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Inequalities for Traces on von Neumann Algebras

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Abstract. A number of useful inequalities, which are known for the trace on a separable Hilbert space, are extended to traces on von Neumann algebras. In particular, we prove the Golden rule, Hölder inequality, and some convexity statements.

A number of useful inequalities relating the traces of operators on a Hilbert space are known[1] when the trace is defined in the usual way. In this paper, we consider generalizations of some of these inequalities to traces on von Neumann algebras. In a subsequent paper, we will discuss applications to entropy and statistical mechanics.

In what follows τ will always be a normal, faithful[2] semifinite trace on a von Neumann algebra, $\mathfrak{A}$, of operators on a Hilbert space $\mathscr{H}$. This means that τ is a function, defined on $\mathfrak{A}^+ = \{A : A \geqq 0\}$ and extended to the 2-sided ideal, M, whose positive part is $M^+ = \{A : A \geqq 0 \text{ and } \tau(A) < \infty\}$ with the following properties[3]:

a) $\tau(A) \geqq 0$ if $A \geqq 0$. (1)

b) $\tau(A + \lambda B) = \tau(A) + \lambda\tau(B)$ if (2)
  i) λ in $\boldsymbol{C}$; A, B in M or,
  ii) $\lambda \geqq 0$; $A, B \geqq 0$.

c) $\tau(A) = \tau(UAU^*)$ if (3)
  $A \geqq 0$; U is unitary.

d) $\tau(AB) = \tau(BA)$ if (4)
  i) A in M, B in $\mathfrak{A}$ or,
  ii) $B = A^*$ in $\mathfrak{A}$.

e) (Normal): If $\{A_i\}$ is a bounded increasing net of positive operators, then $\sup_i \tau(A_i) = \tau\left(\sup_i A_i\right)$ (5)

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[1] See, for example [1–4].

[2] The restriction to faithful traces is not really necessary, (see [5], Corollary 2, p. 83) but simplifies things slightly.

[3] Properties (a), (b ii), and (c) suffice to define a trace (see [5], p. 81).

f) (Semi-finite)[4]: If $A \geqq 0$ and $\tau(A)=\infty$, then there exists a B such that $0<B<A$ and $\tau(B)<\infty$.

g) (Faithful): $\tau(A)=0$ and $A \geqq 0 \Rightarrow A=0$. In addition, one can show that τ has the following useful properties[5]:

h) $A \to \tau(AB)$ is ultraweakly continuous for A in $\mathfrak{A}$, B in M.

i) $|\tau(A^*B)|^2 \leqq \tau(A^*A)\,\tau(B^*B)$ if A^*B is in M. (6)

j) There exists a family, (x_i), of vectors in $\mathscr{H}$ such that $\tau(A)=\sum_k \langle x_k A x_k\rangle$ if $A \geqq 0$. (7)

k) $|\tau(AB) \leqq \tau(|AB|) \leqq \|A\|\,\tau(|B|)$ if A in $\mathfrak{A}$, B in M. (8)

There are five topologies which one uses frequently on von Neumann algebras: norm, ultrastrong, strong, ultraweak, and weak. Some of the properties of these topologies simplify on norm-bounded sets. We will have occasion to make use of the following facts[6], which are not true on unbounded sets.

a) On norm-bounded sets, multiplication is continuous in the strong topology.

b) On norm-bounded sets a sequence is ultrastrongly convergent

$$\begin{aligned} &\Leftrightarrow \text{strongly convergent},\\ &\Rightarrow \text{ultraweakly convergent},\\ &\Leftrightarrow \text{weakly convergent}. \end{aligned}$$

In particular, we note that Property (h) now implies that $A \to \tau(AB)$ is strongly continuous on norm-bounded sets.

The first theorem can be thought of as a special case of Fatou's Lemma.

Theorem 1. *Let (A_n) be a sequence of positive operators converging weakly to A.*

Then:

$$\tau(A) \leqq \varliminf_{n\to\infty} \tau(A_n). \tag{9}$$

Proof. This follows immediately from Property (j) and Fatou's Lemma as stated in Theorem A.2 of the Appendix.

The next three Theorems generalize results of Golden [2] and Thompson [3].

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[4] This definition of semi-finiteness is valid only for normal traces.

[5] See [5]; Proposition 1, p. 82; Theorem 2, p. 88; Corollary, p. 85; Theorem 8, p. 106.

[6] See [6]; pp. 171–176 or [5].

Theorem 2.
$$\tau[(CD)^{2^{p+1}}] \leqq \tau[(CD)^{2^p}(DC)^{2^p}] \tag{10}$$
$$\leqq \tau[(C^2D^2)^{2^p}], \tag{11}$$
$$\leqq \tau[C^{2^{p+1}}D^{2^{p+1}}], \tag{12}$$

if $C, D \geqq 0$, *and* $\tau(C)$ *or* $\tau(D) < \infty$.

Proof. First note that, e.g.

$$0 \leqq \tau[D^{1/2}(CD)^{2^p}CD^{1/2}] = \tau[(CD)^{2^p+1}] \tag{13}$$

so that all terms are real and positive. Now note that the theorem readily follows from (4) and (6) when $p=0$, i.e.,

$$\begin{aligned}\tau[(CD)^2] &= \tau[(CD)(CD)] \\ &\leqq \tau[(CD)(DC)] = \tau[C^2D^2]\end{aligned}$$

It is easy to see that (10) is a special case of (6) and that (12) follows from repeated application of (11). Therefore it suffices to give an inductive proof of (11) under the assumption that (12) is true for all lower p. To do this we introduce the notation:

$$\alpha_m = (CD)^{2^m}(DC)^{2^m}, \tag{14}$$
$$\beta_m = (DC)^{2^m}(CD)^{2^m} \tag{15}$$

and note that:

$$\begin{aligned}\tau[(\alpha_m)^r] &= \tau[(\beta_m)^r] \\ &= \tau[(\alpha_{m-1}\beta_{m-1})^r]. \end{aligned} \tag{16}$$

Now one has:

$$\begin{aligned}\tau[(CD)^{2^{p+1}}] &= \tau[(CD)^{2^p}(CD)^{2^p}] \\ &\leqq \tau[(CD)^{2^p}(DC)^{2^p}] \\ &= \tau(\alpha_p)\end{aligned} \tag{17}$$

and

$$\tau[(C^2D^2)^{2^p}] = \tau[(\alpha_0)^{2^p}]. \tag{18}$$

Thus (11) is satisfied if

$$\tau(\alpha_p) \leqq \tau[(\alpha_0)^{2^p}]. \tag{19}$$

Now we prove (11) by proving (19) under the inductive assumption that (11) and (12) are true for all $k<p$. From (16) we have

$$\begin{aligned}\tau[(\alpha_{p-k})^{2^k}] &= \tau[(\alpha_{p-k-1}\beta_{p-k-1})^{2^k}] \\ &\leqq \tau[(\alpha_{p-k-1})^{2^k}(\beta_{p-k-1})^{2^k}] \\ &\leqq \tau[(\alpha_{p-(k+1)})^{2^{k+1}}] \qquad k<p,\end{aligned}$$

where the first inequality follows from (12) and the second from (6) and (16). This shows that $\tau[(\alpha_{p-k})^{2^k}]$ increases with k so that (19) is satisfied as required.

Theorem 3.

$$\tau[(CD^{2^p}(DC)^{2^p}]\leqq\tau[C^{2^p}D^{2^{p+1}}C^{2^p}] \quad \text{if} \quad C,D\geqq 0\,. \tag{20}$$

Proof. Let (D_n) be an increasing net of positive operators converging strongly to D, such that $\tau(D_n)<\infty\,\forall n$[7]. Then

$$\begin{aligned}\tau[(CD)^{2^p}(DC)^{2^p}]&\leqq\lim_{n\to\infty}\tau[(CD_n)^{2^p}(D_nC)^{2^p}]\\&\leqq\lim_{n\to\infty}\tau[C^{2^p}D_n^{2^{p+1}}C^{2^p}]\\&=\tau[C^{2^p}D^{2^{p+1}}C^{2^p}]\,,\end{aligned}$$

where the first inequality follows from Theorem 1, the second from Theorem 2 and the last from normality of the trace.

Theorem 4.

$$\tau(e^{A+B})\leqq\tau(e^{A/2}e^Be^{A/2}) \tag{21}$$

if a) *A,B are self-adjoint operators, bounded above, and* b) *$A+B$ is essentially self-adjoint.*

Further, if $\tau(e^A)<\infty$ or $\tau(e^B)<\infty$ then

$$\tau(e^{A+B})\leqq\tau(e^Ae^B)\,. \tag{22}$$

Proof. Let $X_p=(e^{A/2^{p+1}}e^{B/2^{p+1}})^{2^p}(e^{B/2^{p+1}}e^{A/2^{p+1}})^{2^p}$.

It then follows from the Trotter formula (33) that $X_p\to e^{A+B}$ strongly. Theorem 3 implies that $\tau(X_p)\leqq\tau(X_0)$ for all p. Applying Theorem 1, one thus gets:

$$\begin{aligned}\tau(e^{A+B})&\leqq\lim_{p\to\infty}\tau(X_p)\\&\leqq\tau(X_0)\\&=\tau(e^{A/2}e^Be^{A/2})\,.\end{aligned}$$

If $\tau(e^B)$ or $\tau(e^A)<\infty$, (22) follows from (4).

The next few theorems are closely related to the inequalities in Section 2.5 of [1].

Theorem 5. (Hölder Inequality).

$$\begin{aligned}|\tau(AB)|&\leqq\tau(|AB|) &(23)\\&\leqq[\tau(|A|^{1/\alpha})]^\alpha[\tau(|B|^{1/1-\alpha})]^{1-\alpha}\qquad(0<\alpha<1)\,. &(24)\end{aligned}$$

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[7] Let (E_n) be the net in Theorem A.5 and $D_n=D^{1/2}E_nD^{1/2}$.

Further, whenever the right hand side of (24) *is finite AB is in M, and*

$$\tau(AB)=\tau(BA). \tag{25}$$

Proof. This is just a special case of more general theorems proven by Kunze and Ogasawara[8]. Since their proofs are somewhat complicated and incomplete, we give a simple complete proof in Appendix B.

Theorem 6. *The function $f(x)=\log\tau(e^{A+xB})$ is convex on $(-\infty,\infty)$ if:*

a) *B is a bounded, self-adjoint operator,*

b) *A is a self-adjoint operator, bounded above, and*

c) $\tau(e^A)<\infty$.

Proof. First note that Theorem 4 implies that

$$\tau(e^{A+xB})\leqq\tau(e^A e^{xB})<\infty \quad \text{whenever} \quad \tau(e^A)<\infty .$$

We first prove the theorem under the assumption $\tau(e^{A/2})<\infty$. Then w.l.o.g., we can choose $\alpha>1/2$ so that $\tau(e^{\alpha A})<\infty$ and $\tau(e^{\alpha(A+xB)})<\infty$. Now

$$\begin{aligned} f(\alpha x+(1-\alpha)\,y) &= \log\tau\left[e^{\alpha(A+xB)+(1-\alpha)(A+yB)}\right] \\ &\leqq \log\tau\left[e^{\alpha(A+xB)}\,e^{(1-\alpha)(A+yB)}\right] \\ &\leqq \log\left[\tau(e^{A+xB})\right]^{\alpha}\left[\tau(e^{A+yB})\right]^{1-\alpha} \\ &= \alpha f(x)+(1-\alpha)f(y), \end{aligned}$$

where the first inequality follows from Theorem 4 and the second from Theorem 5. To prove the theorem in general, let the spectral decomposition of A be

$$\int_{-\infty}^{\|A\|} \lambda\, dE(\lambda)$$

and define:

$$A_k=\int_{-\infty}^{-k}\frac{\lambda}{\alpha}\,dE(\lambda)+\int_{-k}^{\|A\|}\lambda\,dE(\lambda). \tag{26}$$

Then

$$\begin{aligned} \tau(e^{\alpha A_k}) &= \tau\left[\int_{-\infty}^{k} e^{\lambda}\,dE(\lambda)+\int_{-k}^{\|A\|} e^{\lambda\alpha}\,dE(\lambda)\right] \\ &\leqq \tau(e^A)+e^{\alpha\|A\|}\,\tau(I-E(-k)) \\ &\leqq \left[1+e^{\alpha\|A\|}\,e^{k}\right]\tau(e^A) \\ &< \infty \quad \text{for all} \quad k . \end{aligned}$$

Further $A_k\uparrow A$ strongly, $e^{A_k}\uparrow e^A$ ultrastrongly and $e^{A_k-A}\to I$ ultrastrongly. W.l.o.g. we can choose $\alpha x+(1-\alpha)\,y=0$. Let $f_k(x)=\tau(e^{A_k+xB})$. It then

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[8] See [7], Lemma 1.5, and [8] Lemma 3.1.

follows from normality that

$$f(0)=\tau(e^{A})=\lim_{k\to\infty}\tau(e^{A_k})=\lim_{k\to\infty}f_k(0)\,,$$

and from the preceeding argument that

$$f_k(0)\leqq\alpha f_k(x)+(1-\alpha)f_k(y)\,. \tag{27}$$

Using Theorem 4 again, one gets

$$f_k(x)\leqq\log\tau\,[e^{A_k-A}\,e^{A+xB}]\,.$$

Since $e^{A_k-A}\to I$, ultrastrongly, Property (h) implies

$$\lim_{k\to\infty}f_k(x)\leqq\log\tau(e^{A+xB})=f(x)\,.$$

Combining, one gets

$$\begin{aligned}f(0)&\leqq\lim_{k\to\infty}[\alpha f_k(x)+(1-\alpha)f_k(y)]\\&\leqq\alpha f(x)+(1-\alpha)f(y)\end{aligned}\tag{28}$$

as required.

Theorem 7. (Peierls-Bogolyubov Inequality).

$$\log\left[\frac{\tau(e^{A+B})}{\tau(e^{A})}\right]\geqq\frac{\tau(e^{A}B)}{\tau(e^{A})} \tag{29}$$

if A, B satisfy the conditions of Theorem 6.

Proof. If f is a convex function and $a<b$, then

$$f'(a)\leqq\frac{f(b)-f(a)}{b-a}\,.$$

Let $f(x)$ be as in Theorem 6 and $a=0$, $b=1$. To compute $f'(0)$, let $g(x)=\log\tau(e^{A}e^{xB})$ and note that $f(x)\leqq g(x)$ for all x, $f(0)=g(0)$, and

$$g'(x)=\frac{\tau(e^{A}Be^{xB})}{\tau(e^{A}e^{xB})}\,.$$

Then

$$\begin{aligned}f'(0)=g'(0)&=\frac{\tau(e^{A}B)}{\tau(e^{A})}\\&\leqq f(1)-f(0)\\&=\log\left[\frac{\tau(e^{A+B})}{\tau(e^{A})}\right].\end{aligned}$$

Appendix A

Theorem A.1 (Fatou's Lemma)[9]. *If (f_n) is a sequence of non-negative μ-measurable functions, $f_n(x) \to f(x)$ a.e., and μ is a positive measure, then*

$$\int f\, d\mu \leqq \liminf \int f_n d\mu . \tag{30}$$

Theorem A.2.

$$\sum_k \lim_{n\to\infty} c_{kn} \leqq \lim_{n\to\infty} \sum_k c_{kn} \tag{31}$$

if $c_{kn} \geqq 0$ for all k, n and the indicated sums and limits exist.

Proof. This is just a special case of Theorem A.1 with μ a discrete measure.

Theorem A.3 (Trotter formula)[10].

$$e^{A+B} = \underset{n\to\infty}{s\text{-lim}}\, (e^{A/n} e^{B/n})^n \tag{32}$$

if a) *A, B are self-adjoint operators bounded above, and* b) *$A+B$ is essentially self-adjoint.*

Theorem A.4.

$$e^{A+B} = \underset{n\to\infty}{s\text{-lim}}\, (e^{A/n} e^{B/n})^{n/2} (e^{B/n} e^{A/n})^{n/2} \tag{33}$$

if A, B are as in Theorem A.3.

Proof. Apply (32) to $e^{(A+B)/2}$. Taking the subsequence $k=2n$ gives:

$$e^{(A+B)/2} = \underset{k\to\infty}{s\text{-lim}}\, (e^{A/k} e^{B/k})^{k/2} . \tag{34}$$

Since A is bounded above, e^A is a bounded, self-adjoint operator and

$$\|e^{A/k}\| = (\|e^A\|)^{1/k} .$$

Therefore,

$$\|e^{A/k} e^{B/k}\|^{k/2} \leqq (\|e^A\|\ \|e^B\|)^{1/2} ,$$

and the sequence in (34) is bounded independently of k. Then reversing the roles of A and B in (32), and using the fact that multiplication is continuous on bounded sets in the strong topology gives the desired result.

Theorem A.5. *The set of all projections in M forms an increasing net which converges strongly to the identity.*

Proof. To show that the projections in M form a net we must verify that $E \vee F$ is in M whenever E and F are. Let $F' = E \vee F - E$. There

[9] See, for example, Royden [9], p. 113.

[10] See [10], p. 109; [11], Theorem VIII.31, p. 295.

exists[11] a partial isometry W such that $W^*W = F'$ and $WW^* \leqq F$. Thus

$$\begin{aligned}\tau(E \vee F) &= \tau(E) + \tau(F') \\ &= \tau(E) + \tau(W^*W) \\ &= \tau(E) + \tau(WW^*) \\ &\leqq \tau(E) + \tau(F) < \infty ,\end{aligned}$$

and $E \vee F$ is in M. Since M is strongly dense in $\mathfrak{A}$[12], this net must converge to the identity.

Appendix B

Proof of Theorem 5. a) Note that the first inequality is just a special case of (8).

b) We show[13] that $\tau(|AB|) \leqq 1$ if

$$\|A\|_p = \|B\|_q = 1 , \quad \text{and} \quad \tau(E) < \infty$$

where:

$$\|A\|_p = [\tau(|A|^p)]^{1/p}, \quad 1/p + 1/q = 1 , \quad \text{and}$$

$E =$ projection on the range of $|A|$.

Using the polar decomposition, one can write

$$\begin{aligned}|AB| &= V_{AB}^* AB , \\ A &= |A| \, V_A , \\ B &= |B| \, V_B\end{aligned}$$

and

$$\tau(|AB|) = \tau(E|A| \, W|B| \, X)$$

where $W = V_A$ and $X = V_B V_{AB}^*$ are partial isometries. Now let

$$f_\varepsilon(z) = \tau[E(\varepsilon I + |A|^p)^z W(\varepsilon I + |B|^q)^{1-z} X] .$$

Then:

1) $\lim_{\varepsilon \to 0} \mathrm{f}(1/p) = \tau(|AB|)$.

2) $f_\varepsilon(z)$ is entire.

3) $f_\varepsilon(z)$ is bounded on $0 \leqq \mathrm{Re}\, z \leqq 1$ by $\tau(E)(1+\varepsilon)^2$ since
$$\|A\|_p = 1 \Rightarrow \|A\| \leqq 1 .$$

4) $$\begin{aligned}|f_\varepsilon(iy)| &= |\tau[E(\varepsilon + |A|^p)^{iy} \, W(\varepsilon + |B|^q)(\varepsilon + |B|^q)^{-iy} X]| \\ &= |\tau([\text{p.I.}] \, |B|^q) + \varepsilon\tau([\text{p.I.}] \, E)|^{14} \\ &\leqq \tau(|B|^q) + \varepsilon\tau(E) = 1 + \varepsilon\tau(E).\end{aligned}$$

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[11] See [6], p. 200.

[12] See [5], and the construction in Corollary 2, p. 83. E is the identity for a normal, faithful, semi-finite trace.

[13] The argument in Part (b) of this proof was communicated to the author by O. E. Lanford.

[14] [p.I.] stands for a partial isometry.

5) $|f_\varepsilon(1+iy)| = |\tau[E(\varepsilon+|A|^p)(\varepsilon+|A|^p)^{iy}W(\varepsilon+|B|)^{-iy}X]|$
$\leqq 1+\varepsilon\tau(E)$.

Therefore $|f_\varepsilon(z)| \leqq 1+\varepsilon\tau(E)$ on $0 \leqq \mathrm{Re}\, z \leqq 1$, and

$$\tau(|AB|) = \lim_{\varepsilon\to 0} f(1/p) \leqq 1 + \lim_{\varepsilon\to 0} \varepsilon\tau(E) = 1 .$$

c) By considering $A/\|A\|_p$ and $B/\|B\|_q$, one can drop the restriction $\|A\|_p = \|B\|_q = 1$.

d) We now show that one can drop the restriction $\tau(E) < \infty$. Write the spectral decomposition of $|A|$ as $|A| = \int_0^{\|A\|} \lambda\, dF(\lambda)$. Let $E_k = 1 - F(1/k)$. Then $\tau(E_k) \leqq k^p \tau(|A|^p)$ so that $\|A\|_p < \infty \Rightarrow \tau(E_k) < \infty$. Then the projection on the range of $E_k A$ has finite trace so that

$$\tau(|E_k AB|) \leqq \|E_k A\|_p \|B\|_q .$$

Let $X_k = |E_k AB|$. Then

$$X_k^2 = B^* A^* E_k AB , \qquad \underset{k\to\infty}{s\text{-}\lim}\, X_k^2 = B^* A^* AB = X^2$$

and (X_k^2) is an increasing, uniformly bounded sequence of positive operators. Therefore[15] $\underset{k\to\infty}{s\text{-}\lim}\, X_k = |AB|$. Thus, using Theorem 1, one finds

$$\begin{aligned} \tau(|AB|) &\leqq \lim_{k\to\infty} \tau(|E_k AB|) \\ &\leqq \lim_{k\to\infty} \|E_k A\|_p \|B\|_q \\ &= \|A\|_p \|B\|_q . \end{aligned}$$

e) Clearly $|AB|$ is in M^+. Since $AB = Y|AB|$ and M is a 2-sided ideal, AB is in M.

f) To prove cyclicity, we can assume w.l.o.g. that $A > 0$. Let $A_k = E_k A$ with E_k as in part (d) above.

$\tau(A_k) \leqq \|A\|\, \tau(E_k) < \infty$. Therefore, by Property (d)

$$\tau(A_k B) = \tau(B A_k) \qquad \forall k .$$

But

$$\begin{aligned} |\tau[(A - A_k)B]| &\leqq \|A - A_k\|_p \|B\|_q \\ &= \left(\tau\left[\int_0^{1/k} \lambda^p\, dF(\lambda)\right]\right)^{1/p} \|B\|_q \xrightarrow[k\to\infty]{} 0 . \end{aligned}$$

[15] By the Stone-Weierstrass Theorem, $t \to t^{1/2}$ is uniformly approximable by polynomials on $[0, \|X^2\|]$. Since the sequence (X_k^2) is bounded, $\underset{k\to\infty}{s\text{-}\lim}\, P(X_k^2) = P(X^2)$ for any polynomial P. Therefore $\underset{k\to\infty}{s\text{-}\lim}\, X_k = X$.

Similarly

$$\lim_{k\to\infty} |\tau[B(A-A_k)]| = 0\,.$$

Thus

$$\tau(AB) = \lim_{k\to\infty} \tau(A_k B) = \lim_{k\to\infty} \tau(BA_k) = \tau(BA)\,.$$

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