Z. Wahrscheinlichkeitstheorie verw. Gebiete
50, 53–83 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Convergence of Integrated Processes of Arbitrary Hermite Rank* **

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Dedicated to Professor Leopold Schmetterer
on occasion of his 60th Birthday

Summary. Let $\{X(s),\ -\infty<s<\infty\}$ be a normalized stationary Gaussian process with a long-range correlation. The weak limit in $C[0,1]$ of the integrated process $Z_x(t)=\frac{1}{d(x)}\int\limits_0^{xt} G(X(s))\,ds$, $x\to\infty$, is investigated. Here $d(x)=x^H L(x)$ with $\frac{1}{2}<H<1$ and $L(x)$ is a slowly varying function at infinity. The function G satisfies $EG(X(s))=0$, $EG^2(X(s))<\infty$ and has arbitrary Hermite rank $m\geqq 1$. (The *Hermite rank* of G is the index of the first non-zero coefficient in the expansion of G in Hermite polynomials.) It is shown that $Z_x(t)$ converges for all $m\geqq 1$ to some process $\bar{Z}_m(t)$ that depends essentially on m. The limiting process $\bar{Z}_m(t)$ is characterized through various representations involving multiple Itô integrals. These representations are all equivalent in the finite-dimensional distributions sense. The processes $\bar{Z}_m(t)$ are non-Gaussian when $m\geqq 2$. They are self-similar, that is, $\bar{Z}_m(at)$ and $a^H\bar{Z}_m(t)$ have the same finite-dimensional distributions for all $a>0$.

§ 1. Introduction

Self-similar processes have recently attracted the attention of mathematicians such as Sinaï (1976) and Dobrushin (1979), and physicists such as Jona-Lasinio (1977), because of their relevance to the renormalization group approach in physics. Self-similar processes are also of interest in hydrology where they account for the so-called "Hurst effect". See Jona-Lasinio (1977) for a review of the physics literature, and Lawrance and Kottegoda (1977) for a review of the literature in hydrology.

Dobrushin (1979) has introduced a general framework for the study of stationary self-similar random fields. They are defined as generalized random

* Research supported by the National Science Foundation grants MCS 77-03543 and ENG 78-11454.

** This paper contains results closely connected to those of the paper by Dobrushin and Major, Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 27–52 (1979). The investigations were done independently and at about the same time. Different methods were used

0044-3719/79/0050/0053/$06.20

functions on the Schwarz space $S(\mathbb{R}^\nu)$ of test functions or on some subset thereof. In some cases, the domain of definition of these fields can be extended to include indicator functions of rectangles in $\mathbb{R}^\nu$, of intervals when $\nu=1$. In the latter case, the self-similar random fields become self-similar stochastic processes with stationary increments. Stochastic processes of this type are central to this paper.

Definition 1.1. The stochastic process $\{Z(t), -\infty<t<\infty\}$ is *self-similar with parameter H*, if it satisfies the scaling condition

$$Z(at)\stackrel{\Delta}{=}a^H Z(t), \qquad a\geqq 0, \tag{1.1}$$

where $\stackrel{\Delta}{=}$ denotes equality of the finite-dimensional distributions.

For instance, Brownian motion is self-similar with parameter $H=\frac{1}{2}$, and more generally, a stable process is self-similar with parameter $H=\frac{1}{\alpha}$, $0<\alpha\leqq 2$. In such examples, the increments $Z(t+1)-Z(t)$ are independent over disjoint intervals. But there are also self-similar processes whose increments exhibit a *long-range dependence.*

Indeed, if $\frac{1}{2}<H<1$, and if $Z(t)$ has stationary increments and satisfies $Z(0)=0$, $EZ(t)=0$, $EZ^2(1)=1$, then, necessarily, $EZ^2(t)=|t|^{2H}$,

$$EZ(t_1)Z(t_2)=\tfrac{1}{2}\{|t_1|^{2H}+|t_2|^{2H}-|t_1-t_2|^{2H}\}$$

and thus, the correlations of the increments $Z(t+1)-Z(t)$ decrease slowly to zero,

$$E(Z(t+1)-Z(t))(Z(s+t+1)-Z(s+t))\sim H(2H-1)s^{2H-2}$$

as the lag s tends to infinity ($a_k\sim b_k$ means $a_k/b_k\to 1$ as $k\to\infty$). This slow decrease of the correlations is an expression of the long-range dependence of the increments. The Gaussian fractional Brownian motion (see Mandelbrot and Van Ness (1968)) and the non-Gaussian Rosenblatt process (see Taqqu (1975)) are two examples of self-similar processes whose increments exhibit a long-range dependence. More examples can be found in Dobrushin (1979).

There are two important open problems concerning self-similar processes:

1) the characterization of all self-similar processes.

2) given a self-similar processes, find its domain of attraction.

Dobrushin (1979) deals directly with the first problem. Davydov (1971), Taqqu (1975), and recently Dobrushin and Major (1979), characterized stationary sequences of random variables whose normalized sums converge weakly to self-similar processes.

In this paper, we consider a continuous version of the problem attacked by Dobrushin and Major (1979). We introduce the integrated process

$$Z_x(t)=\frac{1}{d(x)}\int_0^{xt} G(X(s))\,ds, \qquad x>0 \tag{1.2}$$

and prove, using a time-indexed representation for $Z_x(t)$, that, as $x\to\infty$, $Z_x(t)$ converges weakly in $C[0,1]$ to a self-similar process. $C[0,1]$ denotes the space of continuous functions on $[0,1]$ with the sup-norm topology. The function G satisfies $EG(X(s))=0$ and $EG^2(X(s))<\infty$ and $\{X(s), -\infty<s<\infty\}$ is a continuous parameter, stationary, normalized Gaussian process, exhibiting a long-range dependence (see the following section for precise assumptions on $X(s)$). The normalization factor $d(x)$ is chosen such $d^2(x)\sim E\left(\int_0^x G(x(s))\,ds\right)^2$ as $x\to\infty$. Because of the strong dependence, $Z_x(t)$ does not converge to Brownian motion. The weak limit $Z_x(t)$ depends on the *Hermite rank* m of G, that is, on the first non-zero index in the expansion of G in Hermite polynomials.

We prove here that $Z_x(t)$ converges weakly for all $m\geqq 1$. The limiting process $\bar Z_m(t)$, which depends on the Hermite rank m, can be represented as

$$\bar Z_m(t)=K(m,H_0)\left\{\int_{-\infty}^{+\infty} dB(\xi_1)\int_{-\infty}^{\xi_1} dB(\xi_2)\dots\int_{-\infty}^{\xi_{m-1}} dB(\xi_m)\right.$$
$$\left.\cdot\int_0^t\prod_{i=1}^m (s-\xi_i)^{H_0-\frac{3}{2}}1(\xi_i<s)\,ds\right\}, \tag{1.3}$$

that is,

$$\bar Z_m(t)=K(m,H_0)\left\{\int_{-\infty}^{0} dB(\xi_1)\int_{-\infty}^{\xi_1} dB(\xi_2)\dots\int_{-\infty}^{\xi_{m-1}} dB(\xi_m)\int_0^t\prod_{i=1}^m (s-\xi_i)^{H_0-\frac{3}{2}}\,ds\right.$$
$$\left.+\int_0^t dB(\xi_1)\int_{-\infty}^{\xi_1} dB(\xi_2)\dots\int_{-\infty}^{\xi_{m-1}} dB(\xi_m)\int_{\xi_1}^t\prod_{i=1}^m (s-\xi_i)^{H_0-\frac{3}{2}}\,ds\right\}. \tag{1.4}$$

In (1.3) and (1.4), B represents the standard real-valued Gaussian white noise measure,

$$1-\frac{1}{2m}<H_0<1, \tag{1.5}$$

is a parameter, and

$$K(m,H_0)=\left\{\frac{m!(m(H_0-1)+1)(2m(H_0-1)+1)}{\left(\int_0^\infty (u+u^2)^{H_0-\frac{3}{2}}du\right)^m}\right\}^{1/2}$$
$$=\left\{\frac{m!(m(H_0-1)+1)(2m(H_0-1)+1)(\Gamma(\frac{3}{2}-H_0))^m}{(\Gamma(H_0-\frac{1}{2})\Gamma(2-2H_0))^m}\right\}^{1/2} \tag{1.6}$$

is a normalization coefficient ensuring $E\bar Z_m^2(1)=1$.

The representation (1.4) of the process $\bar Z_m(t)$, $m\geqq 1$ was introduced in Taqqu (1978a). The $\bar Z_m(t)$'s belong to $L^2(P)$, the Hilbert space of functions that are square integrable with respect to P, P denoting the measure of the underlying probability space supporting B. The $\bar Z_m(t)$'s have stationary increments and they satisfy $\bar Z_m(0)=0$, $E\bar Z_m(t)=0$ and $E\bar Z_m^2(t)=|t|^{2H}$. They are non-Gaussian when

$m \geqq 2$. Explicit formulas for their moments can be found in Theorem 3.1 of Taqqu (1978a)[1,2]. These moments may not necessarily characterize a unique distribution in the case $m \geqq 3$, and therefore the method followed in this paper avoids the use of moments higher than 2.

The processes $\bar{Z}_m(t)$ are self-similar with parameter

$$H = m(H_0 - 1) + 1, \tag{1.7}$$

and since $1 - \frac{1}{2m} < H_0 < 1$, the parameter H satisfies

$$\tfrac{1}{2} < H < 1 \tag{1.8}$$

for all $m \geqq 1$. $\bar{Z}_1(t)$ is fractional Brownian motion and $\bar{Z}_2(t)$ is the Rosenblatt process.

By specializing our theorems to the discrete parameter case $\frac{1}{d(N)} \sum_{i=1}^{[Nt]} G(X_i)$, we recover some of the results obtained by Dobrushin and Major (1979). Dobrushin and Major work in the spectral domain and exploit the fact that the spectral measure of the underlying Gaussian sequence has bounded support. The limiting process that they obtain is characterized through a spectral representation, of a type introduced by Dobrushin (1979) in his Theorem 6.3. That representation is shown in Sect. 6 to be equivalent to (1.3).

Assumptions on $X(s)$ are listed in Sect. 2. The fractional Gaussian noise process is introduced in Sect. 3 as an example of a possible $X(s)$. Section 4 contains preliminary lemmas. The main results about weak convergence are found in Sect. 5. Section 6 contains various equivalent representations for $\bar{Z}_m(t)$.

Some Remarks on the Notation

The three fundamental parameters are m, H_0 and H. $m \geqq 1$ is the Hermite rank, H_0, which is required to satisfy (1.5), involves the underlying Gaussian process $X(s)$, and H, is the self-similarity parameter of the limiting self-similar process. m, H_0 and H are related through (1.7) and one always has $\frac{1}{2} < H < 1$. H and H_0 are identical when $m = 1$.

We now relate these parameters to those used in other papers.

The parameter D in Taqqu (1975; 1977; 1978a) is here $D = 2 - 2H_0$.

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1 A more specific evaluation of some of the moments can be found in Taqqu (1977), p. 228, after setting $D = 2 - 2H_0$ and

$$E\bar{Z}_m(t_1)\bar{Z}_m(t_2)\ldots\bar{Z}_m(t_p) = \frac{\mu_p(1, 1, \ldots, 1)}{(\mu_2(1, 1))^{1/2}}$$

2 *Errata*. In the statement of Theorem 3.1 of Taqqu (1978a), the right hand side of $C(m, D)$ should be multiplied by $(m!)^{1/2}$ and the right hand side of K should be replaced by its square root. Also, line 9, p. 62, should read "$m!\,t^{-mD+2}/C^2(m, D) < \infty$". The constant $C(m, D)$ in that paper is identical to our $K(m, H_0)$ with $D = 2 - 2H_0$

The parameters ν, n, r and κ in Dobrushin (1979) are here $\nu=1$, $n=m$, and, either $r=1$ and $\kappa=-H$, or $r=0$ and $\kappa=1-H$.

The parameters ν, k, α of Dobrushin and Major (1979) are here $\nu=1$, $k=m$ and $\alpha=2-2H_0$.

The parameters d, n, γ_n and α in Sinaï (1976) are here $d=1$, $n=m$, $\gamma_n=H$, and α equals $2H_0$ in some contexts and $2H$ in others.

Finally, note that the processes $\bar{Z}_m(t)$ are defined as in Taqqu (1978a). They are normalized. The processes $\bar{Z}_m(t)$ of Taqqu (1975; 1977) are not normalized.

§ 2. The Underlying Gaussian Process

We define here the stationary Gaussian process $X(s)$, $-\infty<s<\infty$ that appears as the argument of the function G in (1.2). We shall impose conditions on $X(s)$ which depend on a parameter m. This parameter will be identified in Sect. 5 as the Hermite rank of G.

Thus, let $m\geqq 1$ be a given integer and set

$$1-\frac{1}{2m}<H_0<1. \tag{2.1}$$

Let $L(x)$ be a slowly varying function at infinity, defined on $(0,\infty)$, that is bounded on bounded intervals and let C be a positive constant.

Let $e(u)$, $-\infty<u<\infty$, be a measurable function satisfying the following conditions:

(A1) $\sigma^2=\int_{-\infty}^{+\infty} e^2(u)\,du<\infty$.

(A2) $|e(u)|\leqq C u^{H_0-\frac{3}{2}} L(u)$

for almost all $u>0$.

(A3) $e(u)\sim u^{H_0-\frac{3}{2}} L(u)$

as $u\to\infty$.

(A4) There exists a constant γ satisfying

$$0<\gamma<\min\left\{H_0-\left(1-\frac{1}{2m}\right), 1-H_0\right\}$$

such that

$$\int_{-\infty}^{0} |e(u)\,e(xy+u)|\,du=o(x^{2H_0-2}L^2(x))\,y^{2H_0-2-2\gamma}$$

as $x\to\infty$, uniformly in $y\in(0,t]$, for a given $t>0$.

Finally, define

$$X(s)=\frac{1}{\sigma}\int_{-\infty}^{\infty} e(s-\xi)\,dB(\xi), \qquad -\infty<s<\infty \tag{2.2}$$

where B is the standard Gaussian white noise measure satisfying $EB(\Delta)=0$ and $EB^2(\Delta)=|\Delta|$ for Borel sets Δ of finite Lebesgue measure $|\Delta|$. $X(s)$ is thus Gaussian, stationary, and satisfies $EX(s)=0$ and $EX^2(s)=1$.

The condition (A1) ensures that the process $X(s)$, $-\infty<s<\infty$, is well defined in $L^2(P)$.

(A2) controls the behavior of $|e(u)|$ for small positive u. It is a relatively weak condition because the behavior of $L(u)$ around the origin can be modified without affecting the results of the paper.

(A3) ensures that $X(s)$ exhibits a long-range dependence (see relation (2.3) below).

Since $X(s)$ is expressed as a moving average, its spectral distribution function is absolutely continuous. If $e(u)=0$ when $u\leq 0$, then the moving average becomes one-sided, and $X(s)$ is then purely non-deterministic (i.e. regular). In any case, the condition (A4) ensures that the "forward" contribution of $e(u)$ is ultimately negligible as the following computation suggests:

$$\begin{aligned}
R(x)&=EX(s)X(s+x)\\
&=\frac{1}{\sigma^2}\int_{-\infty}^{\infty} e(s-\xi)\,e(s+x-\xi)\,d\xi\\
&=\frac{1}{\sigma^2}\int_{-\infty}^{+\infty} e(u)\,e(x+u)\,du\\
&=o(x^{2H_0-2}L^2(x))+\frac{x}{\sigma^2}\int_0^{\infty} e(xu)\,e(x(1+u))\,du
\end{aligned}$$

because of (A4). An application of Corollary 4.3 below (see Sect. 4), shows that

$$R(x)=EX(s)X(s+x)\sim\left(\frac{1}{\sigma^2}\int_0^{\infty}(u+u^2)^{H_0-\frac{3}{2}}\,du\right)x^{2H_0-2}L^2(x) \tag{2.3}$$

as $x\to\infty$. Note also that if H_m denotes the Hermite polynomial of index m (refer to Sect. 5 for a precise definition), then the variance of the integrated process $\int_0^x H_m(X(s))\,ds$ satisfies

$$\begin{aligned}
E\left(\int_0^x H_m(X(s))\,ds\right)^2&=m!\int_0^x\int_0^x R^m(s_1-s_2)\,ds_1\,ds_2\\
&=2m!\int_0^x ds_2\int_0^{s_2} R^m(y)\,dy\\
&=2m!\left(\frac{1}{\sigma^2}\int_0^{\infty}(u+u^2)^{H_0-\frac{3}{2}}\,du\right)^m\int_0^x ds\int_0^s y^{2H-2}L^{2m}(y)\,dy\\
&\sim\frac{(m!)^2}{\sigma^{2m}K^2(m,H_0)}\,x^{2H}L^{2m}(x)
\end{aligned} \tag{2.4}$$

as $x \to \infty$, with $K(m, H_0)$ defined as in (1.6) and H defined as in (1.7). The square root of expression (2.4) will provide the normalization factor $d(x)$ that enters in (1.2).

A possible $X(s)$ is the fractional Gaussian noise process, defined in the following section.

§ 3. An Example: Fractional Gaussian Noise

The Gaussian process $\bar{Z}_1(t)$, defined in (1.4), is called *fractional Brownian motion* and is commonly denoted $B_{H_0}(t)$. It is self-similar with parameter $\frac{1}{2} < H_0 < 1$. Using (1.4), we have

$$\begin{aligned} B_{H_0}(t) &= K(1, H_0)\left\{\int_{-\infty}^{0} dB(\xi)\int_0^t (s-\xi)^{H_0-\frac{3}{2}}\,ds + \int_0^t dB(\xi)\int_\xi^t (s-\xi)^{H_0-\frac{3}{2}}\,ds\right\} \\ &= \frac{K(1, H_0)}{H_0-\frac{1}{2}}\left\{\int_{-\infty}^{0} dB(\xi)((t-\xi)^{H_0-\frac{1}{2}} - (-\xi)^{H_0-\frac{1}{2}}) + \int_0^t dB(\xi)(t-\xi)^{H_0-\frac{1}{2}}\right\}, \end{aligned} \tag{3.1}$$

with

$$K(1, H_0) = \left(\frac{H_0(2H_0-1)}{\int_0^\infty (u+u^2)^{H_0-\frac{3}{2}}\,du}\right)^{1/2}. \tag{3.2}$$

Recall that $B_{H_0}(0)=0$, $EB_{H_0}(t)=0$, $EB_{H_0}(t)=|t|^{2H_0}$, and that $B_{H_0}(t)$ has stationary increments. Other properties of $B_{H_0}(t)$ are derived in Mandelbrot and Van Ness (1968).

Now define the *fractional Gaussian noise* process as

$$X(s) = B_{H_0}(s) - B_{H_0}(s-1), \qquad -\infty < s < \infty \tag{3.3}$$

This process satisfies the conditions of the preceding section with $m=1$. Indeed, it can be expressed as

$$X(s) = \frac{1}{\sigma}\int_{-\infty}^{s} e(s-\xi)\,dB(\xi) \tag{3.4}$$

with

$$e(u) = \begin{cases} 0 & \text{when } u \leqq 0 \\ u^{H_0-\frac{1}{2}} & \text{when } 0 \leqq u \leqq 1 \\ u^{H_0-\frac{1}{2}} - (u-1)^{H_0-\frac{1}{2}} & \text{when } u \geqq 1 \end{cases} \tag{3.5}$$

and

$$\sigma = \frac{H_0-\frac{1}{2}}{K(1, H_0)}. \tag{3.6}$$

$\sigma^2 = \int_0^\infty e^2(u)\,du$ ensures that $EX^2(s)=1$. The kernel $e(u)$, $-\infty < u < \infty$, satisfies the conditions of the preceding section, because we can write

$$e(u)=u^{H_0-\frac{3}{2}}L(u)$$

when $u>0$, where

$$L(u)=\begin{cases}u, & 0<u\leqq 1\\ u^{\frac{3}{2}-H_0}(u^{H_0-\frac{1}{2}}-(u-1)^{H_0-\frac{1}{2}}), & u\geqq 1.\end{cases} \tag{3.7}$$

$L(u)$ is thus a slowly varying function at infinity, defined on $(0,\infty)$, bounded on bounded intervals, and such that

$$\lim_{u\to\infty} L(u)=\lim_{u\to\infty} u\left(1-\left(1-\frac{1}{u}\right)^{H_0-\frac{1}{2}}\right)=H_0-\tfrac{1}{2}. \tag{3.8}$$

The correlation $R(x)=EX(s)X(s+x)$ equals $\int_0^\infty e(u)\,e(x+u)\,du$. The direct evaluation of this integral for all $x\geqq 0$ is delicate. It is more convenient to use the fact that $X(s)$ is the increment of $B_{H_0}(t)$. This immediately leads to

$$\begin{aligned}R(x)&=EB_{H_0}(1)(B_{H_0}(x)-B_{H_0}(x-1))\\&=\tfrac{1}{2}\{(|x|+1)^{2H_0}-2|x|^{2H_0}+||x|-1|^{2H_0}\}.\end{aligned} \tag{3.9}$$

It is then easy to verify directly that

$$R(x)\sim H_0(2H_0-1)\,x^{2H_0-2} \tag{3.10}$$

as $x\to\infty$. This is consistent with the result (2.3) of the preceding section, because by (3.8),

$$\left(\frac{1}{\sigma^2}\int_0^\infty (u+u^2)^{H_0-\frac{3}{2}}\,du\right)x^{2H_0-2}L^2(x)\sim H_0(2H_0-1)\,x^{2H_0-2}$$

as $x\to\infty$.

§ 4. Preliminary Results

In this section, we establish several lemmas of a technical nature. We conclude the section with a theorem about convergence to $\bar{Z}_m(t)$ in $L^2(P)$.

Lemma 4.1. *Let*

$$V(x)=x^{\rho}L(x)$$

as $x\to\infty$, where $-\infty<\rho<\infty$ and where $L(x)$ is a slowly varying function, defined on $(0,\infty)$ and bounded on bounded intervals. Then $\forall\gamma>0$, $\forall u_0>0$ and $\forall\varepsilon>0$, $\exists x_0=x_0(\varepsilon)$ such that the following relations hold for all $x>x_0$:

$$(\varepsilon-u_0^{\gamma})\,u^{\rho-\gamma}<\frac{V(xu)}{V(x)}<(\varepsilon+u_0^{\gamma})\,u^{\rho-\gamma} \tag{4.1}$$

for all $u\in(0,u_0]$ and,

$$(\varepsilon - u_0^{-\gamma})\, u^{\rho+\gamma} < \frac{V(xu)}{V(x)} < (\varepsilon + u_0^{-\gamma})\, u^{\rho+\gamma} \tag{4.2}$$

for all $u \in [u_0, \infty)$.

Proof. To prove (4.1), note that

$$\frac{V(xu)}{V(x)} = u^{\rho} \frac{L(xu)}{L(x)}$$
$$= u^{\rho-\gamma} \frac{(xu)^{\gamma} L(xu)}{x^{\gamma} L(x)}.$$

Since $\gamma > 0$, $\frac{(xu)^{\gamma} L(xu)}{x^{\gamma} L(x)}$ tends to u^{γ} as $x \to \infty$, uniformly in $u \in (0, u_0]$ (De Haan (1970), p. 21). Therefore for any $\varepsilon > 0$, there is an $x_0(\varepsilon)$ such that for $x > x_0(\varepsilon)$

$$\varepsilon - u_0^{\gamma} < \varepsilon - u^{\gamma} < \frac{(xu)^{\gamma} L(xu)}{x^{\gamma} L(x)} < \varepsilon + u^{\gamma} < \varepsilon + u_0^{\gamma}.$$

Similarly, to prove (4.2), write

$$\frac{V(xu)}{V(x)} = u^{\rho+\gamma} \frac{(xu)^{-\gamma} L(xu)}{x^{-\gamma} L(x)}$$

and use the fact that $\frac{(xu)^{-\gamma} L(xu)}{x^{-\gamma} L(x)}$ tends to $u^{-\gamma}$ as $x \to \infty$, uniformly in $u \in [u_0, \infty)$. This concludes the proof. □

The following lemma provides useful estimates. First, some notation. Let

$$V_1(x) = x^{H_0 - \frac{3}{2}} L_1(x), \tag{4.3}$$
$$V_2(x) = x^{H_0 - \frac{3}{2}} L_2(x) \tag{4.4}$$

where $\frac{1}{2} < H_0 < 1$ and where $L_1(x)$ and $L_2(x)$ are slowly varying functions at infinity, defined on $(0, \infty)$ and bounded on bounded intervals.

Let C be a positive constant and let $e_1(x)$ and $e_2(x)$ be two measurable functions satisfying

$$|e_i(x)| \leqq C V_i(x) \tag{4.5}$$

for almost all x, and

$$e_i(x) \sim V_i(x) \tag{4.6}$$

as $x \to \infty$, for $i = 1, 2$.

Lemma 4.2. *Let* $t > 0$, $0 < \alpha \leqq t \leqq \beta$ *and let* $e_1(x)$, $e_2(x)$ $V_1(x)$, $V_2(x)$ *be defined as above. Let also* $0 < y \leqq t$ *and*

$$0 < \gamma < \min(H_0 - \tfrac{1}{2}, 1 - H_0). \tag{4.7}$$

Then for all large enough x, there are positive constants M_1, M_2 and M_3, independent of α, β and γ such that

$$\begin{aligned}\frac{1}{V_1(x)V_2(x)}&\int_0^\alpha |e_1(xu)\,e_2(x(y+u))|\,du\\&\leqq M_1 y^{2H_0-2-2\gamma}\int_0^{\alpha/y}(u+u^2)^{H_0-\frac{3}{2}-\gamma}\,du,\end{aligned}\tag{4.8}$$

$$\begin{aligned}\frac{1}{V_1(x)V_2(x)}&\int_\beta^\infty |e_1(xu)\,e_2(x(y+u))|\,du\\&\leqq M_2 y^{2H_0-2+2\gamma}\int_{\beta/y}^{\infty}(u+u^2)^{H_0-\frac{3}{2}+\gamma}\,du\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}\frac{1}{V_1(x)V_2(x)}&\int_0^\infty |e_1(xu)\,e_2(x(y+u))|\,du\\&\leqq M_3 \max(y^{2H_0-2-2\gamma}, y^{2H_0-2+2\gamma}).\end{aligned}\tag{4.10}$$

Proof. By Lemma 4.1, $\forall\,\varepsilon>0$, $\exists\, x_0(\varepsilon)$ such that for all $x>x_0(\varepsilon)$,

$$\begin{aligned}\frac{1}{V_1(x)V_2(x)}&\int_0^\alpha |e_1(xu)\,e_2(x(y+u))|\,du\\&\leqq C^2\int_0^\alpha \frac{V_1(xu)}{V_1(x)}\frac{V_2(x(y+u))}{V_2(x)}\,du\\&\leqq C^2(\varepsilon+\alpha^\gamma)(\varepsilon+(y+\alpha)^\gamma)\int_0^\alpha (u(y+u))^{H_0-\frac{3}{2}-\gamma}\,du\\&\leqq C^2(\varepsilon+(2t)^\gamma)\,y^{2H_0-2-2\gamma}\int_0^{\alpha/y}(u+u^2)^{H_0-\frac{3}{2}-\gamma}\,du\end{aligned}$$

because $\alpha\geqq t$ and $y\leqq t$. The integral converges because $H_0-\frac{1}{2}-\gamma>0$. Setting $M_1=C^2(\varepsilon+(2t)^\gamma)$ proves relation (4.8).

Similarly,

$$\begin{aligned}\frac{1}{V_1(x)V_2(x)}&\int_\beta^\infty |e_1(xu)\,e_2(x(y+u))|\,du\\&\leqq C^2(\varepsilon+\beta^{-\gamma})(\varepsilon+(y+\beta)^{-\gamma})\int_\beta^\infty (u(y+u))^{H_0-\frac{3}{2}+\gamma}\,du\\&\leqq C^2(\varepsilon+t^{-\gamma})^2\,y^{2H_0-2+2\gamma}\int_{\beta/y}^\infty (u+u^2)^{H_0-\frac{3}{2}+\gamma}\,du\end{aligned}$$

because $\beta\geqq t$. The integral converges because $H_0-1+\gamma<0$. Setting $M_2=C^2(\varepsilon+t^{-\gamma})^2$ proves relation (4.9).

Finally, let

$$M_3=\max\left\{M_1\int_0^\infty (u+u^2)^{-\gamma+H_0-\frac{3}{2}}\,du,\ M_2\int_0^\infty (u+u^2)^{\gamma+H_0-\frac{3}{2}}\,du\right\}.$$

It is easy to check that $M_3<\infty$. Relation (4.10) results from (4.8) and (4.9) with $\alpha=\beta=t$. This proves the lemma. □

Corollary 4.3. *For all $y>0$,*

$$\lim_{x\to\infty}\frac{1}{V_1(x)V_2(x)}\int_0^\infty e_1(xu)\,e_2(x(y+u))\,du=y^{2H_0-2}\int_0^\infty(u+u^2)^{H_0-\frac{3}{2}}\,du.$$

Proof. Relation (4.10) justifies the application of the dominated convergence theorem:

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{V_1(x)V_2(x)}\int_0^\infty e_1(xu)\,e_2(x(y+u))\,du&=\int_0^\infty u^{H_0-\frac{3}{2}}(y+u)^{H_0-\frac{3}{2}}\,du\\&=y^{2H_0-2}\int_0^\infty(u+u^2)^{H_0-\frac{3}{2}}\,du.\quad\square\end{aligned}$$

Lemma 4.4. *Let $m\geqq 1$, $1\leqq p\leqq m$, $1-\frac{1}{2m}<H_0<1$ and*

$$0<\gamma<\min\left\{H_0-\left(1-\frac{1}{2m}\right),1-H_0\right\}.$$

Then

$$\int_0^t ds_1\int_0^t ds_2\,|s_1-s_2|^{(2H_0-2-2\gamma)m}\left\{\int_0^{\alpha/|s_1-s_2|}(u+u^2)^{H_0-\frac{3}{2}-\gamma}\,du\right\}^p=o(1)\tag{4.11}$$

as $\alpha\to 0$. Also,

$$\begin{aligned}&\int_0^t ds_1\int_0^t ds_2\,|s_1-s_2|^{(2H_0-2+2\gamma)m}\left\{\int_{\beta/|s_1-s_2|}^\infty(u+u^2)^{H_0-\frac{3}{2}+\gamma}\,du\right\}^p\\&\quad=O(\beta^{2H_0-2+2\gamma})\\&\quad=o(1)\end{aligned}\tag{4.12}$$

as $\beta\to\infty$.

Proof. We first prove (4.11). Let $\eta=\eta(\alpha)$ and assume $0<\eta<t$. Then the left hand side of (4.11) is bounded above by $J_1(t,\alpha)+J_2(t,\alpha)$ where

$$J_1(t,\alpha)=\int_0^t ds_1\int_0^t ds_2|s_1-s_2|^{(2H_0-2-2\gamma)m}1_{(0,\eta)}(|s_1-s_2|)\left\{\int_0^\infty(u+u^2)^{H_0-\frac{3}{2}-\gamma}\right\}^p$$

and

$$J_2(t,\alpha)=\int_0^t ds_1\int_0^t ds_2|s_1-s_2|^{(2H_0-2-2\gamma)m}1_{(\eta,t)}(|s_1-s_2|)\left\{\int_0^{\alpha/\eta}(u+u^2)^{H_0-\frac{3}{2}-\gamma}\right\}^p$$

and where $1_A(\cdot)$ denotes the indicator function of the set A. Let $C_1=\left\{\int_0^\infty(u+u^2)^{H_0-\frac{3}{2}-\gamma}\right\}^p<\infty$. Then

$$J_1(t,\alpha)\leqq 2C_1 \iint_{\substack{0<s_2<s_1<t\\ 0<s_1-s_2<\eta}} (s_1-s_2)^{(2H_0-2-2\gamma)m}\,ds_1\,ds_2$$
$$=2C_1\int_0^t ds_1 \int_0^{s_1\wedge\eta} u^{(2H_0-2-2\gamma)m}\,du$$
$$=O(\eta^{(2H_0-2-2\gamma)m+1})$$

as $\eta\to 0$. This tends to 0 since $(2H_0-2-2\gamma)m+1>0$.

Now,

$$J_2(t,\alpha)\leqq C_2\left\{\int_0^{\alpha/\eta} (u+u^2)^{H_0-\frac{3}{2}-\gamma}\right\}^p$$

where $C_2=\int_0^t ds_1\int_0^t ds_2|s_1-s_2|^{(2H_0-2-2\gamma)m}<\infty$. Then, as $\frac{\alpha}{\eta}\to 0$,

$$J_2(t,\alpha)=O\left(\int_0^{\alpha/\eta} u^{H_0-\frac{3}{2}-\gamma}\,du\right)^p$$
$$=O\left(\frac{\alpha}{\eta}\right)^{H_0-\frac{1}{2}-\gamma}$$

tends to 0 because $H_0-\frac{1}{2}-\gamma>0$. Choosing $\eta=\sqrt{\alpha}$ for example, proves (4.11).

To prove (4.12), bound the left hand side of (4.12) by $C_2\left(\int_{\beta/t}^{\infty}(u+u^2)^{H_0-\frac{3}{2}+\gamma}\right)^p$ which is $O(\beta^{2H_0-2+2\gamma})=o(1)$ as $\beta\to\infty$, because $p\geqq 1$ and $2H_0-2+2\gamma<0$.

This completes the proof. □

The preceding lemmas yield the following result.

Lemma 4.5. *Let* $t\geqq 0$, $m\geqq 1$, $1-\dfrac{1}{2m}<H_0<1$. *Suppose that* $e(u)$, $0<u<\infty$ *satisfies the conditions* (A2) *and* (A3) *of Sect.* 2. *Let* $V(x)=x^{H_0-\frac{3}{2}}L(x)$. *Then*

$$\lim_{x\to\infty}\int_0^t ds_1\int_0^t ds_2 \int_{-\infty}^{s_1\wedge s_2} d\xi_1 \int_{-\infty}^{s_1\wedge s_2} d\xi_2 \ldots \int_{-\infty}^{s_1\wedge s_2} d\xi_m$$
$$\cdot\left|\left(\prod_{i=1}^m \frac{e(x(s_1-\xi_i))}{V(x)}-\prod_{i=1}^m (s_1-\xi_i)^{H_0-\frac{3}{2}}\right)\right.$$
$$\cdot\left.\left(\prod_{i=1}^m \frac{e(x(s_2-\xi_i))}{V(x)}-\prod_{i=1}^m (s_2-\xi_i)^{H_0-\frac{3}{2}}\right)\right|=0.$$

Proof. Assume $0<t\leqq 1$ without loss of generality. Let

$$Q(s_1,s_2;x)=\int_{-\infty}^{s_1\wedge s_2} d\xi_1 \int_{-\infty}^{s_1\wedge s_2} d\xi_2 \ldots \int_{-\infty}^{s_1\wedge s_2} d\xi_m$$
$$\cdot\left|\left\{\prod_{i=1}^m \frac{e(x(s_1-\xi_i))}{V(x)}-\prod_{i=1}^m (s_1-\xi_i)^{H_0-\frac{3}{2}}\right\}\right.$$
$$\cdot\left.\left\{\prod_{i=1}^m \frac{e(x(s_2-\xi_i))}{V(x)}-\prod_{i=1}^m (s_2-\xi_i)^{H_0-\frac{3}{2}}\right\}\right|$$

$$=\int_0^\infty du_1 \int_0^\infty du_2 \dots \int_0^\infty du_m \left| \left\{ \prod_{i=1}^m \frac{e(xu_i)}{V(x)} - \prod_{i=1}^m u_i^{H_0-\frac{3}{2}} \right\} \cdot \left\{ \prod_{i=1}^m \frac{e(x(|s_1-s_2|+u_i)}{V(x)} - \prod_{i=1}^m (|s_1-s_2|+u_i)^{H_0-\frac{3}{2}} \right\} \right|.$$

Let $0<\alpha\leqq t\leqq\beta$ and let $1_A(u)$ denote the indicator function of the set A. Then

$$Q(s_1,s_2;x)\leqq \int_0^\infty du_1 \int_0^\infty du_2 \dots \int_0^\infty du_m \left| \left\{ \prod_{i=1}^m \frac{e(xu_i)}{V(x)} - \prod_{i=1}^m u_i^{H_0-\frac{3}{2}} \right\} \cdot \left\{ \prod_{i=1}^m \frac{e(x(|s_1-s_2|+u_i)}{V(x)} - \prod_{i=1}^m (|s_1-s_2|+u_i)^{H_0-\frac{3}{2}} \right\} \right| \cdot \left\{ \sum_{\substack{I\\|I|\geqq 1}} \prod_{i\in I} 1_{(0,\alpha)}(u_i) + \sum_{\substack{I\\|I|\geqq 1}} \prod_{i\in I} 1_{(\beta,\infty)}(u_i) + \prod_{i=1}^m 1_{[\alpha,\beta]}(u_i) \right\}$$

where the set of indices I is a subset of $\{1,2,\dots,m\}$, and where $\sum_I$ denotes a sum running over all possible such subsets.

To simplify the notation, set

$$F^{(1)}(u,y,x)=\left|\frac{e(xu)}{V(x)}\frac{e(x(y+u))}{V(x)}\right|,$$

$$F^{(2)}(u,y,x)=\left|\frac{e(xu)}{V(x)}(y+u)^{H_0-\frac{3}{2}}\right|,$$

$$F^{(3)}(u,y,x)=\left|u^{H_0-\frac{3}{2}}\frac{e(x(y+u))}{V(x)}\right|,$$

$$F^{(4)}(u,y,x)=u^{H_0-\frac{3}{2}}(y+u)^{H_0-\frac{3}{2}}$$

for $y>0$. Then

$$Q(s_1,s_2;x)\leqq A(\alpha,|s_1-s_2|,x)+B(\beta,|s_1-s_2|,x)+C(\alpha,\beta,|s_1-s_2|,x) \tag{4.13}$$

where

$$A(\alpha,|s_1-s_2|,x)=\sum_{j=1}^4 \int_0^\infty du_1 \int_0^\infty du_2 \dots \int_0^\infty du_m \left\{ \prod_{i=1}^m F^{(j)}(u_i,|s_1-s_2|,x) \right\} \cdot \{ \sum_{\substack{I\\|I|\geqq 1}} \prod_{i\in I} 1_{(0,\alpha)}(u_i) \},$$

$$B(\beta,|s_1-s_2|,x)=\sum_{j=1}^4 \int_0^\infty du_1 \int_0^\infty du_2 \dots \int_0^\infty du_m \left\{ \prod_{i=1}^m F^{(j)}(u_i,|s_1-s_2|,x) \right\} \cdot \{ \sum_{\substack{I\\|I|\geqq 1}} \prod_{i\in I} 1_{(\beta,\infty)}(u_i) \},$$

and

$$C(\alpha, \beta, |s_1-s_2|, x)=\int_\alpha^\beta du_1 \int_\alpha^\beta du_2 \dots \int_\alpha^\beta du_m \left|\left\{\prod_{i=1}^m \frac{e(xu_i)}{V(x)}-\prod_{i=1}^m u_i^{H_0-\frac{3}{2}}\right\}\right.$$
$$\left.\cdot\left\{\prod_{i=1}^m \frac{e(x(|s_1-s_2|+u_i))}{V(x)}-\prod_{i=1}^m (|s_1-s_2|+u_i)^{H_0-\frac{3}{2}}\right\}\right|.$$

We first evaluate

$$\limsup_{x\to\infty} \int_0^t \int_0^t A(\alpha, |s_1-s_2|, x)\, ds_1\, ds_2$$

and

$$\limsup_{x\to\infty} \int_0^t \int_0^t B(\beta, |s_1-s_2|, x)\, ds_1\, ds_2.$$

We have

$$A(\alpha, |s_1-s_2|, x)$$
$$=\sum_{\substack{I\\|I|\geqq 1}} \sum_{j=1}^4 \left\{\int_0^\alpha F^{(j)}(u, |s_1-s_2|, x)\, du\right\}^{|I|} \left\{\int_0^\infty F^{(j)}(u, |s_1-s_2|, x)\, du\right\}^{m-|I|}$$

and it follows from Lemma 4.2 that for large enough x, there are constants $M_1^{(j)}$, $M_3^{(j)}$, $j=1,2,3,4$, independent of α and of $|s_1-s_2|$ such that

$$A(\alpha, |s_1-s_2|, x)$$
$$\leqq \sum_{\substack{I\\|I|\geqq 1}} \sum_{j=1}^4 \left\{M_1^{(j)}|s_1-s_2|^{2H_0-2-2\gamma} \int_0^{\alpha/|s_1-s_2|} (u+u^2)^{H_0-\frac{3}{2}-\gamma}\, du\right\}^{|I|}$$
$$\cdot\{M_3^{(j)} \max(|s_1-s_2|^{2H_0-2+2\gamma}, |s_1-s_2|^{2H_0-2-2\gamma})\}^{m-|I|},$$

where γ is an arbitrary number satisfying

$$0<\gamma<\min\{H_0-\tfrac{1}{2}, 1-H_0\}. \tag{4.14}$$

Since $0\leqq s_1, s_2\leqq t\leqq 1$ and $\gamma>0$, we have for large x

$$A(\alpha, |s_1-s_2|, x)$$
$$\leqq M \sum_{\substack{I\\|I|\geqq 1}} |s_1-s_2|^{(2H_0-2-2\gamma)m}\left\{\int_0^{\alpha/|s_1-s_2|} (u+u^2)^{H_0-\frac{3}{2}-\gamma}\, du\right\}^{|I|}$$

where $M>0$ is some new constant.

Now restrict the values of γ further, by requiring $0<(2H_0-2-2\gamma)m<-1$, that is, choose γ such that

$$0<\gamma<\min\left\{H_0-\left(1-\frac{1}{2m}\right), 1-H_0\right\}. \tag{4.15}$$

From Lemma 4.4, we obtain that

$$\limsup_{x\to\infty} \int_0^t ds_1 \int_0^t ds_2 A(\alpha, |s_1-s_2|, x)=o(1) \tag{4.16}$$

as $\alpha\to 0$.

Similarly, applying again Lemma 4.2, we get

$$\begin{aligned}
&B(\beta, |s_1-s_2|, x)\\
&\quad\leqq M \sum_{\substack{I\\ |I|\geqq 1}} |s_1-s_2|^{(2H_0-2-2\gamma)m+4\gamma|I|}\left\{\int_{\beta/|s_1-s_2|}^{\infty} (u+u^2)^{H_0-\frac{3}{2}+\gamma}du\right\}^{|I|}\\
&\quad\leqq M \sum_{\substack{I\\ |I|\geqq 1}} |s_1-s_2|^{(2H_0-2-2\gamma)m}\left\{\int_{\beta/|s_1-s_2|}^{\infty} (u+u^2)^{H_0-\frac{3}{2}+\gamma}du\right\}^{|I|}
\end{aligned}$$

for large enough x, for some new constant M independent of β and $|s_1-s_2|$ and for γ satisfying (4.15). By Lemma 4.4, we have

$$\limsup_{x\to\infty} \int_0^t ds_1 \int_0^t ds_2 B(\beta, |s_1-s_2|, x)=o(1) \tag{4.17}$$

as $\beta\to\infty$.

We now prove that $\lim_{x\to\infty} \int_0^t ds_1 \int_0^t ds_2\, C(\alpha, \beta, |s_1-s_2|, x)=0$.

Let $0\leqq y\leqq t$. We have

$$\frac{e(x(y+u))}{V(x)}=(y+u)^{H_0-\frac{3}{2}}\frac{e(x(y+u))}{V(x(y+u))}\frac{L(x(y+u))}{L(x)}.$$

As $x\to\infty$, $\dfrac{e(x(y+u))}{V(x(y+u))}$ tends to 1 uniformly in $y+u\in[\alpha,\infty)$, and $\dfrac{L(x(y+u))}{L(x)}$ tends to 1 uniformly in $y+u\in[\alpha, t+\beta]$. Thus, $\forall\varepsilon>0$, $\exists x_0(\varepsilon)$ such that for $x>x_0(\varepsilon)$ and $y+u\in[\alpha, t+\beta]$,

$$(1-\varepsilon)(y+u)^{H_0-\frac{3}{2}}\leqq\frac{e(x(y+u))}{V(x)}\leqq(1+\varepsilon)(y+u)^{H_0-\frac{3}{2}}$$

and hence,

$$\begin{aligned}
((1-\varepsilon)^m-1)\prod_{i=1}^{m}(y+u_i)^{H_0-\frac{3}{2}}&\leqq\prod_{i=1}^{m}\frac{e(x(y+u_i))}{V(x)}-\prod_{i=1}^{m}(y+u_i)^{H_0-\frac{3}{2}}\\
&\leqq((1+\varepsilon)^m-1)\prod_{i=1}^{m}(y+u_i)^{H_0-\frac{3}{2}}.
\end{aligned}$$

Thus, for $x > x_0(\varepsilon)$,

$$\begin{aligned}&\int_0^t ds_1 \int_0^t ds_2\, C(\alpha, \beta, |s_1 - s_2|, x)\\ &\quad \leqq O(\varepsilon) \int_0^t ds_1 \int_0^t ds_2\, |s_1 - s_2|^{(2H_0-2)m} \left\{ \int_0^\infty (u+u^2)^{H_0 - \frac{3}{2}}\, du \right\}^m\\ &\quad = O(\varepsilon)\, t^{(2H_0-2)m+2}.\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \int_0^t ds_1 \int_0^t ds_2\, C(\alpha, \beta, |s_1 - s_2|, x) = 0. \tag{4.18}$$

From (4.13), (4.16), (4.17), (4.18) it follows that

$$\limsup_{x\to\infty} \int_0^t \int_0^t Q(s_1, s_2; x)\, ds_1\, ds_2 = o_\alpha(1) + o_\beta(1)$$

as $\alpha \to 0$ and $\beta \to \infty$. Since α and β are arbitrary numbers satisfying $0 < \alpha \leqq t < \beta$, letting $\alpha \to 0$ and $\beta \to \infty$ concludes the proof. □

The next lemma involves condition (A4) of Sect. 2.

Lemma 4.6. *Let $t \geqq 0$, $m \geqq 1$, $1 - \dfrac{1}{2m} < H_0 < 1$. Let $e(u)$, $-\infty < u < \infty$ be defined as in Sect. 2 and let $V(x) = x^{H_0 - \frac{3}{2}} L(x)$. Then*

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{V^{2m}(x)} \int_0^t \int_0^t &\left\{ \int_{-\infty}^0 |e(xu)\, e(x(|s_1 - s_2| + u))|\, du \right\}\\ &\cdot \left\{ \int_{-\infty}^{+\infty} |e(xu)\, e(x(|s_1 - s_2| + u))|\, du \right\}^{m-1} ds_1\, ds_2 = 0.\end{aligned}$$

Proof. Let

$$J(t, x) = \int_0^t \int_0^t A(|s_1 - s_2|, x)\, \{A(|s_1 - s_2|, x) + B(|s_1 - s_2|, x)\}^{m-1}\, ds_1\, ds_2$$

where

$$A(|s_1 - s_2|, x) = \int_{-\infty}^0 |e(xu)\, e(x(|s_1 - s_2| + u))|\, du$$

and

$$B(|s_1 - s_2|, x) = \int_0^\infty |e(xu)\, e(x(|s_1 - s_2| + u))|\, du.$$

Now, $A(A+B)^{m-1} \leqq A 2^{m-1}(A^{m-1} + B^{m-1}) = 2^{m-1}(A^m + AB^{m-1})$ and by Hölder inequality,

$$\int AB^{m-1} \leqq \left(\int A^m\right)^{\frac{1}{m}} \left(\int B^{(m-1)\frac{m}{m-1}}\right)^{1-\frac{1}{m}} = \left(\int A^m\right)^{\frac{1}{m}} \left(\int B^m\right)^{1-\frac{1}{m}}.$$

Therefore,

$$J(t,x)\leqq 2^{m-1}\int_0^t\int_0^t A^m(|s_1-s_2|,x)\,ds_1\,ds_2+2^{m-1}\left\{\int_0^t\int_0^t A^m(|s_1-s_2|,x)\,ds_1\,ds_2\right\}^{\frac{1}{m}}$$
$$\cdot\left\{\int_0^t\int_0^t B^m(|s_1-s_2|,x)\,ds_1\,ds_2\right\}^{1-\frac{1}{m}}.$$

By assumption (A4) of Sect. 2,

$$A(|s_1-s_2|,x)=\frac{1}{x}\int_{-\infty}^{0}|e(u)\,e(x|s_1-s_2|+u)|\,du$$
$$=o(V^2(x))\,|s_1-s_2|^{2H_0-2-2\gamma}$$

as $x\to\infty$, uniformly in $0<|s_1-s_2|<t$, where

$$0<\gamma<\min\left\{H_0-\left(1-\frac{1}{2m}\right),\ 1-H_0\right\}.$$

Using also (4.10) of Lemma 4.2, we get

$$J(t,x)\leqq o(V^{2m}(x))\int_0^t\int_0^t |s_1-s_2|^{(2H_0-2-2\gamma)m}\,ds_1\,ds_2$$
$$+o(V^2(x))\left\{\int_0^t\int_0^t |s_1-s_2|^{(2H_0-2-2\gamma)m}\,ds_1\,ds_2\right\}^{\frac{1}{m}}(V^{2m}(x))^{1-\frac{1}{m}}$$
$$\cdot\left\{\int_0^t\int_0^t \max(|s_1-s_2|^{(2H_0-2-2\gamma)m},|s_1-s_2|^{(2H_0-2+2\gamma)m}\,ds_1\,ds_2\right\}^{1-\frac{1}{m}},$$

and therefore

$$\lim_{x\to\infty}\frac{J(t,x)}{V^{2m}(x)}=0.$$

This completes the proof. □

We now introduce, for each $m\geqq 1$, a collection of processes $\{Y_m(t,x),\ x>0\}$ and a process $\bar{Y}_m(t)$.

Let $1-\frac{1}{2m}<H_0<1$ and let $e(u)$ be a function satisfying the conditions of Sect. 2.

For each $x>0$, define

$$Y_m(t,x)=\int_0^t ds\int_{-\infty}^{+\infty} e(x(s-\xi_1))\,dB(\xi_1)$$
$$\cdot\int_{-\infty}^{\xi_1} e(x(s-\xi_2))\,dB(\xi_2)\ldots\int_{-\infty}^{\xi_{m-1}} e(x(s-\xi_m))\,dB(\xi_m). \tag{4.19}$$

By hypothesis on $e(u)$

$$\int_{-\infty}^{+\infty} e^2(x(s-\xi))\,d\xi=\frac{1}{x}\int_{-\infty}^{\infty} e^2(u)\,du<\infty$$

for all $x>0$, and therefore $Y_m(t,x)$ is well-defined in $L^2(P)$ for each $x>0$.

Define also the process

$$\bar{Y}_m(t)=\frac{1}{K(m,H_0)}\bar{Z}_m(t) \tag{4.20}$$

where $K(m,H_0)$ and $\bar{Z}_m(t)$ were introduced in Sect. 1. Theorem 3.1 of Taqqu (1978a), ensures that $\bar{Y}_m(t)$ is well defined in $L^2(P)$.

Now set

$$V(x)=x^{H_0-\frac{3}{2}}L(x).$$

Theorem 4.7. *For each* $t\geqq 0$,

$$\lim_{x\to\infty} E\left(\frac{Y_m(t,x)}{V^m(x)}-\bar{Y}_m(t)\right)^2=0.$$

Proof. Set $Y_m(t,x)=Y_m^{(1)}(t,x)+Y_m^{(2)}(t,x)$, where

$$\begin{aligned}Y_m^{(1)}(t,x)=&\int_0^t ds \int_{-\infty}^s e(x(s-\xi_1))\,dB(\xi_1)\int_{-\infty}^{\xi_1} e(x(s-\xi_2))\,dB(\xi_2)\\ &\cdot\int_{-\infty}^{\xi_2} e(x(s-\xi_3))\,dB(\xi_3)\dots\int_{-\infty}^{\xi_{m-1}} e(x(s-\xi_m))\,dB(\xi_m)\end{aligned}$$

and

$$\begin{aligned}Y_m^{(2)}(t,x)=&\int_0^t ds \int_s^\infty e(x(s-\xi_1))\,dB(\xi_1)\int_{-\infty}^{\xi_1} e(x(s-\xi_2))\,dB(\xi_2)\\ &\cdot\int_{-\infty}^{\xi_2} e(x(s-\xi_3))\,dB(\xi_3)\dots\int_{-\infty}^{\xi_{m-1}} e(x(s-\xi_m))\,dB(\xi_m).\end{aligned}$$

Now,

$$E\left(\frac{Y_m(t,x)}{V^m(x)}-\bar{Y}_m(t)\right)^2\leqq 2\left\{E\left(\frac{Y_m^{(1)}(t,x)}{V^m(x)}-\bar{Y}_m(t)\right)^2+E\left(\frac{Y_m^{(2)}(t,x)}{V^m(x)}\right)^2\right\}.$$

But

$$\begin{aligned}E\left(\frac{Y_m^{(1)}(t,x)}{V^m(x)}-\bar{Y}_m(t)\right)^2=&\int_{-\infty}^t d\xi_1\int_{-\infty}^{\xi_1} d\xi_2\dots\int_{-\infty}^{\xi_{m-1}} d\xi_m\\ &\cdot\left[\int_0^t ds\left(\prod_{i=1}^m\frac{e(x(s-\xi_i))}{V(x)}-\prod_{i=1}^m(s-\xi_i)^{H_0-\frac{3}{2}}\right)\prod_{i=1}^m 1(\xi_i<s)\right]^2\\ =&\int_0^t ds_1\int_0^t ds_2\int_{-\infty}^{s_1\wedge s_2} d\xi_1\int_{-\infty}^{s_1\wedge s_2} d\xi_2\dots\int_{-\infty}^{s_1\wedge s_2} d\xi_m\\ &\cdot\left|\left(\prod_{i=1}^m\frac{e(x(s_1-\xi_i))}{V(x)}-\prod_{i=1}^m(s_1-\xi_i)^{H_0-\frac{3}{2}}\right)\right.\\ &\cdot\left.\left(\prod_{i=1}^m\frac{e(x(s_2-\xi_i))}{V(x)}-\prod_{i=1}^m(s_2-\xi_i)^{H_0-\frac{3}{2}}\right)\right|.\end{aligned}$$

By Lemma 4.5, this tends to 0 as $x\to\infty$.

Now let

$$\bar{e}(x(s-\xi_1)) = e(x(s-\xi_1))\, 1_{(s,\infty)}(\xi_1).$$

Then

$$\begin{aligned}
&E(Y_m^{(2)}(t,x))^2 \\
&= \int_0^t ds_1 \int_0^t ds_2 \int_{-\infty}^{+\infty} \bar{e}(x(s_1-\xi_1))\,\bar{e}(x(s_2-\xi_1)\, d\xi_1 \\
&\quad \cdot \int_{-\infty}^{\xi_1} e(x(s_1-\xi_2))\, e(x(s_2-\xi_2))\, d\xi_2 \ldots \int_{-\infty}^{\xi_{m-1}} e(x(s_1-\xi_m))\, e(x(s_2-\xi_m))\, d\xi_m \\
&\leqq \int_0^t ds_1 \int_0^t ds_2 \left\{ \int_{s_1 \wedge s_2}^{\infty} |e(x(s_1-\xi))\, e(x(s_2-\xi))|\, d\xi \right\} \\
&\quad \cdot \left\{ \int_{-\infty}^{+\infty} |e(x(s_1-\xi)\, e(x(s_2-\xi))|\, d\xi \right\}^{m-1} \\
&= \int_0^t ds_1 \int_0^t ds_2 \left\{ \int_{-\infty}^{0} |e(xu)\, e(x(|s_1-s_2|+u)|\, du \right\} \\
&\quad \cdot \left\{ \int_{-\infty}^{+\infty} |e(xu)\, e(x(|s_1-s_2|+u)|\, du \right\}^{m-1} \\
&= o(V^{2m}(x))
\end{aligned}$$

as $x \to \infty$, by Lemma 4.6. This concludes the proof. □

§ 5. Weak Convergence

Let $X(s)$, $-\infty < s < \infty$, be the normalized stationary Gaussian process defined in Sect. 2. The parameter H_0 that enters in the definition of $X(s)$ is required to satisfy

$$1 - \frac{1}{2m} < H_0 < 1 \tag{5.1}$$

where $m \geqq 1$ is an integer.

Introduce the Hermite polynomials

$$H_q(x) = (-1)^q\, e^{\frac{x_2}{2}} \frac{d^q}{dx^q} e^{-\frac{x_2}{2}}, \qquad q = 0, 1, \ldots.$$

The first few are $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$.

We first study the convergence of the finite-dimensional distributions of

$$\frac{1}{d(x)} \int_0^{xt} H_m(X(s))\, ds,$$

$t \geqq 0$, as $x \to \infty$, for arbitrary $m \geqq 1$.

As normalization factor, we choose

$$d(x) \sim \frac{m!}{\sigma^m K(m, H_0)} x^H L^m(x) \tag{5.2}$$

as $x \to \infty$, where

$$H=(H_0-1)m+1$$

and where the constant $K(m, H_0)$ is defined as in (1.6). This choice ensures that

$$E\left(\int_0^x H_m(X(s))\,ds\right)^2=\int_0^x\int_0^x m!(EX(s_1)X(s_2))^m\,ds_1\,ds_2 \sim d^2(x) \tag{5.3}$$

as $x \to \infty$ (see (2.4)).

Lemma 5.1. *Let $Y_m(t,x)$ be defined as in* (4.19). *Then, for each $x>0$*

$$\int_0^{xt} H_m(X(s))\,ds \overset{\Delta}{=} \frac{m!}{\sigma^m} x^{1+\frac{m}{2}} Y_m(t,x),$$

where $\overset{\Delta}{=}$ indicates equality of the finite-dimensional distributions.

Proof. First note that, by McKean (1973) for example,

$$\begin{aligned} H_m(X(s)) &= H_m\left(\frac{1}{\sigma}\int_{-\infty}^{\infty} e(s-\xi)\,dB(\xi)\right)\\ &= \frac{m!}{\sigma^m}\int_{-\infty}^{\infty} e(s-\xi_1)\,dB(\xi_1)\int_{-\infty}^{\xi_1} e(s-\xi_2)\,dB(\xi_2)\dots\int_{-\infty}^{\xi_{m-1}} e(s-\xi_m)\,dB(\xi_m). \end{aligned}$$

Now let $a_1, a_2, \dots, a_p$ be $p \geqq 1$ arbitrary constants and let $t_1, t_2, \dots, t_p \geqq 0$. Then

$$\begin{aligned} \sum_{j=1}^p a_j \int_0^{xt_j} H_m(X(s))\,ds &= x\sum_{j=1}^p a_j \int_0^{t_j} H_m(X(xs))\,ds\\ &= \frac{m!}{\sigma^m} x \sum_{j=1}^p a_j \int_0^{t_j} ds \int_{-\infty}^{+\infty} e(x(s-\xi_1))\,dB(x\xi_1)\\ &\quad\cdot \int_{-\infty}^{\xi_1} e(x(s-\xi_2))\,dB(x\xi_2)\dots\int_{-\infty}^{\xi_{m-1}} e(x(s-\xi_m))\,dB(x\xi_m) \end{aligned}$$

in the $L^2(P)$ sense. Using an argument similar to the one used in the proof of Theorem 3.1 of Taqqu (1978a) (formally, $dB(x\xi) \overset{\Delta}{=} x^{1/2}\,dB(\xi)$), we get

$$\sum_{j=1}^p a_j \int_0^{xt_j} H_m(X(s))\,ds \overset{\Delta}{=} \sum_{j=1}^p a_j \frac{m!}{\sigma^m} x^{1+\frac{m}{2}} Y_m(t_j, x)$$

for each $x>0$. The lemma follows because the a_j are arbitrary. $\square$

Theorem 5.2. *As $x \to \infty$,*

$$\frac{1}{d(x)}\int_0^{xt} H_m(X(s))\,ds \Rightarrow \bar{Z}_m(t)$$

in the sense of convergence of the finite-dimensional distributions.

Proof. Let $a_1, a_2, \dots, a_p$ be arbitrary constants and suppose without loss of generality that $t_1, t_2, \dots, t_p > 0$. Then by Lemma 5.1, for each $x > 0$,

$$\sum_{j=1}^{p} a_j \frac{1}{d(x)} \int_0^{xt_j} H_m(X(s))\, ds \stackrel{\Delta}{=} \sum_{j=1}^{p} a_j \frac{m!\, x^{1+\frac{m}{2}}}{\sigma^m d(x)} Y_m(t_j, x).$$

But by (5.2),

$$\begin{aligned}\frac{m!\, x^{1+\frac{m}{2}}}{\sigma^m d(x)} &\sim \frac{m!}{\sigma^m} \frac{\sigma^m K(m, H_0)}{m!} \frac{x^{1+\frac{m}{2}}}{x^{m(H_0-1)+1} L^m(x)} \\ &= \frac{K(m, H_0)}{V^m(x)}\end{aligned}$$

as $x \to \infty$, where $V(x) = x^{H_0 - \frac{1}{2}} L(x)$. Also, by (4.20),

$$\bar{Z}_m(t) = K(m, H_0)\, \bar{Y}_m(t),$$

Therefore, as $x \to \infty$.

$$\sum_{j=1}^{p} a_j \frac{m!\, x^{1+\frac{m}{2}}}{\sigma^m d(x)} Y_m(t_j, x) \sim \sum_{j=1}^{p} a_j \bar{Z}_m(t_j) + K(m, H_0) \sum_{j=1}^{p} a_j \left(\frac{Y_m(t_j, x)}{V^m(x)} - \bar{Y}_m(t_j) \right).$$

But

$$\begin{aligned}&\lim_{x \to \infty} E \left\{ \sum_{j=1}^{p} a_j \left(\frac{Y_m(t_j, x)}{V^m(x)} - \bar{Y}_m(t_j) \right) \right\}^2 \\ &\leqq \sum_{j=1}^{p} a_j^2 \sum_{j=1}^{p} \lim_{x \to \infty} E \left(\frac{Y_m(t_j, x)}{V^m(x)} - \bar{Y}_m(t_j) \right)^2 \\ &= 0\end{aligned}$$

by Theorem 4.7. This concludes the proof. □

$H_m(X(s))$ is a new process, obtained from the Gaussian process $X(s)$ through a non-linear transformation. A more general non-linear transformation would lead to the process $G(X(s))$. We now choose G to be an arbitrary function satisfying $EG(X(s)) = 0$ and $EG^2(X(s)) < \infty$, and we study the weak convergence in $C[0, 1]$ of

$$\frac{1}{d(x)} \int_0^{xt} G(X(s))\, ds$$

as $x \to \infty$, where $d(x)$ is defined as in (5.2). $C[0, 1]$ is the space of continuous functions on $[0, 1]$ with the sup-norm topology.

Let X denote an $N(0, 1)$ random variable, and as in Taqqu (1975), let

$$\mathscr{G} = \{G : EG(X) = 0, EG^2(X) < \infty\}. \tag{5.4}$$

The *Hermite rank* of a function $G \in \mathscr{G}$ is the index of the first non-zero coefficient in the expansion of G in Hermite polynomials. Let $\mathscr{G}_m$ be the subset of $\mathscr{G}$ that contains all functions with Hermite rank m. Then

$$\mathscr{G}=\left(\bigcup_{i=1}^{\infty}\mathscr{G}_i\right)\cup\mathscr{G}_\infty,$$

where $\mathscr{G}_\infty$ contains the function 0, and,

$$\mathscr{G}_i\cap\mathscr{G}_j=\emptyset$$

when $i\neq j$.

Let $\sum_{q=m}^{\infty}\frac{J(q)}{q!}H_q(\cdot)$ denote the Hermite expansion of $G(\cdot)\in\mathscr{G}_m$. Since

$$EH_{q_1}(X)\,H_{q_2}(X)=q_1!\,\delta_{q_1,q_2},$$

the coefficients $J(q)$ satisfy

$$J(q)=EG(X)\,H_q(X),$$

and the series $\sum_{q=m}^{\infty}\frac{J(q)}{q!}H_q(X)$ converges to $G(X)$ in $L^2\left(\mathbb{R}^1,\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right)$. By Parserval's relation,

$$EG^2(X)=\sum_{q=m}^{\infty}\frac{J^2(q)}{q!}<\infty.$$

Lemma 5.3. *Let $G\in\mathscr{G}_m$. Then as $x\to\infty$,*

$$\frac{1}{d(x)}\int_0^{xt}G(X(s))\,ds\Rightarrow\frac{J(m)}{m!}\bar{Z}_m(t)$$

in the sense of convergence of the finite-dimensional distributions.

Proof. Let

$$G^*(X(s))=G(X(s))-\frac{J(m)}{m!}H_m(X(s)). \tag{5.4}$$

Then

$$\int_0^{xt}G(X(s))\,ds=\frac{J(m)}{m!}\int_0^{xt}H_m(X(s))+\int_0^{xt}G^*(X(s))\,ds.$$

Proceeding as in the proof of Corollary 3.1 of Taqqu (1975), one can show that

$$\begin{aligned}E\left(\int_0^x G^*(X(s))\,ds\right)^2 &\leq\int_0^x\int_0^x|EG^*(X(s_1))\,G^*(X(s_2))|\,ds_1\,ds_2\\ &=o(d^2(x))\end{aligned} \tag{5.5}$$

as $x\to\infty$, and therefore the limiting finite-dimensional distributions of $\frac{1}{d(x)}\int_0^{xt}G(X(s))\,ds$ are the same as those of $\frac{J(m)}{m!}\frac{1}{d(x)}\int_0^{xt}H_m(X(s))\,ds$. But by Theorem 5.2, they are those of $\frac{J(m)}{m!}\bar{Z}_m(t)$. This concludes the proof. □

To establish tightness, we will use

Lemma 5.4. *Let $G \in \mathscr{G}_m$. Then for large enough x, there exists positive constants C and δ, independent of t, such that, for all $t \geqq 0$,*

$$E\left(\frac{1}{d(x)} \int_0^{xt} G(X(s))\, ds\right)^2 \leqq C t^{1+\delta}.$$

Here $0 < \delta < 2H - 1$.

Proof. Suppose without loss of generality that $t > 0$, and let

$$Q(t, x) = E\left(\frac{1}{d(x)} \int_0^{xt} G(X(s))\, ds\right)^2 . \tag{5.6}$$

By (5.4), (5.5), (5.3) and (5.2),

$$\begin{aligned} Q(t, x) &\sim \frac{J^2(m)}{m!} E\left(\frac{1}{d(x)} \int_0^{xt} H_m(X(s))\, ds\right)^2 \\ &\sim \frac{J^2(m)}{m!} \frac{d^2(xt)}{d^2(x)} \\ &\sim \frac{J^2(m)}{m!} \frac{(xt)^{2H} L^m(xt)}{x^{2H} L^m(x)} \end{aligned}$$

as $xt \to \infty$. Thus, $\forall \varepsilon > 0$, $\exists \eta(\varepsilon)$ and a constant $C_1 = C_1(\varepsilon, m)$ such that for all $xt > \eta(\varepsilon)$,

$$Q(t, x) \leqq C_1 t^{2H} \frac{L^m(xt)}{L^m(x)}. \tag{5.7}$$

Choose now $0 < \delta < 2H - 1$ and consider two cases:

i) When $xt > \eta(\varepsilon)$, it follows from (5.7) that

$$Q(t, x) \leqq C_1 \frac{(xt)^{2H-1-\delta} L^{2m}(xt)}{x^{2H-1-\delta} L^{2m}(x)} t^{1+\delta}.$$

Since $2H - 1 - \delta > 0$, $\dfrac{(xt)^{2H-1-\delta} L^{2m}(xt)}{x^{2H-1-\delta} L^{2m}(x)}$ tends to 1 as $x \to \infty$, uniformly in $0 \leqq t \leqq 1$, and thus, for x large enough, there exists a constant $C_2 > 0$, independent of t, such that

$$Q(t, x) \leqq C_2 t^{1+\delta}.$$

ii) When $xt \leqq \eta(\varepsilon)$, use (5.6) and the fact that $EG^2(X(s)) = C_3 < \infty$, to get

$$Q(t, x) \leqq C_3 \frac{x^2 t^2}{d^2(x)}.$$

Since $d(x)$ is asymptotically proportional to $x^{2H} L^m(x)$ as $x \to \infty$, there exists for large enough x, a constant $C_4 > 0$, such that

$$Q(t, x) \leqq C_4 \frac{x^2 t^2}{x^{2H} L^{2m}(x)}.$$

But $(xt)^{1-\delta} < \eta^{1-\delta}$ where δ is defined as above. Therefore

$$\begin{aligned} Q(t,x) &\leqq C_4 \eta^{1-\delta} \frac{x^{1+\delta-2H}}{L^{2m}(x)} t^{1+\delta} \\ &\leqq C_5 t^{1+\delta} \end{aligned}$$

for large enough x, because $1+\delta-2H<0$. This concludes the proof. □

As a consequence of Lemmas 5.3 and 5.4, we obtain

Theorem 5.5. *Let $G \in \mathscr{G}_m$. Then as $x \to \infty$*

$$\frac{1}{d(x)} \int_0^{xt} G(X(s))\, ds \Rightarrow \frac{J(m)}{m!} \bar{Z}_m(t)$$

in the sense of weak convergence in $C[0,1]$.

We now turn to the *discrete parameter case.*

Let $D[0,1]$ be the space of functions that are right-continuous and have left limits, and suppose that $D[0,1]$ has the Skorokhod topology. The following theorem is based on an important result recently obtained by Dobrushin and Major (1979).

Theorem 5.6. *Let $G \in \mathscr{G}_m$. Suppose that $\{X_i, i \geqq 1\}$ is a stationary normalized Gaussian sequence with $EX_i X_{i+k} \sim k^{2H_0-2} L^2(k)$ as $k \to \infty$, where $1-\frac{1}{2m} < H_0 < 1$ and where $L(x)$ is a slowly varying function at infinity, bounded on bounded intervals. Then*

$$\frac{1}{d(N)} \sum_{i=1}^{[Nt]} G(X_i) \Rightarrow \frac{J(m)}{m!} \bar{Z}_m(t)$$

in the sense of weak convergence in $D[0,1]$.

Proof. Dobrushin and Major (1979) have shown that under the assumptions of this theorem, the finite-dimensional distributions of $\frac{1}{d(N)} \sum_{i=1}^{[Nt]} G(X_i)$ converge as $N \to \infty$. In fact, they converge to the finite-dimensional distributions of $\frac{J(m)}{m!} \bar{Z}_m(t)$. This follows from Theorem 5.5. by specializing $\{X_i, i \geqq 1\}$ to a sequence that admits a weighted average representation consistent with the assumptions of Sect. 2 (see Taqqu (1978b) for details). It also follows from the identification (in the finite-dimensional distributions sense) of $\frac{J(m)}{m!} \bar{Z}_m(t)$ with the limit obtained by Dobrushin and Major (see Theorem 6.3 below). The tightness of the $D[0,1]$ sequence $\left\{\frac{1}{d(N)} \sum_{i=1}^{[Nt]} G(X_i),\ N=1,2,\dots\right\}$ can be established by suitable modifications of the proof of Lemma 5.4. □

See Taqqu (1975) for an alternative proof of this theorem in the cases $m=1$ and $m=2$.

§ 6. The Wiener-Itô-Dobrushin Representation for $\bar{Z}_m(t)$.

$\bar{Z}_m(t)$ admits various representations, equivalent in the finite-dimensional distributions sense.

Consider first the definition of $\bar{Z}_m(t)$ given in (1.3). The integrand

$$\int_0^t \prod_{j=1}^m (s-\xi_j)^{H_0-3/2} 1(\xi_j<s)\, ds$$

is a symmetric function of $\xi_1, \ldots, \xi_m$, and therefore one can write

$$\bar{Z}_m(t) = \frac{K(m, H_0)}{m!} \int'_{\mathbb{R}^m} \left\{ \int_0^t \prod_{j=1}^m (s-\xi_j)^{H_0-\frac{3}{2}} 1(\xi_j<s)\, ds \right\} dB(\xi_1) \ldots dB(\xi_m) \tag{6.1}$$

where $\int'_{\mathbb{R}^m}$ is the Wiener-Itô multiple integral on $\mathbb{R}^m$. See Itô (1951) for a precise definition. Heuristically, $\int'_{\mathbb{R}^m}$ denotes integration over $\mathbb{R}^m$, disregarding integration over the hyperplanes $\xi_i=\xi_j$, $i \neq j$, $i,j=1,2,\ldots,m$. The integral $\int'_{\mathbb{R}^m}$ is defined through an isometric mapping from the Hilbert space of square integrable functions $f(\xi_1,\ldots,\xi_m)$ into $L^2(P)$.

Let now W and $\int''_{\mathbb{R}^m}$ be defined as in Sect. 4 of Dobrushin (1979). These definitions involve a modification of the definition of the Wiener-Itô multiple integral. W is a Gaussian "white noise" (complex) random spectral measure that satisfies $W\left(\sum_{i=1}^n \Delta_i\right) = \sum_{i=1}^n W(\Delta_i)$, $W(\Delta) = \overline{W(-\Delta)}$ and $EW(\Delta_1)\overline{W(\Delta_2)} = |\Delta_1 \cap \Delta_2|$ for Borel sets of $\mathbb{R}^1 \setminus \{0\}$ that have finite Lebesgue measure $|\cdot|$. The real and imaginary parts of $W(\Delta)$ are independent normal random variables with mean 0 and variance $\frac{1}{2}|\Delta|$. To define $\int''_{\mathbb{R}^m}$. one introduces $\mathcal{H}_m$, the real Hilbert space of complex-valued symmetric functions $f(\lambda_1,\ldots,\lambda_m)$ of $\lambda_1,\ldots,\lambda_m \in R^1$ that are even, i.e. $f(\lambda_1,\ldots,\lambda_m) = \overline{f(-\lambda_1,\ldots,-\lambda_m)}$, and that have a square integrable modulus. The integral $\int''_{\mathbb{R}^m}$ is defined through an isometric mapping

$$\mathcal{H}_m \to L^2(P'') \text{ (into)}$$
$$f \mapsto I(f) = \int''_{\mathbb{R}^m} f(\lambda_1,\ldots,\lambda_m)\, W(d\lambda_1) \ldots W(d\lambda_m).$$

P'' is the probability measure of the space on which W is defined and $L^2(P'')$ is the real Hilbert space of real-valued functions that are square integrable with respect to P''. The mapping is defined in such a way, that heuristically, one disregards integration over the hyperplanes $\xi_i=\xi_j$ and $\xi_i=-\xi_j$, $i \neq j$, $i,j=1,2,\ldots,m$. The fact that both f and W are even ensures that $I(f)$ is a real-valued random variable.

Lemma 6.1. *Let $A(\xi_1,\ldots,\xi_m)$ be a real-valued function in $L^2(\mathbb{R}^m)$ which is invariant under the permutation of its indices, and let*

$$\tilde{A}(\lambda_1, \ldots, \lambda_m) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{i\sum_{j=1}^{m} \xi_j \lambda_j} A(\xi_1, \ldots, \xi_m)\, d\xi_1 \ldots d\xi_m$$

be its Fourier transform. Then

$$\int_{\mathbb{R}^m}' A(\xi_1 \ldots \xi_m)\, dB(\xi_1) \ldots dB(\xi_m) \stackrel{\Delta}{=} \int_{\mathbb{R}^m}'' \tilde{A}(\lambda_1, \ldots, \lambda_m)\, W(d\lambda_1) \ldots W(d\lambda_m) \tag{6.2}$$

Proof. Let $\psi_0, \psi_1, \psi_2, \ldots$ be a complete orthornormal set of functions in $L^2(\mathbb{R}^1)$. Then

$$A(\xi_1, \ldots, \xi_m) = \sum_{k_1, \ldots, k_m \geq 0} c_{k_1, \ldots, k^m} \psi_{k_1}(\xi) \ldots \psi_{k_m}(\xi_m). \tag{6.3}$$

Since A is symmetric,

$$\begin{aligned}
&\int' A(\xi_1, \ldots, \xi_m)\, dB(\xi_1) \ldots dB(\xi_m) \\
&= \int' \frac{1}{m!} \sum_{(i_1, \ldots, i_m) \in \pi_m} \sum_k c_{k_1, \ldots, k_m} \psi_{k_1}(\xi_{i_1}) \ldots \psi_{k_m}(\xi_{i_m})\, dB(\xi_1) \ldots dB(\xi_m) \\
&= \sum_k c_{k_1, \ldots, k_m} \int' \frac{1}{m!} \sum_{(i_1, \ldots, i_m) \in \pi_m} \psi_{k_{i_1}}(\xi_1) \ldots \psi_{k_{i_m}}(\xi_m)\, dB(\xi_1) \ldots dB(\xi_m) \\
&= \sum_k c_{k_1, \ldots, k_m} H_{j_1}\left(\int \phi_1(\xi)\, dB(\xi)\right) H_{j_2}\left(\int \phi_2(\xi)\, dB(\xi)\right) \ldots H_{j_m}\left(\int \phi_m(\xi)\, dB(\xi)\right)
\end{aligned} \tag{6.4}$$

by applying Theorem 3.1 of Itô (1951). Here π_m is the set of all $m!$ permutations of the indices $(1, \ldots, m)$, the H_j's are the Hermite polynomials defined in the preceding section and the set of functions $\{\psi_{k_1}, \psi_{k_2}, \ldots, \psi_{k_m}\}$ is identical to the set of functions $\{\phi_1, \ldots, \phi_1, \phi_2, \ldots, \phi_2, \ldots, \phi_m, \ldots, \phi_m\}$, there being j_1 identical functions (denoted ϕ_1), j_2 other identical functions (denoted ϕ_2), $\ldots, j_m$ other identical functions (denoted ϕ_m), with $0 \leqq j_1, \ldots, j_m \leqq m$ and $j_1 + \ldots + j_m = m$. Each index $j_1, \ldots, j_m$ and each function $\phi_1, \ldots, \phi_m$ depends on $k_1, k_2, \ldots, k_m$.

The Fourier transform $\tilde{A}(\lambda_1, \ldots, \lambda_m)$ belongs to $\mathscr{H}_m$. Using (6.3), we obtain

$$\tilde{A}(\lambda_1, \ldots, \lambda_m) = \sum_k c_{k_1, \ldots, k_m} \tilde{\psi}_{k_1}(\lambda_1) \ldots \tilde{\psi}_{k_m}(\lambda_m) \tag{6.5}$$

where $\tilde{\psi}(\lambda) = (2\pi)^{-1/2} \int e^{i\xi\lambda} \psi(\xi)\, d\xi$. By Parseval's identity, $\int |\psi|^2 = \int |\tilde{\psi}|^2$, and hence $\{\tilde{\psi}_k, k = 0, 1, 2, \ldots\}$ is a complete orthonormal set in $\mathscr{H}_m$.

Consider now $\int'' \tilde{A}(\lambda_1, \ldots, \lambda_m)\, W(d\lambda_1) \ldots W(d\lambda_m)$. It can also be expressed in terms of one-dimensional integrals. Indeed, starting with (6.5), using formulae (4.14) and (4.15) of Dobrushin (1979), and proceeding as above, we get

$$\begin{aligned}
&\int'' \tilde{A}(\lambda_1, \ldots, \lambda_m)\, W(d\lambda_1) \ldots W(d\lambda_m) \\
&= \sum_k c_{k_1, \ldots, k_m} H_{j_1}\left(\int \tilde{\phi}_1(\lambda)\, W(d\lambda)\right) H_{j_2}\left(\int \tilde{\phi}_2(\lambda)\, W(d\lambda)\right) \ldots H_{j_m}\left(\int \tilde{\phi}_m(\lambda)\, W(d\lambda)\right)
\end{aligned} \tag{6.6}$$

where the indices $j_1, \ldots, j_m$ and the functions $\phi_1, \ldots, \phi_m$ are defined as in (6.4).

To compare (6.4) with (6.6), we note that for any $s \geqq 0$, the random vectors $(X_k = \int \psi_k(\xi)\, dB(\xi),\ k = 0, 1, \ldots, s)$ and $(Y_k = \int \tilde{\psi}_k(\lambda)\, W(d\lambda),\ k = 0, 1, \ldots, s)$ are both multivariate normal, with mean 0 and covariances

$$EX_iX_j=\int \psi_i(\xi)\psi_j(\xi)\,d\xi=\int \tilde{\psi}_i(\lambda)\overline{\tilde{\psi}_j(\lambda)}\,d\lambda=EY_i\bar{Y}_j,$$

$i, j=0, 1, \ldots, s$. They are identically distributed, and therefore

$$\int' A(\xi_1,\ldots,\xi_m)\,dB(\xi_1)\ldots dB(\xi_m)\stackrel{\Delta}{=}\int'' \tilde{A}(\lambda_1,\ldots,\lambda_m)\,W(d\lambda_1)\ldots W(d\lambda_m). \quad \square$$

Remark 6.1. Relation (6.2) can be interpretated as Parseval's identity where the complex even measure $W(\cdot)$ is viewed as equivalent (in the finite-dimensional distribution sense) to the Fourier transform of the real measure $B(\cdot)$. $E|W(\Delta)|^2=E(B(\Delta))^2=|\Delta|$ follows from the L^2 isometry of the Fourier transform, $\int'$ ensures the independence of $dB(\xi_1),\ldots,dB(\xi_m)$ and $\int''$ ensures the independence of $W(d\lambda_1),\ldots,W(d\lambda_m)$.

Remark 6.2. Lemma 6.1 also holds if $\xi_1,\ldots,\xi_m\in\mathbb{R}^\nu$ and $\lambda_1,\ldots,\lambda_m\in\mathbb{R}^\nu$, $\nu\geqq 1$. In that case, define $A(\lambda_1,\ldots,\lambda_m)$ as the $\mathbb{R}^{m\nu}$-Fourier transform of $A(\xi_1,\ldots,\xi_m)$ and replace $\mathbb{R}^m$ by $\mathbb{R}^{m\nu}$ in (6.2).

Lemma 6.2. *Let*

$$A(\xi_1,\ldots,\xi_m)=\int_{-\infty}^{+\infty}\phi(s)\prod_{j=1}^{m}(s-\xi_j)^{H_0-\frac{3}{2}}1(\xi_j<s)\,ds$$

where $\frac{1}{2}<H_0<1$ *and* $\phi(s)$ *is any integrable function in* $\mathbb{R}^1$ *such that*

$$\int_{\mathbb{R}^m}|A(\xi_1,\ldots,\xi_m)|^2\,d\xi_1\ldots d\xi_m<\infty.$$

Let $\tilde{\phi}(\lambda)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}e^{i\lambda\xi}\phi(\xi)\,d\xi$. *Then,*

$$\begin{aligned}&\int'_{\mathbb{R}^m}A(\xi_1,\ldots,\xi_m)\,dB(\xi_1)\ldots dB(\xi_m)\\ &\quad\stackrel{\Delta}{=}\left(\frac{\Gamma(H_0-\frac{1}{2})}{\sqrt{2\pi}}\right)^m\int''_{\mathbb{R}^m}\sqrt{2\pi}\,\tilde{\phi}(\lambda_1+\ldots+\lambda_m)\,|\lambda_1|^{\frac{1}{2}-H_0}\\ &\quad\quad\ldots|\lambda_m|^{\frac{1}{2}-H_0}\,W(d\lambda_1)\ldots W(d\lambda_m).\end{aligned}$$

Proof. We first evaluate the Fourier transform of $A(\xi_1,\ldots,\xi_m)$. Some care is needed because the function $u^{H_0-\frac{3}{2}}1(u>0)$ belongs neither to $L^1(\mathbb{R}^1)$ nor to $L^2(\mathbb{R}^1)$. Introduce

$$A_T(\xi_1,\ldots,\xi_m)=\begin{cases}A(\xi_1,\ldots,\xi_m) & \text{if } |\xi_k|<T, \quad k=1,\ldots,m\\ 0 & \text{otherwise.}\end{cases}$$

If $\tilde{A}_T(\lambda_1,\ldots,\lambda_m)=(2\pi)^{-m/2}\int e^{i\sum_{j=1}^{m}\xi_j\lambda_j}A_T(\xi_1,\ldots,\xi_m)\,d\xi_1\ldots d\xi_m$, which belongs to $L^1(\mathbb{R}^m)\cap L^2(\mathbb{R}^m)$, converges, as $T\to\infty$, to $\tilde{A}(\lambda_1,\ldots,\lambda_m)$ almost everywhere pointwise, then, by a classical argument, $\tilde{A}(\lambda_1,\ldots,\lambda_m)$ is almost everywhere equal to the $L^2(\mathbb{R}^m)$-Fourier transform of $A(\xi_1,\ldots,\xi_m)$.

Introduce

$$B_\lambda(a,b)=\frac{1}{\sqrt{2\pi}}\int_a^b e^{-iu\lambda}u^{H_0-\frac{3}{2}}\,du$$

for $0\leqq a\leqq b<\infty$, and $B_\lambda(a,\infty)=\lim_{b\to\infty}B_\lambda(a,b)$. Since $H_0>\frac{1}{2}$, one has $|B_\lambda(0,1)|\leqq(2\pi)^{-1/2}\int_0^1 u^{H_0-\frac{3}{2}}\,du=(2\pi)^{-1/2}(H_0-\frac{1}{2})^{-1}$. By the second mean-value theorem, for $\lambda\neq 0$ and $a>0$, $B_\lambda(a,b)=(2\pi)^{-1/2}a^{H_0-\frac{3}{2}}\int_a^{u'}e^{-iu\lambda}\,du$ where $a\leqq u'\leqq b$ and therefore $|B_\lambda(a,b)|\leqq\frac{2a^{H_0-\frac{3}{2}}}{(2\pi)^{1/2}|\lambda|}$. Thus

$$\sup_{0\leqq a\leqq b}|B_\lambda(a,b)|\leqq\frac{1}{(2\pi)^{1/2}}\left(\frac{1}{H_0-\frac{1}{2}}+\frac{2}{|\lambda|}\right)$$

Now,

$$\begin{aligned}&\tilde{A}_T(\lambda_1,\dots,\lambda_m)\\&\quad=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}e^{i\sum_{j=1}^m\xi_j\lambda_j}\int_{-\infty}^{+\infty}\phi(s)\prod_{j=1}^m(s-\xi_j)^{H_0-\frac{3}{2}}1(\xi_j<s)1(|\xi_j|<T)\,ds\,d^m\xi\\&\quad=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}e^{-i\sum_{j=1}^m u_j\lambda_j}\int_{-\infty}^{+\infty}\phi(s)\,e^{is(\lambda_1+\dots+\lambda_m)}\prod_{j=1}^m u_j^{H_0-\frac{3}{2}}1(u_j>0)\\&\qquad\cdot 1(s-T<u_j<s+T)\,ds\,d^m u.\end{aligned}$$

Suppose $\lambda_1,\dots,\lambda_m$ different from zero. Then

$$\begin{aligned}|\tilde{A}_T(\lambda_1,\dots,\lambda_m)|&\leqq\int_{-\infty}^{+\infty}|\phi(s)|\prod_{j=1}^m|B_{\lambda_j}(\max(0,s-T),\max(0,s+T))|\,ds\\&\leqq\left(\int_{-\infty}^{+\infty}|\phi(s)|\,ds\right)\prod_{j=1}^m\frac{1}{(2\pi)^{1/2}}\left(\frac{1}{H_0-\frac{1}{2}}+\frac{2}{|\lambda_j|}\right)\end{aligned}$$

is finite and uniformly bounded in T. Thus,

$$\begin{aligned}\tilde{A}(\lambda_1,\dots,\lambda_m)&=\lim_{T\to\infty}\tilde{A}_T(\lambda_1,\dots,\lambda_m)\\&=\int_{-\infty}^{+\infty}e^{is(\lambda_1+\dots+\lambda_m)}\phi(s)\prod_{j=1}^m\lim_{T\to\infty}B_{\lambda_j}(\max(0,s-T),\max(0,s+T))\,ds\\&=\sqrt{2\pi}\,\tilde{\phi}(\lambda_1+\dots+\lambda_m)\prod_{j=1}^m\left\{\frac{1}{\sqrt{2\pi}}\int_0^{\to\infty}e^{-iu\lambda_j}u^{H_0-\frac{3}{2}}\,du\right\},\end{aligned}$$

where $\int_0^{\to\infty}$ denotes an improper Riemann integral. After a change of variables,

$$\begin{aligned}&\tilde{A}(\lambda_1,\dots,\lambda_m)\\&\quad=\sqrt{2\pi}\,\tilde{\phi}(\lambda_1+\dots+\lambda_m)\prod_{j=1}^m\left\{|\lambda_j|^{\frac{1}{2}-H_0}\frac{1}{\sqrt{2\pi}}\int_0^{\to\infty}e^{-iu\,\mathrm{sign}\,\lambda_j}u^{H_0-\frac{3}{2}}\,du\right\}\\&\quad=\sqrt{2\pi}\,\tilde{\phi}(\lambda_1+\dots+\lambda_m)\prod_{j=1}^m\{|\lambda_j|^{\frac{1}{2}-H_0}(2\pi)^{-1/2}\Gamma(H_0-\tfrac{1}{2})\,C(\lambda_j)\}\end{aligned}$$

where $C(\lambda)=e^{-i\frac{\pi}{2}(H_0-\frac{1}{2})}$ for $\lambda>0$, $C(-\lambda)=\overline{C(\lambda)}$ and thus $|C(\lambda)|=1$ for all $\lambda\neq 0$.

Applying Lemma 6.1, we get

$$\int'_{\mathbb{R}^m} A(\xi_1, \ldots, \xi_m)\, dB(\xi_1) \ldots dB(\xi_m) \stackrel{\Delta}{=} \int''_{\mathbb{R}^m} \tilde{A}(\lambda_1, \ldots, \lambda_m)\, W(d\lambda_1) \ldots W(d\lambda_m)$$

$$= \left(\frac{\Gamma(H_0-\frac{1}{2})}{\sqrt{2\pi}}\right)^m \int''_{\mathbb{R}^m} \sqrt{2\pi}\, \tilde{\phi}(\lambda_1 + \ldots + \lambda_m)$$

$$\cdot \left\{\prod_{j=1}^{m} |\lambda_j|^{\frac{1}{2}-H_0}\right\} C(\lambda_1)\, W(d\lambda_1) \ldots C(\lambda_m)\, W(d\lambda_m)$$

$$\stackrel{\Delta}{=} \left(\frac{\Gamma(H_0-\frac{1}{2})}{\sqrt{2\pi}}\right)^m \int''_{\mathbb{R}^m} \sqrt{2\pi}\, \tilde{\phi}(\lambda_1 + \ldots + \lambda_m) \left\{\prod_{j=1}^{m} |\lambda_j|^{\frac{1}{2}-H_0}\right\} W(d\lambda_1) \ldots W(d\lambda_m)$$

after applying the change of variables formula for Wiener-Itô-Dobrushin integrals: formally, $C(\lambda_j)\, W(d\lambda_j) \stackrel{\Delta}{=} W(d\lambda_j)$ (see Dobrushin (1979) Proposition 4.2). □

Theorem 6.3. *$\bar{Z}_m(t)$, defined in* (1.3), *admits also the following representation*

$$\bar{Z}_m(t) \stackrel{\Delta}{=} K_1(m, H_0) \int''_{\mathbb{R}^m} \frac{e^{i(\lambda_1 + \ldots + \lambda_m)t} - 1}{i(\lambda_1 + \ldots + \lambda_m)} \frac{1}{|\lambda_1|^{H_0-\frac{1}{2}}} \cdots \frac{1}{|\lambda_m|^{H_0-\frac{1}{2}}} W(d\lambda_1) \ldots W(d\lambda_m)$$

where

$$K_1(m, H_0) = \left\{\frac{(m(H_0-1)+1)(2m(H_0-1)+1)}{m!\{2\Gamma(2-2H_0)\sin(H_0-\frac{1}{2})\pi\}^m}\right\}^{1/2}.$$

Proof. Let $p \geqq 1$ *and let* $a_1, \ldots, a_p$ be arbitrary real numbers. Let $\bar{Z}_m(t)$ be defined as in (1.3). Then by (6.1),

$$\sum_{k=1}^{p} a_k \bar{Z}_m(t_k)$$

$$= \frac{K(m, H_0)}{m!} \int_{\mathbb{R}^m} \left\{\sum_{k=1}^{p} a_k \int_0^{t_k} \prod_{j=1}^{m} (s-\xi_j)^{H_0-\frac{3}{2}} 1(\xi_j < s)\right\} dB(\xi_1) \ldots dB(\xi_m)$$

$$= \frac{K(m, H_0)}{m!} \int'_{\mathbb{R}^m} A(\xi_1, \ldots, \xi_m)\, dB(\xi_1) \ldots dB(\xi_m)$$

where we set

$$\phi(s) = \sum_{k=1}^{p} a_k 1_{(0, t_k)}(s)$$

and

$$A(\xi_1, \ldots, \xi_m) = \int_{-\infty}^{+\infty} \phi(s) \prod_{j=1}^{m} (s-\xi_j)^{H_0-\frac{3}{2}} 1(\xi_j < s)\, ds.$$

Naturally, $A \in L^2(\mathbb{R}^m)$. Applying Lemma 6.2, we get

$$\sum_{k=1}^{p} a_k \bar{Z}_m(t_k) \stackrel{\Delta}{=} K_1(m, H_0) \int''_{\mathbb{R}^m} \sqrt{2\pi}\, \tilde{\phi}(\lambda_1 + \ldots + \lambda_m)$$

$$\cdot \left\{\prod_{j=1}^{m} |\lambda_j|^{\frac{1}{2}-H_0}\right\} W(d\lambda_1) \ldots W(d\lambda_m),$$

where

$$K_1(m, H_0) = \frac{K(m, H_0)(\Gamma(H_0 - \frac{1}{2}))^m}{m!\,(2\pi)^{m/2}}.$$

Here, $\sqrt{2\pi}\,\tilde{\phi}(\lambda) = \sum_{k=1}^{p} a_k \int_0^{t_k} e^{i\xi\lambda} d\xi = \sum_{k=1}^{p} a_k \frac{e^{i\lambda t_k} - 1}{i\lambda}$. The constants $a_1, \ldots, a_p$ being arbitrary, we conclude that

$$\tilde{Z}_m(t) \stackrel{\Delta}{=} K_1(m, H_0) \int''_{\mathbb{R}^m} \frac{e^{i(\lambda_1 + \ldots + \lambda_m)t} - 1}{i(\lambda_1 + \ldots + \lambda_m)} |\lambda_1|^{\frac{1}{2} - H_0} \ldots |\lambda_m|^{\frac{1}{2} - H_0} W(d\lambda_1) \ldots W(d\lambda_m).$$

It remains to evaluate the constant $K_1(m, H_0)$. Using the expression for $K(m, H_0)$ given in (1.6), we get

$$K_1(m, H_0) = \left\{ \frac{m!(m(H_0 - 1) + 1)(2m(H_0 - 1) + 1)(\Gamma(H_0 - \frac{1}{2}))^{2m}}{(2\pi)^m (m!)^2 \left(\int_0^\infty (u + u^2)^{H_0 - \frac{3}{2}} du \right)^m} \right\}^{1/2}.$$

But

$$\frac{(\Gamma(H_0 - \frac{1}{2}))^2}{2\pi \int_0^\infty (u + u^2)^{H_0 - \frac{3}{2}} du} = \frac{(\Gamma(H_0 - \frac{1}{2}))^2\, \Gamma(\frac{3}{2} - H_0)}{2\pi \Gamma(H_0 - \frac{1}{2})\, \Gamma(2 - 2H_0)}$$

$$= \frac{\pi}{2\pi \Gamma(2 - 2H_0) \sin(H_0 - \frac{1}{2})\pi}$$

since $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$. Therefore,

$$K_1(m, H_0) = \left\{ \frac{(m(H_0 - 1) + 1)(2m(H_0 - 1) + 1)}{m!\{2\Gamma(2 - 2H_0) \sin(H_0 - \frac{1}{2})\pi\}^m} \right\}^{1/2}.$$

This concludes the proof. □

Acknowledgement. I would like to thank P. Major for his useful suggestions about this last section.

References

1. Davydov, Yu.A.: The invariance principle for stationary processes. Theor. Probability Appl. **15**, 487–498 (1970)
2. De Haan, L.: On regular variation and its application to the weak convergence of sample extremes. Math. Centre Tracts **32**, Math. Centre, Amsterdam (1970)
3. Dobrushin, R.L.: Gaussian and their subordinated self-similar random generalized fields. Ann. Probability **7**, 1–28 (1979)
4. Dobrushin, R.L., Major, P.: Non-central limit theorems for non-linear functionals of Gaussian fields. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 27–52 (1979)
5. Itô, K.: Multiple Wiener integral. J. Math. Soc. Japan **3**, 157–164 (1951)

6. Jona-Lasinio, G.: Probabilistic approach to critical behavior. In: New Developments in Quantum Field Theory and Statistical Mechanics, Cargese 1976 (ed: M. Lévy and P. Mitter), pp. 419–446. NewYork: Plenum 1977
7. Lawrance, A.J., Kottegoda, N.T.: Stochastic modelling of riverflow time series. J. Royal Statist. Soc. Ser. A **140**, Part 1, 1–47 (1977)
8. McKean, H.P.: Geometry of differential space. Ann. Probability **1**, 197–206 (1973)
9. Mandelbrot, B., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**, 422–437 (1968)
10. Sinaï, Ya.G.: Self-similar probability distributions. Theor. Probability Appl. **21**, 64–80 (1976)
11. Taqqu, M.S.: Weak convergence to fractional Brownian motion and to the Rosenblatt process. Z. Wahrscheinlichkeitstheorie verw. Gebiete **31**, 287–302 (1975)
12. Taqqu, M.S.: Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. Z. Wahrscheinlichkeitstheorie verw. Gebiete **40**, 203–238 (1977)
13. Taqqu, M.S.: A representation for self-similar processes. Stochastic Processes Appl. **7**, 55–64 (1978a)
14. Taqqu, M.S.: Weak convergence at all Hermite ranks. Techn. Report No. 389, August 3, 1978. School of Operations Research, Cornell University (1978b)

Received November 6, 1978, in revised form April 11, 1979

Note Added in Proof. A review of recent results concerning self-similar processes and a list of open problems appears in

Taqqu, M.S.: Self-similar processes and related ultraviolet and infrared catastrophes. Tech. Report No. 423, School of Operations Research, Cornell University (1979). Presented at the International Colloquium on "Random Fields: Rigorous Results in Statistical Mechanics and Quantum Field Theory," Esztergom, Hungary, June 24–30, 1979. To appear in 1980 in a joint edition published by the Janós Bolyai Mathematical Society and North Holland Publishing Company.

In the appendix of that paper, results of this paper and of Dobrushin and Major (1979) are combined to show that Theorem 5.5 holds under weaker conditions on $X(s)$. A multivariate version of the theorem is stated.

Statistical techniques for detecting self-similarity and estimating the self-similarity parameter are discussed in

Mandelbrot, B.B., Taqqu, M.S.: Robust R/S analysis of long-run serial correlation. IBM Technical Report (1979). To appear in the *Proceedings of the 42^nd^ Session of the International Statistical Institute, Manila, Philippines.*