Quadratic hedging in affine stochastic volatility models

Jan Kallsen Richard Vierthauer
Christian-Albrechts-Universität zu Kiel*

Abstract

We determine the variance-optimal hedge for a subset of affine processes including a number of popular stochastic volatility models. This framework does not require the asset to be a martingale. We obtain semiexplicit formulas for the optimal hedging strategy and the minimal hedging error by applying general structural results and Laplace transform techniques. The approach is illustrated numerically for a Lévy-driven stochastic volatility model with jumps as in Carr et al. (2003).

Key words: Mean-variance hedging, affine processes, stochastic volatility, Laplace transform

1 Introduction

A classical problem in Mathematical Finance is how to hedge the risk from selling a contingent claim. Since perfect hedging strategies do not exist in incomplete markets, we focus on variance-optimal hedging strategies in this paper. This concept has been studied intensively in the literature (cf. [28, 26] for an overview). The idea is to choose an initial endowment w^* and a self-financing strategy ϑ^* such that the expected squared hedging error

$$R(w, \vartheta) := E\left(\left(w + \int_0^T \vartheta(t)dS(t) - H\right)^2\right)$$

is minimized over all such endowments w and strategies ϑ. Here, S denotes the discounted price process of a stock, T the time horizon, and H the discounted payoff of the option that is to be hedged. The minimal hedging error $R(w^*, \vartheta^*)$ quantifies the residual risk that cannot be avoided. It may enter the premium that the issuer charges the buyer.

*Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Christian-Albrechts-Platz 4, 24098 Kiel, Germany (e-mail: kallsen@math.uni-kiel.de, vierthauer@math.uni-kiel.de)

We consider the above problem for European-style options in stochastic volatility models of affine structure. This class generalizes Lévy processes and includes e.g. the Heston [15] model. Another example is the stochastic volatility model of Barndorff-Nielsen and Shephard [1, henceforth BNS], which is of the form $S = \exp(z)$ with

$$\begin{aligned} dz(t) &= \mu v(t-)dt + \sqrt{v(t-)}dW(t), \\ dv(t) &= -\lambda v(t-)dt + dr(t). \end{aligned} \tag{1.1}$$

Here, μ, λ denote constants, W a Wiener process and r a subordinator, i.e. an increasing Lévy process. A generalization of Carr et al. [2] allows for jumps in the stock price as well.

This paper rests on several pillars. One main ingredient is a general characterization of the variance-optimal hedge in [4]. Moreover, results of [9, 10] on affine processes are used on the way to concrete solutions. These cannot be expressed in closed form. But using the integral transform approach of [16] or similarly [3], we can derive semiexplicit representations which allow for straightforward numerical implementation. The problem has been attacked in [5] for the Heston model and in [25, 22, 21] for the case that the discounted stock is a martingale. A different approach is taken in [14] and [7], which rely on partial differential equation and partly simulation methods.

The structure of the paper is as follows. We start by introducing the general affine setup and the mean-variance hedging problem. Section 3 contains the solution in integral form. It is applied to the time-changed Lévy process model of [2] in the subsequent section. The final Section 5 provides numerical illustration. The appendix contains proofs and some results on the calculus of semimartingale characteristics, which constitutes the main technical tool of this paper.

Unexplained notation is used as in [17, 4]. Mathematical formalism is treated liberally in this study. E.g. we do not state and verify technical conditions concerning interchanging the order of integration in Fubinis theorem, uniform integrability of local martingales, admissibility of trading strategies, existence of analytic extensions of complex functions etc. Arguments and conditions of this type have been worked out in related setups by [16, 25, 22, 21, 13].

2 General setup

We generally work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ with fixed time horizon $T \in \mathbb{R}_+$. The initial σ-field $\mathscr{F}_0$ is assumed to be trivial. The discounted stock price S is supposed to be of the form

$$S(t) = e^{z(t)},$$

where the *return process* z is a component of a bivariate affine semimartingale $X = (v, z)$ in a sense made precise below. The second component v plays the role of stochastic volatility or, more accurately, *stochastic activity* in the model.

The key tool in this paper is the concept of semimartingale characteristics (cf. [17, 19] for details). We call a predictable triplet (b, c, F) *local* or *differential characteristics* of an $\mathbb{R}^d$-valued special semimartingale X if the triplet (B, C, ν) defined via

$$
\begin{aligned}
B(t) &:= \int_0^t b(s)ds, \\
C(t) &:= \int_0^t c(s)ds, \\
\nu([0,t] \times A) &:= \int_0^t F(s, A)ds \text{ for } t \in [0, T], A \in \mathscr{B}^d
\end{aligned}
$$

constitute semimartingale characteristics in the sense of [17] for $h(x) := x$. This in turn means that $X - B$ is a local martingale, $C = \langle X^c, X^c \rangle$ is the matrix-valued quadratic variation process of the continuous martingale part of X, and ν is the compensator of the random measure of jumps of X. On an intuitive level, b stands for the local drift rate of X, c for the local covariance matrix of the continuous part, and F for the local Lévy measure of jumps. For a Lévy process the triplet (b, c, F) is deterministic and constant. It coincides with the Lévy-Khintchine triplet of X. For Itô processes

$$
dX(t) = \mu(t)dt + \sigma(t)dW(t)
$$

we have $b(t) = \mu(t)$, $c(t) = \sigma(t)^2$ and $F = 0$. Some rules on the calculus of local characteristics are summarized in the appendix.

Recall that $z = \log(S)$ denotes the return process and v some kind of activity process that needs to be specified further. From now on we assume that $X = (v, z)$ is an $\mathbb{R}_+ \times \mathbb{R}$-valued special semimartingale having local characteristics (b, c, F) of the form

$$
\begin{aligned}
b(t) &= \beta_0(t) + v(t-)\beta_1(t), && (2.1) \\
c(t) &= \gamma_0(t) + v(t-)\gamma_1(t), && (2.2) \\
F(t, dx) &= \varphi_0(t, dx) + v(t-)\varphi_1(t, dx) && (2.3)
\end{aligned}
$$

with deterministic Lévy-Khintchine triplets $(\beta_0, \gamma_0, \varphi_0), (\beta_1, \gamma_1, \varphi_1)$ which are continuous in t in the sense of [10]. (2.1–2.3) imply that X is a (*time-inhomogeneous*) *affine Markov process* in the sense of [10]. We will mostly focus on the *time-homogeneous* case where $(\beta_0, \gamma_0, \varphi_0), (\beta_1, \gamma_1, \varphi_1)$ do not depend on t. Time-homogeneous affine processes are discussed in [9].

Our results below are expressed in terms of the *Lévy exponents*

$$
\psi_i^X(t, u) := u^\top \beta_i(t) + \frac{1}{2} u^\top \gamma_i(t) u + \int \left(e^{u^\top x} - 1 - u^\top x \right) \varphi_i(t, dx)
$$

for $i = 0, 1,\ t \in [0, T]$, and $u \in \mathbb{C}^2$ such that the integral exists, i.e. for

$$
(t, u) \in D_i := \left\{ (\tilde{t}, \tilde{u}) \in [0, T] \times \mathbb{C}^2 : \int_{|x| \geq 1} e^{\mathrm{Re}(\tilde{u})^\top x} \varphi_i(\tilde{t}, dx) < \infty \right\}.
$$

We call

$$\psi^X(t,u) := \psi_0^X(t,u) + v(t-)\psi_1^X(t,u)$$

Lévy exponent of X, which is random due to the presence of $v(t-)$. If X is time-homogeneous, the argument t in ψ_0^X, ψ_1^X etc. will be omitted.

The solution to the quadratic hedging problem will be expressed in terms of

$$\begin{aligned}\kappa_i(t,x,y_1,y_2,\hat{y}_1,\hat{y}_2) &:= \psi_i^X(t,x+y_1+y_2,\hat{y}_1+\hat{y}_2) - \psi_i^X(t,x+y_1,\hat{y}_1) \\ &\quad - \psi_i^X(t,x+y_2,\hat{y}_2) + \psi_i^X(t,x,0),\end{aligned}$$

for $i=0,1$ and $t\in[0,T]$ and $x,y_1,y_2,\hat{y}_1,\hat{y}_2\in\mathbb{C}$ such that the Lévy exponents exist.

Example 2.1 The BNS model (1.1) is time-homogeneous. The associated Lévy-Khintchine triplets $(\beta_i,\gamma_i,\varphi_i), i=0,1$, are given by

$$\begin{aligned}\beta_0 = \begin{pmatrix} b^r \\ 0 \end{pmatrix}, \ \gamma_0 &= 0, \quad \varphi_0(A) = \int 1_A(x,0)F^r(dx) \quad \forall A\in\mathscr{B}^2, \\ (\beta_1,\gamma_1,\varphi_1) &= \left(\begin{pmatrix} -\lambda \\ \mu \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0\right),\end{aligned}$$

where $(b^r,0,F^r)$ denotes the Lévy-Khintchine triplet of r, cf. [19, Section 4.3]. The corresponding Lévy exponents are

$$\begin{aligned}\psi_0^X(u_1,u_2) &= b^r u_1 + \int (e^{u_1x} - 1 - u_1x)\,F^r(dx), \\ \psi_1^X(u_1,u_2) &= -\lambda u_1 + \mu u_2 + \frac{1}{2}u_2^2.\end{aligned}$$

In order to solve the hedging problem for S, we need the following additional assumptions:

1. $\psi_0^X(t,0,u_2)=0$ for any $(t,u_2)\in[0,T]\times i\mathbb{R}$. This condition means that the local dynamics of z depend in a linear (rather than an affine) fashion on $v(t-)$. This restriction is imposed to obtain semiexplicit solutions in the case where S is not a martingale. It is not needed if S is a martingale, cf. [22, 21].

2. S is a locally square-integrable semimartingale (cf. [4, Definition A.1]), which is needed for quadratic hedging to make sense. This condition holds if $[0,T]\times[0,\varepsilon)\times(-\varepsilon,2+\varepsilon)\subset D_0\cap D_1$ for some $\varepsilon>0$.

3. $\psi_1^X(t,0,2)\neq 2\psi_1^X(t,0,1)$ for any $t\in[0,T]$. This condition essentially rules out constant stock price processes.

4. There exists some equivalent local martingale measure $Q\sim P$ for S such that we have $E\left(\frac{dQ}{dP}\right)^2<\infty$. This can be interpreted as a kind of no-arbitrage condition needed in the framework of quadratic hedging.

5. We assume $\varphi_1(t, \mathbb{R} \times \{x\}) = 0$ for any $t \in [0,T], x \in \mathbb{R}$. This condition means roughly speaking that the jumps of z have a diffuse law. It holds immediately if the involved Lévy measures has a Lebesgue density.

We turn now to the European-style option that is to be hedged. Its payoff is denoted as

$$H = f(\log(S(T))) = f(z(T))$$

with some function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(s) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{sp} \hat{f}(p) dp,$$

where $R \in \mathbb{R}$ and $\hat{f} : R + i\mathbb{R} \to \mathbb{C}$ denotes another function. For the European call with strike K we have $f(s) = (e^s - K)^+$ and

$$\hat{f}(p) = \frac{K^{1-p}}{p(p-1)} \tag{2.4}$$

for arbitrary $R > 1$, cf. [16]. For the put we have $f(s) = (K - e^s)^+$ and the same $\hat{f}$ as in (2.4) but with $R < 0$. The integral representation of further payoff functions can be found in [16]. $\hat{f}$ is generally obtained as bilateral Laplace transform of f.

As noted in the introduction, the goal is to minimize the expected squared hedging error

$$R(w, \vartheta) := E\left(\left(w + \int_0^T \vartheta(t) dS(t) - H\right)^2\right) \tag{2.5}$$

over all $w \in \mathbb{R}$ and all admissible trading strategies ϑ. For the proper notion of admissibility in the context of quadratic hedging we refer to [4, Definition 2.2]. It means essentially that the gains of ϑ can be approximated in an L^2-sense by the gains of a sequence of simple, i.e. bounded and piecewise constant strategies.

3 Main results

In this section we solve the hedging problem (2.5) in the affine setup of the previous section. The approach relies on the general structural results of [4]. A two-step procedure is applied in order to solve the problem. First, the auxiliary problem of optimal investment under quadratic utility is solved. Its solution enters the hedging problem in the second step.

3.1 Pure investment problem

The pure investment problem in [4] involves an *opportunity process* L, an *adjustment process* a, and an *opportunity-neutral measure* $P^\star$. The process L satisfies

$$L(t) = \inf_{\vartheta} E\left(\left(1 - \int_t^T \vartheta(s) dS(s)\right)^2 \middle| \mathscr{F}_t\right), \tag{3.1}$$

i.e. it stands for the maximal quadratic utility between t and T of an investor with negative quadratic utility function. The adjustment process is related to the optimizer $\vartheta^{(t)}$ in (3.1). Specifically, we have

$$\vartheta^{(t)}(s) = \left(1 - \int_t^{s-} \vartheta^{(t)}(r)dS(r)\right) a(s)$$

for $s \in [t,T]$. In particular, the strategy $\vartheta^{(0)}(t) = a(t)\mathscr{E}(-\int_0^\cdot a(s)dS(s))(t-)$ minimizes

$$E\left(\left(1 - \int_0^T \vartheta(t)dS(t)\right)^2\right).$$

Hence it is directly related to dynamic portfolio optimization in a Markowitz sense. The opportunity-neutral measure is harder to motivate. It helps to simplify the structure of the hedging problem if S fails to be a martingale. The solution to the pure investment problem in our present setup reads as follows.

Theorem 3.1 *Set*

$$\begin{aligned} \bar{a}(t,x) &:= \frac{\psi_1^X(t,x,1) - \psi_1^X(t,x,0)}{\kappa_1(t,x,0,0,1,1)}, \\ \bar{\alpha}(t,x) &:= \bar{a}(t,x)(\psi_1^X(t,x,1) - \psi_1^X(t,x,0)). \end{aligned}$$

Let $\alpha_1 : [0,T] \to \mathbb{R}$ be the solution to the terminal value problem

$$\alpha_1'(t) = -\psi_1^X(t,\alpha_1(t),0) + \bar{\alpha}(t,\alpha_1(t)), \quad \alpha_1(T) = 0 \tag{3.2}$$

and set

$$\alpha_0(t) := \int_t^T \psi_0^X(s,\alpha_1(s),0)ds. \tag{3.3}$$

Then the opportunity process L and the adjustment process a in the sense of [4] satisfy

$$L(t) = \exp(\alpha_0(t) + \alpha_1(t)v(t)), \tag{3.4}$$

$$a(t) = \frac{\bar{a}(t,\alpha_1(t))}{S(t-)}. \tag{3.5}$$

PROOF. In the appendix. □

The opportunity process L allows for a multiplicative decomposition $L = L(0)ZA$ with a martingale Z and a predictable process of finite variation A satisfying $Z(0) = A(0) = 1$. The martingale $Z^{P^\star} := Z$ can be used as density process of some probability measure $P^\star \sim P$. This so-called *opportunity-neutral measure* plays a crucial role for determining the variance-optimal hedge. According to [4], we have $A = \exp(\int_0^t b^{\mathscr{L}(L)}(s)ds)$, where $b^{\mathscr{L}(L)}$ denotes the drift part in the local characteristics of the stochastic logarithm $\mathscr{L}(L) := \int_0^\cdot \frac{1}{L(t-)}dL(t)$.

3.2 Variance-optimal hedging

We turn now to the hedging problem introduced at the end of Section 2. According to [4], a first step consists in determining the mean value process V of the option defined as

$$V(t) := E_{P^\star}\left(H\mathscr{E}\left(N^\star - (N^\star)^t\right)(T)\middle|\,\mathscr{F}_t\right), \tag{3.6}$$

where $N^\star$ denotes the $P^\star$-martingale part of the special semimartingale $-\int_0^\cdot a(t)dS(t)$ and $(N^*)^t$ the process $N^\star$ stopped at t. In particular, the optimal endowment w^* in (2.5) is given by $V(0)$, cf. [4, Theorem 4.10].

Theorem 3.2 (Mean value process) *For fixed* $p \in R + i\mathbb{R}$ *let* $\Phi_1(\cdot, p) : [0,T] \to \mathbb{C}$ *denote the solution to the terminal value problem* $\Phi_1(T,p) = 0$*,*

$$\begin{aligned}\partial_1\Phi_1(t,p) &= -\psi_1^X(t,\Phi_1(t,p)+\alpha_1(t),p) + \psi_1^X(t,\alpha_1(t),0) \\ &\quad + \bar{a}(t,\alpha_1(t))\kappa_1(t,\alpha_1(t),\Phi_1(t,p),0,p,1).\end{aligned} \tag{3.7}$$

Moreover, set

$$\Phi_0(t,p) := \int_t^T \Big(\psi_0^X(s,\Phi_1(s,p)+\alpha_1(s),0) - \psi_0^X(s,\alpha_1(s),0)\Big)ds \tag{3.8}$$

and

$$V_p(t) := S(t)^p \exp(\Phi_0(t,p) + \Phi_1(t,p)v(t)) \tag{3.9}$$

for $t \in [0,T], p \in R + i\mathbb{R}$*. Then the mean value process* V *from (3.6) satisfies*

$$V(t) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} V_p(t)\hat{f}(p)dp. \tag{3.10}$$

PROOF. In the appendix. □

In some concrete models as in Section 4, the ordinary differential equation (3.7) and the integral (3.8) can be calculated in closed form. In this case the mean value process (3.10) is obtained by a single numerical integration.

The next step on our way is to determine the optimal hedging strategy ϑ^* in (2.5). According to [4, Theorem 4.10] it is specified by its feedback form

$$\vartheta^*(t) = \xi(t) - \left(V(0) + \int_0^{t-}\vartheta^*(s)dS(s) - V(t-)\right)a(t),$$

where ξ denotes the so-called *pure hedge coefficient*, which solves

$$\langle S, V\rangle^{P^\star}(t) = \int_0^t \xi(s)d\langle S,S\rangle^{P^\star}(s). \tag{3.11}$$

It is yet to be determined, whereas V and a are already computed in (3.10) and (3.5).

Theorem 3.3 (Hedging strategy) *The pure hedge coefficient is of the form*

$$\xi(t) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} \xi_p(t) \hat{f}(p) dp,$$

where

$$\xi_p(t) := \frac{V_p(t-)}{S(t-)} \frac{\kappa_1(t, \alpha_1(t), \Phi_1(t,p), 0, p, 1)}{\kappa_1(t, \alpha_1(t), 0, 0, 1, 1)}.$$

PROOF. In the appendix. □

Finally, we can use the results of [4] in order to determine the minimal hedging error in (2.5). According to [4, Theorem 4.12] it satisfies

$$R(w^*, \vartheta^*) = E\left(\int_0^T L(t) d \left\langle V - \int_0^{\cdot} \xi(s) dS(s), V - \int_0^{\cdot} \xi(s) dS(s) \right\rangle^{P^\star} (t) \right).$$

Its integral representation in the present setup reads as follows.

Theorem 3.4 (Hedging error) *For fixed $p, q \in \mathbb{C}, t \in [0,T]$ let $\Upsilon_1(\cdot, q, p, t) \to \mathbb{C}$ denote the solution to the terminal value problem*

$$\partial_1 \Upsilon_1(s, q, p, t) = -\psi_1^X(s, \Upsilon_1(s, q, p, t), p), \quad \Upsilon_1(t, q, p, t) = q.$$

Moreover, set

$$\Upsilon_0(s, q, p, t) := \int_s^t \psi_0^X(r, \Upsilon_1(r, q, p, t), p) dr$$

for $0 \leq s \leq t \leq T$. For $t \in [0,T], p_1, p_2 \in R + i\mathbb{R}$ define

$$\begin{aligned} \varrho(t, p_1, p_2) \quad := \quad & S(0)^p \exp(\hat{\Phi}_0 + \Upsilon_0(0, \hat{\Phi}_1, p, t) + \Upsilon_1(0, \hat{\Phi}_1, p, t) v(0)) \times \\ & \times \left(\varrho_0 + \varrho_1 \left(\partial_2 \Upsilon_0(0, \hat{\Phi}_1, p, t) + \partial_2 \Upsilon_1(0, \hat{\Phi}_1, p, t) v(0) \right) \right) \end{aligned} \quad (3.12)$$

with

$$\begin{aligned} p \quad &:= \quad p_1 + p_2, \\ \hat{\Phi}_i \quad &:= \quad \hat{\Phi}_i(t, p_1, p_2) := \alpha_i(t) + \Phi_i(t, p_1) + \Phi_i(t, p_2), \; i = 0, 1, \\ \varrho_0 \quad &:= \quad \varrho_0(t, p_1, p_2) := \kappa_0(t, \alpha_1(t), \Phi_1(t, p_1), \Phi_1(t, p_2), p_1, p_2), \\ \varrho_1 \quad &:= \quad \varrho_1(t, p_1, p_2) := \kappa_1(t, \alpha_1(t), \Phi_1(t, p_1), \Phi_1(t, p_2), p_1, p_2) \\ & \qquad - \frac{\kappa_1(t, \alpha_1(t), \Phi_1(t, p_1), 0, p_1, 1) \kappa_1(t, \alpha_1(t), \Phi_1(t, p_2), 0, p_2, 1)}{\kappa_1(t, \alpha_1(t), 0, 0, 1, 1)}. \end{aligned}$$

Then the minimal expected squared hedging error satisfies

$$R(w^*, \vartheta^*) = -\frac{1}{4\pi^2} \int_0^T \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \varrho(t, p_1, p_2) \hat{f}(p_1) \hat{f}(p_2) dp_1 dp_2 dt.$$

PROOF. In the appendix. □

Remark 3.5 As special cases of Theorems 3.2–3.4 we recover results from the literature. If X is time-homogeneous and $\psi_j(u,0) = 0$ for $j = 0,1, u \in i\mathbb{R}$, the stock price follows an exponential Lévy process. The results of Theorems 3.2–3.4 correspond to Theorems 3.1, 3.2 in [16]. If, instead, we suppose that X is time-homogeneous and $\psi_1(0,1) = 0$, then S is a martingale and Theorems 3.2–3.4 correspond to [22, Theorems 4.1, 4.2].

4 Time-changed Lévy processes

We apply the general results from Section 3 to a class of stochastic volatility models considered in [2]. In this framework the return process z is modelled by a time-changed Lévy process. Choosing an integrated Ornstein-Uhlenbeck (OU) process as time change leads to the following model for z and the activity process v:

$$\begin{aligned} z(t) &= z(0) + \ell(\hat{v}(t)), & (4.1)\\ d\hat{v}(t) &= v(t-)dt, & (4.2)\\ dv(t) &= -\lambda v(t-)dt + dr(t) & (4.3) \end{aligned}$$

with $\lambda > 0$ and independent Lévy processes r, ℓ such that r is increasing. Their Lévy-Khintchine triplets and Lévy exponents are denoted by $(b^r, c^r, F^r), \psi^r$ and $(b^\ell, c^\ell, F^\ell), \psi^\ell$, respectively. If ℓ is chosen as Brownian motion with drift, one obtains the dynamics of the BNS model (1.1). It is shown in [19] that $X = (v, z)$ is an affine process with triplets $(\beta_i, \gamma_i, \varphi_i), i = 0,1$ of the form

$$\beta_0 = \begin{pmatrix} b^r \\ 0 \end{pmatrix}, \quad \gamma_0 = 0, \quad \varphi_0(A) = \int 1_A(x,0)F^r(dx) \quad \forall A \in \mathscr{B}^2,$$

$$\beta_1 = \begin{pmatrix} -\lambda \\ b^\ell \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & c^\ell \end{pmatrix}, \quad \varphi_1(A) = \int 1_A(0,x)F^\ell(dx) \quad \forall A \in \mathscr{B}^2.$$

The corresponding Lévy exponents are

$$\begin{aligned} \psi_0^X(u_1,u_2) &= \psi^r(u_1), \\ \psi_1^X(u_1,u_2) &= -\lambda u_1 + \psi^\ell(u_2). \end{aligned}$$

Applying the results from the previous section we obtain the solution to the quadratic hedging problem.

Proposition 4.1 *For the present setup (4.1–4.3) the functions in Theorems 3.1–3.4 read as follows.*

$$\Upsilon_0(s,q,p,t) = \int_s^t \psi^r(\Upsilon_1(\tilde{s},q,p,t))d\tilde{s}, \tag{4.4}$$

$$
\begin{aligned}
\Upsilon_1(s,q,p,t) &= qe^{\lambda(s-t)} - \frac{\psi^\ell(p)}{\lambda}\left(e^{\lambda(s-t)}-1\right),\\
\partial_2\Upsilon_1(s,q,p,t) &= e^{\lambda(s-t)},\\
\Phi_0(t,p) &= \int_t^T \Big(\psi^r(\Phi_1(s,p)+\alpha_1(s)) - \psi^r(\alpha_1(s))\Big)ds, \qquad (4.5)\\
\Phi_1(t,p) &= \frac{\bar a\bar\kappa_1(p,1)-\psi^\ell(p)}{\lambda}\left(e^{\lambda(t-T)}-1\right),\\
\alpha_0(t) &= \int_t^T \psi^r(\alpha_1(s))ds, \qquad (4.6)\\
\alpha_1(t) &= \frac{\bar\alpha}{\lambda}\left(e^{\lambda(t-T)}-1\right),\\
\varrho_0 &= \bar\kappa_0(\alpha_1(t),\Phi_1(t,p_1),\Phi_1(t,p_2)),\\
\varrho_1 &= \bar\kappa_1(p_1,p_2) - \frac{\bar\kappa_1(p_1,1)\bar\kappa_1(p_2,1)}{\bar\kappa_1(1,1)},\\
\bar\alpha &= \bar a\psi^\ell(1),\\
\bar a &= \frac{\psi^\ell(1)}{\bar\kappa_1(1,1)},\\
\kappa_1(t,x,y_1,y_2,\hat y_1,\hat y_2) &= \bar\kappa_1(\hat y_1,\hat y_2),\\
\bar\kappa_1(\hat y_1,\hat y_2) &:= \psi^\ell(\hat y_1+\hat y_2)-\psi^\ell(\hat y_1)-\psi^\ell(\hat y_2),\\
\kappa_0(t,x,y_1,y_2,\hat y_1,\hat y_2) &= \bar\kappa_0(x,y_1,y_2),\\
\bar\kappa_0(x,y_1,y_2) &:= \psi^r(x+y_1+y_2)-\psi^r(x+y_1)-\psi^r(x+y_2)+\psi^r(x).
\end{aligned}
$$

PROOF. One easily verifies that the candidates for $\alpha_1, \Phi_1, \Upsilon_1$ satisfy the corresponding terminal value problems. Moreover, the candidates for $\alpha_0, \Phi_0, \Upsilon_0$ have the proper derivative. □

In some cases the integrals in (4.4–4.6) and the derivative $\partial_2\Upsilon_0$ in (3.12) can be expressed in closed form.

Example 4.2 (Carr et al. [2] model with Γ-OU subordinator) Suppose that the Lévy exponent of r equals $\psi^r(u) = \frac{\lambda\zeta u}{\eta-u}$ with constants $\zeta,\eta > 0$. In this case the Ornstein-Uhlenbeck type process v has a stationary gamma law, cf. [6, Example 15.1]. Note that the integrals in (4.4–4.6) are all of the form

$$G(q) := \int_{t_1}^{t_2} \psi^r(g(t,q))dt$$

with $g(t,q) = w(\exp(\lambda(t-\tilde t))-1) + q\exp(\lambda(t-\tilde t))$ for some constants $w\in\mathbb{C}, t_1, t_2, \tilde t \in [0,T]$ with $t_1 \le t_2$. One easily verifies that this integral has the closed-form representation

$$G(q) = \begin{cases} \frac{-\zeta}{\eta+w}\left(\lambda(t_2-t_1)w - \eta\log\left(\frac{g(t_1,q)-\eta}{g(t_2,q)-\eta}\right)\right) & \text{if } \eta+w\neq 0,\\ \zeta\left(\lambda(t_1-t_2)+\frac{\eta}{\eta-q}\left(e^{-\lambda(t_1-\tilde t)}-e^{-\lambda(t_2-\tilde t)}\right)\right) & \text{if } \eta+w=0. \end{cases}$$

Here, $\log$ denotes the *distinguished logarithm* in the sense of [27], i.e. the branch is chosen such that the resulting function is continuous in t. The derivative of G is needed for $\partial_2\Upsilon_0$ in (3.12) and it satisfies

$$G'(q) = \begin{cases} \frac{\zeta\eta}{\eta+w}\left(\frac{e^{\lambda(t_1-\tilde{t})}}{g(t_1,q)-\eta} - \frac{e^{\lambda(t_2-\tilde{t})}}{g(t_2,q)-\eta}\right) & \text{if } \eta+w\neq 0, \\ \frac{\zeta\eta}{(\eta-q)^2}\left(e^{-\lambda(t_1-\tilde{t})} - e^{-\lambda(t_2-\tilde{t})}\right) & \text{if } \eta+w=0. \end{cases}$$

Remark 4.3 From Proposition 4.1 one can deduce that some expressions can be simplified in the BNS model (1.1), namely

$$\begin{aligned} \xi_p(t) &= \frac{pV_p(t-)}{S(t)}, \\ \varrho_1 &= 0, \end{aligned}$$

cf. also [7, Section 4]. This structure for ξ_p is obtained for any model in our affine setup such that the asset price process S is continuous and the *local independence condition*

$$\psi_j^X(t,u_1,u_2) = \psi_j^X(t,u_1,0) + \psi_j^X(t,0,u_2), \quad j=0,1$$

is satisfied (e.g. the Heston [15] model with independent Brownian motions). By (3.9, 3.10), $V_p(t)$ and hence $V(t)$ are deterministic functions of asset price and volatility. If we differentiate this function with respect to the asset price, we obtain $\xi_p(t)$ resp. $\xi(t)$. Therefore the pure hedge ξ can be viewed as a *delta hedge* if V is interpreted as price process of the claim. Local independence implies that volatility risk cannot be hedged by trading in the stock.

Barndorff-Nielsen and Shephard [1] consider also superpositions of Lévy-driven Ornstein-Uhlenbeck processes. This extension can be treated along the same lines as in the present paper if v is replaced by a multivariate process. For ease of notation we do not consider this generalization here.

5 Numerical illustration

We demonstrate the approach numerically in some concrete stochastic volatility models. We use the parameters estimated in [24] for German stock index data. We compare four different models.

1. Firstly, we consider the model in (4.1–4.3) with a Γ-OU subordinator (cf. Example 4.2) and a normal inverse Gaussian (NIG) Lévy process ℓ. Its Lévy exponent ψ^ℓ is given by

$$\psi^\ell(u) = u\mu + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+u)^2}\right)$$

with constants $\alpha,\delta>0$, $\mu,\beta\in\mathbb{R}$ such that $|\beta|<\alpha$. The estimated parameters in [24] are $\hat{\alpha}=90.1$, $\hat{\beta}=-16.0$, $\hat{\delta}=85.9$, $\hat{\mu}=15.5$, $\hat{\lambda}=2.54$, $\hat{\zeta}=0.847$, and $\hat{\eta}=17.5$.

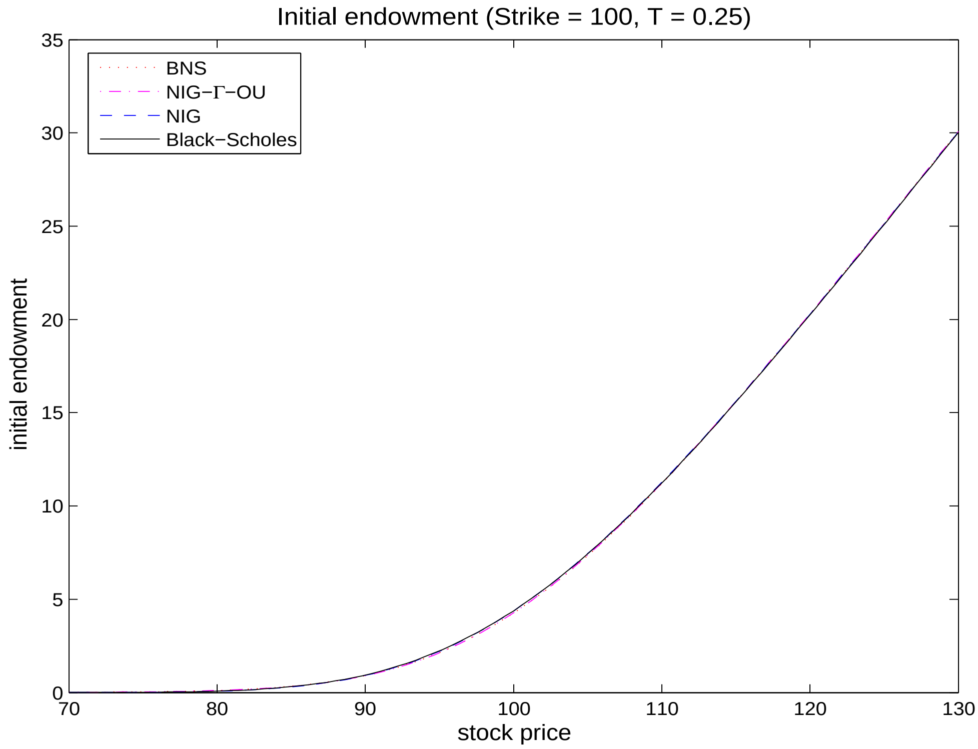

Figure 1: Optimal initial endowment

2. In the BNS model (1.1) with a Γ-OU process as in Example 4.2, [24] obtains the estimate $\hat{\mu} = 0.904$ for the drift parameter of the return process. The estimation procedure in [24] for the volatility process delivers the same values for λ, ζ and η as in the above NIG-Γ-OU case. In both stochastic volatility models we set $v(0) := \hat{\zeta}/\hat{\eta} = 0.0484$, which coincides with the expectation of the stationary law of the activity process.

3. For a comparison with [16] we consider an exponential Lévy model. Specifically, the return process is chosen as $z = z(0) + \ell$ with a NIG Lévy process ℓ. The corresponding formulas follow from Section 3 or from [16, Theorems 3.1, 3.2]. The estimated parameters are $\hat{\alpha} = 53.0, \hat{\beta} = -5.09, \hat{\delta} = 2.53, \hat{\mu} = 0.288$.

4. Finally we consider a Black-Scholes model with estimated variance parameter $\hat{\sigma}^2 = 0.0484$.

Note that time is measured in years in the above parameterization.

We compute the solution to the mean-variance hedging problem by evaluating the formulas of the previous sections numerically for a European call with strike $K = 100$ and a maturity of three months, i.e. $T = 0.25$. Figure 1 shows the optimal initial endowment w^* as a function of the initial asset price $S(0)$. The results are remarkably similar for the four models. From Figure 2 we see that the same is true for the variance-optimal hedging

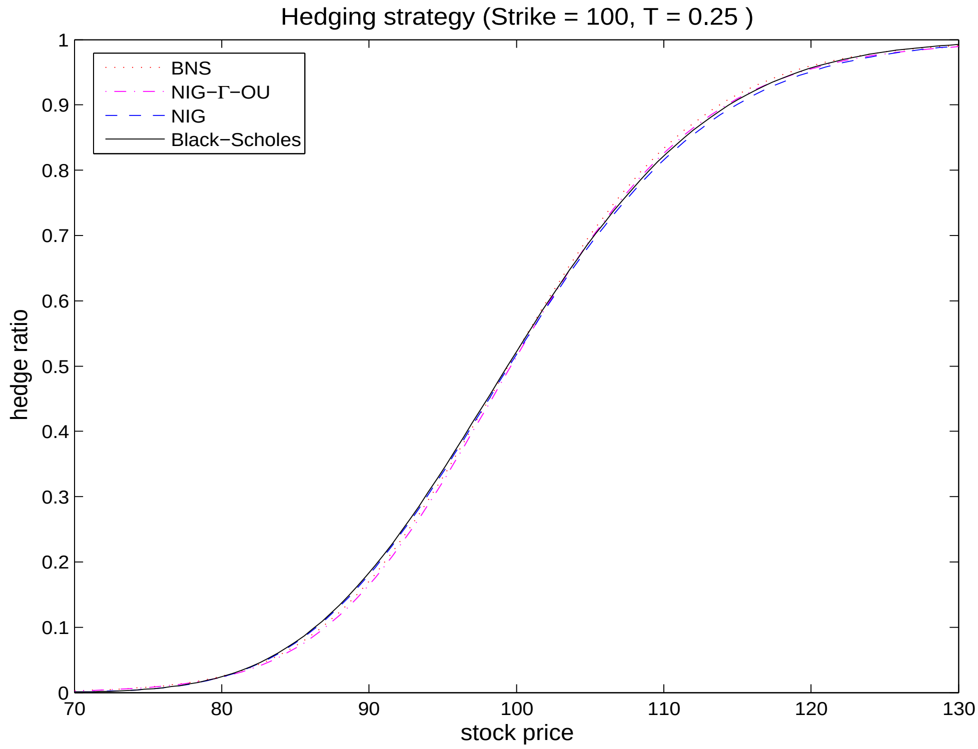

Figure 2: Variance-optimal hedging strategy at $t = 0$

strategy ϑ^* at $t = 0$. This is in line with similar observations for exponential Lévy processes [16] or in the martingale case [21]. In Figure 3 the resulting hedging errors are shown as a function of $S(0)$. Here the models differ substantially from each other. From an intuitive perspective, incompleteness may result from both jumps in the stock and stochastic volatility. Therefore it does not come as a surprise that a model allowing for both yields the highest expected squared hedging error. Note that the hedging error vanishes in the Black-Scholes case because variance-optimal hedging is perfect.

A Appendix

We start with facts on semimartingale calculus and affine processes. The proofs of the statements in Section 3 are given in Section A.3.

A.1 Semimartingale calculus

This paper relies heavily on the calculus of semimartingale characteristics. For the reader's convenience we summarize some rules which can be found e.g. in [17, 19, 18]. As in the whole paper we use the truncation fucntion $h(x) = x$ for ease of notation. This implicitly means that we assume all relevant semimartingales to be special. The version for general

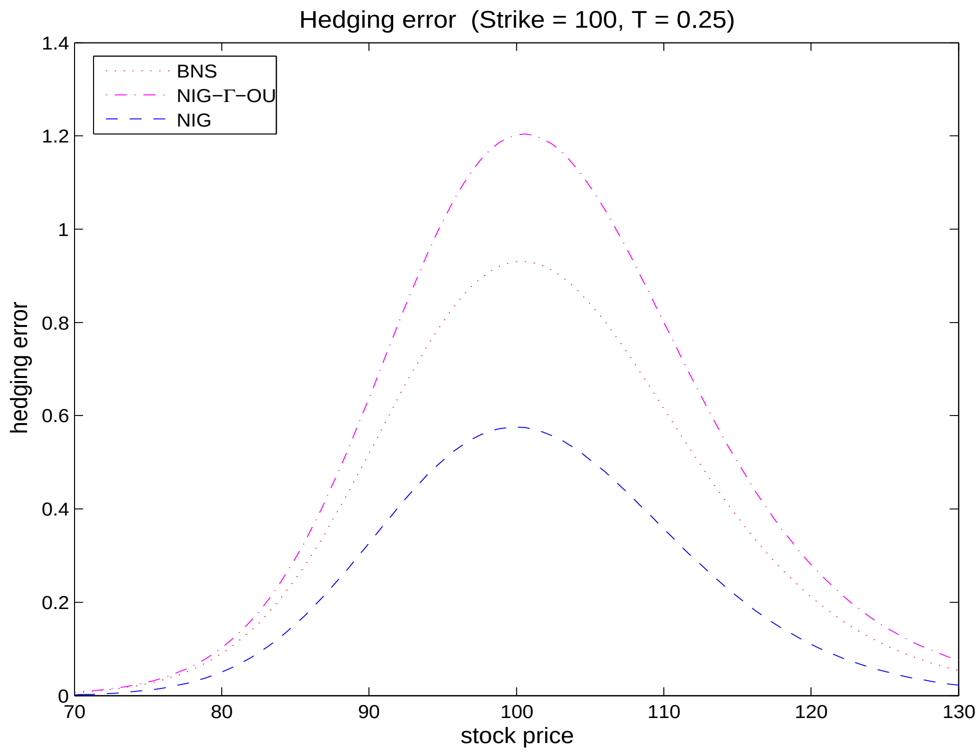

Figure 3: Minimal expected squared hedging error

semimartingales involving bounded truncation functions can be found in the cited references.

Definition A.1 Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b, c, F). Then the $\mathbb{R}^{d\times d}$-valued predictable process

$$\tilde{c}(t) := c(t) + \int xx^\top F(t, dx)$$

is called *modified second characteristic* of X if the integral exists.

This modified second characteristic appears in the context of predictable covariation processes.

Lemma A.2 *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b, c, F) and modified second characteristics $\tilde{c}$. If the corresponding integral exists, we have*

$$\langle X_i, X_j\rangle(t) = \int_0^t \tilde{c}_{ij}(s)ds$$

for the predictable covariation process of the components of $X = (X_1, \ldots, X_d)$.

PROOF. [4, Proposition 1.2] □

Lemma A.3 (Integration) *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b,c,F) and H an $\mathbb{R}^{n\times d}$-valued predictable process which is integrable with respect to X. The local characteristics $(\hat{b},\hat{c},\hat{F})$ of the $\mathbb{R}^n$-valued integral process*

$$\int_0^\cdot H(t)dX(t) := \left(\int_0^\cdot H^{j\cdot}(t)dX(t)\right)_{j=1,\dots,n}$$

are of the form

$$\begin{aligned}\hat{b}(t) &= H(t)b(t),\\ \hat{c}(t) &= H(t)c(t)H(t)^\top,\\ \hat{F}(t,A) &= \int 1_A(H(t)x)F(t,dx) \quad \forall A\in\mathscr{B}^n \text{ with } 0\notin A.\end{aligned}$$

PROOF. [23, Lemma 3] □

Lemma A.4 (C^2-functions) *Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics (b,c,F). Suppose that $f:U\to\mathbb{R}^n$ is twice continuously differentiable on some open subset $U\subset\mathbb{R}^d$ such that X,X_- are U-valued. Then the local characteristics $(\hat{b},\hat{c},\hat{F})$ of the $\mathbb{R}^n$-valued semimartingale $f(X)$ are given by*

$$\begin{aligned}\hat{b}_i(t) &= \sum_{k=1}^d \partial_k f_i(X(t-))b_k(t) + \frac{1}{2}\sum_{k,l=1}^d \partial_{kl}f_i(X(t-))c_{kl}(t)\\ &\quad + \int\left(f_i(X(t-)+x) - f_i(X(t-)) - \sum_{k=1}^d \partial_k f_i(X(t-))x_k\right)F(t,dx),\\ \hat{c}_{ij}(t) &= \sum_{k,l=1}^d \partial_k f_i(X(t-))c_{kl}(t)\partial_l f_j(X(t-)),\\ \hat{F}(t,A) &= \int 1_A\left(f(X(t-)+x) - f(X(t-))\right)F(t,dx) \quad \forall A\in\mathscr{B}^n \text{ with } 0\notin A.\end{aligned}$$

PROOF. [12, Corollary A.6] □

Lemma A.5 (Change of measure) *Let X be an $\mathbb{R}^d$-valued semimartingale and $P^\star\sim P$ a probability measure with density process Z. Denote the stochastic logarithm of Z by $\mathscr{L}(Z) := \int_0^\cdot \frac{1}{Z(t-)}dZ(t)$. If $(X,\mathscr{L}(Z))$ admits local characteristics (b,c,F), the $P^\star$-characteristics $(b^\star,c^\star,F^\star)$ of $(X,\mathscr{L}(Z))$ are given by*

$$\begin{aligned}b_i^\star(t) &= b_i(t) + c_{i,d+1}(t) + \int x_i x_{d+1}F(t,dx), \quad i=1,\dots,d+1,\\ c^\star(t) &= c(t),\\ F^\star(t,A) &= \int 1_A(x)(1+x_{d+1})F(t,dx).\end{aligned}$$

PROOF. [18, Lemma 5.1] $\square$

The following result is needed in the proof of Theorem 3.2.

Lemma A.6 *Let $X = (X_1, \dots, X_d)$ be an $\mathbb{R}^d$-valued semimartingale such that X_d does not have jumps of size -1. If (b, c, F) denote the local characteristics of X, then the local characteristics $(\hat{b}, \hat{c}, \hat{F})$ of the $\mathbb{R}^{d+1}$-valued semimartingale*

$$Y = (Y_1, \dots, Y_d, Y_{d+1}) := \left(X_1, \dots, X_{d-1}, \log |\mathscr{E}(X_d)|, \sum_{s \leq \cdot} 1_{\{\Delta X_d < -1\}} \right)$$

are given by

$$\begin{aligned}
\hat{b}_i(t) &= b_i(t), \quad i = 1, \dots, d-1, \\
\hat{b}_d(t) &= b_d(t) - \frac{1}{2} c_{dd}(t) + \int (\log |1 + x_d| - x_d) F(t, dx), \\
\hat{b}_{d+1}(t) &= \int 1_{(-\infty,-1)}(x_d) F(t, dx), \\
\hat{c}_{ij}(t) &= c_{ij}(t), \quad i, j = 1, \dots, d, \\
\hat{c}_{d+1,i}(t) &= 0, \quad i = 1, \dots, d+1, \\
\hat{F}(t, A) &= \int 1_A \left(x_1, \dots, x_{d-1}, \log |1 + x_d|, 1_{(-\infty,-1)}(x_d) \right) F(t, dx) \quad \forall A \in \mathscr{B}^{d+1}.
\end{aligned}$$

PROOF. Denote by (B, C, ν) the (integral) characteristics of X in the sense of [17, Definition II.2.6]. The canonical representation of Y_d and Y_{d+1} in the sense of [17, Theorem II.2.34] equals

$$\begin{aligned}
Y_d &= X_d^c + \log |1 + x_d| * (\mu^X - \nu) \\
&\quad + B_d - \frac{1}{2} \langle X_d^c, X_d^c \rangle + (\log |1 + x_d| - x_d) * \nu, \\
Y_{d+1} &= 1_{(-\infty,-1)}(x_d) * (\mu^X - \nu) + 1_{(-\infty,-1)}(x_d) * \nu.
\end{aligned}$$

This yields the local characteristics of Y above. $\square$

A.2 Multivariate affine processes

Another key role in this paper is played by affine Markov processes and their characterization in terms of generalized Riccati equations. They are studied in depth in [10] and [9]. We focus here on the subclass needed for our purposes.

Definition A.7 Let $X = (v, z)$ an $\mathbb{R}_+ \times \mathbb{R}^d$ -valued semimartingale. We call X *time-inhomogeneous affine process* if its local characteristics (b, c, F) are affine functions of $v(t-)$

$$\begin{aligned}
b(t) &= \beta_0(t) + v(t-)\beta_1(t), \\
c(t) &= \gamma_0(t) + v(t-)\gamma_1(t), \\
F(t, A) &= \varphi_0(t, A) + v(t-)\varphi_1(t, A) \quad \forall A \in \mathscr{B}^{d+1},
\end{aligned}$$

where $(\beta_i(t), \gamma_i(t), \varphi_i(t))$, $i = 0, 1$ are strongly admissible Lévy-Khintchine triplets on $\mathbb{R}^{d+1}$ in the sense of [20].

As in Section 2 we denote the *Lévy exponents* corresponding to $(\beta_i(t), \gamma_i(t), \varphi_i(t))$ for $i = 0, 1$ by

$$\psi_i^X(t, u) = u^\top \beta_i(t) + \frac{1}{2} u^\top \gamma_i(t) u + \int \left(e^{u^\top x} - 1 - u^\top x \right) \varphi_i(t, dx).$$

Moreover, we call

$$\psi^X(t, u) := \psi_0^X(t, u) + v(t-)\psi_1^X(t, u)$$

Lévy exponent of X. In [10] a generalized Riccati equation is derived for the conditional characteristic or moment generating function of an affine process.

Lemma A.8 *Let X be a time-inhomogeneous affine process as in Definition A.7. The conditional exponential moment $E(e^{u^\top X(t)}|\mathscr{F}_s)$ for $s \leq t$ and reasonable $u \in \mathbb{C}^{d+1}$ is given by*

$$E(e^{u^\top X(t)}|\mathscr{F}_s) = \exp\left(\Psi_0(s, t, u) + \Psi_1(s, t, u) X_1(s) + \sum_{i=2}^{d+1} X_i(s) u_i \right),$$

where Ψ_1 is the solution of the terminal value problem

$$\partial_1 \Psi_1(s, t, u) = -\psi_1^X(s, \Psi_1(s, t, u), u_2, \ldots, u_{d+1}), \quad \Psi_1(t, t, u) = u_1$$

and

$$\Psi_0(s, t, u) := \int_s^t \left(\psi_0^X(r, \Psi_1(r, t, u), u_2, \ldots, u_{d+1}) \right) dr.$$

PROOF. This is proved in [10]; for a reformulation in the above sense cf. [20]. The extension to $u \notin i\mathbb{R}^d$ requires sufficient regularity, cf. [10, Lemma 6.5], [20, Theorem 5.1], [11, Theorem 3.3], [8, (10.8.2)] in this respect. $\square$

A.3 Proofs of Section 3

In the following we use the notation

$$(b^{X_1,\ldots,X_d}, c^{X_1,\ldots,X_d}, F^{X_1,\ldots,X_d}) = \left(\begin{pmatrix} b^{X_1} \\ \vdots \\ b^{X_d} \end{pmatrix}, \begin{pmatrix} c^{X_1} & \cdots & c^{X_1 X_d} \\ \vdots & \ddots & \vdots \\ c^{X_d X_1} & \cdots & c^{X_d} \end{pmatrix}, F^{X_1,\ldots,X_d} \right)$$

for the local characteristics of an $\mathbb{R}^d$-valued semimartingale $X = (X_1, \ldots, X_d)$. An additional star indicates that the local characteristics refer to measure $P^\star$.

PROOF OF THEOREM 3.1. We show that L given by (3.4) meets the conditions of Theorem 3.25 in [4] (up to uniform integrability and admissibility). $L(T)=1$ and $L>0$ are obvious. Note that $\bar{\alpha}(t,\alpha_1(t))\geq 0$ in (3.2). Moreover, the modified terminal value problem

$$\tilde{\alpha}'(t)=-\psi_1^X(t,\tilde{\alpha}(t),0),\quad \tilde{\alpha}(T)=0$$

is solved by $\tilde{\alpha}=0$. A comparison argument yields that α_1 and hence also α_0 are nonpositive. It remains to prove that

$$b^L(t)=L(t-)\frac{\bar{b}(t)^2}{\bar{c}(t)} \tag{A.1}$$

where

$$\begin{aligned}
\bar{b}(t) &:= b^S(t)+\frac{c^{SL}(t)}{L(t-)}+\int x_1\frac{x_2}{L(t-)}F^{S,L}(t,dx),\\
\bar{c}(t) &:= c^S(t)+\int x_1^2\left(1+\frac{x_2}{L(t-)}\right)F^{S,L}(t,dx).
\end{aligned}$$

The local characteristics of (v,z) are given. They lead immediately to the characteristics of the process (v,z,I) where $I(t):=t$ denotes the identity process. Application of Lemma A.4 yields the local characteristics $(b^{S,L},c^{S,L},F^{S,L})$ of (S,L), namely

$$\begin{aligned}
b^S(t) &= S(t-)v(t-)\psi_1^X(t,0,1),\\
b^L(t) &= L(t-)(\alpha_0'(t)+\alpha_1'(t)v(t-)+\psi^X(t,\alpha_1(t),0)),\\
c^{S,L}(t) &= \begin{pmatrix} S(t-)^2c^z(t) & S(t-)L(t-)\alpha_1(t)c^{vz}(t)\\ S(t-)L(t-)\alpha_1(t)c^{vz}(t) & L(t-)^2\alpha_1(t)^2c^v(t)\end{pmatrix},\\
F^{S,L}(t,A) &= \int 1_A(S(t-)(e^{x_2}-1),L(t-)(e^{\alpha_1(t)x_1}-1))F(t,dx)\\
&\quad \forall A\in\mathscr{B}^2 \text{ with } 0\notin A.
\end{aligned}$$

We obtain

$$\begin{aligned}
\bar{b}(t) &= S(t-)v(t-)\left(\psi_1^X(t,\alpha_1(t),1)-\psi_1^X(t,\alpha_1(t),0)\right),\\
\bar{c}(t) &= S(t-)^2v(t-)\kappa_1(t,\alpha_1(t),0,0,1,1).
\end{aligned}$$

By (3.2, 3.3) we have that (A.1) holds. According to [4, Theorem 3.25] the adjustment process is given by $\bar{b}/\bar{c}$, which coincides with (3.5). □

PROOF OF THEOREM 3.2. Since

$$H=\frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty}e^{z(T)p}\hat{f}(p)dp,$$

Fubinis theorem yields

$$V(t)=\frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty}V_p(t)\hat{f}(p)dp$$

with

$$V_p(t) := E_{P^\star}\left(e^{pz(T)}\mathscr{E}(N^\star - (N^\star)^t)(T)\middle|\mathscr{F}_t\right).$$

From the proof of Theorem 3.1 we know the local characteristics of (S, L). This immediately yields an expression for the drift coefficient $b^{\mathscr{L}(L)}(t) = \frac{1}{L(t-)}b^L(t)$ (cf. Lemma A.3). Recall that $Z^{P^\star}(t) = \exp(-\int_0^t b^{\mathscr{L}(L)}(s)ds)\frac{L(t)}{L(0)}$. By definition we have

$$N^\star(t) = -\int_0^t a(s)dS(s) + \int_0^t a(s)b^{S\star}(s)ds,$$

where $b^{S\star}$ denotes the drift coefficient in the local characteristics of S relative to probability measure $P^\star$. Fix $t \in [0, T]$. Since $\int_0^\cdot a(s)b^{S\star}(s)ds$ is continuous and of finite variation, we have

$$\mathscr{E}(N^\star - (N^\star)^t)(T) = \mathscr{E}(U_1 + U_2)(T) = \mathscr{E}(U_1)(T)\mathscr{E}(U_2)(T) = \mathscr{E}(U_1)(T)\exp(U_2(T))$$

where $U_1(s) := -\int_{t\wedge s}^s a(r)dS(r)$ and $U_2(s) := \int_{t\wedge s}^s a(r)b^{S\star}(r)dr$. The conditions on ψ_0^X and the Lévy measure φ_1 imply that U_1 has almost surely no jumps of size -1, which implies that its stochastic exponential $\mathscr{E}(U_1)$ does not vanish. By [17, I.4.64], $\mathscr{E}(U_1)$ changes its sign whenever $\Delta U_1(s) < -1$, i.e. we have

$$\mathscr{E}(U_1)(s) = \exp\left(\log|\mathscr{E}(U_1)(s)| + i\pi\sum_{r\leq s} 1_{\{\Delta U_1(r)<-1\}}\right).$$

By successive application of Lemmas A.3–A.6 we can now determine the local characteristics of the semimartingale $Y := (v, z, \log|\mathscr{E}(U_1)|, \sum_{s\leq\cdot} 1_{\{\Delta U_1(s)<-1\}}, U_2)$ relative to $P^\star$. Indeed, we start with the local characteristics of (v, z, I) relative to P. Lemma A.3 yields those of $(v, z, I, \int_0^\cdot b^{\mathscr{L}(L)}(s)ds)$, Lemma A.4 those of $(v, z, I, Z^{P^\star})$, again Lemma A.3 those of $(v, z, I, \mathscr{L}(Z^{P^\star}))$. Lemma A.5 now leads to the $P^\star$-local characteristics of $(v, z, I, \mathscr{L}(Z^{P^\star}))$. Lemma A.4 yields those of (v, z, S, I), Lemma A.3 those of (v, z, U_1, U_2) and finally Lemma A.6 the $P^\star$-local characteristics of $Y = (Y_1, Y_2, Y_3, Y_4, Y_5)$. Very lengthy but straightforward calculations yield that they are given by

$$\begin{aligned}
b^{Y_1\star}(s) &= b^v(s) + c^v(s)\alpha_1(s) + \int x_1(e^{\alpha_1(s)x_1} - 1)F(s, dx),\\
b^{Y_2\star}(s) &= b^z(s) + c^{vz}(s)\alpha_1(s) + \int x_2(e^{\alpha_1(s)x_1} - 1)F(s, dx),\\
b^{Y_3\star}(s) &= -v(s-)\bar{a}(s, \alpha_1(s)) - \frac{1}{2}\bar{a}^2c^z(s)\\
&\quad + \int(\log|1 - \bar{a}(e^{x_2} - 1)| + \bar{a}(e^{x_2} - 1))e^{\alpha_1(s)x_1}F(s, dx),\\
b^{Y_4\star}(s) &= \int 1_{(1,\infty)}(\bar{a}(e^{x_2} - 1))e^{\alpha_1(s)x_1}F(s, dx),\\
b^{Y_5\star}(s) &= v(s-)\bar{a}(s, \alpha_1(s)),
\end{aligned}$$

$$
\begin{aligned}
c^{Y_1\star}(s) &= c^v(s),\\
c^{Y_1\star Y_2\star}(s) &= c^{vz}(s),\\
c^{Y_2\star}(s) &= c^z(s),\\
c^{Y_1Y_3\star}(s) &= -\bar{a}c^{vz}(s),\\
c^{Y_2Y_3\star}(s) &= -\bar{a}c^{z}(s),\\
c^{Y_3\star}(s) &= \bar{a}^2c^{z}(s),\\
c^{Y_iY_j\star}(s) &= 0 \quad \text{for } i=1,\dots,5 \text{ and } j=4,5,\\
F^{Y\star}(s,A) &= \int 1_A\big(x_1,x_2,\log|1-\bar{a}(e^{x_2}-1)|,1_{(1,\infty)}(\bar{a}(e^{x_2}-1)),0\big)\, e^{\alpha_1(s)x_1}F(s,dx)
\end{aligned}
$$

for any $A\in\mathscr{B}^5$ and $s\in[t,T]$, where

$$\bar{a}:=\bar{a}(s,\alpha_1(s)).$$

In particular, Y is a time-inhomogeneous affine process whose characteristic function is obtained from Lemma A.8. Moreover, we have that

$$e^{z(T)p}\mathscr{E}(N^\star-(N^\star)^t)(T)=\exp(pY_2(T)+Y_3(T)+i\pi Y_4(T)+Y_5(T)).$$

Hence

$$V_p(t)=E_{P^\star}(\exp(pY_2(T)+Y_3(T)+i\pi Y_4(T)+Y_5(T))|\mathscr{F}_t)\,,$$

which is obtained from the generalized characteristic or moment generating function of Y (cf. Lemma A.8). Another lengthy but straightforward calculation shows that

$$
\begin{aligned}
\psi_1^Y(s,q,p,1,i\pi,1) &= \psi_1^X(s,q+\alpha_1(s),p)-\psi_1^X(s,\alpha_1(s),0)\\
&\quad -\bar{a}(s,\alpha_1(s))\kappa_1(s,\alpha_1(s),q,0,p,1),\\
\psi_0^Y(s,q,p,1,i\pi,1) &= \psi_0^X(s,q+\alpha_1(s),p)-\psi_0^X(s,\alpha_1(s),0).
\end{aligned}
$$

This yields that $V_p(t)$ is of the form stated in the assertion. □

PROOF OF THEOREM 3.3. In view of (3.11) we need to compute $\langle S,V\rangle^{P^\star}$ and $\langle S,S\rangle^{P^\star}$. Linearity of the predictable covariation yields

$$\langle S,V\rangle^{P^\star}(t)=\frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty}\langle S,V_p\rangle^{P^\star}(t)\hat{f}(p)dp. \tag{A.2}$$

In an intermediate step in the proof of Theorem 3.2 we have determined the local characteristics of (v,z,I) relative to $P^\star$. Lemma A.4 yields those of (S,V_p). This leads to the modified second $P^\star$-characteristics

$$\tilde{c}^{S,V_p\star} = \begin{pmatrix} \tilde{c}^{S\star} & \tilde{c}^{SV_p\star}\\ \tilde{c}^{SV_p\star} & \tilde{c}^{V_p\star}\end{pmatrix}$$

of (S, V_p), which satisfy

$$\tilde{c}^{S\star}(t) = S(t-)^2 \left(c^z(t) + \int (e^{x_2} - 1)^2 e^{\alpha_1(t)x_1} F(t, dx) \right), \tag{A.3}$$

$$\begin{aligned} \tilde{c}^{SV_p\star}(t) &= V_p(t-)S(t-)\Big(pc^z(t) + \Phi_1(t,p)c^{vz}(t) \\ &\quad + \int (e^{x_2} - 1)(e^{\Phi_1(t,p)x_1 + px_2} - 1) e^{\alpha_1(t)x_1} F(t, dx) \Big). \end{aligned} \tag{A.4}$$

By

$$\begin{aligned} \langle S, S \rangle^{P^\star}(t) &= \int_0^t \tilde{c}^{S\star}(t) ds, \\ \langle S, V_p \rangle^{P^\star}(t) &= \int_0^t \tilde{c}^{SV_p\star}(t) ds \end{aligned}$$

and using (3.11, A.2), the assertion follows. □

PROOF OF THEOREM 3.4. From Lemma A.8 we obtain for $p, q \in \mathbb{C}$ the moment generating function of the affine process $X = (v, z)$. It is of the form

$$E\big(e^{qv(t)+pz(t)}\big) = \exp(\Upsilon_0(0,q,p,t) + \Upsilon_1(0,q,p,t)v(0) + pz(0)) \tag{A.5}$$

with Υ_0, Υ_1 as in the assertion. Differentiation relative to q yields

$$\begin{aligned} E\big(e^{qv(t)+pz(t)}v(t)\big) &= \exp(\Upsilon_0(0,q,p,t) + \Upsilon_1(0,q,p,t)v(0) + pz(0)) \times \\ &\quad \times (\partial_2\Upsilon_0(0,q,p,t) + \partial_2\Upsilon_1(0,q,p,t)v(0)) . \end{aligned} \tag{A.6}$$

An alternative representation of the expected squared hedging error in [4, Theorem 4.12] is

$$R(w^*, \vartheta^*) = E\bigg(\int_0^T \left(\tilde{c}^{V\star}(t) - \frac{(\tilde{c}^{SV\star}(t))^2}{\tilde{c}^{S\star}(t)} \right) L(t) dt \bigg),$$

where we use the notation of the proof of Theorem 3.3 with V instead of V_p. Bilinearity of the predictable covariation yields

$$\tilde{c}^{V\star}(t) = -\frac{1}{4\pi^2} \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \tilde{c}^{V_{p_1}V_{p_2}\star}(t) \hat{f}(p_1) \hat{f}(p_2) dp_1 dp_2.$$

The modified second characteristics $\tilde{c}^{V_{p_1}V_{p_2}\star}$ can be calculated similarly as in the proof of Theorem 3.3. In particular, we obtain

$$\begin{aligned} \tilde{c}^{V_{p_1}V_{p_2}\star}(t) &= V_{p_1}(t-)V_{p_2}(t-)\Big((p_1\Phi_1(t,p_2) + p_2\Phi_1(t,p_1))c^{vz}(t) \\ &\quad + p_1p_2c^z(t) + \Phi_1(t,p_1)\Phi_1(t,p_2)c^v(t) \\ &\quad + \int (e^{\Phi_1(t,p_1)x_1 + p_1x_2} - 1)(e^{\Phi_1(t,p_2)x_1 + p_2x_2} - 1) e^{\alpha_1(t)x_1} F(t, dx) \Big). \end{aligned} \tag{A.7}$$

It remains to determine

$$E\left(\left(\tilde{c}^{V_{p_1}V_{p_2}\star}(t) - \frac{\tilde{c}^{SV_{p_1}\star}(t)\tilde{c}^{SV_{p_2}\star}(t)}{\tilde{c}^{S\star}(t)}\right) L(t)\right).$$

From (A.3, A.4, A.7) and $v(t) = v(t-)$ almost surely we obtain after some calculations

$$\left(\tilde{c}^{V_{p_1}V_{p_2}\star}(t) - \frac{\tilde{c}^{SV_{p_1}\star}(t)\tilde{c}^{SV_{p_2}\star}(t)}{\tilde{c}^{S\star}(t)}\right) L(t) = e^{\hat{\Phi}_0+\hat{\Phi}_1 v(t)+pz(t)} \left(\varrho_0 + \varrho_1 v(t)\right) \quad \text{a.s.}$$

with $\hat{\Phi}_0$, $\hat{\Phi}_1$, p, ϱ_0, ϱ_1 as defined in the assertion. By (A.5, A.6) we have

$$E\left(\left(\tilde{c}^{V_{p_1}V_{p_2}\star}(t) - \frac{\tilde{c}^{SV_{p_1}\star}(t)\tilde{c}^{SV_{p_2}\star}(t)}{\tilde{c}^{S\star}(t)}\right) L(t)\right) = \varrho(t, p_1, p_2).$$

This yields the assertion. □

Acknowledgements

The authors gratefully acknowledge financial support through *Sachbeihilfe KA 1682/2-1* of the *Deutsche Forschungsgemeinschaft*.

References

[1] O. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society, Series B*, 63:167–241, 2001.

[2] P. Carr, H. Geman, D. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13:345–382, 2003.

[3] A. Černý. Optimal continuous-time hedging with leptokurtic returns. *Mathematical Finance*, 17:175–203, 2007.

[4] A. Černý and J. Kallsen. On the structure of general mean-variance hedging strategies. *The Annals of Probability*, 35:1479–1531, 2007.

[5] A. Černý and J. Kallsen. Mean-variance hedging and optimal investment in Heston's model with correlation. *Mathematical Finance*, 18:473–492, 2008.

[6] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, Boca Raton, 2004.

[7] R. Cont, P. Tankov, and E. Voltchkova. Hedging with options in models with jumps. In F. Benth et al., editor, *Stochastic analysis and applications. The Abel symposium 2005*, pages 197–217, Berlin, 2007. Springer.

[8] J. Dieudonné. *Foundations of modern analysis*. Academic Press, New York, 1960.

[9] D. Duffie, D. Filipovic, and W. Schachermayer. Affine processes and applications in finance. *The Annals of Applied Probability*, 13:984–1053, 2003.

[10] D. Filipović. Time-inhomogeneous affine processes. *Stochastic Processes and their Applications*, 115:639–659, 2005.

[11] D. Filipovic and E. Mayerhofer. Affine diffusion processes: theory and applications. Preprint, 2009.

[12] T. Goll and J. Kallsen. Optimal portfolios for logarithmic utility. *Stochastic Processes and their Applications*, 89:31–48, 2000.

[13] C. Gourieroux, J. Laurent, and H. Pham. Mean-variance hedging and numéraire. *Mathematical Finance*, 8:179–200, 1998.

[14] D. Heath, E. Platen, and M. Schweizer. A comparison of two quadratic approaches to hedging in incomplete markets. *Mathematical Finance*, 11(4):385–413, 2001.

[15] S. Heston. A closed-form solution for options with stochastic volatilities with applications to bond and currency options. *The Review of Financial Studies*, 6:327–343, 1993.

[16] F. Hubalek, L. Krawczyk, and J. Kallsen. Variance-optimal hedging for processes with stationary independent increments. *The Annals of Applied Probability*, 16:853–885, 2006.

[17] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Berlin, second edition, 2003.

[18] J. Kallsen. σ-localization and σ-martingales. *Theory of Probability and Its Applications*, 48:152–163, 2004.

[19] J. Kallsen. A didactic note on affine stochastic volatility models. In Yu. Kabanov, R. Liptser, and J. Stoyanov, editors, *From Stochastic Calculus to Mathematical Finance*, pages 343–368. Springer, Berlin, 2006.

[20] J. Kallsen and J. Muhle-Karbe. Exponentially affine martingales, affine measure changes and exponential moments of affine processes. Preprint, 2008.

[21] J. Kallsen and A. Pauwels. Variance-optimal hedging for time-changed Lévy processes. Preprint, 2009.

[22] J. Kallsen and A. Pauwels. Variance-optimal hedging in general affine stochastic volatility models. Preprint, 2009.

[23] J. Kallsen and A. Shiryaev. Time change representation of stochastic integrals. *Theory of Probability and Its Applications*, 46:522–528, 2002.

[24] J. Muhle-Karbe. *On Utility-Based Investment, Pricing and Hedging in Incomplete Markets*. Ph.D. dissertation (TU München), München, 2009.

[25] A. Pauwels. *Varianz-optimales Hedging in affinen Volatilitätsmodellen*. Ph.D. dissertation (TU München), München, 2007.

[26] H. Pham. On quadratic hedging in continuous time. *Mathematical Methods of Operations Research*, 51:315–339, 2000.

[27] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.

[28] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, Cambridge, 2001.