BMO-ESTIMATION AND ALMOST EVERYWHERE EXPONENTIAL SUMMABILITY OF QUADRATIC PARTIAL SUMS OF DOUBLE FOURIER SERIES

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ABSTRACT. It is proved a BMO-estimation for quadratic partial sums of two-dimensional Fourier series from which it is derived an almost everywhere exponential summability of quadratic partial sums of double Fourier series.

1. INTRODUCTION

Let $\mathbb{T} := [-\pi, \pi) = \mathbb{R}/2\pi$ and $\mathbb{R} := (-\infty, \infty)$. We denote by $L_1(\mathbb{T})$ the class of all measurable functions f on \mathbb{R} that are 2π -periodic and satisfy

$$\|f\|_1 := \int_{\mathbb{T}} |f| < \infty.$$

The Fourier series of the function $f \in L_1(\mathbb{T})$ with respect to the trigonometric system is the series

(1)
$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx},$$

where

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

are the Fourier coefficients of f.

Denote by $S_n(x, f)$ the partial sums of the Fourier series of f and let

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{k=0}^n S_k(x, f)$$

be the (C, 1) means of (1). Fejér [1] proved that $\sigma_n(f)$ converges to f uniformly for any 2π -periodic continuous function. Lebesgue in [15] established

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almost everywhere convergence of (C, 1) means if $f \in L_1(\mathbb{T})$. The strong summability problem, i.e. the convergence of the strong means

(2)
$$\frac{1}{n+1}\sum_{k=0}^{n}|S_{k}(x,f)-f(x)|^{p}, \quad x \in \mathbb{T}, \quad p > 0,$$

was first considered by Hardy and Littlewood in [11]. They showed that for any $f \in L_r(\mathbb{T})$ $(1 < r < \infty)$ the strong means tend to 0 a.e., if $n \to \infty$. The trigonometric Fourier series of $f \in L_1(\mathbb{T})$ is said to be (H, p)-summable at $x \in T$, if the values (2) converge to 0 as $n \to \infty$. The (H, p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [17] for p = 2, and later by Zygmund [26] for the general case $1 \le p < \infty$. K. I. Oskolkov in [19] proved the following

Theorem A. Let $f \in L_1(\mathbb{T})$ and let Φ be a continuous positive convex function on $[0, +\infty)$ with $\Phi(0) = 0$ and

(3)
$$\ln \Phi(t) = O(t/\ln \ln t) \quad (t \to \infty).$$

Then for almost all x

(4)
$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(|S_k(x, f) - f(x)|\right) = 0.$$

It was noted in [19] that V. Totik announced the conjecture that (4) holds almost everywhere for any $f \in L_1(\mathbb{T})$, provided

(5)
$$\ln \Phi (t) = O(t) \quad (t \to \infty).$$

In [20] V.Rodin proved

Theorem B. Let $f \in L_1(\mathbb{T})$. Then for any A > 0

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(\exp\left(A \left| S_k(x, f) - f(x) \right| \right) - 1 \right) = 0$$

for a. e. $x \in \mathbb{T}$.

G. Karagulyan [12] proved that the following is true.

Theorem C. Suppose that a continuous increasing function $\Phi : [0, \infty) \rightarrow [0, \infty), \Phi(0) = 0$, satisfies the condition

$$\limsup_{t \to +\infty} \frac{\log \Phi(t)}{t} = \infty$$

Then there exists a function $f \in L_1(\mathbb{T})$ for which the relation

$$\limsup_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(|S_k(x, f)| \right) = \infty$$

holds everywhere on \mathbb{T} .

In fact, Rodin in [20] has obtained a BMO estimate for the partial sums of Fourier series and his theorem stated above is obtained from that estimate by using John-Nirenberg theorem. Recall the definition of BMO [0, 1] space. It is the Banach space of functions $f \in L_1[0, 1]$ with the norm

$$||f||_{\text{BMO}} = \Re(f) + \left| \int_0^1 f(t) dt \right|$$

where

$$\Re(f) = \sup_{I} (|f - f_I|)_I, f_I = \frac{1}{|I|} \int_I f(t) dt$$

and the supremum is taken over all intervals $I \subset [0,1]$ ([4], chap. 6). Let $\{\xi_n : n = 0, 1, 2, ...\}$ be an arbitrary sequence of numbers. Taking $\delta_k^n = [k/(n+1), (k+1)/(n+1)]$, we define

BMO
$$[\xi_n] = \sup_{0 \le n < \infty} \left\| \sum_{k=0}^n \xi_k \mathbb{I}_{\delta_k^n}(t) \right\|_{BMO}$$

where $\mathbb{I}_{\delta_k^n}(t)$ is the characteristic function of δ_k^n . Notice that the expressions

(6) BMO
$$\left[\widetilde{S}_{n}(x,f)\right]$$
, BMO $\left[S_{n}(x,f)\right]$, $f \in L_{1}(\mathbb{T}), x \in \mathbb{T}$

define a sublinear operators, where $\widetilde{S}_n(x, f)$ is the conjugate partial sum. The following theorem is proved by Rodin in [20].

Theorem D. The operators (6) are of weak type (1,1), i.e. the inequalities

(7)
$$|\{x \in \mathbb{T} : \text{BMO} [S_n(x, f)] > \lambda\}| \le \frac{c}{\lambda} \int_{\mathbb{T}} |f(t)| dt$$

and

(8)
$$|\{x \in \mathbb{T} : \text{BMO} [\widetilde{S}_n(x, f)] > \lambda\}| \le \frac{c}{\lambda} \int_{\mathbb{T}} |f(t)| dt$$

hold for any $f \in L_1(\mathbb{T})$.

In this paper we study the question of exponential summability of quadratic partial sums of double Fourier series. Let $f \in L_1(\mathbb{T}^2)$, be a function with Fourier series

(9)
$$\sum_{m,n=-\infty}^{\infty} \widehat{f}(m,n) e^{i(mx+ny)},$$

where

$$\widehat{f}(m,n) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(x,y) e^{-i(mx+ny)} dx dy$$

are the Fourier coefficients of the function f. The rectangular partial sums of (9) are defined as follows:

$$S_{MN}(x, y, f) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \widehat{f}(m, n) e^{i(mx+ny)}.$$

We denote by $L \log L(\mathbb{T}^2)$ the class of measurable functions f, with

$$\iint_{\mathbb{T}^2} |f| \log^+ |f| < \infty,$$

where $\log^+ u := \mathbb{I}_{(1,\infty)} \log u$. For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [18] has proved, that if $f \in L \log L (\mathbb{T}^2)$, then

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(S_{kk} \left(x, y, f \right) - f \left(x, y \right) \right) = 0$$

for a. e. $(x, y) \in \mathbb{T}^2$. L. Zhizhiashvili [24] improved this result showing that class $L \log L(\mathbb{T}^2)$ can be replaced by $L_1(\mathbb{T}^2)$.

From a result of S. Konyagin [14] it follows that for every $\varepsilon > 0$ there exists a function $f \in L \log^{1-\varepsilon} (\mathbb{T}^2)$ such that

(10)
$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |S_{kk}(x, y, f) - f(x, y)| \neq 0 \quad \text{for a. e. } (x, y) \in \mathbb{T}^2.$$

The main result of the present paper is the following.

Theorem 1. If $f \in L \log L(\mathbb{T}^2)$, then

(11)
$$\left| \{ (x,y) \in \mathbb{T}^2 : \text{BMO}\left[S_{nn}(f,x,y)\right] > \lambda \} \right|$$
$$\leq \frac{c}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f| \log^+ |f| \right)$$

for any $\lambda > 0$, where c is an absolute positive constant.

The following theorem shows that the quadratic sums of two-dimensional Fourier series of a function $f \in L \log L(\mathbb{T}^2)$ are almost everywhere exponentially summable to the function f. It will be obtained from the previous theorem by using John-Nirenberg theorem.

Theorem 2. Suppose that $f \in L \log L(\mathbb{T}^2)$. Then for any A > 0

$$\lim_{m \to \infty} \frac{1}{m+1} \sum_{n=0}^{m} \left(\exp\left(A \left| S_{nn}\left(x, y, f\right) - f\left(x, y\right) \right| \right) - 1 \right) = 0$$

for a. e. $(x, y) \in \mathbb{T}^2$.

According to a Lemma of L. D. Gogoladze [9], this theorem can be formulated in more general settings. **Theorem 3.** Let $\psi : [0, \infty) \to [0, \infty)$ be a increasing function, satisfying the conditions

$$\lim_{u \to 0} \psi(u) = \psi(0) = 0, \lim_{u \to \infty} \sup \frac{\log \psi(u)}{u} < \infty.$$

Then for any $f \in L \log L(\mathbb{T}^2)$ we have

$$\lim_{m \to \infty} \frac{1}{m+1} \sum_{n=0}^{m} \psi\left(|S_{nn}(x, y, f) - f(x, y)| \right) = 0$$

almost everywhere on \mathbb{T}^2 .

The results on Marcinkiewicz type strong summation for the Fourier series have been investigated in [2, 3, 10, 6, 7, 5, 8, 16, 23, 27, 28, 24]

2. NOTATIONS AND LEMMAS

The relation $a \leq b$ below stands for $a \leq c \cdot b$, where c is an absolute constant. The conjugate function of a given $f \in L_1(\mathbb{T})$ is defined by

$$\tilde{f}(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+t)}{2\tan(t/2)} dt = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon < |t| < \pi} \frac{f(x+t)}{2\tan(t/2)} dt.$$

According to Kolmogorov's and Zygmund's inequalities (see [26], chap. 7), we have

(12)
$$|\{x \in \mathbb{T} : |\tilde{f}(x)| > \lambda\}| \lesssim \frac{\|f\|_{L_1(\mathbb{T})}}{\lambda},$$

(13)
$$\int_{\mathbb{T}} |\tilde{f}(x)| dx \lesssim 1 + \int_{\mathbb{T}} |f(x)| \log^+ |f(x)| dx.$$

It will be used two simple properties of BMO norm below. First one says, if $\xi_n = c, n = 1, 2, ...$, then BMO $[\xi_n] = |c|$. The second one is, the bound

$$BMO\left[\xi_n\right] \le 3\sup_n |\xi_n|.$$

We shall consider the operators

$$U_n(x, f) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{\cos nt}{2\tan(t/2)} f(x+t) dt.$$

The following lemma is an immediate consequence of Theorem D.

Lemma 1. The inequality

$$|\{x \in \mathbb{T} : BMO[U_n(x, f)] > \lambda\}| \lesssim \frac{\|f\|_{L_1(\mathbb{T})}}{\lambda}$$

holds for any $f \in L_1(\mathbb{T})$.

Proof. For the conjugate Dirichet kernel we have

(14)
$$\tilde{D}_n(t) = \frac{\cos(t/2) - \cos(n+1/2)t}{2\sin(t/2)} \\ = \frac{1}{2\tan(t/2)} + \frac{\sin nt}{2} - \frac{\cos nt}{2\tan(t/2)}$$

and we get

$$\tilde{S}_n(x,f) = \frac{1}{\pi} \int_{\mathbb{T}} \tilde{D}_n(t) f(x+t) dt$$
$$= \tilde{f}(x) + \frac{1}{2\pi} \int_{\mathbb{T}} f(x+t) \sin nt dt - U_n(x,f).$$

Thus, applying simple properties of BMO norm, we obtain

BMO
$$[U_n(x, f)] \le |\tilde{f}(x)| + \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)| dt + BMO \left[\tilde{S}_n(x, f) \right]$$

Applying the bound (12) and Theorem D, the last inequality completes the proof of lemma. $\hfill \Box$

We consider the square partial sums

(15)

$$S_{nn}(x,y,f) = \frac{1}{\pi^2} \iint_{\mathbb{T}^2} \frac{\sin(n+1/2)t\sin(n+1/2)s}{4\sin(t/2)\sin(s/2)} f(x+t,y+s) dt ds$$

and their modification, defined by

$$S_{nn}^{*}(x,y,f) = \frac{1}{\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin nt \sin ns}{4 \tan (t/2) \tan (s/2)} f(x+t,y+s) \, dt ds.$$

Lemma 2. If $f \in L \log L(\mathbb{T}^2)$, then

$$\iint_{\mathbb{T}^2} \sup_n |S_{nn}(x,y,f) - S^*_{nn}(x,y,f)| \, dxdy \lesssim 1 + \iint_{\mathbb{T}^2} |f| \log^+ |f|.$$

Proof. Substituting the expression for Dirichlet kernel

$$D_n(t) = \frac{\sin(n+1/2)t}{2\sin t/2} = \frac{\sin nt}{2\tan(t/2)} + \frac{\cos nt}{2}$$

in (15), we get

$$\begin{split} S_{nn}\left(x,y,f\right) &- S_{nn}^{*}(x,y,f) \\ &= \frac{1}{\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\sin nt \cdot \cos ns}{4 \tan(t/2)} f\left(x+t,y+s\right) dt ds \\ &+ \frac{1}{\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos nt \cdot \sin ns}{4 \tan(s/2)} f\left(x+t,y+s\right) dt ds \\ &+ \frac{1}{4\pi^{2}} \iint_{\mathbb{T}^{2}} \cos nt \cdot \cos ns \cdot f\left(x+t,y+s\right) dt ds \\ &= S_{nn}^{(1)}\left(x,y,f\right) + S_{nn}^{(2)}\left(x,y,f\right) + S_{nn}^{(3)}\left(x,y,f\right). \end{split}$$

It is clear, that

(16)
$$|S_{nn}^{(3)}(x,y,f)| \lesssim ||f||_{L^1(\mathbb{T}^2)} \lesssim 1 + \iint_{\mathbb{T}^2} |f| \log^+ |f|.$$

Everywhere below the notation

$$\text{p.v.} \iint_{\mathbb{T}^2} f(t,s) dt ds$$

stands for either

p.v.
$$\int_{\mathbb{T}} \left(\text{p.v.} \int_{\mathbb{T}} f(t,s) dt \right) ds$$
, or p.v. $\int_{\mathbb{T}} \left(\text{p.v.} \int_{\mathbb{T}} f(t,s) ds \right) dt$

and in each cases we have equality of these two iterated integrals. To observe that we will need just the fact that $f \in L \log L(\mathbb{T})$ implies $\tilde{f} \in L_1(\mathbb{T})$. Hence,

making simple transformations and then changing the variables, we get

$$\begin{aligned} &(17)\\ S_{nn}^{(1)}\left(x,y,f\right) \\ &= \mathrm{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin n(t+s)}{2\tan(t/2)} f\left(x+t,y+s\right) ds dt \\ &+ \mathrm{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin n(t-s)}{2\tan(t/2)} f\left(x+t,y+s\right) ds dt \\ &= \mathrm{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin nu \cdot f\left(x+v,y+u-v\right)}{2\tan(v/2)} dv du \quad (u=t+s,\,v=t) \\ &+ \mathrm{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\sin nu \cdot f\left(x+v,y+v-u\right)}{2\tan(v/2)} dv du \quad (u=t-s,\,v=t) \\ &= \frac{1}{2\pi} \iint_{\mathbb{T}} \sin nu \left(\mathrm{p.v.} \frac{1}{\pi} \iint_{\mathbb{T}} \frac{f\left(x+v,y+u-v\right)}{2\tan(v/2)} dv \right) du \\ &+ \frac{1}{2\pi} \iint_{\mathbb{T}} \sin nu \left(\mathrm{p.v.} \frac{1}{\pi} \iint_{\mathbb{T}} \frac{f\left(x+v,y+v-u\right)}{2\tan(v/2)} dv \right) du. \end{aligned}$$

Observe, that the functions

$$F_1(x, y, u) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+u-v)}{2\tan(v/2)} dv$$
$$F_2(x, y, u) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v, y+v-u)}{2\tan(v/2)} dv$$

are defined for almost all triples (x, y, u). Moreover, we shall prove that

(18)
$$\iiint_{\mathbb{T}^3} |F_i(x, y, u)| dx dy du \lesssim 1 + \iint_{\mathbb{T}^2} |f| \log^+ |f|, \quad i = 1, 2.$$

Consider the function h(t, s, u) := f(t + s, t + u - s). Substituting x = t + sand y = t - s in the expression of F_1 , we get

$$F_1(t+s, t-s, u) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{h(t, s+v, u)}{2 \tan(v/2)} dv.$$

Thus, first using the inequality (13) for variable s, then integrating by t and u, we obtain

$$\iiint_{\mathbb{T}^3} |F_1(t+s,t-s,u)| ds dt du \lesssim 1 + \iiint_{\mathbb{T}^3} |h(t,s,u)| |\log^+ |h(t,s,u)| dt ds du.$$

After the change of variables t = (x+y)/2 and s = (x-y)/2 in the integrals, we get (18) in the case i = 1. The case i = 2 may be proved similarly. On the other hand, from (17) it follows that

$$|S_{nn}^{(1)}(x,y,f)| \le \frac{1}{2\pi} \int_{\mathbb{T}} |F_1(x,y,u)| du + \frac{1}{2\pi} \int_{\mathbb{T}} |F_2(x,y,u)| du.$$

Combining this inequality with (18), we obtain

(19)
$$\iint_{\mathbb{T}^2} \sup_{n} |S_{nn}^{(1)}(x, y, f)| dx dy \lesssim 1 + \iint_{\mathbb{T}^2} |f| \log^+ |f|$$

Similarly we can get the same bound for $S_{nn}^{(2)}(x, y, f)$, which together with (16) completes the proof of lemma.

3. Proof of Theorems

Proof of Theorem 1. From Lemma 2 we obtain

$$|S_{nn}(x, y, f) - S_{nn}^*(x, y, f)| \le \phi(x, y), \quad n = 1, 2, \dots,$$

where the function $\phi(x, y) \ge 0$ satisfies the bound

$$\iint_{\mathbb{T}^2} \phi(x,y) dx dy \lesssim 1 + \iint_{\mathbb{T}^2} |f| \log^+ |f|.$$

Thus we get

BMO
$$[S_{nn}(x, y, f)] \leq$$
 BMO $[S_{nn}^*(x, y, f)] + 3\phi(x, y)$

Hence, the theorem will be proved, if we obtain BMO weak (1, 1) estimate for modified partial sums. We have

$$\begin{split} S_{nn}^{*}\left(x,y,f\right) &= \frac{1}{2\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n(t-s) \cdot f\left(x+t,y+s\right)}{4\tan\left(t/2\right)\tan\left(s/2\right)} dt ds \\ &- \frac{1}{2\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos n(t+s) \cdot f\left(x+t,y+s\right)}{4\tan\left(t/2\right)\tan\left(s/2\right)} dt ds \\ &= \frac{1}{2\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos nu \cdot f\left(x+u+v,y+v\right)}{4\tan\left((u+v)/2\right)\tan\left(v/2\right)} du dv \quad (u=t-s,v=s) \\ &- \frac{1}{2\pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\cos nu \cdot f\left(x+u+v,y-v\right)}{4\tan\left((u+v)/2\right)\tan\left(v/2\right)} du dv \quad (u=t+s,v=-s) \\ &= I_{n}(x,y,f) - J_{n}(x,y,f). \end{split}$$

Using a simple and an important identity

(20)
$$\frac{1}{\tan((u+v)/2)\tan(v/2)} = \frac{1}{\frac{1}{\tan(u/2)\tan(v/2)}} - \frac{1}{\tan(u/2)\tan((u+v)/2)} - 1,$$

we obtain

$$\begin{split} I_n(x,y,f) &= \mathrm{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\cos nu \cdot f\left(x+u+v,y+v\right)}{4\tan(u/2)\tan(v/2)} du dv \\ &- \mathrm{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} \frac{\cos nu \cdot f\left(x+u+v,y+v\right)}{4\tan(u/2)\tan((u+v)/2)} du dv \\ &- \frac{1}{2\pi^2} \iint_{\mathbb{T}^2} f\left(x+t,y+s\right) dt ds \\ &= \mathrm{p.v.} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\cos nu}{2\tan(u/2)} \left(\mathrm{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f\left(x+u+v,y+v\right)}{2\tan(v/2)} dv \right) du \\ &- \mathrm{p.v.} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\cos nu}{2\tan(u/2)} \left(\mathrm{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f\left(x+u+v,y+v\right)}{2\tan((u+v)/2)} dv \right) du \\ &- \mathrm{p.v.} \frac{1}{2\pi^2} \iint_{\mathbb{T}} f\left(t,s\right) dt ds = I_n^{(1)}(x,y,f) - I_n^{(2)}(x,y,f) - I^{(0)}, \end{split}$$

where

(21)
$$|I^{(0)}| = \frac{1}{2\pi^2} \left| \iint_{\mathbb{T}^2} f(t,s) dt ds \right| \lesssim 1 + \iint_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dx dy.$$

Observe that

$$I_n^{(1)}(x, y, f) = \frac{1}{2} \cdot U_n(x, A(\cdot, y))$$

where

$$A(x,y) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+v,y+v)}{2\tan(v/2)} dv.$$

Denoting g(t,s) := f(t+s,t-s) and substituting x = t+s and y = t-s we get

$$A(t+s,t-s) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{g(t+v,s)}{2\tan(v/2)} dv.$$

Using the inequality (13) for variable t and then integrating by s, we obtain

$$\iint_{\mathbb{T}^2} |A(t+s,t-s)| ds dt \lesssim 1 + \iint_{\mathbb{T}^2} |g(t,s)| |\log^+ |g(t,s)| dt ds$$

After the changing back of variables t = (x + y)/2 and s = (x - y)/2 we get

(22)
$$\iint_{\mathbb{T}^2} |A(x,y)| \, dx dy \lesssim 1 + \iint_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| \, dx dy.$$

Hence, applying the Lemma 1, we conclude

(23)
$$|\{(x,y) \in \mathbb{T}^2 : BMO[I_n^{(1)}(x,y,f)] > \lambda\}|$$

$$\lesssim \frac{1}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dx dy\right).$$

After the changing of variable $u+v\to\nu$ in the inner integral of the expression of $I_n^{(2)}(x,y,f)$ we get

$$I_n^{(2)}(x,y,f) = \text{p.v.} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\cos nu}{2\tan(u/2)} \left(\text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f\left(x+\nu, y+\nu-u\right)}{2\tan(\nu/2)} d\nu \right) du,$$

and then analogously we can prove that

(24)
$$|\{(x,y) \in \mathbb{T}^2 : BMO[I_n^{(2)}(x,y,f)] > \lambda\}|$$

$$\lesssim \frac{1}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dx dy\right).$$

Hence, using (21), (23) and (24), we obtain

$$\begin{split} |\{(x,y) \in \mathbb{T}^2 : \, \mathrm{BMO}\left[I_n(x,y,f)\right] > \lambda\}| \\ \lesssim \frac{1}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dx dy\right). \end{split}$$

Using the absolutely same process we may get the analogous estimate for $J_n(x, y, f)$ and therefore for $S_{nn}^*(x, y, f)$. The theorem is proved.

Let X be either [0,1] or \mathbb{T}^2 and $L_M = L_M(X)$ is the Orlicz space of functions on X, generated by Young function M, i. e. M is convex continuous even function such that M(0) = 0 and

$$\lim_{t \to 0+} \frac{M(t)}{t} = \lim_{t \to \infty} \frac{t}{M(t)} = 0.$$

It is well known that L_M is a Banach space with respect to Luxemburg norm

$$\|f\|_{(M)} := \inf\left\{\lambda: \lambda > 0, \int_{X} M\left(\frac{|f|}{\lambda}\right) \le 1\right\} < \infty.$$

We will need some basic properties of Orlicz spaces (see [13]).

1) According to a theorem from ([13], chap. 2, theorem 9.5) we have

(25)
$$||f||_{(M)} \le 1 \Rightarrow \int_{X} M(|f|) \le ||f||_{(M)}$$

2) From this fact we may deduce, that

(26)
$$0,5\left(1+\int_{X} M(|f|)\right) \le \|f\|_{(M)} \le 1+\int_{X} M(|f|)$$

provided $||f||_{(M)} = 1.$

3) From the definition of norm $\|\cdot\|_{(M)}$ immediately follows that $|f(x)| \leq |g(x)|$ implies $\|f\|_{(M)} \leq \|g\|_{(M)}$. Besides, for any measurable set E we have

 $\|\mathbb{I}_E\|_{(M)} = o(1) \text{ as } |E| \to 0 \ ([13], (9.23)).$

4) If M satisfies Δ_2 -condition, that is

$$M\left(2t\right) \le cM\left(t\right), t > t_0,$$

and $X = \mathbb{T}^2$, then the set of two variable trigonometric polynomials on \mathbb{T}^2 is dense in L_M ([13], §10).

5) From (25) it follows that for any sequence of functions f_n the condition $||f_n||_{(M)} \to 0$ implies $\int_{Y} M(|f_n|) \to 0$.

Proof of Theorem 2. We will deal with two M-functions

$$\Phi(t) = t \log^+ t,$$

$$\Psi(t) = \exp t - 1.$$

We consider two Orlicz spaces $L_{\Phi} = L_{\Phi}(\mathbb{T}^2)$ and $L_{\Psi} = L_{\Psi}(0, 1)$. Combining (26) with Theorem 1, we may obtain

(27)
$$|\{(x,y) \in \mathbb{T}^2 : \text{BMO}\left[S_{nn}(x,y,f)\right] > \lambda\} \lesssim \frac{\|f\|_{(\Phi)}}{\lambda}.$$

Indeed, at first we deduce the case when $||f||_{(\Phi)} = 1$, then, using a linearity principle, we get the inequality in the general case.

The inequality

$$(28) ||f||_{(\Psi)} \lesssim ||f||_{BMO}$$

proved in [20]. It is an immediate consequence of the John-Nirenberg theorem. Denote

(29)
$$\mathcal{B}f(x,y) = \sup_{0 \le n < \infty} \left\| \sum_{k=0}^n S_{kk}(x,y,f) \mathbb{I}_{\delta_k^n}(t) \right\|_{(\Psi)}.$$

Notice, that by the definition we have

BMO
$$[S_{nn}(f, x, y)] = \sup_{0 \le n < \infty} \left\| \sum_{k=0}^{n} S_{kk}(x, y, f) \mathbb{I}_{\delta_k^n}(t) \right\|_{BMO}$$
.

So, taking into account (27) and (28) we obtain

(30)
$$|\{(x,y) \in \mathbb{T}^2 : \mathcal{B}f(x,y) > \lambda\} \lesssim \frac{\|f\|_{(\Phi)}}{\lambda}.$$

On the other hand we have

$$\frac{1}{n+1} \sum_{k=0}^{n} (\exp A|S_{kk}(x,y,f) - f(x,y)| - 1)$$

= $\frac{1}{n+1} \sum_{k=0}^{n} \Psi(A|S_{kk}(x,y,f) - f(x,y)|)$
= $\int_{0}^{1} \Psi\left(A \sum_{k=0}^{n} |S_{kk}(x,y,f) - f(x,y)| \mathbb{I}_{\delta_{k}^{n}}(t)\right) dt.$

Thus, according the property 5) of Orlicz spaces, to prove the theorem it is enough to prove that

(31)
$$\left\|\sum_{k=0}^{n} (S_{kk}(x,y,f) - f(x,y))\mathbb{I}_{\delta_{k}^{n}}(t)\right\|_{(\Psi)} \to 0,$$

almost everywhere on \mathbb{T}^2 as $n \to \infty$, for any $f \in L_{\Phi}$. It is easy to observe, that (31) holds if f is a real trigonometric polynomial in two variables. Indeed, if P(x, y) is a polynomial of degree m, then we have

$$S_{kk}(x, y, P) - P(x, y) \equiv 0, \quad k \ge m.$$

Therefore, if $n \geq m$, then we get

$$\left|\sum_{k=0}^{n} (S_{kk}(x, y, P) - P(x, y)) \mathbb{I}_{\delta_{k}^{n}}(t)\right| \leq C \cdot \mathbb{I}_{[0, m/(n+1)]}(t),$$

where C is a constant, depending on P. Then, applying the property 3) of Orlicz spaces, we conclude that (31) holds if f = P. To prove the general

case, we consider the set

(32)
$$G_{\lambda} = \{(x,y) \in \mathbb{T}^2 : \lim_{n \to \infty} \left\| \sum_{k=0}^n (S_{kk}(x,y,f) - f(x,y)) \mathbb{I}_{\delta_k^n}(t) \right\|_{(\Psi)} > \lambda \}.$$

To complete the proof of theorem, it enough to prove that $|G_{\lambda}| = 0$ if $\lambda > 0$. It is easy to check that $\Phi(t)$ satisfies the Δ_2 -condition. Therefore, according the property 4), we may chose a polynomial P(x, y) such that $||f - P||_{(\Phi)} < \varepsilon$. Using the definition of (Φ) -norm, we get

$$\int_{\mathbb{T}^2} \Phi\left(\left| \frac{f-P}{\varepsilon} \right| \right) < 1.$$

From Chebishev's inequality, one can easily deduce

$$|\{(x,y) \in \mathbb{T}^2 : |f(x,y) - P(x,y)| > \lambda\}| \le \frac{1}{\Phi(\lambda/\varepsilon)}, \quad \lambda > 0.$$

Thus, using (30) for any $\lambda > 0$ we get

$$\begin{split} |G_{\lambda}| &= |\{(x,y) \in \mathbb{T}^{2}:\\ \limsup_{n \to \infty} \left\| \sum_{k=0}^{n} (S_{kk}(x,y,f-P) - f(x,y) + P(x,y)) \mathbb{I}_{\delta_{k}^{n}}(dt) \right\|_{(\Psi)} > \lambda \}|\\ &\leq |\{\mathcal{B}(f-P)(x,y) + c | f(x,y) - P(x,y)| > \lambda \}|\\ &\lesssim \frac{\|f-P\|_{(\Phi)}}{\lambda} + \frac{1}{\Phi(\lambda/\varepsilon)} \leq \frac{\varepsilon}{\lambda} + \frac{1}{\Phi(\lambda/\varepsilon)}. \end{split}$$

Since $\varepsilon > 0$ may be taken sufficiently small, we conclude $|G_{\lambda}| = 0$ if $\lambda > 0$.

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