# BMO-ESTIMATION AND ALMOST EVERYWHERE EXPONENTIAL SUMMABILITY OF QUADRATIC PARTIAL SUMS OF DOUBLE FOURIER SERIES 

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#### Abstract

It is proved a BMO-estimation for quadratic partial sums of two-dimensional Fourier series from which it is derived an almost everywhere exponential summability of quadratic partial sums of double Fourier series.


## 1. Introduction

Let $\mathbb{T}:=[-\pi, \pi)=\mathbb{R} / 2 \pi$ and $\mathbb{R}:=(-\infty, \infty)$. We denote by $L_{1}(\mathbb{T})$ the class of all measurable functions $f$ on $\mathbb{R}$ that are $2 \pi$-periodic and satisfy

$$
\|f\|_{1}:=\int_{\mathbb{T}}|f|<\infty
$$

The Fourier series of the function $f \in L_{1}(\mathbb{T})$ with respect to the trigonometric system is the series

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{i n x} \tag{1}
\end{equation*}
$$

where

$$
\widehat{f}(n):=\frac{1}{2 \pi} \int_{\mathbb{T}} f(x) e^{-i n x} d x
$$

are the Fourier coefficients of $f$.
Denote by $S_{n}(x, f)$ the partial sums of the Fourier series of $f$ and let

$$
\sigma_{n}(x, f)=\frac{1}{n+1} \sum_{k=0}^{n} S_{k}(x, f)
$$

be the $(C, 1)$ means of (11). Fejér [1] proved that $\sigma_{n}(f)$ converges to $f$ uniformly for any $2 \pi$-periodic continuous function. Lebesgue in [15] established

[^0]almost everywhere convergence of $(C, 1)$ means if $f \in L_{1}(\mathbb{T})$. The strong summability problem, i.e. the convergence of the strong means
\[

$$
\begin{equation*}
\frac{1}{n+1} \sum_{k=0}^{n}\left|S_{k}(x, f)-f(x)\right|^{p}, \quad x \in \mathbb{T}, \quad p>0 \tag{2}
\end{equation*}
$$

\]

was first considered by Hardy and Littlewood in [11]. They showed that for any $f \in L_{r}(\mathbb{T})(1<r<\infty)$ the strong means tend to 0 a.e., if $n \rightarrow \infty$. The trigonometric Fourier series of $f \in L_{1}(\mathbb{T})$ is said to be $(H, p)$-summable at $x \in T$, if the values (2) converge to 0 as $n \rightarrow \infty$. The $(H, p)$-summability problem in $L_{1}(\mathbb{T})$ has been investigated by Marcinkiewicz [17] for $p=2$, and later by Zygmund [26] for the general case $1 \leq p<\infty$. K. I. Oskolkov in [19] proved the following

Theorem A. Let $f \in L_{1}(\mathbb{T})$ and let $\Phi$ be a continuous positive convex function on $[0,+\infty)$ with $\Phi(0)=0$ and

$$
\begin{equation*}
\ln \Phi(t)=O(t / \ln \ln t) \quad(t \rightarrow \infty) \tag{3}
\end{equation*}
$$

Then for almost all $x$

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(\left|S_{k}(x, f)-f(x)\right|\right)=0 \tag{4}
\end{equation*}
$$

It was noted in 19 that V. Totik announced the conjecture that (4) holds almost everywhere for any $f \in L_{1}(\mathbb{T})$, provided

$$
\begin{equation*}
\ln \Phi(t)=O(t) \quad(t \rightarrow \infty) \tag{5}
\end{equation*}
$$

In 20] V.Rodin proved
Theorem B. Let $f \in L_{1}(\mathbb{T})$. Then for any $A>0$

$$
\lim _{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^{n}\left(\exp \left(A\left|S_{k}(x, f)-f(x)\right|\right)-1\right)=0
$$

for $a$. e. $x \in \mathbb{T}$.
G. Karagulyan [12] proved that the following is true.

Theorem C. Suppose that a continuous increasing function $\Phi:[0, \infty) \rightarrow$ $[0, \infty), \Phi(0)=0$, satisfies the condition

$$
\limsup _{t \rightarrow+\infty} \frac{\log \Phi(t)}{t}=\infty
$$

Then there exists a function $f \in L_{1}(\mathbb{T})$ for which the relation

$$
\limsup _{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(\left|S_{k}(x, f)\right|\right)=\infty
$$

holds everywhere on $\mathbb{T}$.

In fact, Rodin in [20] has obtained a BMO estimate for the partial sums of Fourier series and his theorem stated above is obtained from that estimate by using John-Nirenberg theorem. Recall the definition of BMO $[0,1]$ space. It is the Banach space of functions $f \in L_{1}[0,1]$ with the norm

$$
\|f\|_{\text {Вмо }}=\mathfrak{R}(f)+\left|\int_{0}^{1} f(t) d t\right|
$$

where

$$
\mathfrak{R}(f)=\sup _{I}\left(\left|f-f_{I}\right|\right)_{I}, f_{I}=\frac{1}{|I|} \int_{I} f(t) d t
$$

and the supremum is taken over all intervals $I \subset[0,1]$ ([4], chap. 6). Let $\left\{\xi_{n}: n=0,1,2, \ldots\right\}$ be an arbitrary sequence of numbers. Taking $\delta_{k}^{n}=$ $[k /(n+1),(k+1) /(n+1)]$, we define

$$
\mathrm{BMO}\left[\xi_{n}\right]=\sup _{0 \leq n<\infty}\left\|\sum_{k=0}^{n} \xi_{k} \mathbb{I}_{k}^{n}(t)\right\|_{\mathrm{BMO}}
$$

where $\mathbb{I}_{\delta_{k}^{n}}(t)$ is the characteristic function of $\delta_{k}^{n}$. Notice that the expressions

$$
\begin{equation*}
\operatorname{BMO}\left[\widetilde{S}_{n}(x, f)\right], \quad \operatorname{BMO}\left[S_{n}(x, f)\right], \quad f \in L_{1}(\mathbb{T}), x \in \mathbb{T} \tag{6}
\end{equation*}
$$

define a sublinear operators, where $\widetilde{S}_{n}(x, f)$ is the conjugate partial sum. The following theorem is proved by Rodin in [20].

Theorem D. The operators (6) are of weak type $(1,1)$, i.e. the inequalities

$$
\begin{equation*}
\left|\left\{x \in \mathbb{T}: \operatorname{BMO}\left[S_{n}(x, f)\right]>\lambda\right\}\right| \leq \frac{c}{\lambda} \int_{\mathbb{T}}|f(t)| d t \tag{7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\left\{x \in \mathbb{T}: \operatorname{BMO}\left[\widetilde{S}_{n}(x, f)\right]>\lambda\right\}\right| \leq \frac{c}{\lambda} \int_{\mathbb{T}}|f(t)| d t \tag{8}
\end{equation*}
$$

hold for any $f \in L_{1}(\mathbb{T})$.
In this paper we study the question of exponential summability of quadratic partial sums of double Fourier series. Let $f \in L_{1}\left(\mathbb{T}^{2}\right)$, be a function with Fourier series

$$
\begin{equation*}
\sum_{m, n=-\infty}^{\infty} \widehat{f}(m, n) e^{i(m x+n y)} \tag{9}
\end{equation*}
$$

where

$$
\widehat{f}(m, n)=\frac{1}{4 \pi^{2}} \iint_{\mathbb{T}^{2}} f(x, y) e^{-i(m x+n y)} d x d y
$$

are the Fourier coefficients of the function $f$. The rectangular partial sums of (9) are defined as follows:

$$
S_{M N}(x, y, f)=\sum_{m=-M}^{M} \sum_{n=-N}^{N} \widehat{f}(m, n) e^{i(m x+n y)}
$$

We denote by $L \log L\left(\mathbb{T}^{2}\right)$ the class of measurable functions $f$, with

$$
\iint_{\mathbb{T}^{2}}|f| \log ^{+}|f|<\infty
$$

where $\log ^{+} u:=\mathbb{I}_{(1, \infty)} \log u$. For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [18] has proved, that if $f \in L \log L\left(\mathbb{T}^{2}\right)$, then

$$
\lim _{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^{n}\left(S_{k k}(x, y, f)-f(x, y)\right)=0
$$

for a. e. $(x, y) \in \mathbb{T}^{2}$. L. Zhizhiashvili 24 improved this result showing that class $L \log L\left(\mathbb{T}^{2}\right)$ can be replaced by $L_{1}\left(\mathbb{T}^{2}\right)$.

From a result of S . Konyagin [14] it follows that for every $\varepsilon>0$ there exists a function $f \in L \log ^{1-\varepsilon}\left(\mathbb{T}^{2}\right)$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^{n}\left|S_{k k}(x, y, f)-f(x, y)\right| \neq 0 \quad \text { for a. e. } \quad(x, y) \in \mathbb{T}^{2} \tag{10}
\end{equation*}
$$

The main result of the present paper is the following.
Theorem 1. If $f \in L \log L\left(\mathbb{T}^{2}\right)$, then

$$
\begin{align*}
& \left|\left\{(x, y) \in \mathbb{T}^{2}: \operatorname{BMO}\left[S_{n n}(f, x, y)\right]>\lambda\right\}\right|  \tag{11}\\
\leq & \left.\left.\frac{c}{\lambda}\left(1+\iint_{\mathbb{T}^{2}}|f| \log ^{+} \mid f\right) \right\rvert\,\right)
\end{align*}
$$

for any $\lambda>0$, where $c$ is an absolute positive constant.
The following theorem shows that the quadratic sums of two-dimensional Fourier series of a function $f \in L \log L\left(\mathbb{T}^{2}\right)$ are almost everywhere exponentially summable to the function $f$. It will be obtained from the previous theorem by using John-Nirenberg theorem.
Theorem 2. Suppose that $f \in L \log L\left(\mathbb{T}^{2}\right)$. Then for any $A>0$

$$
\lim _{m \rightarrow \infty} \frac{1}{m+1} \sum_{n=0}^{m}\left(\exp \left(A\left|S_{n n}(x, y, f)-f(x, y)\right|\right)-1\right)=0
$$

for a. e. $(x, y) \in \mathbb{T}^{2}$.
According to a Lemma of L. D. Gogoladze [9], this theorem can be formulated in more general settings.

Theorem 3. Let $\psi:[0, \infty) \rightarrow[0, \infty)$ be a increasing function, satisfying the conditions

$$
\lim _{u \rightarrow 0} \psi(u)=\psi(0)=0, \limsup _{u \rightarrow \infty} \frac{\log \psi(u)}{u}<\infty .
$$

Then for any $f \in L \log L\left(\mathbb{T}^{2}\right)$ we have

$$
\lim _{m \rightarrow \infty} \frac{1}{m+1} \sum_{n=0}^{m} \psi\left(\left|S_{n n}(x, y, f)-f(x, y)\right|\right)=0
$$

almost everywhere on $\mathbb{T}^{2}$.
The results on Marcinkiewicz type strong summation for the Fourier series have been investigated in [2, 3, 10, 6, 7, 5, 8, 16, 23, 27, 28, 24]

## 2. Notations and lemmas

The relation $a \lesssim b$ bellow stands for $a \leq c \cdot b$, where $c$ is an absolute constant. The conjugate function of a given $f \in L_{1}(\mathbb{T})$ is defined by

$$
\tilde{f}(x)=\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+t)}{2 \tan (t / 2)} d t=\lim _{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{\varepsilon<|t|<\pi} \frac{f(x+t)}{2 \tan (t / 2)} d t .
$$

According to Kolmogorov's and Zygmund's inequalities (see [26], chap. 7), we have

$$
\begin{array}{r}
|\{x \in \mathbb{T}:|\tilde{f}(x)|>\lambda\}| \lesssim \frac{\|f\|_{L_{1}(\mathbb{T})}}{\lambda}, \\
\int_{\mathbb{T}}|\tilde{f}(x)| d x \lesssim 1+\int_{\mathbb{T}}|f(x)| \log ^{+}|f(x)| d x . \tag{13}
\end{array}
$$

It will be used two simple properties of BMO norm below. First one says, if $\xi_{n}=c, n=1,2, \ldots$, then BMO $\left[\xi_{n}\right]=|c|$. The second one is, the bound

$$
\mathrm{BMO}\left[\xi_{n}\right] \leq 3 \sup _{n}\left|\xi_{n}\right| .
$$

We shall consider the operators

$$
U_{n}(x, f)=\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{\cos n t}{2 \tan (t / 2)} f(x+t) d t
$$

The following lemma is an immediate consequence of Theorem D.
Lemma 1. The inequality

$$
\left|\left\{x \in \mathbb{T}: \operatorname{BMO}\left[U_{n}(x, f)\right]>\lambda\right\}\right| \lesssim \frac{\|f\|_{L_{1}(\mathbb{T})}}{\lambda}
$$

holds for any $f \in L_{1}(\mathbb{T})$.

Proof. For the conjugate Dirichet kernel we have

$$
\begin{align*}
\tilde{D}_{n}(t) & =\frac{\cos (t / 2)-\cos (n+1 / 2) t}{2 \sin (t / 2)}  \tag{14}\\
& =\frac{1}{2 \tan (t / 2)}+\frac{\sin n t}{2}-\frac{\cos n t}{2 \tan (t / 2)}
\end{align*}
$$

and we get

$$
\begin{aligned}
\tilde{S}_{n}(x, f) & =\frac{1}{\pi} \int_{\mathbb{T}} \tilde{D}_{n}(t) f(x+t) d t \\
& =\tilde{f}(x)+\frac{1}{2 \pi} \int_{\mathbb{T}} f(x+t) \sin n t d t-U_{n}(x, f)
\end{aligned}
$$

Thus, applying simple properties of BMO norm, we obtain

$$
\mathrm{BMO}\left[U_{n}(x, f)\right] \leq|\tilde{f}(x)|+\frac{1}{2 \pi} \int_{\mathbb{T}}|f(t)| d t+\operatorname{BMO}\left[\tilde{S}_{n}(x, f)\right]
$$

Applying the bound (12) and Theorem D, the last inequality completes the proof of lemma.

We consider the square partial sums

$$
\begin{equation*}
S_{n n}(x, y, f)=\frac{1}{\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin (n+1 / 2) t \sin (n+1 / 2) s}{4 \sin (t / 2) \sin (s / 2)} f(x+t, y+s) d t d s \tag{15}
\end{equation*}
$$

and their modification, defined by

$$
S_{n n}^{*}(x, y, f)=\frac{1}{\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin n t \sin n s}{4 \tan (t / 2) \tan (s / 2)} f(x+t, y+s) d t d s
$$

Lemma 2. If $f \in L \log L\left(\mathbb{T}^{2}\right)$, then

$$
\iint_{\mathbb{T}^{2}} \sup _{n}\left|S_{n n}(x, y, f)-S_{n n}^{*}(x, y, f)\right| d x d y \lesssim 1+\iint_{\mathbb{T}^{2}}|f| \log ^{+}|f|
$$

Proof. Substituting the expression for Dirichlet kernel

$$
D_{n}(t)=\frac{\sin (n+1 / 2) t}{2 \sin t / 2}=\frac{\sin n t}{2 \tan (t / 2)}+\frac{\cos n t}{2}
$$

in (15), we get

$$
\begin{aligned}
S_{n n}(x, y, f) & -S_{n n}^{*}(x, y, f) \\
& =\frac{1}{\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin n t \cdot \cos n s}{4 \tan (t / 2)} f(x+t, y+s) d t d s \\
& +\frac{1}{\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n t \cdot \sin n s}{4 \tan (s / 2)} f(x+t, y+s) d t d s \\
& +\frac{1}{4 \pi^{2}} \iint_{\mathbb{T}^{2}} \cos n t \cdot \cos n s \cdot f(x+t, y+s) d t d s \\
& =S_{n n}^{(1)}(x, y, f)+S_{n n}^{(2)}(x, y, f)+S_{n n}^{(3)}(x, y, f) .
\end{aligned}
$$

It is clear, that

$$
\begin{equation*}
\left|S_{n n}^{(3)}(x, y, f)\right| \lesssim\|f\|_{L^{1}\left(\mathbb{T}^{2}\right)} \lesssim 1+\iint_{\mathbb{T}^{2}}|f| \log ^{+}|f| \tag{16}
\end{equation*}
$$

Everywhere below the notation

$$
\text { p.v. } \iint_{\mathbb{T}^{2}} f(t, s) d t d s
$$

stands for either

$$
\text { p.v. } \int_{\mathbb{T}}\left(\text { p.v. } \int_{\mathbb{T}} f(t, s) d t\right) d s, \text { or p.v. } \int_{\mathbb{T}}\left(\text { p.v. } \int_{\mathbb{T}} f(t, s) d s\right) d t
$$

and in each cases we have equality of these two iterated integrals. To observe that we will need just the fact that $f \in L \log L(\mathbb{T})$ implies $\tilde{f} \in L_{1}(\mathbb{T})$. Hence,
making simple transformations and then changing the variables, we get

$$
\begin{align*}
S_{n n}^{(1)} & (x, y, f)  \tag{17}\\
& =\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin n(t+s)}{2 \tan (t / 2)} f(x+t, y+s) d s d t \\
& +\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin n(t-s)}{2 \tan (t / 2)} f(x+t, y+s) d s d t \\
& =\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin n u \cdot f(x+v, y+u-v)}{2 \tan (v / 2)} d v d u \quad(u=t+s, v=t) \\
& + \text { p.v. } \frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin n u \cdot f(x+v, y+v-u)}{2 \tan (v / 2)} d v d u \quad(u=t-s, v=t) \\
& =\frac{1}{2 \pi} \int_{\mathbb{T}} \sin n u\left(\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+u-v)}{2 \tan (v / 2)} d v\right) d u \\
& +\frac{1}{2 \pi} \int_{\mathbb{T}} \sin n u\left(\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+v-u)}{2 \tan (v / 2)} d v\right) d u .
\end{align*}
$$

Observe, that the functions

$$
\begin{aligned}
& F_{1}(x, y, u)=\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+u-v)}{2 \tan (v / 2)} d v \\
& F_{2}(x, y, u)=\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+v-u)}{2 \tan (v / 2)} d v
\end{aligned}
$$

are defined for almost all triples $(x, y, u)$. Moreover, we shall prove that

$$
\begin{equation*}
\iiint_{\mathbb{T}^{3}}\left|F_{i}(x, y, u)\right| d x d y d u \lesssim 1+\iint_{\mathbb{T}^{2}}|f| \log ^{+}|f|, \quad i=1,2 \tag{18}
\end{equation*}
$$

Consider the function $h(t, s, u):=f(t+s, t+u-s)$. Substituting $x=t+s$ and $y=t-s$ in the expression of $F_{1}$, we get

$$
F_{1}(t+s, t-s, u)=\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{\pi} \int_{\mathbb{T}} \frac{h(t, s+v, u)}{2 \tan (v / 2)} d v
$$

Thus, first using the inequality (13) for variable $s$, then integrating by $t$ and $u$, we obtain

$$
\iiint_{\mathbb{T}^{3}}\left|F_{1}(t+s, t-s, u)\right| d s d t d u \lesssim 1+\iiint_{\mathbb{T}^{3}}|h(t, s, u)|\left|\log ^{+}\right| h(t, s, u) \mid d t d s d u .
$$

After the change of variables $t=(x+y) / 2$ and $s=(x-y) / 2$ in the integrals, we get (18) in the case $i=1$. The case $i=2$ may be proved similarly. On the other hand, from (17) it follows that

$$
\left|S_{n n}^{(1)}(x, y, f)\right| \leq \frac{1}{2 \pi} \int_{\mathbb{T}}\left|F_{1}(x, y, u)\right| d u+\frac{1}{2 \pi} \int_{\mathbb{T}}\left|F_{2}(x, y, u)\right| d u
$$

Combining this inequality with (18), we obtain

$$
\begin{equation*}
\iint_{\mathbb{T}^{2}} \sup _{n}\left|S_{n n}^{(1)}(x, y, f)\right| d x d y \lesssim 1+\iint_{\mathbb{T}^{2}}|f| \log ^{+}|f| \tag{19}
\end{equation*}
$$

Similarly we can get the same bound for $S_{n n}^{(2)}(x, y, f)$, which together with (16) completes the proof of lemma.

## 3. Proof of Theorems

Proof of Theorem 1. From Lemma 2 we obtain

$$
\left|S_{n n}(x, y, f)-S_{n n}^{*}(x, y, f)\right| \leq \phi(x, y), \quad n=1,2, \ldots
$$

where the function $\phi(x, y) \geq 0$ satisfies the bound

$$
\iint_{\mathbb{T}^{2}} \phi(x, y) d x d y \lesssim 1+\iint_{\mathbb{T}^{2}}|f| \log ^{+}|f|
$$

Thus we get

$$
\mathrm{BMO}\left[S_{n n}(x, y, f)\right] \leq \mathrm{BMO}\left[S_{n n}^{*}(x, y, f)\right]+3 \phi(x, y)
$$

Hence, the theorem will be proved, if we obtain BMO weak $(1,1)$ estimate for modified partial sums. We have

$$
\begin{aligned}
& S_{n n}^{*}(x, y, f) \\
& \quad=\frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n(t-s) \cdot f(x+t, y+s)}{4 \tan (t / 2) \tan (s / 2)} d t d s \\
& \quad-\frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n(t+s) \cdot f(x+t, y+s)}{4 \tan (t / 2) \tan (s / 2)} d t d s \\
& \quad=\frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n u \cdot f(x+u+v, y+v)}{4 \tan ((u+v) / 2) \tan (v / 2)} d u d v \quad(u=t-s, v=s) \\
& \\
& -\frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n u \cdot f(x+u+v, y-v)}{4 \tan ((u+v) / 2) \tan (v / 2)} d u d v \quad(u=t+s, v=-s) \\
& \quad=I_{n}(x, y, f)-J_{n}(x, y, f)
\end{aligned}
$$

Using a simple and an important identity

$$
\begin{align*}
& \frac{1}{\tan ((u+v) / 2) \tan (v / 2)}=  \tag{20}\\
& \qquad \frac{1}{\tan (u / 2) \tan (v / 2)}-\frac{1}{\tan (u / 2) \tan ((u+v) / 2)}-1
\end{align*}
$$

we obtain

$$
\begin{aligned}
& I_{n}(x, y, f) \\
& =\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n u \cdot f(x+u+v, y+v)}{4 \tan (u / 2) \tan (v / 2)} d u d v \\
& -\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n u \cdot f(x+u+v, y+v)}{4 \tan (u / 2) \tan ((u+v) / 2)} d u d v \\
& -\frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} f(x+t, y+s) d t d s \\
& =\mathrm{p} . \mathrm{v} \cdot \frac{1}{2 \pi} \int_{\mathbb{T}} \frac{\cos n u}{2 \tan (u / 2)}\left(\mathrm{p} . \mathrm{v} \cdot \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+u+v, y+v)}{2 \tan (v / 2)} d v\right) d u \\
& \text {-p.v. } \frac{1}{2 \pi} \int_{\mathbb{T}} \frac{\cos n u}{2 \tan (u / 2)}\left(\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+u+v, y+v)}{2 \tan ((u+v) / 2)} d v\right) d u \\
& -\frac{1}{2 \pi^{2}} \iint_{\mathbb{T}^{2}} f(t, s) d t d s=I_{n}^{(1)}(x, y, f)-I_{n}^{(2)}(x, y, f)-I^{(0)},
\end{aligned}
$$

where

$$
\begin{equation*}
\left|I^{(0)}\right|=\frac{1}{2 \pi^{2}}\left|\iint_{\mathbb{T}^{2}} f(t, s) d t d s\right| \lesssim 1+\iint_{\mathbb{T}^{2}}|f(x, y)| \log ^{+}|f(x, y)| d x d y \tag{21}
\end{equation*}
$$

Observe that

$$
I_{n}^{(1)}(x, y, f)=\frac{1}{2} \cdot U_{n}(x, A(\cdot, y))
$$

where

$$
A(x, y)=\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+v)}{2 \tan (v / 2)} d v
$$

Denoting $g(t, s):=f(t+s, t-s)$ and substituting $x=t+s$ and $y=t-s$ we get

$$
A(t+s, t-s)=\mathrm{p} \cdot \mathrm{v} \cdot \frac{1}{\pi} \int_{\mathbb{T}} \frac{g(t+v, s)}{2 \tan (v / 2)} d v
$$

Using the inequality (13) for variable $t$ and then integrating by $s$, we obtain

$$
\iint_{\mathbb{T}^{2}}|A(t+s, t-s)| d s d t \lesssim 1+\iint_{\mathbb{T}^{2}}|g(t, s)|\left|\log ^{+}\right| g(t, s) \mid d t d s
$$

After the changing back of variables $t=(x+y) / 2$ and $s=(x-y) / 2$ we get

$$
\begin{equation*}
\iint_{\mathbb{T}^{2}}|A(x, y)| d x d y \lesssim 1+\iint_{\mathbb{T}^{2}}|f(x, y)| \log ^{+}|f(x, y)| d x d y \tag{22}
\end{equation*}
$$

Hence, applying the Lemma 1 , we conclude

$$
\begin{align*}
& \left|\left\{(x, y) \in \mathbb{T}^{2}: \operatorname{BMO}\left[I_{n}^{(1)}(x, y, f)\right]>\lambda\right\}\right|  \tag{23}\\
& \quad \lesssim \frac{1}{\lambda}\left(1+\iint_{\mathbb{T}^{2}}|f(x, y)| \log ^{+}|f(x, y)| d x d y\right)
\end{align*}
$$

After the changing of variable $u+v \rightarrow \nu$ in the inner integral of the expression of $I_{n}^{(2)}(x, y, f)$ we get

$$
I_{n}^{(2)}(x, y, f)=\text { p.v. } \frac{1}{2 \pi} \int_{\mathbb{T}} \frac{\cos n u}{2 \tan (u / 2)}\left(\text { p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+\nu, y+\nu-u)}{2 \tan (\nu / 2)} d \nu\right) d u
$$

and then analogously we can prove that

$$
\begin{align*}
& \left|\left\{(x, y) \in \mathbb{T}^{2}: \operatorname{BMO}\left[I_{n}^{(2)}(x, y, f)\right]>\lambda\right\}\right|  \tag{24}\\
& \quad \lesssim \frac{1}{\lambda}\left(1+\iint_{\mathbb{T}^{2}}|f(x, y)| \log ^{+}|f(x, y)| d x d y\right)
\end{align*}
$$

Hence, using (21), (23) and (24), we obtain

$$
\begin{aligned}
\mid\left\{(x, y) \in \mathbb{T}^{2}: \operatorname{BMO}\left[I_{n}(x, y, f)\right]\right. & >\lambda\} \mid \\
& \lesssim \frac{1}{\lambda}\left(1+\iint_{\mathbb{T}^{2}}|f(x, y)| \log ^{+}|f(x, y)| d x d y\right)
\end{aligned}
$$

Using the absolutely same process we may get the analogous estimate for $J_{n}(x, y, f)$ and therefore for $S_{n n}^{*}(x, y, f)$. The theorem is proved.

Let $X$ be either $[0,1]$ or $\mathbb{T}^{2}$ and $L_{M}=L_{M}(X)$ is the Orlicz space of functions on $X$, generated by Young function $M$, i. e. $M$ is convex continuous even function such that $M(0)=0$ and

$$
\lim _{t \rightarrow 0+} \frac{M(t)}{t}=\lim _{t \rightarrow \infty} \frac{t}{M(t)}=0 .
$$

It is well known that $L_{M}$ is a Banach space with respect to Luxemburg norm

$$
\|f\|_{(M)}:=\inf \left\{\lambda: \lambda>0, \int_{X} M\left(\frac{|f|}{\lambda}\right) \leq 1\right\}<\infty .
$$

We will need some basic properties of Orlicz spaces (see [13] ).

1) According to a theorem from (13], chap. 2, theorem 9.5) we have

$$
\begin{equation*}
\|f\|_{(M)} \leq 1 \Rightarrow \int_{X} M(|f|) \leq\|f\|_{(M)}, \tag{25}
\end{equation*}
$$

2) From this fact we may deduce, that

$$
\begin{equation*}
0,5\left(1+\int_{X} M(|f|)\right) \leq\|f\|_{(M)} \leq 1+\int_{X} M(|f|) \tag{26}
\end{equation*}
$$

provided $\|f\|_{(M)}=1$.
3) From the definition of norm $\|\cdot\|_{(M)}$ immediately follows that $|f(x)| \leq$ $|g(x)|$ implies $\|f\|_{(M)} \leq\|g\|_{(M)}$. Besides, for any measurable set $E$ we have

$$
\left\|\mathbb{I}_{E}\right\|_{(M)}=o(1) \text { as }|E| \rightarrow 0 \quad([13],(9.23)) .
$$

4) If $M$ satisfies $\Delta_{2}$-condition, that is

$$
M(2 t) \leq c M(t), t>t_{0},
$$

and $X=\mathbb{T}^{2}$, then the set of two variable trigonometric polynomials on $\mathbb{T}^{2}$ is dense in $L_{M}([13, \S 10)$.
5) From (25) it follows that for any sequence of functions $f_{n}$ the condition $\left\|f_{n}\right\|_{(M)} \rightarrow 0$ implies $\int_{X} M\left(\left|f_{n}\right|\right) \rightarrow 0$.

Proof of Theorem [2. We will deal with two $M$-functions

$$
\begin{gathered}
\Phi(t)=t \log ^{+} t \\
\Psi(t)=\exp t-1
\end{gathered}
$$

We consider two Orlicz spaces $L_{\Phi}=L_{\Phi}\left(\mathbb{T}^{2}\right)$ and $L_{\Psi}=L_{\Psi}(0,1)$. Combining (26) with Theorem 1, we may obtain

$$
\begin{equation*}
\left\lvert\,\left\{(x, y) \in \mathbb{T}^{2}: \operatorname{BMO}\left[S_{n n}(x, y, f)\right]>\lambda\right\} \lesssim \frac{\|f\|_{(\Phi)}}{\lambda}\right. \tag{27}
\end{equation*}
$$

Indeed, at first we deduce the case when $\|f\|_{(\Phi)}=1$, then, using a linearity principle, we get the inequality in the general case.

The inequality

$$
\begin{equation*}
\|f\|_{(\Psi)} \lesssim\|f\|_{\text {BMO }} \tag{28}
\end{equation*}
$$

proved in [20]. It is an immediate consequence of the John-Nirenberg theorem. Denote

$$
\begin{equation*}
\mathcal{B} f(x, y)=\sup _{0 \leq n<\infty}\left\|\sum_{k=0}^{n} S_{k k}(x, y, f) \mathbb{I}_{\delta_{k}^{n}}(t)\right\|_{(\Psi)} \tag{29}
\end{equation*}
$$

Notice, that by the definition we have

$$
\mathrm{BMO}\left[S_{n n}(f, x, y)\right]=\sup _{0 \leq n<\infty}\left\|\sum_{k=0}^{n} S_{k k}(x, y, f) \mathbb{I}_{\delta_{k}^{n}}(t)\right\|_{\mathrm{BMO}}
$$

So, taking into account (27) and (28) we obtain

$$
\begin{equation*}
\left\lvert\,\left\{(x, y) \in \mathbb{T}^{2}: \mathcal{B} f(x, y)>\lambda\right\} \lesssim \frac{\|f\|_{(\Phi)}}{\lambda}\right. \tag{30}
\end{equation*}
$$

On the other hand we have

$$
\begin{aligned}
& \frac{1}{n+1} \sum_{k=0}^{n}\left(\exp A\left|S_{k k}(x, y, f)-f(x, y)\right|-1\right) \\
& =\frac{1}{n+1} \sum_{k=0}^{n} \Psi\left(A\left|S_{k k}(x, y, f)-f(x, y)\right|\right) \\
& =\int_{0}^{1} \Psi\left(A \sum_{k=0}^{n}\left|S_{k k}(x, y, f)-f(x, y)\right| \mathbb{I}_{\delta_{k}^{n}}(t)\right) d t
\end{aligned}
$$

Thus, according the property 5) of Orlicz spaces, to prove the theorem it is enough to prove that

$$
\begin{equation*}
\left\|\sum_{k=0}^{n}\left(S_{k k}(x, y, f)-f(x, y)\right) \mathbb{I}_{\delta_{k}^{n}}(t)\right\|_{(\Psi)} \rightarrow 0 \tag{31}
\end{equation*}
$$

almost everywhere on $\mathbb{T}^{2}$ as $n \rightarrow \infty$, for any $f \in L_{\Phi}$. It is easy to observe, that (31) holds if $f$ is a real trigonometric polynomial in two variables. Indeed, if $P(x, y)$ is a polynomial of degree $m$, then we have

$$
S_{k k}(x, y, P)-P(x, y) \equiv 0, \quad k \geq m
$$

Therefore, if $n \geq m$, then we get

$$
\left|\sum_{k=0}^{n}\left(S_{k k}(x, y, P)-P(x, y)\right) \mathbb{I}_{\delta_{k}^{n}}(t)\right| \leq C \cdot \mathbb{I}_{[0, m /(n+1)]}(t)
$$

where $C$ is a constant, depending on $P$. Then, applying the property 3) of Orlicz spaces, we conclude that (31) holds if $f=P$. To prove the general
case, we consider the set

$$
\begin{align*}
& G_{\lambda}=\left\{(x, y) \in \mathbb{T}^{2}:\right.  \tag{32}\\
&\left.\limsup _{n \rightarrow \infty}\left\|\sum_{k=0}^{n}\left(S_{k k}(x, y, f)-f(x, y)\right) \mathbb{I}_{\delta_{k}^{n}}(t)\right\|_{(\Psi)}>\lambda\right\}
\end{align*}
$$

To complete the proof of theorem, it enough to prove that $\left|G_{\lambda}\right|=0$ if $\lambda>0$. It is easy to check that $\Phi(t)$ satisfies the $\Delta_{2}$-condition. Therefore, according the property 4), we may chose a polynomial $P(x, y)$ such that $\|f-P\|_{(\Phi)}<\varepsilon$. Using the definition of $(\Phi)$-norm, we get

$$
\int_{\mathbb{T}^{2}} \Phi\left(\left|\frac{f-P}{\varepsilon}\right|\right)<1
$$

From Chebishev's inequality, one can easily deduce

$$
\left|\left\{(x, y) \in \mathbb{T}^{2}:|f(x, y)-P(x, y)|>\lambda\right\}\right| \leq \frac{1}{\Phi(\lambda / \varepsilon)}, \quad \lambda>0
$$

Thus, using (30) for any $\lambda>0$ we get

$$
\begin{aligned}
& \left|G_{\lambda}\right|=\mid\left\{(x, y) \in \mathbb{T}^{2}:\right. \\
& \left.\quad \limsup _{n \rightarrow \infty}\left\|\sum_{k=0}^{n}\left(S_{k k}(x, y, f-P)-f(x, y)+P(x, y)\right) \mathbb{I}_{\delta_{k}^{n}}(d t)\right\|_{(\Psi)}>\lambda\right\} \mid \\
& \quad \leq|\{\mathcal{B}(f-P)(x, y)+c|f(x, y)-P(x, y)|>\lambda\}| \\
& \quad \\
& \quad \frac{\|f-P\|_{(\Phi)}}{\lambda}+\frac{1}{\Phi(\lambda / \varepsilon)} \leq \frac{\varepsilon}{\lambda}+\frac{1}{\Phi(\lambda / \varepsilon)}
\end{aligned}
$$

Since $\varepsilon>0$ may be taken sufficiently small, we conclude $\left|G_{\lambda}\right|=0$ if $\lambda>$ 0 .

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