMs is $O(\sqrt{n}L)$, where n denotes the dimension of the problem (in standard from), and L the binary input size of the problem. In 1996, Todd and Ye showed that $O(\sqrt[3]{n}L)$ is a lower bound for the iteration complexity of IPMs [267]. It is well known that the iteration complexity highly depends on the curliness of the central path, and that the presence of redundancy may severely affect this curliness. Deza et al. [61] showed that by adding enough redundant constraints to the Klee-Minty example of dimension n, the central path may be forced to visit all 2^n vertices of the Klee-Minty cube. An enhanced version of the same example, where the number of inequalities is $N = O(2^{2n}n^3)$, yields an $O(\sqrt{N}/\log N)$ lower bound for the iteration complexity, thus almost closing (up to a factor of $\log N$) the gap with the best worst case iteration bound for IPMs [62].

Instructors adapting the book as textbook in a course may contact the authors at <terlaky@mcmaster.ca> for obtaining the "Solution Manual" for the exercises and getting access to a user forum.

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[7] MOSEK: `http://www.mosek.com`

[8] SeDuMi: `http://sedumi.mcmaster.ca`

[9] CPLEX: `http://cplex.com`

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To reach a unified presentation of many diverse results, it did not suffice to make a bundle of existing papers. It was necessary to recast completely the form in which these results found their way into the journals. This was a very time-consuming task: we want to thank our universities for giving us the opportunity to do this job.

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