Ref.TH.1706-CERN

ON THE EQUIVALENCE OF THE EUCLIDEAN AND WIGHTMAN FORMULATION OF FIELD THEORY

V. Glaser
CERN - Geneva

A B S T R A C T

A mistake in paper [1] on the "Axioms for Euclidean Green's Functions" is corrected in the following sense: thanks to these axioms the Euclidean Schwinger functions S_n can be analytically continued to the corresponding Wightman functions W_n possessing all the correct analyticity properties and satisfying a generalized positivity condition in the complex domain. It is however suggested by the proof that their tempered behaviour near the Minkowski points cannot be guaranteed without additional assumptions.

Ref.TH.1706-CERN
17 July 1973

1. INTRODUCTION

The very interesting paper on the "Axioms for Euclidean Green's functions" by Osterwalder and Schrader [1] claims to prove the equivalence of the usual Wightman axioms with axioms for the Euclidean Green's functions as formulated in 1). Unfortunately the crucial Lemma 8.8 of that paper turns out to be wrong *).

Some years ago, the present author had studied the inter-relation between the positivity condition and the analyticity properties of the Green's functions in momentum space [2]. As pointed out in that paper, the theorems proved there could be easily translated into analogous theorems on the Wightman functions in x space. It turns out that these theorems are essentially sufficient to prove the statements made in 1); but in a restricted sense. The Euclidean Green's functions satisfying the Osterwalder-Schrader postulates can be shown to be restrictions of functions analytic in the whole Wightman causal domain and to satisfy the positivity condition there in a sense to be presently explained. The author has, however, not been able to show the tempered growth of these analytic functions near the real Minkowski space boundary and he believes at present that this is impossible to achieve without further assumptions on the growth properties of the Schwinger functions S_n with respect to the index n. This is suggested by the fact that in order to reach the real Minkowski space by analytic completion for a given S_n an infinite number of steps are required, each of which involves the other functions S_m via the Schwartz inequality with higher and higher values of m. The problem of temperedness is intimately related to the old unanswered question of whether the Wightman axioms with $W_n \in \mathcal{D}'$ imply $W_n \in \mathcal{S}'$ as a consequence of the positivity condition and the Bochner-Schwartz theorem.

Since the boundary value of a function analytic in a tube, whatever be its growth, is always well defined in the sense of hyperfunctions, we may say that a set of Schwinger functions satisfying the O.S. postulates gives rise to a generalized Wightman field theory, the field being an operator valued hyperfunction. In view of the fact that hyperfunctions still admit the notion of support, and hence of causality, it would be of interest to investigate whether such a field theory still allows the construction of asymptotic states.

---

*) This fact has been first discovered by the authors themselves, as pointed to me by R. Sénéor and Ph. Blanchard.

2. A REFORMULATION OF THE PROBLEM

The Wightman axioms can be formulated in terms of analytic functions in the following way. Consider the Hilbert space vectors

$$\phi_n(\underline{z}) = A(z_1)\,A(z_2)\cdots A(z_n)\,\Omega \qquad \Omega = \text{vacuum} \tag{1}$$

From the spectral condition and translational covariance of the field operator A it follows [cf., Ref. 3)] that they are "tempered" analytic functions in the tubes

$$\sigma_n = \left\{ \underline{z} \in \mathbb{C}^{4n} \mid \mathrm{Im}\, z_1 \in V_+ ,\ \mathrm{Im}(z_k - z_{k-1}) \in V_+ \quad k = 2 \ldots n \right\} \tag{2}$$

Their boundary values on $\mathbb{R}^{4n}$ exist therefore in the sense of vector valued tempered distributions. Local commutativity and the edge-of-the-wedge theorem imply that they can be analytically continued into the domain

$$\Sigma_n = \mathcal{H}\Big(\bigcup_{\pi \in P_n} \sigma_n^{\pi} \cup \mathcal{N}(\mathcal{R}) \Big) \equiv \mathcal{H}(\Sigma_n^{\circ}) \tag{3}$$

which is the envelope of holomorphy of the union of all the permuted tubes σ_n^{π} with a complex neighbourhood of the real spacelike points.

From this follows the positivity condition

$$\sum_{n,m} \int A_{nm}(\underline{z}', \underline{z})\, f_n(\underline{z}')\, \bar{f}_m(\underline{z})\, d\mu_n(\underline{z}')\, d\mu_m(\underline{z}) \geq 0 \tag{4}$$

where

$$A_{nm}(\underline{z}', \underline{z}) = (\phi_n(\underline{z}'), \phi_m(\underline{z})) \equiv \mathcal{W}_{n+m}(\overleftarrow{\underline{z}}', \underline{z}) \tag{5}$$

is the $n+m^{\text{th}}$ Wightman function, $\overleftarrow{\underline{z}}' = (\bar{z}_n', \bar{z}_{n-1}', \ldots, \bar{z}_1')$ analytic in Σ_m with respect to $\underline{z}$ and antianalytic in Σ_n with respect to $\underline{z}'$. To begin with, (4) is certainly true for, say, all terminating sequences $\{f_n\}_0^{\infty}$ of test functions $f_n \in \mathcal{S}(\mathbb{C}^{4n})$, the $d\mu_n$ being any positive measures on $\mathbb{C}^{4n}$ with compact support in Σ_n. In order to display the tempered growth of the analytic functions $\mathcal{W}_{n+m}$ near the boundary values in the sense of distributions, it is, however, more appropriate to take $d\mu_n$ of the form

$$d\mu_n(\underline{\xi}) = e^{-\varphi_n(\underline{\xi})}\, d\lambda_n(\underline{\xi}) \tag{6}$$

where $d\lambda_n$ is the Legesgue measure in $\mathbb{C}^{4n}$ and the φ_n are appropriately chosen C^∞ positive functions defined in Σ_n of at most logarithmic growth near the boundary $\partial\Sigma_n$ chosen in such a way that

$$tr\, A \equiv \sum_{n=0}^{\infty} \int_{\Sigma_n} A_{nn}(\bar{\underline{\xi}}, \underline{\xi})\, d\mu_n(\underline{\xi}) < \infty \tag{7}$$

Here $A = \left\{ A_{nm}(\bar{\underline{Z}}', \underline{Z}) \right\}$ is the operator which acts in an obvious way on the Hilbert space of sequences $\underline{f} = (f_0, f_1, f_2, \ldots)$ of functions f_n <u>analytic</u> in Σ_n equipped with the norm

$$(\underline{f}, \underline{f}) = \sum_{n=0}^{\infty} \int_{\Sigma_n} |f_n(\underline{\xi})|^2\, d\mu_n(\underline{\xi}) < \infty \tag{8}$$

Conversely, given such a measure $d\mu = (d\mu_0, d\mu_1, \ldots)$ on $\bigoplus_0^\infty \Sigma_n$, every positive operator A acting on the Bergman Hilbert space $\mathcal{H}(d\mu)$ of analytic functions defined by (8) will have as its matrix elements $A_{nm}(\bar{\underline{Z}}', \underline{Z})$ functions anti-analytic-analytic in $\Sigma_n \times \Sigma_m$ with growth properties completely controlled by the measures (6). Since the envelopes of holomorphy Σ_n are explicitly unknown we may, without any loss, replace them with the corresponding primitive domains Σ_n^o (3), in which the admissible forms of the weight functions φ_n (6) can, in principle, be explicitly determined with the help of the edge-of-the-wedge theorem.

This version of the positivity condition, which is a translation of [Ref. 1), pp. 83-85] from momentum into x space, trivializes in a certain sense the causality condition. The hard part remains, of course, the translational invariance as indicated by the last equality in (5).

Turning back to formulas (4) and (5), one may try to concentrate the positive measures $d\mu_n$ on $\Sigma_n \cap \mathbb{E}_n$, where $\mathbb{E}_n = \{ \underline{Z} \in \mathbb{C}^{4n} : Z_i = (ix_i^o, \vec{x}_i) \equiv \equiv \underset{\sim}{x}_i$, x_i^o and $\vec{x}_i$ real, $i = 1, \ldots, n \}$ is the Euclidean subspace of $\mathbb{C}^{4n}$. By doing so, one gets from the Wightman postulates the Euclidean Osterwalder-Schrader postulates for the Schwinger functions

$$S_m(x_1, \ldots x_m) = W_m(z_1, \ldots z_m)\Big|_{\underline{z} \in \mathbb{E}_m} \tag{9}$$

E0 : Temperedness : $\mathcal{S}_0 = 1, \quad \mathcal{S}_n \in \mathring{\mathcal{S}}'(\mathbb{R}^{4n})$

E1 : Euclidean invariance

E2 : Positivity

E3 : Symmetry : $\mathcal{S}_n(x_{\pi 1},\ldots,x_{\pi n}) = \mathcal{S}_n(x_1,\ldots,x_n)$

E4 : Cluster property

What we want to show here is that E0-E3 conversely imply the existence of the Lorentz invariant functions $\mathcal{W}_n$ analytic in the correct primitive Wightman domain satisfying the positivity condition (4), (5) as well as the restriction relation (9). For that purpose, it is enough, as it was shown in Ref. 1), to restrict the positivity condition to the subsets $\sigma_n \cap \mathbb{E}_n$ of $\Sigma_n \cap \mathbb{E}_n$, where σ_n is the unpermuted tube (2), and to prove then that E0, E1 and E2 imply the analytic continuation of the distributions

$$S_n(x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}) = \mathcal{S}_n(x_1, \ldots, x_n) \tag{10}$$

for $x_1^0 < x_2^0 < \ldots < x_n^0$, to the tubes

$$\mathcal{T}_{n-1} = \left\{ \mathrm{Im}(z_k - z_{k-1}) \in V_+ , \; k = 2 \ldots n \right\} \tag{11}$$

The Schwinger functions S_n and $\mathcal{S}_n$ are related to each other in the same way as the Wightman functions W_n and $\mathcal{W}_n$, viz.,

$$W_n(z_2 - z_1, \ldots, z_n - z_{n-1}) = \mathcal{W}_n(z_1, \ldots, z_n) \tag{10'}$$

After having obtained analyticity in the tubes τ_{n-1}, the analytic continuation to the extended tubes τ'_{n-1} is immediate due to E1, and the symmetry E3 then implies analyticity in the union of the permuted extended tubes, which is equivalent to causality by very well-known theorems [cf., Ref. 3), p.85].

The invariance by translations as expressed by (10') leads to the following form of the positivity condition (4) on the unpermuted tubes σ_n :

$$\sum_{n,m} \int W_{n+m}(\overleftarrow{\underline{\xi}}', z - \bar{z}', \underline{\xi}) \, f_n(z', \underline{\xi}') \, \bar{f}_m(z, \underline{\xi}) \, d\mu_n(z', \underline{\xi}') \, d\mu_m(z, \underline{\xi}) \geq 0 \tag{12}$$

with

$$W_{n+m}(\overleftarrow{\underline{\xi}}', z-\bar{z}', \underline{\xi}) = W_{n+m}(-\bar{\xi}'_{n-1}, \dots, -\bar{\xi}'_1, z-\bar{z}', \xi_1 \dots \xi_{m-1})$$

In this formula the positive measures $d\mu_n$ must have their support in τ_+^{xn}, where $\tau_+ = \{ Z \in \mathbb{C}^4 : \mathrm{Im}\, Z \in V_+ \}$. The Euclidean positivity condition (4.3) of Ref. 1) is obtained in our notation by concentrating the measures $d\mu_n$ on $\sigma_n \cap \mathbb{E}_n$. In order to simplify the notation, we shall suppose space-time to be one-dimensional for the purpose of the next section, since the space components of the four vectors would remain passive spectators anyway. The expression (4.3) of Ref. 1) is then obtained by inserting into (12)

$$d\mu_n(z, \underline{\xi}) = \delta(x)\, \delta_{n-1}(\underline{\xi})\, dx\, dy\, d^{n-1}\xi\, d^{n-1}\eta \qquad (13)$$

where $Z = x+iy$, $\underline{\zeta} = \underline{\xi}+i\underline{\eta}$, and taking

$$f_n(z, \underline{\xi}) = \varphi_n(y, \underline{\eta}) \in \mathcal{S}(\mathbb{R}_+^n) \qquad (14)$$

where $\mathcal{S}(\mathbb{R}_+^n)$ is the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of elements φ_n with support in $\{ y \geq 0,\ y_1 \geq 0, \dots, y_{n-1} \geq 0 \} = \mathbb{R}_+^n$. Under these conditions we may substitute into (12)

$$W_{n+m}(i\eta'_{n-1}, \dots, i\eta'_1, i(y+y'), i\eta_1 \dots i\eta_{m-1}) = \qquad (15)$$
$$= S_{n+m}(\eta'_{n-1}, \dots \eta'_1, y+y', \eta_1 \dots \eta_{m-1})$$

where according to Ref. 1), S_{n+m} is to be considered as a tempered distribution in $\mathcal{S}(\mathbb{R}^{n+m-1})$ with support in $\mathbb{R}_+^{n+m-1}$.

We are now ready to prove :

Theorem 1

Conditions (12)-(15) imply the analytic continuation of $S_n(x_1, \dots, x_{n-1})$ into the topological product of the right complex half planes

$$\mathcal{P}_+^{n-1} = \{ \underline{z} = \underline{x} + i\underline{y} \in \mathbb{C}^{n-1} \mid \underline{x} \in \mathbb{R}_+^{n-1} \}$$

or, equivalently, of $W_n(iy_1, \dots, iy_{n-1})$ into $i\mathcal{P}_+^{n-1}$.

This replaces the too strong Lemma 8.8 of Ref. 1).

3. PROOF OF THEOREM 1

From (12)-(15) it follows, as it was shown in Section 4.1 of Ref. 1), that the distribution W_{n+1} (15) can be continued analytically in each variable separately to the whole upper complex half-plane. More precisely, the n functions

$$f_\nu = W_{n+1}(iy_1 \cdots iy_{\nu-1}, z_\nu, iy_{\nu+1}, \cdots iy_n) \tag{16}$$

are tempered analytic in $\mathrm{Im}\, z_\nu > 0$ and take their values in tempered distribution with support in $y_k \geq 0$ with respect to the variables y_k, $k = 1,\ldots,\hat{\nu},\ldots,n$. This follows also from Corollary 1 to Theorem 3, p. 86, of Ref. 2). We mention it because Th.3, being a purely local theorem, implies (16) also in the case of $S_{n+1} \in \mathcal{D}'(\mathbb{R}^n_+)$ (see the Appendix), while the method used in Ref. 1) necessitates $S_{n+1} \in \mathcal{S}'$.

Since the functions f_ν all analytically continue the same distribution $W_{n+1}(i\underline{y})$, we are faced with a typical degenerate edge-of-the-wedge problem [the case of the Malgrange-Zerner theorem, cf., Ref. 4)]. In the variables

$$w_k \equiv u_k + i v_k = \log(-i z_k) \qquad k = 1 \cdots n \tag{17}$$

we readily see that the functions $\varphi_\nu(\underline{w}) = f_\nu(ie^{w_1},\ldots,ie^{w_n})$ are analytic in the "flat tubes"

$$\mathcal{T}_\nu = \left\{ \underline{w} \in \mathbb{C}^n \mid \mathrm{Im}\, w_k = 0 \quad \forall\, k \neq \nu \quad |\mathrm{Im}\, w_\nu| < \frac{\pi}{2} \right\}$$

and coincide all for $\{ \mathrm{Im}\, w_k = 0, \; k = 1,\ldots,n \}$. The envelope of holomorphy of this problem is given by the convex envelope of $\bigcup_1^n \mathcal{T}_\nu$ and we get :

Lemma 1

The distributions $W_{n+1}(i\underline{y})$, $n = 1,2,\ldots,$ are restrictions to

$$i\,\mathring{\mathbb{R}}^n_+ = \left\{ \mathrm{Im}\, z_k > 0 \qquad \mathrm{Re}\, z_k = 0 \qquad k = 1 \cdots n \right\}$$

of functions $W_{n+1}(\underline{z})$ analytic in

$$\mathcal{K}_n = \left\{ \underline{z} \in \mathbb{C}^n \mid \sum_1^n |\arg z_k - \frac{\pi}{2}| < \frac{\pi}{2} \right\}$$

The distribution part of the Lemma can be easily handled even for the case $W \in \mathcal{D}'$.

$\mathcal{K}_n$ is not the topological product of the upper half-planes $i\mathcal{P}_+^n$ as claimed by Lemma 8.8 of Ref. 1) and therefore a counter-example to it can be readily constructed.

Lemma 1 implies :

Corollary to Lemma 1 :

The tempered distributions $W_{n+1}(i\underline{x}) = S_{n+1}(\underline{x})$ are real analytic functions on $\overset{\circ}{\mathcal{R}}{}_+^n$ and therefore polynomially bounded together with all their derivatives :

$$|D^\alpha S_{n+1}(\underline{x})| < C \frac{(1+|\underline{x}|)^M}{[\min_{1\le k\le n} x_k]^N}$$

C,N,M being some α dependent positive constants.

In order to prove Theorem 1, we need the following two theorems, extracted from Ref. 2) :

Theorem A :

Let the set of functions $A_{nm}(\underline{z}',\underline{z})$, $n,m = 0,1,2,\ldots$, be anti-analytic-analytic in $\Sigma_n \times \Sigma_m$, where Σ_n are connected open sets in $\mathbb{C}^{4n}$ containing the open Euclidean sets $\mathcal{R}_n \subset \mathcal{E}_n$ ($\mathcal{R}_n \subset \Sigma_n$, $n = 0,1,2,\ldots$; $\mathcal{R}_n$ open relatively to the Euclidean subspace $\mathcal{E}_n$ of $\mathbb{C}^{4n}$). Let furthermore the A_{nm} satisfy the positivity condition (4) on the sets $\mathcal{R}_n$ with $d\mu_n$ = the Lebesgue measure concentrated on $\mathcal{E}_n$ for all terminating sequences of C^∞ test functions f_n with compact support in $\mathcal{R}_n$. Then the positivity condition (4) remains valid in all of Σ_n, i.e., for $d\mu_n$ = any positive measure with compact support in Σ_n and for all terminating sequences of test functions $f_n \in C^\infty(\mathbb{C}^{4n})$.

Theorem B : ("Theorem of the diagonal")

Let the set of functions A_{nm} of Theorem A satisfy the positivity condition (4) on the open sets $\Sigma_n \subset \mathbb{C}^{4n}$. Suppose the diagonal elements $A_{nn}(\overline{z},z)$ (both arguments equal !), $n = 0,1,2,\ldots$, which are real analytic functions on Σ_n, can be analytically continued as real analytic functions to some larger open connected sets $\Omega_n \supset \Sigma_n$.

Then all the functions $A_{nm}(\bar{Z}', Z)$ $n,m = 0,1,2,\ldots$, can be continued as antianalytic-analytic functions into the domains $\Omega_n \times \Omega_m$. By Theorem A the positivity condition continues to hold in Ω_n, $n = 0,1,2,\ldots$

Theorem A is a mixture of Theorems 1 and 4 of Ref. 2) and Theorem B is contained in Theorem 4 of Ref. 2).

Remark :

Both theorems are easily proved with the help of the Schwartz inequalities

$$\left| A_{mm}^{\alpha\beta}(\underline{\bar{z}}', \underline{z}) \right|^2 \leq A_{nn}^{\alpha\alpha}(\underline{\bar{z}}', \underline{z}') \, A_{mm}^{\beta\beta}(\underline{\bar{z}}, \underline{z}) \cdot A_{\gamma\gamma}^{\gamma\gamma}(\underline{\bar{z}}, \underline{z}) \geq 0 \tag{18}$$

where

$$A_{\mu\nu}^{\alpha\beta}(\underline{\bar{z}}', \underline{z}) = \left(\frac{\partial}{\partial \underline{\bar{z}}'}\right)^{\alpha} \left(\frac{\partial}{\partial \underline{z}}\right)^{\beta} A_{\mu\nu}(\underline{\bar{z}}', \underline{z})$$

α and β being any multi-indices. These are obtained from (4) by, e.g., choosing for the test functions $\underline{f} \in \mathcal{D}$ sequences which tend in the topology of $\mathcal{D}'$ to appropriate linear combinations of derivatives of the Dirac measure. The theorems are then first proved for the diagonal elements A_{nn} by taking $m = n$ in (18) and then extended to off-diagonal elements A_{nm} by the use of (18). The purpose of this remark is to stress the fact that the analytic behaviour off the diagonal is completely governed by the analytic structure on the diagonal.

In order to prove Theorem 1, we want to enlarge the domains $\mathcal{K}_n$ of Lemma 1 by the repeated use of Theorems A and B. In our case we have

$$A_{nm}(\underline{\bar{z}}', \underline{z}) = W_{n+m}(\underleftarrow{\xi}', z - \bar{z}', \underline{\xi}) \tag{19}$$

$$A_{nn}(\underline{\bar{z}}, \underline{z}) = W_{2n}(\underleftarrow{\xi}, z - \bar{z}, \underline{\xi}) \equiv f(z - \bar{z}) \tag{19'}$$

To start with, (19) is only defined for positive purely imaginary arguments $\underline{\xi}^{(1)} = i\underline{y}^{(1)}$, $Z^{(1)} = iy^{(1)}$. If we assume that the W_{n+1} are real analytic on $i\overset{\circ}{\mathbb{R}}{}_+^n$ - and this we already know by the argument leading to Lemma 1 - (19') is also defined and real analytic for $\operatorname{Im} Z > 0$ since $Z - \bar{Z} = 2i \operatorname{Im} Z$.

For fixed $\underline{\xi} = i\underline{y}$ it follows then from Theorems A and B that $f(Z-\bar{Z})$ is the restriction of a function $f(Z-\bar{Z}')$ antianalytic-analytic in $\{ \operatorname{Im} Z' > 0 \} \times \{ \operatorname{Im} Z > 0 \}$, that is, f is analytic in the whole upper half-plane. Moreover, from (18) with $n = m$ we have the inequalities

$$| f^{(\alpha+\beta)}(z-\bar{z}') |^2 \leq i^{2\alpha} f^{(2\alpha)}(z-\bar{z})\, i^{2\beta} f^{(2\beta)}(z'-\bar{z}') \qquad (20)$$

$$\alpha, \beta = 0, 1, 2 \ldots$$

which imply, in particular,

$$| f(x+iy) | \leq f(iy) \qquad (21)$$

for all $-\infty < x < \infty$, $y > 0$ and

$$0 \leq f(i(y+y')) \leq f^{1/2}(2iy)\, f^{1/2}(2iy') \qquad (21')$$

for all $y, y' > 0$. By the replacement $(y,y') \to (y+\frac{a}{2}, \frac{a}{2})$, $y, a > 0$, (21') becomes

$$0 \leq f(iy+ia) \leq f^{1/2}(2iy+ia)\, f^{1/2}(ia) \leq f^{1/2^n}(2^n iy + ia) \cdot f^{\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}}(ia)$$

Since by the Corollary to Lemma 1 $f(iy)$ is polynomially bounded for $y \to 0$, the right-hand side converges for $n \to \infty$ and we get finally

$$| f(z) | \leq f(i \operatorname{Im} z) \leq f(iy') \qquad (22)$$

for all $\operatorname{Im} Z \geq y' > 0$. [Compare formula (4.9) of Ref. 1).] Similar inequalities hold also for the derivatives of f. The off-diagonal part of Theorems A and B, combined with (22), leads then to the inequality

$$| W_{n+m}(\overleftarrow{\underline{\xi}}', z, \underline{\xi}) | \leq W_{2n}^{1/2}(\overleftarrow{\underline{\xi}}', ia', \underline{\xi}')\, W_{2m}^{1/2}(\overleftarrow{\underline{\xi}}, ia, \underline{\xi}) \qquad (23)$$

valid for all $\operatorname{Im} Z \geq \frac{1}{2}(a+a') > 0$, $a > 0$, $a' > 0$. Here W_{n+m} is analytic in the upper half-plane $\operatorname{Im} Z > 0$ and real analytic in $\underline{\xi}' = i\underline{y}' \in i\, \overset{\circ}{\mathbb{R}}{}_+^{n-1}$ and $\underline{\xi} = i\underline{y} \in i\, \overset{\circ}{\mathbb{R}}{}_+^{m-1}$.

This last inequality tells us that the functions f_ν (16) leading to Lemma 1 are bounded in their domain of analyticity by the functions $W^{\frac{1}{2}}_{2\nu} W^{\frac{1}{2}}_{2(n+1-\nu)}$ of purely imaginary arguments. Therefore the function $W_{n+1}(\underline{Z})$ will be bounded by the same set of functions, $\nu = 1,\ldots,n+1$, in the holomorphy envelope $\mathcal{K}_n$ of Lemma 1. We shall not attempt here to determine the exact law of the propagation of these bounds into $\mathcal{K}_n$. It can, however, be concluded, using the methods of Ref. 5) that W_{n+1} is again polynomially bounded in $\mathcal{K}_n$.

Knowing now that the functions W_{2n} are analytic in $\mathcal{K}_{2n-1}$ by extracting information out of the functions $W_{2\nu}$, $\nu = 1,\ldots,2n-1$, as just described, we conclude that

$$W_{2n}\left(\underline{\xi}, z-\bar{z}, \underline{\xi}\right)$$

is real analytic in $\left\{ \underline{\xi} \in \mathcal{K}_{2(n-1)} \right\} \times \left\{ \mathrm{Im}\, z > 0 \right\}$

The argument leading from (19') to (23) tells us that W_{n+1} is analytic in the domain

$$\mathring{\mathcal{K}}_n^{(1)} = \bigcup_{\nu=1}^{n} \mathcal{T}_\nu^{(1)}$$

with

$$\mathcal{T}_\nu^{(1)} = \left\{ \underline{z} \in \mathbb{C}^n \;\middle|\; (z_1 \ldots z_{n-1}) \in \mathcal{K}_{2(n-1)},\ \mathrm{Im}\, z_\nu > 0,\ (z_{\nu+1} \ldots z_n) \in \mathcal{K}_{2(n-\nu+1)} \right\}$$

Now, the domains $\tau_\nu^{(1)}$, which are again tubes in the angular variables (17), are strictly larger than the corresponding domains τ_ν of the first step and therefore the envelope of holomorphy $\mathcal{H}\left(\mathring{\mathcal{K}}_n^{(1)}\right) = \mathcal{K}_n^{\nu(1)}$, which is simply the convex envelope of $\mathring{\mathcal{K}}_n^{(1)}$ in the angular variables (17), is strictly larger than $\mathcal{K}_n$. By repeating the process, we get a strictly increasing sequence of domains

$$\mathcal{K}_n \subset \mathcal{K}_n^{(1)} \subset \mathcal{K}_n^{(2)} \subset \cdots$$

which are all convex tubes in the angular variables (17), and what remains to be shown is that

$$\lim_{l \to \infty} \mathcal{K}_n^{(l)} = i\,\mathcal{P}_+^n$$

That this is indeed the case may be seen in the following, maybe not the most economic way. Let us first consider the set of functions W_{n+1} with only two complex variables, the rest of the variables being kept purely imaginary :

$$F_{n+1}(z_1,z_2) \equiv W_{n+1}(iy_1 \dots iy_{\mu-1}, z_1, iy_{\mu+1} \dots iy_{\nu-1}, z_2, iy_{\nu+1} \dots, iy_n) \quad (24)$$

Lemma 1 implies that all F_{n+1} are analytic in the domain

$$\mathcal{D}_2 = \{ (z_1,z_2) \in \mathbb{C}^2 \mid |\arg z_k - \tfrac{\pi}{2}| < \tfrac{\pi}{4} \quad k=1,2 \}$$

The repeated use of positivity yields then the domain

$$\mathring{\mathcal{D}}_2^{(1)} = \bigcup_{\nu=1}^{2} \mathcal{T}_\nu^{(1)}$$

$$\mathcal{T}_\nu^{(1)} = \{ |\arg z_\nu - \tfrac{\pi}{2}| < \tfrac{\pi}{2} \quad |\arg z_k - \tfrac{\pi}{2}| < \tfrac{\pi}{4} \quad k \neq \nu \}$$

for all the functions F_{n+1}.

The convex envelope of $\mathring{\mathcal{D}}_2^{(1)}$ contains the domain

$$\mathcal{D}_2^{(1)} = \{ |\arg z_k - \tfrac{\pi}{2}| < \tfrac{\pi}{2}(\tfrac{1}{2}+\tfrac{1}{4}) \quad k=1,2 \}$$

after $\ell+1$ steps, we get the domain

$$\mathcal{D}_2^{(\ell)} = \{ |\arg z_k - \tfrac{\pi}{2}| < \tfrac{\pi}{2}(\tfrac{1}{2}+\tfrac{1}{4}+\dots+\tfrac{1}{2^{\ell+1}}) \quad k=1,2 \}$$

which converges to the topological product of the upper half-planes. We find therefore that all the functions (24) are analytic in $\mathrm{Im}\, Z_k > 0$, $k=1,2$, while remaining real analytic in the rest of the variables. By using positivity just once more with respect to a variable lying between Z_1 and Z_2, we get analyticity of all the W_{n+1} functions in the topological product of the upper half-planes in any three variables, while the others are kept purely imaginary. Making use of this information, we now complexify any four of the variables in W_{n+1}. Again a denumerable number of steps is needed in order to reach the topological product of four complex upper half-planes, while the five-fold topological product is obtained with just one step more. By continuing this procedure, one obtains the proof of Theorem 1. The somewhat tedious geometrical details will be omitted here.

Remark :

In order to reach the domain $\mathcal{H}_n^{(\ell)}$ for the function W_{n+1} the set of functions $W_{2\nu}(i\underline{x})$ $\nu = 1,2,\ldots,2^{\ell}n$ gets involved. While the temperedness of the function W_{n+1} in the domain $\mathcal{H}_n^{(\ell)}$ can be proved, the author does not see how this could continue to be true in the limit $\ell \to \infty$ without some further assumptions on the behaviour of the functions

$$\left| W_{2^{\ell}\nu}(i\underline{x}) \right|^{\frac{1}{2^{\ell}}} \qquad \nu = 1, 2, 3 \ldots \tag{25}$$

when $\ell \to \infty$. It would therefore be very interesting to find physically reasonable conditions on the quantities (25) which indeed guarantee temperedness in $i\mathcal{P}_+^n$.

4. THE ANGULAR CONTINUATION OF $S_{n+1}(\underline{x})$ INTO THE TUBE τ_n

We turn again our attention to the case of four-dimensional space-time and write

$$S_{n+1}(\underline{x}) = W_{n+1}(ix_1^0, \vec{x}_1 \ldots ix_n^0, \vec{x}_n) \equiv S_{n+1}(\underline{x}^0, \underline{\vec{x}}) \tag{26}$$

To start with the S_{n+1} are tempered distribution with support in

$$\mathbb{R}_+^{4n} = \{ \underline{x} \in \mathbb{R}^{4n} \mid x_k^0 \geq 0 \quad k = 1 \ldots n \} \tag{27}$$

The considerations of the last section are now to be applied to the distributions

$$\Delta_{n+1}(\underline{x}^0) = \int S_{n+1}(\underline{x}^0, \underline{\vec{x}}) \prod_{i=1}^{n} \varphi_i(\vec{x}_i)\, d^3\vec{x}_i \tag{28}$$

with fixed $\varphi_i \in \mathcal{S}(\mathbb{R}^3)$, respectively, $W_{n+1}(ix^0) = s_{n+1}(\underline{x}^0)$, which replace the quantities $S_{n+1}(\underline{x}) \neq W_{n+1}(i\underline{x})$, $\underline{x} \in \mathbb{R}_+^n$ of that section.

From the theory of the analytic continuation of analytic functions with values in nuclear linear topological vector spaces [cf., Refs. 3) or 6)] we get the generalization of Lemma 1 :

Lemma 1' :

The distributions $S_{n+1}(\underline{x})$, $n = 1,2,\ldots$, are restrictions of functions $S_{n+1}(\underline{z}^0, \vec{\underline{y}})$ analytic in

$$\mathcal{K}_m = \{ \underline{z}^0 \in \mathbb{C}^m \mid \sum_{k=1}^{m} |\arg z_k^0| < \frac{\pi}{2} \}$$

with values in tempered distributions $\mathcal{S}'(\mathbb{R}^{3n})$.

We want to show first that Lemma 1' and the Euclidean invariance of S_{n+1}, i.e.,

$$S_{n+1}(R\underline{y}) = S_{n+1}(\underline{y}) \tag{29}$$

for all $R \in SO_4$ such that both $\underline{y}$ and $R\underline{y} \in \mathring{\mathbb{R}}_+^{4n}$, imply

Lemma 2 :

$S_{n+1}(\underline{x})$, $n = 1,2,3,\ldots$, is real analytic in

$$\mathring{\mathbb{R}}_+^{4n} = \{ \underline{y} \in \mathbb{R}^{4n} \mid y_k^0 > 0 \quad \forall k \}$$

i.e., S_{n+1} can be analytically continued into a complex neighbourhood

$$\mathcal{N}(\mathring{\mathbb{R}}_+^{4n}) \subset \mathbb{C}^{4n} \quad \text{of} \quad \mathring{\mathbb{R}}_+^{4n}$$

Proof :

Let C be a proper open convex cone in $\mathbb{R}^4$ pointed at the origin of $\mathbb{R}^4$ and

$$\tilde{C} = \{ y \in \mathbb{R}^4 \mid x \cdot y = \sum_{0}^{3} x^r y^r > 0 \quad \forall x \in C \}$$

the dual cone of C. $\tilde{C}$ is then also a proper open convex cone pointed at the origin. Let furthermore C be such that

$$\tilde{C} \subset C \subset \mathring{\mathbb{R}}^4 \tag{30}$$

Note : if $C \to \mathring{\mathbb{R}}^4$ then $\tilde{C} \to \{ \underline{x} \in \mathbb{R}^4 \mid y^0 > 0, \vec{y} = 0 \}$, i.e., to the positive time axis. Consider then the points $\in \mathbb{C}^{4n}$ of the form

$$\underline{z} = \underline{u} + (e\xi_1, e\xi_2, \cdots e\xi_m) = \underline{u} + e\underline{\xi}$$

where e is any fixed vector $\in \tilde{C}$. Equation (29) and Lemma 1' then imply that S_{n+1} can be analytically continued to complex points of the form

$$\underline{z} = \underline{u} + e\,\underset{\sim}{\xi} \qquad \underline{u} \in C^{xn} \qquad e \in \tilde{C} \qquad \underset{\sim}{\xi} \in \mathcal{H}_n$$

Take now a system T of linearly independent vectors $e_\alpha \subset \tilde{C}$, $\alpha = 1,..,4$, and consider the points of the form

$$\underline{z} = \underline{u} + \sum_{\alpha=1}^{4} e_\alpha \underset{\sim}{\xi}_\alpha \qquad \underset{\sim}{\xi}_\alpha \in \mathbb{C}^n$$

For fixed $\underline{u} \in C^{xn}$ and considered as a function of the variables $\underset{\sim}{\xi} = (\underset{\sim}{\xi}_1,..,\underset{\sim}{\xi}_4) \in \mathbb{C}^{4n}$, S_{n+1} can be analytically continued into the union of the four flat domains $\mathcal{T}_\alpha = \{ \underset{\sim}{\xi} \in \mathbb{C}^4 : \underset{\sim}{\xi}_\alpha \in \mathcal{H}_n,\ \underset{\sim}{\xi}_\beta \in \overset{\circ}{\mathbb{R}}{}^n_+ \text{ for } \beta \neq \alpha \}$. In the logarithmic variables $W_{\alpha k} = \ln Z_{\alpha k}$, $\alpha = 1,\ldots,4$; $k = 1,\ldots,n$, we again hit upon a problem of the analytic completion of four flat tubes, which can be simply resolved by convex completion. As a result, S_{n+1} can be analytically continued into the domain

$$\mathcal{A}_n^{\mathcal{H}_n, C, T} = \{ \underline{z} \in \mathbb{C}^{4n} \mid \underline{z} = \underline{u} + \sum_{1}^{4} e_\alpha \underset{\sim}{\xi}_\alpha \quad u \in C^{xn} \quad e_\alpha \in \tilde{C} \quad |\arg \xi_{\alpha k}| < \frac{\pi}{2} \theta_{\alpha k} \quad \forall\, \theta_{\alpha k} \quad \theta_{\alpha k} > 0 \quad \sum_{\alpha=1}^{4} \sum_{k=1}^{n} \theta_{\alpha k} = 1 \} \tag{31}$$

Since the system $T = \{ e_\alpha \}$ is linearly independent, the set (31) is clearly open in $\mathbb{C}^{4n}$, and so is therefore also the set

$$\mathcal{A}_n^{\mathcal{H}_n} = \bigcup_{C,T} \mathcal{A}_n^{\mathcal{H}_n, C, T} \tag{32}$$

We claim that (32) is a complex neighbourhood of $\overset{\circ}{\mathbb{R}}{}^{4n}_+$. For, if we take any $\underline{y} \in \overset{\circ}{\mathbb{R}}{}^{4n}_+$, then there exists a C close enough to $\overset{\circ}{\mathbb{R}}{}^{4}_+$, so that $\underline{y} \in C^{xn}$. Since this is an open set, there exist $\underline{u} \in C$, $\underset{\sim}{\xi}_\alpha \in \overset{\circ}{R}{}^n_+$; $\alpha = 1,\ldots,4$, such that $\underline{y} = \underline{u} + \Sigma e_\alpha \underset{\sim}{\xi}_\alpha$ for any fixed allowed system $T = \{ e_\alpha \}$. But this is a real point of the domain (31) with C, T just chosen, Q.E.D.

From Lemma 2 and the proof of Theorem 1, we now readily get :

Lemma 3 :

The functions S_{n+1}, $n = 1,2,\ldots,$ can be aanlytically continued into the domain

$$\mathcal{A}_n = \bigcup_{C,T} \mathcal{A}_n^{C,T}$$

$$\mathcal{A}_n^{C,T} = \Big\{ \underline{z} \in \mathbb{C}^{4n} \;\Big|\; \underline{z} = \underline{u} + \sum_1^4 e_\alpha \underline{\xi}_\alpha \qquad (32')$$

$$\underline{u} \in C^{*n} \quad e_\alpha \in \tilde{C} \;\; (\alpha = 1..4) \;\; |arg\, \xi_{\alpha k}| < \tfrac{\pi}{2} \theta_\alpha \;\; k = 1..n$$

for all $\theta_\alpha > 0$ such that $\sum_1^4 \theta_\alpha = 1 \Big\}$

$\mathcal{A}_n$ is a complex neighbourhood of the flat tube

$$\mathcal{B}_n = \{ \underline{z} \in \mathbb{C}^{4n} \;|\; Re\, z_k^\circ > 0 \quad Im\, \vec{z}_k = 0 \quad k = 1..n \} \qquad (33)$$

The domains $\mathcal{A}_n^{C,T}$ are the domains (31) with $\mathcal{H}_n$ replaced by $\mathcal{P}_n$. The last assertion of the Lemma is verified as follows. In the limit $C \to \mathbb{R}^{4\,\circ}_{+}$ the vectors e_α collapse to col-linear vectors pointing along the positive time-axis and therefore $\mathcal{A}_n^{C,T}$ collapses to $\mathcal{B}_n$. Now, given any $\underline{Z} \in \mathcal{B}_n$, it can be immediately seen by continuity arguments that there exist a C and a T such that $\underline{Z} \in \mathcal{A}_n^{C,T}$, which is open in $\mathbb{C}^{4n}$.

We are ready to prove the last :

Theorem 2 :

The fun ctions $W_{n+1}(\underline{Z}) = S_{n+1}(i\underline{Z}^\circ, \vec{\underline{Z}})$ can be analytically continued into the tube

$$\mathcal{T}_n = \{ \underline{z} \in \mathbb{C}^{4n} \;|\; Im\, z_k \in V_+ \quad k = 1..n \}$$

Proof :

By standard arguments [cf., Ref. 3)] it can be shown that Euclidean invariance (24) implies

$$S_{n+1}(R\underline{z}) = S_{n+1}(\underline{z}) \qquad (34)$$

for all $R \in SO_4(\mathbb{C})$, whenever $\underline{Z}$ and $R\underline{Z}$ remain in the domain $\mathcal{A}_n$ of Lemma 3. For the functions W_{n+1} this implies

$$W_{n+1}(\Lambda \underline{z}) = W_{n+1}(\underline{z}) \tag{35}$$

for all $\Lambda \in \mathcal{L}_+(\mathbb{C})$, whenever $\underline{Z}$ and $\Lambda\underline{Z}$ remain in the domain $\mathcal{N}(\hat{\mathcal{B}}_n) = \hat{\mathcal{A}}_n$, where $\hat{\mathcal{A}}_n$ is a complex neighbourhood of the flat tube

$$\hat{\mathcal{B}}_n = \{ \underline{z} \in \mathbb{C}^{4n} \mid Im\, z_k^\circ > 0 \quad \vec{z}_k = 0 \quad k = 1 \dots n \}$$

It is the domain (32') expressed in the just adopted variables. By specializing Λ to real Lorentz transformations, (35) implies in particular that W_{n+1} is analytic in

$$\mathcal{H}\Big(\bigcup_{\Lambda \in \mathcal{L}_+^\uparrow} \Lambda \mathcal{N}(\hat{\mathcal{B}}_n) \Big)$$

where $\mathcal{H}(\Omega)$ is the envelope of holomorphy of Ω. Now $\Lambda\hat{\mathcal{B}}_n = \tau^{xn}_{\Lambda e}$, where

$$\mathcal{T}_{\Lambda e} = \{ z \in \mathbb{C}^4 \mid Im\, z = \rho\, \Lambda e \quad \rho > 0 \}$$

is the flat tube along the unit four-vector Λe, $e = (1, \vec{0})$. If we take first the $\Lambda \in \mathcal{L}_+^\uparrow$ in a close enough neighbourhood Ω of the unit element such that

$$\bigcap_{\Lambda \in \Omega} \Lambda \mathcal{N}(\hat{\mathcal{B}}_n) \neq \emptyset$$

we will have

$$\mathcal{H}\Big(\bigcup_{\Lambda \in \Omega} \Lambda \mathcal{N}(\hat{\mathcal{B}}_n) \Big) \supset Ch\Big(\bigcup_{\Lambda \in \Omega} \mathcal{T}^{xn}_{\Lambda e} \Big) = \mathcal{T}_n(\Omega)$$

where Ch denotes the convex hull. Since $\tau_n(\Omega)$ is an open tube in $\mathbb{C}^{4n}$, we are allowed to conclude

$$\mathcal{H}\Big(\bigcup_{\Lambda \in \mathcal{L}_+^\uparrow} \Lambda \mathcal{N}(\hat{\mathcal{B}}_n) \Big) \supset Ch\Big(\bigcup_{\Lambda \in \mathcal{L}_+^\uparrow} \mathcal{T}^{xn}_{\Lambda e} \Big) = \mathcal{T}_n$$

the last equality being easily verified.

Note :

All the tedious exercises of this section were necessary in order to avoid the notion of the real boundary values of the objects W_{n+1} from the flat tubes $\Lambda\hat{\mathcal{B}}_n$. It is possible that the introduction of the notion of hyperfunction, instead of distribution at an early enough stage might avoid this trouble.

ACKNOWLEDGEMENTS

It is a pleasure to thank Drs. Ph. Blanchard, R. Sénéor and R. Stora for very helpful discussion on the subject of this paper.

A P P E N D I X

For the reader's convenience, we want to indicate another way to get from the assumptions (12)-(15) to formula (16). In order to avoid notational complications, we shall restrict our attention to the diagonal elements $A_{nn}(\bar{Z},Z) \equiv A(\bar{Z},Z)$ of the positivity condition (4). In Ref. 2), the following theorem and its corollary have been proved :

Theorem C :

Let the distribution $A(\bar{Z},Z) \in \mathcal{D}'(\Omega)$, Ω an open connected set in $\mathbb{C}^N$, satisfy - in the sense of distributions - the following positivity condition

(a) $$\sum_{\alpha,\beta} \bar{a}_\alpha a_\beta \bar{\partial}^\alpha \partial^\beta A(\bar{z},z) \geqslant 0$$

for all terminating sequences of complex numbers $\{ a_\alpha \}$ everywhere in Ω. Then $A(\bar{Z},Z)$ is the restriction to the "diagonal plane" $Z' = \bar{Z}$ of a function $A(Z',Z)$ analytic in the domain $\Omega^* \times \Omega \in \mathbb{C}^{2N}$. This function satisfies the positivity condition

$$\int A(\bar{z}',z)\, d\bar{\mu}(z')\, d\mu(z) \geqslant 0$$

for all complex measures with compact support in Ω.

Note :

α,β are multi-indices : $\alpha = (\alpha_1,\ldots,\alpha_N)$; $Z = x+iy$, $x,y \in \mathbb{R}^N$, and $A(\bar{Z},Z)$ is to be thought of as a distribution of the real variables $(x,y) \in \mathbb{R}^{2N}$;

$$\bar{\partial}^\alpha = \left\{ \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right\}^\alpha \qquad \partial^\beta = \left\{ \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right\}^\beta$$

Ω^* is the complex conjugate domain of Ω.

Suppose now the distribution A of Theorem C is of the form $A(\bar{Z},Z) = W(Z-\bar{Z}) = W(2iy)$, where $Z = x+iy$. If we set $W(iy) = S(y)$, the positivity condition (a) becomes

(b) $$\sum_{\alpha,\beta} a_\alpha \bar{a}_\beta \frac{\partial^{\alpha+\beta}}{\partial y^{\alpha+\beta}} S(y) \geqslant 0$$

Corollary C :

Let the distribution $S(x) \in \mathcal{D}'(B)$, where B is an open connected cone in $\mathbb{R}^N$, satisfy the positivity condition (b) for all terminating sequences of complex numbers $\{a_\alpha\}$. Then $S(x)$ is the restriction to the real plane $y=0$ of a function $A(x+iy)$ analytic in the tube

$$\mathcal{T}_{\hat{B}} = \{ x+iy \in \mathbb{C}^N \mid x \in \hat{B} = Ch(B) \}$$

$A(z)$ moreover satisfies the positivity condition

(c) $$\int A(z-\bar{z}')\, d\mu(z)\, d\bar{\mu}(z') \geq 0$$

for all complex measures with compact support in $\mathcal{T}_{\hat{B}}$.

We now prove :

Lemma A :

Let the distribution $S \in \mathcal{D}'(B)$, where B is an open convex cone in $\mathbb{R}^N$, satisfy the Osterwalder-Schrader positivity condition

(d) $$\int S(y+y')\, \bar{f}(y)\, f(y')\, d^N y\, d^N y' \geq 0$$

for all test functions $f \in \mathcal{D}(\Omega)$. Then (d) implies the condition (b) in B. Since (b) $\Rightarrow$ (c) $\Rightarrow$ (d) the conditions (b) and (d) are equivalent.

Proof :

Take for f a sequence $f_n \in \mathcal{D}(\Omega)$ such that

$$\lim_{n \to \infty} f_n(y) = \sum_\alpha a_\alpha (-1)^{|\alpha|} D^\alpha \delta_N(y - y_n)$$

in the sense of $\mathcal{D}'(B)$. It is an easy exercise in distribution theory to show that we get in the limit the condition (b), Q.E.D.

REFERENCES

1) K. Osterwalder and R. Schrader, Comm.Math.Phys. 31, 83-112 (1973).

2) V. Glaser, Problems of Theoretical Physics, "Nauka", Moscow (1969), p.69.

3) R. Jost, The General Theory of Quantized Fields, Amer.Math.Soc.Publ., Providence, R.I., (1965).

4) H. Epstein, Axiomatic Field Theory, V1, Gordon and Breach Publ., New York (1966), p.3.

5) H. Epstein, V. Glaser and A. Martin, Comm.Math.Phys. 13, 257-316 (1969).

6) M. Zerner, Les Fonctions Holomorphes à Valeurs Vectorielles et leurs Valeurs aux Bords, mimeographed notes of a series of lectures, Orsay (1961).