Z. Wahrscheinlichkeitstheorie verw. Geb. 17, 113–123 (1971)

Abstract Wiener Processes and Their Reproducing Kernel Hilbert Spaces

G. KALLIANPUR*

1. Introduction

In this paper we study properties of Gaussian probability measures on function spaces that are closely related to the concept of measurable norm recently introduced by L. Gross ([2, 3]). We begin by giving in Section 2 a rather simple proof of Gross's main theorem on abstract Wiener spaces which brings into evidence the important role played by the reproducing kernel Hilbert space (RKHS) in the study of Gaussian processes. In fact, broadly speaking, the aim of the present paper is to explore the relationship between a Gaussian probability measure and the RKHS associated with it. This problem is of particular interest for the wide class of Gaussian measures defined on linear topological spaces which contain, in the set theoretic sense, the associated RKHS. Abstract Wiener spaces belong to this class (as shown by Corollary of Section 2) as do all Gaussian processes given on the Banach space $C(T)$ of real continuous functions on a compact metric space T. The latter processes are considered in Section 3 which contains the main results of the paper concerning the supports of Gaussian measures. In Section 4, these results, viz. Theorems 3 and 4 are extended to arbitrary separable Banach spaces.

We shall assume that the reader is familiar with the notion of cylinder set measure on any locally convex, linear topological space L (see e.g. [3]). An equivalent concept is that of the weak distribution in L. Let L^* be the topological dual of L.

Definition 1. A weak distribution on L is an equivalence class of linear maps F from L^* to the linear space $M(\Omega, \mathscr{A}(\Omega), P)$ of random variables on some probability space $(\Omega, \mathscr{A}(\Omega), P)$ (the choice of which depends on F). Two maps F_1 and F_2 are equivalent if for any finite set $y_1, \ldots, y_n$ in L^* the joint distribution of $F_j(y_1), \ldots, F_j(y_n)$ is the same for $j=1, 2$.

Definition 2. If $L=H$ is a separable Hilbert space, F is called a canonical normal distribution (or simply a canonical distribution) on H if to each h in H^* the real random variable $F(h)$ is normally distributed with mean 0 and variance $\|h\|^2$. (Here $\|h\|$ denotes H^*-norm.) From now on, F will always be a representative of the canonical distribution.

If f is a tame function on L, i.e., a function of the form

$$f(x)=\varphi[\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle] \tag{1.1}$$

* Research supported in part by NSF Grant GP-11888.

where $y_1, \ldots, y_n \in L^*$, φ is a Baire function of n real variables, and $\langle y, x\rangle$ is the value of y at x, then

$$\tilde{f}(\omega) = \varphi[F(y_1)(\omega), \ldots, F(y_n)(\omega)] \qquad (\omega \in \Omega) \tag{1.2}$$

is a random variable on Ω having the same distribution as f under the weak distribution. We shall use Gross's notation to denote by $\tilde{f}$ the r.v. (as described above) corresponding to any tame function on L.

Let H be a separable Hilbert space and let $\mathscr{F}$ be the family of all finite dimensional projections on H.

Definition 3. A norm or semi-norm $\|x\|_1$ on H is said to be measurable if to every $\varepsilon > 0$ there exists a projection $P_\varepsilon \in \mathscr{F}$ such that

$$\operatorname{Prob}(\|Px\|_1^{\sim} > \varepsilon) < \varepsilon \tag{1.3}$$

for all $P \perp P_\varepsilon$, $(P \in \mathscr{F})$.

Gross starts with the following set up which leads to his main theorem which is Theorem 1 given below.

Let H be a separable Hilbert space, F the canonical distribution on H and $\|x\|_1$ a measurable norm on H. Let B be the Banach space which is the completion of H with respect to $\|x\|_1$. Then we may identify B^*, the dual of B with a subset of H^* and write

$$B^* \subset H^* = H \subset B. \tag{1.4}$$

Furthermore, the canonical distribution on H induces a weak distribution on B if we restrict the map F to B^*. Let μ be the cylinder set measure on B determined by this weak distribution. The result of Gross is the following ([3]).

Theorem 1. *Let $\|x\|_1$ be a measurable norm on H and μ the cylinder set measure on B induced by the canonical distribution on H. Then μ extends to a countably additive Gaussian measure on $(B, \mathscr{A}(B))$, where $\mathscr{A}(B)$ is the σ-field of Borel sets in B.*

Before proceeding to give a proof in the next Section we shall give one more definition, that of the RKHS determined by a covariance function.

Definition 4. Let T be a complete separable metric space and let R be a real continuous covariance function on $T \times T$. Then R determines a Hilbert space $H(R)$, called the RKHS of R which has the following properties:

$H(R)$ consists of real functions f on T such that

$$R(\cdot, t) \in H(R), \tag{1.5}$$

$$(f, R(\cdot, t)) = f(t) \tag{1.6}$$

for every t in T.

2. Abstract Wiener Spaces

Proof of Theorem 1. Since $\|x\|_1$ is a measurable norm on H, (1.3) of Definition 3 applies. Choose $\varepsilon = 2^{-n}$ $(n = 1, 2, \ldots)$ and write $P_n = P_{2^{-n}}$. Without loss of generality the projections P_n may be taken to be increasing to the identity operator as $n \to \infty$. Letting

$$Q_n = P_{n+1} - P_n \qquad (n > 1) \tag{2.1}$$

we have $Q_n \in \mathscr{F}$ and $Q_n \perp P_n$ so that from (1.3)

$$\operatorname{Prob}\{\|Q_n x\|_1^{\sim} > 2^{-n}\} < 2^{-n}. \tag{2.2}$$

Let H_n be the range space of P_n. Then $H_n \subset H_{n+1}$, $H_{n+1} \ominus H_n$ is $\perp$ to H_n and we have

$$H = H_1 \oplus \sum_{n=1}^{\infty} (H_{n+1} \ominus H_n). \tag{2.3}$$

Because of the decomposition (2.3) of H we can choose a complete orthonormal system (CONS) $\{e_j\}_1^{\infty}$ in H such that $\{e_j\}_1^{k_1}$ is a CONS in H_1, $\{e_j\}_{k_1+1}^{k_2}$ is a CONS in $H_2 \ominus H_1$, etc. Then for every n, $\{e_j\}_1^{k_n}$ is a CONS in H_n. For each x in H

$$x = \sum_{j=1}^{\infty} (x, e_j)\, e_j, \tag{2.4}$$

and

$$P_n x = \sum_{j=1}^{k_n} (x, e_j)\, e_j. \tag{2.5}$$

Let F be a representative of the canonical distribution in H. Then $(j = 1, 2, \ldots)$

$$\xi_j(\omega) = F(e_j)(\omega) = (x, e_j)^{\sim}(\omega) \tag{2.6}$$

are independent $N(0, 1)$ random variables on some probability space. Since

$$\|P_n x\|_1 = \left\| \sum_{j=1}^{k_n} (x, e_j)\, e_j \right\|_1 \tag{2.7}$$

is a tame function on H it is easy to see that the random variable $\|P_n x\|_1^{\sim}$ is given by

$$\|P_n x\|_1^{\sim}(\omega) = \left\| \sum_{j=1}^{k_n} \xi_j(\omega)\, e_j \right\|_1. \tag{2.8}$$

(2.8) can be seen as follows: If we set

$$\varphi(a_1, \ldots, a_{k_n}) = \left\| \sum_1^{k_n} a_j e_j \right\|_1 \tag{2.9}$$

then clearly φ is a continuous function of the real variables $a_1, \ldots, a_{k_n}$ and hence, of course, a Borel function. (2.8) then follows from (1.1), (1.2), (2.6), and (2.9). For each n define the following mapping of Ω into B.

$$Y_{k_n}(\omega) = \sum_{j=1}^{k_n} \xi_j(\omega)\, e_j. \tag{2.10}$$

It is trivial to verify that Y_{k_n} is a $(\mathscr{A}(\Omega), \mathscr{A}(B))$ measurable mapping of Ω into B. Since we also have

$$\|Q_n x\|_1^{\sim}(\omega) = \left\| \sum_{j=k_n+1}^{k_{n+1}} \xi_j(\omega)\, e_j \right\|_1$$

(2.2) gives

$$P\{\omega : \|Y_{k_{n+1}}(\omega) - Y_{k_n}(\omega)\|_1 > 2^{-n}\} < 2^{-n}. \tag{2.11}$$

Let

$$\mu_{k_n} = P\, Y_{k_n}^{-1}$$

be the (countably additive) probability measure on $(B, \mathscr{A}(B))$ induced by Y_{k_n}. It easily follows from (2.11) that Y_{k_n} converges in probability. Hence the sequence $\{\mu_{k_n}\}$ converges weakly to a measure ν, say on B. To show that ν is Gaussian, we compute the characteristic functional of ν. Let f be any element of B^*. Then

$$\begin{aligned}\int_B e^{i\langle f, x\rangle}\,\nu(dx) &= \lim_{n\to\infty} \int_B e^{i\langle f, x\rangle}\,\mu_{k_n}(dx)\\ &= \lim_{n\to\infty} \int_\Omega e^{i\langle f, Y_{k_n}(\omega)\rangle}\,P(d\omega)\\ &= \lim_{n\to\infty} e^{-\frac{1}{2}\sum_1^{k_n}\langle f, e_j\rangle^2} = e^{-\frac{1}{2}(f,f)_{H^*}},\end{aligned}\tag{2.12}$$

from the Parseval relation since $f \in H^*$ from (1.4). From (2.12) we see that the covariance functional of ν equals the covariance computed under the weak distribution F (see (2.14) below). It follows easily that ν is the extension to $(B, \mathscr{A}(B))$ of the cylinder set measure.

Although we do not need it for the proof of Theorem 1, with a little more work the argument given above can be used to prove that the sequence of sums $Y_N(\omega) = \sum_{j=1}^{N} \xi_j(\omega)\, e_j$ converges in probability.

In view of the later results of the paper, the following remarks are worth making. We state a special case of Theorem 1 which, in fact, can be shown to be equivalent to it.

Let T be a compact metric space.

Theorem 1′. *Let $H = H(R)$, the RKHS of R which is a continuous covariance on $T \times T$. For x in $H(R)$ let*

$$\|x\|_1 = \sup_{t\in T} |x(t)|.$$

Denote by $\overline{H(R)}$ or C_0 the closure of $H(R)$ in $C(T)$ with respect to $\|x\|_1$.

If $\|x\|_1$ is a measurable norm on $H(R)$ and μ is the cylinder set measure on C_0 induced by the canonical distribution on $H(R)$, then μ extends to a countably additive Gaussian measure on $(C_0, \mathscr{A}(C_0))$ where $\mathscr{A}(C_0)$ is the σ-field of Borel sets in C_0.

Theorem 1 can be reduced to Theorem 1′ if we make use of the Banach-Mazur Theorem [6] according to which every separable Banach space B is isometrically isomorphic (congruent) to a closed linear subspace C_0 of the Banach space C of real continuous functions on $[0, 1]$. If ψ represents this congruence it can be shown that H is congruent (under ψ) to a certain RKHS $H(R)$ contained in C_0 and that the measurability of the norm $\|x\|_1$ on H is equivalent to the measurability of the sup-norm on $H(R)$. If ν is the Gaussian measure on C_0 obtained from Theorem 1′, it can be verified that the required countably additive extension $\tilde{\nu}$ to B of the canonical distribution on H is given by the relation $\nu = \tilde{\nu}\,\psi^{-1}$. We shall not go into the details here.

As a corollary to Theorem 1, we have the following result. Let Γ be the continuous covariance functional of the abstract Wiener measure μ on B.

Corollary 1. *The Hilbert space H of Theorem 1 is congruent to the RKHS of the process μ.*

For our next result we assume as in Theorem 1 that $\|x\|_1$ is a measurable norm on H, B is the Banach space which is the completion of H with respect to $\|x\|_1$. μ denotes the Gaussian measure of Theorem 1.

Theorem 2. *For every positive* ε

$$\mu\{x\in B\colon \|x\|_1 \leqq \varepsilon\} > 0. \tag{2.13}$$

In the proof we shall make use of the following fact observed by Gross and contained as part of Corollary 5.4 in [2].

Lemma 1. *If* $\|x\|_1$ *is a measurable norm on* H, *then*

$$\|x\|_1 \leqq A\,\|x\|_H \tag{2.14}$$

for all $x\in H$, *A being a positive number (independent of* x*).*

Proof of Theorem 2. Suppose that for some $\varepsilon>0$

$$\mu\{x\in B\colon \|x\|_1 \leqq \varepsilon\} = 0. \tag{2.15}$$

For $m\in H$ let us denote by σ_m the translation

$$\sigma_m x = x+m \qquad (x\in B). \tag{2.16}$$

Since H is the RKHS of the covariance R of μ, it follows from a well known result (see [7]) that the Gaussian measures $\mu\sigma_m^{-1}$ $(m\in H)$ and μ are mutually absolutely continuous. It follows from (2.15) that

$$\mu\{x\in B\colon \|x-m\|_1 \leqq \varepsilon\} = 0 \tag{2.17}$$

for all $m\in H$.

Let D be a countable dense subset of H, which exists since H is separable. Let $y\in B$ and $\varepsilon>0$ be as in (2.17).

There exists $x\in H$ such that

$$\|y-x\|_1 < \frac{\varepsilon}{2}, \tag{2.18}$$

and an element $m\in D$ such that

$$\|x-m\|_H < \frac{\varepsilon}{2A}, \tag{2.19}$$

A being the constant of Lemma 1. That lemma and (2.19) imply

$$\|x-m\|_1 < \tfrac{1}{2}\varepsilon. \tag{2.20}$$

From (2.18) and (2.20) we have

$$\|y-m\|_1 < \varepsilon. \tag{2.21}$$

Hence D is dense in B relative to $\|x\|_1$ norm, which in turn implies that

$$B \subset \bigcup_{m\in D} \{x\in B\colon \|x-m\|_1 \leqq \varepsilon\}. \tag{2.22}$$

From (2.17) and (2.22) we get $\mu(B)=0$ which is impossible. Hence Theorem 2 is proved.

3. Supports of Gaussian Measures on the Space of Continuous Functions

In this section we shall assume that $C=C(T)$ is the Banach space (with sup norm $\| \ \|_1$) of all real valued continuous functions $x(\cdot)$ on T. The index set T is, for the present, assumed to be a compact metric space. Later we shall specialise to $T=[0,1]$.

Let $\mathscr{A}(C)$ be the σ-field of Borel subsets of C. Then it is easy to see that $\mathscr{A}(C)$ is generated by the cylinder sets of the form

$$\{x \in C\colon [x(t_1), \ldots, x(t_n)] \in B\}$$

where $B \in \mathscr{A}(R^n)$ (σ-field of n-dimensional Borel sets). Suppose that μ is a Gaussian probability measure on $(C, \mathscr{A}(C))$, with mean zero and covariance R given by

$$R(t,s)=\int_C x(t)\,x(s)\,\mu(dx), \qquad (s,t \in T). \tag{3.1}$$

We shall assume that R is continuous on $T \times T$ and denote by $H(R)$ the RKHS of R. The σ-field $\mathscr{A}_\mu(C)$ will denote the completion of $\mathscr{A}(C)$ with respect to μ.

By topological support of μ, written $\operatorname{supp}(\mu)$ we mean the uniquely defined closed set F of C with the following properties:

$$\mu(F)=1, \tag{3.2}$$

for every open set G such that $G \cap F \neq \phi$

$$\mu(G \cap F)>0 \qquad \text{(see, e.g., [4])}. \tag{3.3}$$

Before we give the main theorem of this section we state a lemma given in [5] with modifications made to suit the set up of this paper.

Lemma 2 (Lemma 6, [5]). *Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal set in $H(R)$ and g an $\mathscr{A}_\mu(C)$ measurable real valued function such that for each x in C and every rational r*

$$g(x+r\,e_j)=g(x), \qquad (j=1,2,3,\ldots). \tag{3.4}$$

Then

$$g(x)=\text{constant} \qquad \text{a.s. } \mu. \tag{3.5}$$

Theorem 3. *Let μ be a Gaussian measure on $(C, \mathscr{A}_\mu(C))$ with mean zero and continuous covariance R. Then*

$$\overline{H(R)}=\operatorname{supp}(\mu) \tag{3.6}$$

where $\overline{H(R)}$ is the closure of $H(R)$ in C.

Proof. Let F denote $\operatorname{supp}(\mu)$ and let $f \in F$. Define

$$S_n=\left\{x \in C\colon \|x-f\|_1<\frac{1}{n}\right\}, \qquad (n=1,2,3,\ldots)$$

and

$$A_n=S_n+\overline{H(R)}.$$

Now A_n, being an open set, is $\mathscr{A}_\mu(C)$ measurable and further it is easy to verify that if $m \in H(R)$ then for every $x \in C$

$$x+m \in A_n \qquad \text{if and only if } x \in A_n,$$

i.e.,

$$\chi_{A_n}(x+m)=\chi_{A_n}(x), \tag{3.7}$$

for all $x\in C$ and $m\in H(R)$. Hence from the lemma

$$\chi_{A_n}(x)=\text{constant} \quad \text{a.s. } \mu. \tag{3.8}$$

But since $A_n \supset S_n+0$, we have $\mu(A_n)\geqq\mu(S_n)>0$. Further $S_n\cap F\neq\phi$ and $F=\operatorname{supp}(\mu)$. Hence

$$\chi_{A_n}(x)=1 \quad \text{a.s. } \mu,$$

i.e.,

$$\mu(A_n)=1. \tag{3.9}$$

Since $A_n\supseteq A_{n+1}$, we have

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right)=1.$$

Since $\overline{H(R)}$ is closed it is easy to show that

$$\bigcap_{n=1}^{\infty} A_n=f+\overline{H(R)}$$

and we get

$$\mu(f+\overline{H(R)})=1. \tag{3.10}$$

Since μ is Gaussian with mean zero, it is symmetric and we have

$$\mu(-(f+\overline{H(R)}))=1,$$

i.e.,

$$\mu(-f+\overline{H(R)})=1. \tag{3.11}$$

Hence there exist $g, h\in\overline{H(R)}$ such that $f+g=-f+h$ which shows that $f\in\overline{H(R)}$. We have thus shown that

$$\mu(\overline{H(R)})=1. \tag{3.12}$$

It remains to show that $\overline{H(R)}=\operatorname{supp}(\mu)$.

Suppose $\overline{H(R)}\neq\operatorname{supp}(\mu)$. Then there exists an element $x_0\in\overline{H(R)}$ such that for some integer $n\geqq 1$ $\mu(S_n)=0$ where $S_n=\left\{x\colon \|x-x_0\|_1<\dfrac{1}{n}\right\}$. Since for $m\in H(R)$ the measures $\mu\,\sigma_m^{-1}$ and μ are equivalent, (for the definition of σ_m see (2.18)), we have

$$\mu(S_n+m)=0 \quad \text{for all } m\in H(R).$$

It is clear that the family of open sets $\{S_n+m\}$, $m\in H(R)$, covers $\overline{H(R)}$, and since C is separable, by the Lindelöf theorem, there is a countable subcover $\{S_n+m_i\}_{i=1}^{\infty}$, $m_i\in H(R)$. But $\mu(S_n+m_i)=0$ for each i, hence $\mu(\overline{H(R)})=0$, which contradicts $\mu(\overline{H(R)})=1$. Hence $\overline{H(R)}=\operatorname{supp}(\mu)$. This completes the proof of the theorem.

It is convenient to have the following restatement of Theorem 3.

Theorem 4. *Let μ be a Gaussian measure on $(C, \mathscr{A}_\mu(C))$ with mean zero and continuous covariance. Let $x_0\in C$. Then*

$$\mu(\{x\colon \|x-x_0\|_1<\varepsilon\})>0 \tag{3.13}$$

for all $\varepsilon>0$ if and only if $x_0\in\overline{H(R)}$.

Let C_0 be a closed linear subspace and let $\mathscr{A}(C_0)$ be the family of all Borel subsets of C_0. It is useful to state the previous result for Gaussian measures on $(C_0, \mathscr{A}(C_0))$. It can be deduced easily from Theorem 3 or can be proved independently.

Theorem 5. *Let μ be a Gaussian measure on $(C_0, \mathscr{A}(C_0))$ with continuous covariance and zero mean. Then $\overline{H(R)} \subset C_0$ and*

$$\overline{H(R)} = \operatorname{supp}(\mu), \tag{3.14}$$

i.e., for $x_0 \in C_0$ $\mu(\{x \in C_0 \colon \|x - x_0\| < \varepsilon\}) > 0$ for all $\varepsilon > 0$ if and only if $x_0 \in \overline{H(R)}$.

Proof. Extend the measure μ from C_0 to C by writing

$$\tilde{\mu}(A) = \mu(A \cap C_0), \qquad A \in \mathscr{A}(C). \tag{3.15}$$

Then μ is a Gaussian measure on C with the same RKHS as the measure μ and $\tilde{\mu}(C_0) = \mu(C_0) = 1$. Since C_0 is closed and $\overline{H(R)}$ is $\operatorname{supp}(\tilde{\mu})$ by Theorem 3, it follows that $\overline{H(R)} \subset C_0$. Hence $\overline{H(R)} = \operatorname{supp}(\mu)$. This completes the proof of the theorem.

We conclude this section with a result concerning Gaussian measures on $C[0,1]$, the space of continuous functions on $T = [0,1]$.

Theorem 6. *Let R be a continuous covariance on $[0,1] \times [0,1]$. Then the canonical normal distribution on $H(R)$ extends to a Gaussian (countably additive) probability measure on $\overline{H(R)}$, the closure of $H(R)$ in $C[0,1]$ if and only if $\|x\|_1$ is a measurable norm on $H(R)$.*

Proof. The "if" part of the theorem is simply Theorem 1′. The "only if" part is proved as follows.

For brevity, set $C_0 = \overline{H(R)}$ and let μ denote the countably additive extension to C_0 of the canonical normal distribution. Then μ is a Gaussian measure on $(C_0, \mathscr{A}(C_0))$ with R as covariance. Let F be the representative of the canonical normal distribution in $H(R)$ given by the (continuous) linear map $F \colon H^*(R) \to M[C_0, \mathscr{A}(C_0), \mu]$ (the latter being the linear space of real random variables on $(C_0, \mathscr{A}(C_0), \mu)$), given by

$$F[R(\cdot, t)](x) = x(t), \qquad (x \in C_0, t \in T). \tag{3.16}$$

Let $\{t_j\}$, $t_j = j 2^{-n}$ $(j = 0, \ldots, 2^n)$ be a sequence of partitions of $T (n = 1, 2, \ldots)$. If φ is a real, Borel measurable function of $2^n + 1$ real variables, then

$$G_n(x) = \varphi[x(t_0), \ldots, x(t_{2^n})] \qquad (x \in H(R)) \tag{3.17}$$

is obviously a tame function on $H(R)$. This follows easily if we recall the definition of a tame function given in (1.1) and note that for x in $H(R)$ $x(t_j) = (R(\cdot, t_j), x)$. If $G_n^{\sim}$ denotes the corresponding random variable in $M[C_0, \mathscr{A}(C_0), \mu]$ it is easy to see that $G_n^{\sim}$ is the $\mathscr{A}(C_0)$-measurable function on C_0 given by

$$G_n^{\sim}(x) = \varphi[x(t_0), \ldots, x(t_{2^n})] \qquad (x \in C_0). \tag{3.18}$$

Now let us choose

$$G_n(x) = \sup\{|x(t_j)|, j = 0, \ldots, 2^n\} \qquad (x \in H(R)). \tag{3.19}$$

From (3.18)

$$G_n^{\sim}(x)=\sup\{|x(t_j)|, j=0,\dots,2^n\} \qquad (x\in C_0). \tag{3.20}$$

$G_n(x)$ $(n=1,2,\dots)$ is a sequence of tame (and hence measurable) semi-norms on $H(R)$ and $G_n(x)$ increases to the limit $\|x\|_1=\sup_{0\leqq t\leqq 1}|x(t)|$. The sequence of random variables $G_n^{\sim}(x)$ on $(C_0,\mathscr{A}(C_0),\mu)$ converges for every x in C_0 and hence in μ-probability to $\|x\|_1$. Furthermore from Theorem 4 and (3.13) we have $\mu\{x\in C_0\colon \|x\|_1\leqq\varepsilon\}>0$ for every positive ε. From Corollary 4.4 of [2] it follows that $\|x\|_1$ is a measurable norm on $H(R)$. The proof of Theorem 6 is complete.

4. Gaussian Measures on Separable Banach Spaces

Let B be an arbitrary separable Banach space with norm $\|\cdot\|$ and let μ be a Gaussian measure on $(B,\mathscr{A}(B))$ with mean zero and continuous covariance functional Γ. Then according to the Banach-Mazur theorem alluded to in Section 2, B is isometrically isomorphic, or congruent, to a closed linear subspace, say, C_1 of C, the Banach space of real continuous functions on $[0,1]$. Let ψ denote this congruence from B onto C_1 and let $\|\cdot\|_1$ be the sup norm on C. It is easy to verify that

$$\mathscr{A}(B)=\psi^{-1}(\mathscr{A}(C_1)), \qquad \text{where } \mathscr{A}(C_1)$$

is the σ-field of Borel sets of C_1 and that $\tilde{\mu}$ defined by

$$\tilde{\mu}=\mu\,\psi^{-1} \tag{4.1}$$

is a Gaussian measure on C_1 with covariance function R which is related to the covariance functional Γ of μ in the following way. The proof of the Banach-Mazur theorem given in [6] makes use of the fact that ψ sends x in a one to one manner into the function of t given by

$$y(t)=\psi(x)[t]=\langle f_t, x\rangle \qquad \text{where } f_t \text{ is an element of } S^*$$

the unit ball in B^*. It can then be shown quite easily that

$$R(t,s)=\int_{C_1} y(t)\,y(s)\,\tilde{\mu}(dy)=\Gamma(f_t,f_s) \qquad (0\leqq s,t\leqq 1). \tag{4.2}$$

The continuity of Γ implies that R is continuous on the unit square. Let us define

$$H=\psi^{-1}[H(R)] \tag{4.3}$$

where $H(R)$ is the RKHS of $\tilde{\mu}$ and

$$h_t=\psi^{-1}(R(\cdot,t)) \tag{4.4}$$

for each t in $[0,1]$.

Since $H(R)$ is a linear subspace of C_1 it is clear from (4.3) that H is a linear subspace of B. Furthermore H has Hilbert structure, i.e.,

$$(x_1,x_2)_0=(\psi(x_1),\psi(x_2))_{H(R)} \tag{4.5}$$

makes H a Hilbert space under the inner product $(,)_0$. The Hilbert space H acts as a RKHS for the Gaussian measure μ on B. In fact (we assume here that Γ is strictly positive, i.e., $\Gamma(f,f)=0$ implies $f=0$) from (4.2) and (4.4) we have

$$(h_t, h_s)_0 = \Gamma(f_t, f_s) = \big(\Gamma(\cdot, f_t), \Gamma(\cdot, f_s)\big)_{H(\Gamma)} \tag{4.6}$$

and hence H is congruent to $H(\Gamma)$ which is the RKHS of μ. The congruence of H to $H(\Gamma)$ is not specifically used in the proof of the result given below but it serves to explain the importance of H in studying μ. In this connection, it is perhaps appropriate to adopt a term coined by Schwartz [8] and call H the Hilbert subspace of the Gaussian process $(B, \mathscr{A}(B), \mu)$. We are now in a position to prove the following extension of Theorems 3 and 4.

Theorem 7. *Let B be an arbitrary separable Banach space and let μ be a Gaussian measure on $(B, \mathscr{A}_\mu(B))$ with continuous covariance functional. Then*

$$\overline{H} = \operatorname{supp}(\mu), \tag{4.7}$$

where $\overline{H}$ is the closure (with respect to the norm topology in B) of H in B, and for any x_0 in B

$$\mu\{x \in B \colon \|x - x_0\| < \varepsilon\} > 0 \tag{4.8}$$

for every positive ε if and only if

$$x_0 \in \overline{H}. \tag{4.9}$$

Proof. Applying Theorem 5 of the previous section to the Gaussian measure $\tilde{\mu}$ defined by (4.1) on $(C_1, \mathscr{A}(C_1))$ we have

$$\overline{H(R)} = \operatorname{supp}(\tilde{\mu}), \tag{4.10}$$

$\overline{H(R)}$ being the closure of $H(R)$ in C_1. In particular, $\tilde{\mu}(\overline{H(R)})=1$ immediately gives $\mu(\overline{H})=1$. Let G be any open set in B such that $G \cap \overline{H} \neq \phi$. Since ψ is a congruence from B to C_1 we have $\psi(G \cap \overline{H}) \neq \phi$,

$$\psi(G) \cap \overline{H(R)} \neq \phi. \tag{4.11}$$

In (4.11) it should be noted that $\overline{H}$ is mapped onto $\overline{H(R)}$ by ψ, a fact which is easy to verify. Hence from (4.10) $\psi(G)$ being open in C_1

$$\tilde{\mu}\big(\psi(G) \cap \overline{H(R)}\big) > 0. \tag{4.12}$$

But the left hand side of (4.12) is precisely $\mu(G \cap \overline{H})$. Thus (4.7) is proved. The second assertion of the theorem, (4.8) follows similarly.

In conclusion it may be pointed out that Theorems 4 and 7 contain as special cases the principal conclusions of two results recently obtained by Garsia, Posner and Rodemich in [1] (Theorems 2.1 and 3.1). They consider a separable, mean continuous Gaussian process $x(t)$ $(0 \leqq t \leqq 1)$ with mean zero and covariance R. Their first result concerns the measure on $L_2[0,1]$ induced by $x(t)$. Let L_R be the closed linear subspace of $L_2[0,1]$ spanned by the eigenfunctions $\{\varphi_n(t)\}$ of R.

Theorem 2.1 ([1]). *With probability one $x(t) \in L_R$. For $f_0 \in L_2[0,1]$, the neighborhood $\{f \in L_2 \colon \|f - f_0\|_2 < \varepsilon\}$ has positive probability for all $\varepsilon > 0$ if and only if $f_0 \in L_R$.*

Next, suppose the process $x(t)$ is sample continuous with probability one. Then $x(t)$ induces a measure in the space $C[0,1]$. Define C_R to be the closed linear subspace of $C[0,1]$ generated by finite linear combinations of the $\{\varphi_n\}$ with the metric

$$\|f-g\|_\infty = \sup_{0\leqq t\leqq 1} |f(t)-g(t)|.$$

We now state the relevant part of their second result.

Theorem 3.1 [1]. *If $x(t)$ has continuous paths, then $x(t)\in C_R$ with probability 1. For any function $f_0(t)\in C[0,1]$, the neighborhood*

$$\{f\in C[0,1]: \|f-f_0\|_\infty<\varepsilon\}$$

has positive probability for every $\varepsilon>0$, if and only if $f_0\in C_R$.

Remarks. Theorem 6 of Section 3 furnishes a necessary and sufficient criterion for a separable Gaussian process $x(t)$ $(0\leqq t\leqq 1)$ to have continuous paths.

The Gaussian measure μ of Theorem 7 is an abstract Wiener process.

I would like to thank M. G. Nadkarni for the idea of using Lemma 6 of [5] in the proof of Theorem 3.

References

1. Garsia, Adriano M., Posner, Edward C., Rodemich, Eugene R.: Some properties of the measures on function spaces induced by Gaussian processes. J. math. Analysis Appl. **21**, 150–161 (1968).
2. Gross, Leonard: Measurable functions on Hilbert space. Trans. Amer. math. Soc. **105**, 372–390 (1962).
3. – Abstract Wiener spaces. 5th Berkeley Symposium on Probability and Statistics. University of California, Berkeley 1965.
4. Hewitt, E., Ross, K.A.: Abstract harmonic analysis. New York: Academic Press 1963.
5. Kallianpur, G.: Zero one laws for Gaussian processes. Trans. Amer. math. Soc. **149**, 199–211 (1970).
6. Lyusternik, L.A., Sobolev, V.I.: Elementi Funktianalnogo Analyza, Moscow: GTTI 1950.
7. Parzen, E.: Statistical inference on time series by Hilbert space methods, I. Department of Statistics, Stanford University. Tech. Report No. **23** (1959).
8. Schwartz, L.: Radon measures on arbitrary topological spaces, to be published by the Tata Institute of Fundamental Research, Bombay (preprint prepared by the Department of Mathematics, University of Maryland).

Professor G. Kallianpur
University of Minnesota
School of Mathematics
Minneapolis, Minnesota 55455, USA

(Received September 5, 1969)