The effects of minimal length and maximal momentum on the transition rate of ultra cold neutrons in gravitational field

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Abstract

The existence of a minimum observable length and/or a maximum observable momentum is in agreement with various candidates of quantum gravity such as string theory, loop quantum gravity, doubly special relativity and black hole physics. In this scenario, the Heisenberg uncertainty principle is changed to the so-called Generalized (Gravitational) Uncertainty Principle (GUP) which results in modification of all Hamiltonians in quantum mechanics. In this paper, following a recently proposed GUP which is consistent with quantum gravity theories, we study the quantum mechanical systems in the presence of both a minimum length and a maximum momentum. The generalized Hamiltonian contains two additional terms which are proportional to αp^3 and $\alpha^2 p^4$ where $\alpha \sim 1/M_{Pl}c$ is the GUP parameter. For the case of a quantum bouncer, we solve the generalized Schrödinger equation in the momentum space and find the modified energy eigenvalues and eigenfunctions up to the second-order in GUP parameter. The effects of the GUP on the transition rate of ultra cold neutrons in gravitational spectrometers are discussed finally.

Keywords: Quantum gravity; Generalized uncertainty principle; Quantum bouncer.

1 Introduction

The modification of classical notion of the spacetime is one of the common features of all quantum gravity theories. In these theories, it is assumed that the usual concept of continuity of the spacetime manifold would break down when we probe distances smaller than the Planck length or energies larger than the Planck energy. If this fact is confirmed by future experiments, it could make a deep influence on our understanding about our surrounding universe. On the other hand, it may help us to find the answer of many unsolved problems such as the mechanism of singularity avoidance at early universe and also the black hole spacetime.

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One of the common properties of various candidates of quantum gravity such as string theory, loop quantum gravity and doubly special relativity is the existence of a minimum measurable length. Also, some evidence from black hole physics assert that a minimal length of the order of the Planck length arises naturally from any theory of quantum gravity. In addition, in the context of non-commutativity of the spacetime manifold, we also realize the existence of a minimal measurable length.

Evidently, this assumption is in apparent contradiction with the Heisenberg uncertainty principle which in principle agrees with the measurement of highly accurate results for a particles' positions or momenta, separately. In fact, in the Heisenberg picture, the minimum observable length is actually zero. So, if we are interested in to incorporate the idea of minimal length, we need to modify the Heisenberg uncertainty principle to the so-called Generalized Uncertainty Principle (GUP) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In other words, we should modify the commutation relations between position and momentum operators in the Hilbert space (deformed Heisenberg algebra). Moreover, in doubly special relativity theories, in order to preserve velocity of light and the Planck energy as two invariant quantities, the existence of a maximal momentum is essentially required [17, 18, 19].

In GUP formalism, the idea of a minimum observable length and a maximum observable momentum changes the usual form of all Hamiltonians in quantum mechanics (see ref. [20] and references therein). In fact, the modified Hamiltonians contain additional terms proportional to the powers greater than two of the momentum. So, in the quantum domain, the corresponding generalized Schrödinger equation has a completely different differential structure. More precisely, when we solve a forth-order generalized Schrödinger equation in the position space, some solutions are unphysical which should be discarded. However, if possible, it is more desirable to reduce the order of the differential equation by some methods such as solving the differential equation in the momentum space.

In this paper, we consider a recently proposed GUP which is consistent with string theory, doubly special relativity and black hole physics and predicts both a minimum measurable length and a maximum measurable momentum [21, 22, 23]. For this purpose, first we find the modified Hamiltonian of a general quantum mechanical system up to the second order of the GUP parameter α . Then, for the case of a particle which is bouncing elastically and vertically above a mirror in the Earth's gravitational field we solve the generalized Schrödinger equation in the momentum space and find the corresponding energy eigenvalues and eigenfunctions up to $\mathcal{O}(\alpha^2)$. In particular, we show that the existence of a maximal momentum reduces the effect of a minimal length on the energy spectrum which results in the reduction of the transition rate of ultra cold neutrons in gravitational spectrometers with respect to the case that the assumption of the maximal momentum is absent [24].

2 A generalized uncertainty principle

Recently, a GUP is proposed by Ali *et al.* which is consistent with the existence of the minimal measurable length and the maximal measurable momentum [21, 22]. In this proposal, the spaces of position and momentum are assumed to be commutative separately i.e. $[X_i, X_j] = [P_i, P_j] = 0$. Also, the following deformed Heisenberg algebra are satisfied

$$[X_i, P_j] = i\hbar \left[\delta_{ij} - \alpha \left(P\delta_{ij} + \frac{P_i P_j}{P}\right) + \alpha^2 \left(P^2 \delta_{ij} + 3P_i P_j\right)\right], \tag{1}$$

where $\alpha = \alpha_0/M_{Pl}c = \alpha_0\ell_{Pl}/\hbar$, $P^2 = \sum_{j=1}^3 P_jP_j$, M_{Pl} is the Planck mass, ℓ_{Pl} is the Planck length $\approx 10^{-35}m$, and $M_{Pl}c^2$ is the Planck energy $\approx 10^{19}$ GeV. Using the above commutation relations, we can obtain the generalized uncertainty relation in one-dimension up to the second order of the GUP parameter [21, 22]

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left[1 - 2\alpha \langle P \rangle + 4\alpha^2 \langle P^2 \rangle \right],$$

$$\geq \frac{\hbar}{2} \left[1 + \left(\frac{\alpha}{\sqrt{\langle P^2 \rangle}} + 4\alpha^2 \right) \Delta P^2 + 4\alpha^2 \langle P \rangle^2 - 2\alpha \sqrt{\langle P^2 \rangle} \right]. \tag{2}$$

The above inequality implies both a minimum length and a maximum momentum at the same time, namely [21, 22]

$$\begin{aligned}
\Delta X \ge (\Delta X)_{min} \approx \alpha_0 \ell_{Pl}, \\
\Delta P \le (\Delta P)_{max} \approx \frac{M_{Pl}c}{\alpha_0}.
\end{aligned}$$
(3)

We can also rewrite the position and momentum operators in terms of new variables

$$\begin{cases} X_i = x_i, \\ P_i = p_i \left(1 - \alpha p + 2\alpha^2 p^2 \right), \end{cases}$$

$$\tag{4}$$

where x_i and p_i obey the usual commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. It is straightforward to check that with this definition, eq. (1) is satisfied up to $\mathcal{O}(\alpha^2)$. Therefore, we can interpret p_i and P_i as follows: p_i is the momentum operator at low energies $(p_i = -i\hbar\partial/\partial x_i)$ and P_i is the momentum operator at high energies. Moreover, p is the magnitude of the p_i vector $(p^2 = \sum_{j=1}^3 p_j p_j)$. To study the effects of this kind of GUP on the quantum mechanical systems, let us consider the following general Hamiltonian

$$H = \frac{P^2}{2m} + V(\vec{R}). \tag{5}$$

Now, if we write the high energy momentum in terms of low energy one (4), we obtain

$$H = H_0 + \alpha H_1 + \alpha^2 H_2 + \mathcal{O}(\alpha^3), \tag{6}$$

where $H_0 = \frac{p^2}{2m} + V(\vec{R})$ and

$$H_1 = -\frac{p^3}{m}, \qquad H_2 = \frac{5p^4}{m}.$$
 (7)

Therefore, in the GUP scenario two additional terms proportional to αp^3 and $\alpha^2 p^4$ appear in the modified version of the Hamiltonian which the later is the result of the minimum length assumption and the former is the result of the maximum momentum assumption. In the next section, we consider the problem of a quantum bouncer in GUP formalism and find its modified eigenfunctions and eigenvalues up to $\mathcal{O}(\alpha^2)$ and compare our results with the ones which the second assumption is absent [24]. As an application, we show that GUP will affect the transition rate of ultra cold neutrons bouncing above a mirror in the Earth's gravitational field.

3 Modification of a quantum bouncer's spectrum in GUP scenario

To study the effects of GUP on the spectrum of a quantum bouncer, let us consider a particle of mass m which is bouncing elastically and vertically on an ideal reflecting floor in the earth's gravitational field so that

$$V(X) = \begin{cases} mgX & X > 0, \\ \infty & X \le 0, \end{cases}$$
(8)

where g is the acceleration caused by the gravitational attraction of the Earth. The Hamiltonian of the system is

$$H = \frac{P^2}{2m} + mgX,\tag{9}$$

which using eq. (6) casts in the form of the following generalized Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} - i\alpha\frac{\hbar^3}{m}\frac{\partial^3\psi(x)}{\partial x^3} + 5\alpha^2\frac{\hbar^4}{m}\frac{\partial^4\psi(x)}{\partial x^4} + mgx\psi(x) = E\psi(x).$$
 (10)

This equation is exactly solvable for $\alpha = 0$ and the solutions can be written in the form of the Airy functions. Also, the energy eigenvalues correspond to the zeros of the Airy function. For the case of $\alpha \neq 0$, we encounter a quite different situation. Because, the above equation is a forth-order differential equation which in general admits four independent solutions. However, some of these solutions are unphysical and should be discarded. One way to obtain physical solutions is to reduce the order of the differential equation which is fortunately possible in our case. In fact, if we write the above equation in the momentum space, because of the linear form of the potential term, it can be rewritten as a first order differential equation. Since the first order equation is much easier to handle, we define a new variable $z = x - \frac{E}{mg}$ and rewrite the above equation in the momentum space

$$\frac{p^2}{2m}\phi(p) - \alpha \frac{p^3}{m}\phi(p) + 5\alpha^2 \frac{p^4}{m}\phi(p) + i\hbar m g\phi'(p) = 0,$$
(11)

where $\phi(p)$ is the inverse Fourier transform of $\psi(z)$ and the prime denotes the derivative with respect to p. It is straightforward to check that this equation admits the following solution

$$\phi(p) = \phi_0 \exp\left[\frac{i}{6m^2g\hbar} \left(p^3 - \frac{3}{2}\alpha p^4 + 6\alpha^2 p^5\right)\right].$$
(12)

Since α is a small quantity, we can expand the above solution up to the second-order of α as

$$\phi(p) \simeq \phi_0 \exp\left(\frac{ip^3}{6m^2g\hbar}\right) \left(1 + \frac{i}{m^2g\hbar} \left[-\frac{\alpha p^4}{4} + \alpha^2 \left(p^5 + \frac{ip^8}{32m^2g\hbar}\right)\right] + \mathcal{O}(\alpha^3)\right).$$
(13)

Now, using the Fourier transform, we can obtain the solution in the position space. Before writing the solution, we should be careful about the nature of the terms appear in the above equation. Note that, the terms which obey $\phi^*(p) = \pm \phi(-p)$ result in real and imaginary terms in the wave function, respectively.

So the first term in the square brackets leads to an imaginary term in the wave function. Since the Hamiltonian is hermitian, both the real and imaginary parts of the wave function should satisfy the Schrödinger equation and vanish at x = 0, separately. Because the unperturbed wave function is real and the condition $\text{Im}[\psi(0)] = 0$ does not depend on α , the resulting energy spectrum and consequently the imaginary wave function will not be physical and should be discarded. This is in agreement with the fact that bound states of one-dimensional quantum systems should be real. Note that, the existence of an imaginary part of the wave function is due to the presence of p^3 in the perturbed Hamiltonian. We can also deduce this result by implementing the perturbation analysis. It is straightforward to check that the first-order correction of $H_1 = -\alpha p^3/m$ to the wave function is completely imaginary. However, since the unperturbed eigenfunctions are real functions of x, the first-order correction of H_1 to the energy spectrum is identically zero i.e. $\langle n|\frac{-p^3}{m}|n\rangle = -\frac{i\hbar^3}{m} \int_{-\infty}^{\infty} (\psi_n^0(x))^* \frac{\partial^3}{\partial x^3}(\psi_n^0(x)) dx \equiv 0$. Putting these facts together, we conclude that the effect of GUP on the eigenfunctions is at least second-order in GUP parameter and up to a normalization factor we have

$$\operatorname{Re}[\psi(x)] = \theta(x) \left[\operatorname{Ai} \left[\beta \left(x - \frac{E}{mg} \right) \right] + \alpha^2 m^2 g \left(x - \frac{E}{mg} \right) \times \left\{ 9 \operatorname{Ai} \left[\beta \left(x - \frac{E}{mg} \right) \right] + \left(x - \frac{E}{mg} \right) \operatorname{Ai'} \left[\beta \left(x - \frac{E}{mg} \right) \right] - \frac{1}{4} \beta^3 \left(x - \frac{E}{mg} \right)^3 \operatorname{Ai} \left[\beta \left(x - \frac{E}{mg} \right) \right] \right\} \right],$$
(14)

where $\theta(x)$ is the Heaviside step function, $\beta = \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}$ and the prime denotes derivative with respect to x. Finally, since the potential is infinite for $x \leq 0$, we demand that the wave function should vanish at x = 0. This condition results in the quantization of the particle's energy, namely

Ai
$$\left(-\frac{\beta E_n}{mg}\right) - \alpha^2 m E_n \left[9 \operatorname{Ai} \left(-\frac{\beta E_n}{mg}\right) - \frac{E_n}{mg} \times \right]$$

× Ai' $\left[\beta \left(x - \frac{E_n}{mg}\right)\right] \Big|_{x=0} + \frac{E_n^3}{2mg^2\hbar^2} \operatorname{Ai} \left(-\frac{\beta E_n}{mg}\right) = 0.$ (15)

To proceed further and for the sake of simplicity, let us work in the units of $\hbar = 1$, g = 2, and m = 1/2. In this set of units, the energy eigenvalues are the minus of the roots of the following algebraic equation

$$\operatorname{Ai}(x) + \frac{1}{2}\alpha^{2}x \left[9\operatorname{Ai}(x) + x\operatorname{Ai}'(x) - \frac{1}{4}x^{3}\operatorname{Ai}(x)\right] = 0.$$
(16)

So, the energy eigenvalues will be quantized and result in the following eigenfunctions

$$\psi_n(x) = \operatorname{Ai}(x - E_n) + \frac{1}{2}\alpha^2(x - E_n) \Big[9\operatorname{Ai}(x - E_n) + (x - E_n)\operatorname{Ai}'(x - E_n) - \frac{1}{4}(x - E_n)^3\operatorname{Ai}(x - E_n) \Big],$$
(17)

where E_n should satisfy eq. (16). Figure 1 shows the resulting normalized ground state and first excited state eigenfunctions for perturbed and unperturbed Hamiltonians with $\alpha = 0.1$. Moreover, we present the first ten energy eigenvalues for $\alpha = 0.01$ in table 1. These results, as we have expected, show that in the presence of GUP the energy spectrum slightly increases. So, the assumptions of a minimal length and a maximal momentum result in a positive shift in the energy spectrum of a quantum bouncer. However, as table 1 shows, this positive shift for the case of $H = H_0 + H_1 + H_2$ is smaller with respect to the case that we relax the assumption of a maximum momentum $H = H_0 + H_2$. This is due to the fact that when we impose an upper bound on the momentum, we actually eliminate the contribution of highly excited states. This effect is also observed in the perturbation analysis of a particle in a box and the harmonic oscillator in the GUP formalism [23].

We can also use these results for the case of ultra cold neutrons in an experiment with high precision neutron gravitational spectrometer which has been demonstrated few years ago [25, 26, 27]. In fact, the observation of spontaneous decay of an excited state and graviton emission in this experiment would be a Planck-scale physics effect [28]. The transition probability in the quadrupole approximation and in the presence of GUP is [24, 28]

$$\Gamma_{k \to n}^{\text{GUP}} \simeq \left(1 + \frac{5\Delta\lambda_{kn}}{\lambda_k - \lambda_n} \right) \Gamma_{k \to n},\tag{18}$$

where $-\lambda_n$ are the zeros of the Airy function, $\Delta\lambda_n = \frac{E_n}{E_0} - \lambda_n$, $E_0 = mg/\beta$, $\Delta\lambda_{kn} = \Delta\lambda_k - \Delta\lambda_n$, and

$$\Gamma_{k \to n} = \frac{512}{5(\lambda_k - \lambda_n)^3} \left(\frac{m}{M_{Pl}}\right)^2 \frac{E_0^5 c}{\beta^4 (\hbar c)^5}.$$
(19)

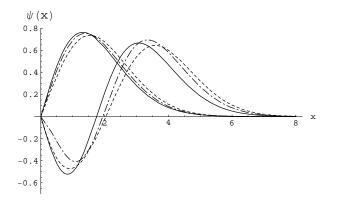


Figure 1: The normalized ground state and first excited state eigenfunctions of a quantum bouncer in the framework of the generalized commutation relation (1) for H_0 (solid line), $H_0 + H_2$ (dashed line), $H_0 + H_1 + H_2$ (dot-dashed line), with $\hbar = 1$, g = 2, m = 1/2, and $\alpha = 0.1$.

For instance, the probability of the spontaneous graviton emission from the first excited state to the ground state in the absence of GUP is $\Gamma_{1\to0} \sim 10^{-77} \text{s}^{-1}$ [28]. Incorporation of the GUP effect causes a shift in transition rate that are summarized in the last column of table 1. So it is essentially possible to find the effects of the generalized uncertainty principle on the transition rate of neutrons bouncing above a mirror in the Earth's gravitational field. Now, using table 1, we can compare the effect of H_1 [24] and $H_1 + H_2$ on the transition probability. Since for all states we have $\Delta \lambda_{kn}^{(02)} - \Delta \lambda_{kn}^{(012)} > 0$, we find that the existence of both a minimal length and a maximal momentum reduces the transition rate with respect to the presence of a minimal length alone.

To show the consistency of our approach with other quantum gravity models, let us consider the Hamiltonian of a (1+1)-dimensional quantum gravity model in the post-Newtonian approximation as $H = H^0 + H'$ where [29]

$$H^{0} = \frac{p^{2}}{m} + 2\pi G m^{2} |r|, \qquad (20)$$

and

$$H' = -\frac{p^4}{4m^3c^2} + \frac{4\pi G}{c^2}|r|p^2.$$
 (21)

Note that the second part of (20) has the form V(x) = mgx upon choosing $g \to 2\pi Gm$ and the first part of (21) has the form of $\alpha^2 H_2$ upon choosing $\alpha^2 \to \frac{1}{20m^2c^2}$. At this point, let us derive a relation for $\left|\frac{\Delta E_n}{E_n}\right|$, where $\Delta E_n = E_n - E_n^0$. By expanding eq. (15)

n	H_0	$H_0 + H_2$	$\Delta \lambda_{n(n-1)}^{(02)}$	$H_0 + H_1 + H_2$	$\Delta \lambda_{n(n-1)}^{(012)}$
0	2.3381	2.3392	-	2.3384	-
1	4.0879	4.0913	2.3×10^{-3}	4.0888	0.6×10^{-3}
2	5.5206	5.5267	2.7×10^{-3}	5.5221	0.6×10^{-3}
3	6.7867	6.7960	3.2×10^{-3}	6.7891	0.9×10^{-3}
4	7.9441	7.9569	3.5×10^{-3}	7.9475	1.0×10^{-3}
5	9.0226	9.0391	3.7×10^{-3}	9.0271	1.1×10^{-3}
6	10.040	10.061	4.5×10^{-3}	10.046	1.5×10^{-3}
7	11.008	11.033	4.0×10^{-3}	11.016	2.0×10^{-3}
8	11.936	11.965	4.0×10^{-3}	11.946	2.0×10^{-3}
9	12.829	12.862	4.0×10^{-3}	12.841	2.0×10^{-3}

Table 1: The first ten quantized energies of a quantum bouncer in GUP formalism for $\hbar = 1$, g = 2, m = 1/2, and $\alpha = 0.01$.

around the unperturbed solutions, the first and the last terms in the square bracket are negligible in comparison with the second term and using

$$\operatorname{Ai}\left(-\frac{\beta}{mg}E_n\right) \simeq -\frac{E_n - E_n^0}{mg}\operatorname{Ai}'\left[\beta\left(x - \frac{E_n}{mg}\right)\right]\Big|_{x=0} + \dots$$
(22)

we find

$$\left|\frac{\Delta E_n}{E_n}\right| \sim m\alpha^2 E_n,\tag{23}$$

which is in agreement with the general result of ref. [23]. Also from eq. (15) we have $E_n \approx -\beta^{-1}mga_n$ where a_n are the zeros of the Airy function and $\beta = \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}$. Thus, we find $\left|\frac{\Delta E_n}{E_n}\right| \sim m^2 \alpha^2 g \left(\frac{\hbar^2}{m^2g}\right)^{1/3} (-a_n)$. Finally, by choosing $g \to 2\pi Gm$ and $\alpha^2 \to \frac{1}{20m^2c^2}$ we obtain

$$\left| \frac{\Delta E_n}{E_n} \right| \sim \frac{1}{20} \frac{(2\pi\hbar G)^{2/3}}{c^2} (-a_n), \tag{24}$$

which agrees with the results of ref. [29].

As the final remark, let us estimate the actual magnitude of the GUP corrections to the quantum systems. To do this end, we need to use the numerical values of the fundamental constants c, \hbar and G and the neutron's mass (~ 10^{-27} Kg) and energy (~ 10^{-12} eV) [25, 26, 27] in the calculations presented above i.e.

$$\left|\frac{\Delta E_n}{E_n}\right| \sim \alpha_0^2 \frac{\ell_{Pl}^2}{\hbar^2} m E_n \sim 10^{-60} \alpha_0^2.$$
⁽²⁵⁾

It is usually assumed that the dimensionless parameter α_0 is of the order of unity [21]. In this case, the minimal measurable length is the Planck length ℓ_{pl} (3). Therefore, the α -dependent terms are important only when energies (lengths) are comparable to the Planck energy (length). So if we assume $\alpha_0 \sim 1$, the relative change in the Neutron's energy is of the order of $\mathcal{O}(10^{-60})$ which as we have expected is very tiny. However, if we relax this assumption, since the accuracy of Nesvizhevsky experiments is about $\frac{\Delta z}{z} \sim \frac{\Delta E}{E} \sim 10\%$ [25, 26, 27], where Δz denotes the uncertainty of the Neutron's position and E = mgz, the upper bound of α_0 would be

$$\alpha_0 \le 10^{29},\tag{26}$$

which is weaker than that predicted by the electroweak scale $\alpha_0 \leq 10^{17}$ [16, 21]. Therefore, the more accurate measurements indeed reduce the upper bound on α_0 or show the effects of GUP on the spectrum of the ultra cold neutrons.

4 Conclusions

In this paper we have studied the effects of a recently proposed Generalized Uncertainty Principle on quantum mechanical systems. This form of GUP is consistent with various candidates of quantum gravity such as string theory, doubly special relativity and black hole physics which also implies a maximum observable momentum. We showed that the presence of a minimal length and a maximal momentum results in the modification of all Hamiltonians in quantum mechanics. In fact, the modified Hamiltonians contain two additional terms proportional to αp^3 and $\alpha^2 p^4$ which result in a fourth-order generalized Schrödinger equation. For the case of a quantum bouncer, to avoid unphysical solutions, we solved it in the momentum space as a first-order differential equation and obtained the energy eigenvalues and eigenfunctions up to the second order of the GUP parameter. We showed that, although the additional term H_1 has no first-order contribution in the solutions, it has a second-order contribution and reduces the effect of the second term H_2 on the energy spectrum. In other words, the upper limit on the momentum excludes the contribution of highly excited states. This result is also in agreement with previous perturbative studies regarding other quantum mechanical systems. Moreover, the presence of a maximal momentum reduces the transition rate of ultra cold neutrons bouncing above a mirror in the Earth's gravitational field in comparison with the case that only H_2 is present. We note that if these effects be confirmed by future experiments, they could make a deep influence on our understanding about our surrounding universe and also on ultimate formulation of the quantum gravity proposal.

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