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Spectral Representations of Infinitely Divisible Processes

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Summary. The spectral representations for arbitrary discrete parameter infinitely divisible processes as well as for (centered) continuous parameter infinitely divisible processes, which are separable in probability, are obtained. The main tools used for the proofs are (i) a "polar-factorization" of an arbitrary Lévy measure on a separable Hilbert space, and (ii) the Wiener-type stochastic integrals of non-random functions relative to arbitrary "infinitely divisible noise".

0. Introduction

For the analysis of many statistical and probabilistic problems for stationary Gaussian processes, a significant tool is provided by the spectral representations of these processes in terms of the "Gaussian noise". Motivated by these considerations, many authors advocated the need to develop similar spectral representations for symmetric stable processes in terms of the "stable noise" and to apply these to study the analogous problems for these processes; and such representations were in fact developed by several authors (Schilder [27], Kuelbs [13], Bretagnolle et al. [2] and Schriber [28]). With the same motivation, recently spectral representations of symmetric semistable processes in terms of the "semistable noise" are also obtained (Rajput, Rama-Murthy [20]) which are shown to be valid for non-symmetric semistable processes as long as α, the index of the process, is not 1; more recently, a similar result for non-symmetric stable processes with index $\alpha \neq 1$ is also obtained (Hardin [7]). Already, the spectral representations of symmetric stable processes have successfully been used to solve the prediction and interpolation problems (e.g., Cambanis, Soltani [3], Cambanis, Miamee [4], Hosoya [9]) and to study the structural and path properties (e.g., Cambanis, Hardin and Weron [5], Rootzen [22], Rosinski [25], and Rosinski and Woyczynski [26]) for certain subclasses of these processes.

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In working with Gaussian and symmetric stable processes $X=\{X_t\colon t\in T\}$ and their spectral representations $\{\int f_t\, d\Lambda\}$, one discerns two main reasons which make these representations useful in solving various questions about the processes X: (a) Many problems of interest about X can be meaningfully reformulated in terms of the non-random functions f_t and the corresponding "noise" Λ (or sometimes in terms of certain parameters characterizing Λ, e.g., its control measure). (b) These reformulated questions can be effectively solved by making use of the rich structure of the metric linear space of functions generated by $\{f_t\}$ and the fact that Λ enjoys properties very similar to X but, at the same time, admits much simpler probabilistic structure. In view of this observation and the remarks made in the previous paragraph, it is thus tempting to suggest that one should develop spectral representations for each subclass of infinitely divisible processes X in terms of the non-random functions f_t belonging to a "nice space" and the "noise" Λ which exhibits properties similar to that of X. But, since different methods of proof may be required to obtain spectral representations for different subclasses of infinitely divisible processes, it may lead to an unending process; and thus a better question would be to ask: Is it possible to develop *one* procedure whereby, for any given infinitely divisible process X, one can choose non-random functions f_t and "an infinitely divisible noise" Λ such that $X\overset{d}{=}\{\int f_t\, d\Lambda\}$ and, additionally, the following criteria are met?

(i) The "noise" Λ retains properties similar to X; for example, if X belongs to a known class such as α-stable or self-decomposable processes, then Λ belongs to the corresponding class of "noises".

(ii) The functions f_t belong to a linear topological space which is "similar" in its structure to that of the linear space of the process X.

The *main theme* of this paper is to provide an "essentially" complete affirmative answer to this question. This is accomplished in two steps: first, we obtain the spectal representations for arbitrary discrete parameter infinitely divisible processes; and then, using this and some limiting arguments, we obtain the representations for continuous parameter infinitely divisible processes which are separable in probability. We reiterate that the representing "noise" Λ and the representing functions f_t chosen for the representations do meet the critera (i) and (ii), respectively. In fact, as regards to (ii), we show that the space L generated by $\{f_t\}$ is a subspace of a suitable Musielak-Orlicz space, which is continuously (and linearly) embedded in the linear space $L(X)$ of X. Further, if X satisfies some additional conditions (like the ones mentioned above in the continuous parameter case), then we show that L is in fact topologically and linearly isomorphic to $L(X)$. In addition to the above representations which are valid only in law, we also obtain spectral representations which are valid almost surely; this, however, requires that the process be redefined on a slightly larger probability space. Before we end this paragraph we would like to make a few more points: First we note that "integral" representations (in law) of an arbitrary infinitely divisible process in terms of the "Poisson noise" are known (Maruyama [15]); but, as neither the noise nor the representing functions necessarily meet the requirements we ask for, these representations do not fall in the category of the spectral representations we are interested in this paper. Second we point out that our spectral representations (in law) of infinitely divisible processes, when specialized to stable and semistable processes, yield, in a unified way, all known spectral representations for these processes mentioned in the first paragraph above. Finally, we mention the papers (Cambanis [6], Rajput, Rama-Murthy [21] and Hardin [8]) which have some relevance to the spectral representations we have discussed above.

Besides the spectral representations noted above, we also present several other results which fall in two broad categories. All of these play a crucial role for our proofs of the spectral representation theorems, but we also feel that these will be of independent interest. In one category of these results, we obtain a "polar factorization" of an arbitrary Lévy measure on l_2 in terms of a finite measure on the boundary of the unit sphere of l_2 and a family of Lévy measures on the real line. This factorization is similar in spirit to the known factorization of a symmetric stable Lévy measure on $\mathbf{R}^n$ (Lévy [14]) and on l_2 (Kuelbs [13]); and plays an analogous role in the development of the spectral representations here as did the factorization of a symmetric stable Lévy measure for the proofs of the spectral representations of symmetric stable processes in [2, 13, 27, 28]. The results in the other category are concerned with a systematic study of Wiener-type integrals $\int f dA$ of non-random functions with respect to an arbitrary "infinitely divisible noise" A. The main results we present here are: (a) a characterization of A-integrable functions in terms of certain parameters of A; (b) the identification of the space of A-integrable functions as a certain Musielak-Orlicz space; and (c) an isomorphism theorem between this Musielak-Orlicz space and a suitable subspace of L_p-space of random variables. The theory of Wiener-type integrals under various hypotheses on the "noise" A has a long history (e.g., Urbanik, Woyczynski [30], Urbanik [29], Rosinski [23, 24], Schilder [27] and Rajput and Rama-Murthy [20]); the development of these integrals presented here is the most general in the sense that we require minimal hypotheses both on the "noise" A and the space on which integrands and A are defined.[1]

The organization of the rest of the paper is as follows: Sect. 1 contains the preliminaries; Sect. 2 contains the development of stochastic integrals relative to the "infinitely divisible noise" A and a characterization of A-integrable functions. Sect. 3 is concerned with the identification of the space of A-integrable functions as a certain Musielak-Orlicz space and its isomorphism with the subspaces of L_p-space of random variables. Sect. 4 contains, the spectral representation results (in law) for the discrete and the continuous parameter infinitely divisible processes; Sect. 4, also contains the "polar factorization" result of Lévy measures on l_2. Sect. 6 is concerned with the spectral representation of infinitely divisible process which hold almost surely.

[1] Recently the authors have received a manuscript by Kwapien and Woyczynski entitled *Semimartingale integrals via decoupling inequalities and tangent processes*. In this paper, they give a characterization of previsible stochastic processes that are integrable relative to semimartingales. As a necessary first step to obtain this result, they also characterize non-random functions that are integrable relative to general "independent increment noise". This later result, obtained independently of ours, has some overlap with our Theorems 3.3 and 3.4 when specialized to $S=[0, \infty)$ and $p=0$

I. Preliminaries and Some Notations

In this section, we recall some definitions and known facts; also we fix some notations and conventions which we shall use throughout the paper.

Let H be a real (finite or infinite dimensional) separable Hilbert space and let μ be an infinitely divisible (**ID**) prob. measure on H (i.e., μ has a unique n-th root for each $n=1, 2, 3\ldots$). As is well-known, for every **ID** prob. measure μ, $\{\mu^s\colon s>o\}$, the set of s-th roots of μ, forms a continuous (in the weak topology) semigroup under convolution, which is also tight on every finite interval of $R^+=(0, \infty)$. Using this semigroup, we shall now define Gaussian, stable and semistable prob. measures on H. These definitions are non-standard but are equivalent to the traditional definitions which are usually given in terms of weak limits of certain normed sums. We adopted this route mainly because we make use of these defining properties of these prob. measures. Before we record these definitions, we introduce a few notations: For a measure ν on H and a nonzero a in $\mathbf{R}$ (the reals), we denote by $a\cdot\nu$, the measure defined by $a\cdot\nu(B)=\nu(a^{-1}B)$, for every Borel set B of H; further, we shall use the notations $\mathbf{S}(\alpha)$, $\mathbf{S}(r, \alpha)$ and **SD** for the phrases "stable of index α", "semistable of index (r, α)" and "self-decomposable", respectively, where $0<\alpha<2$ and $0<r<1$. Let now μ be a prob. measure on H, we say μ is a $\mathbf{S}(\alpha)$ (resp. a $\mathbf{S}(r, \alpha)$) *prob. measure* if μ is **ID** and

$$\mu^t = t^{\frac{1}{\alpha}}\cdot\mu*\delta_{x(t)}, \qquad \text{for all } t\in(0, 1], \tag{1.1}$$

$$(\text{resp. } \mu^r = r^{\frac{1}{\alpha}}\cdot\mu*\delta_{x(r)}), \tag{1.2}$$

where $\delta_{x(t)}$ and $\delta_{x(r)}$ denote the Dirac measures at the elements $x(t)$ and $x(r)$ of H, respectively, and $*$ denotes the usual convolution operation. If $x(t)$ in (1.1) (resp. $x(r)$ in (1.2)) is θ, the zero element of H and $\alpha\neq 1$, then we say μ is a *centered* $\mathbf{S}(\alpha)$ (resp. a *centered* $\mathbf{S}(r, \alpha)$) *prob. measure*. If $\alpha=1$, then we say μ is a *centered* $\mathbf{S}(1)$ (resp. a *centered* $\mathbf{S}(r, 1)$) *prob. measure* only in the case when μ is a *symmetric* $\mathbf{S}(1)$ (resp. $\mathbf{S}(r, 1)$) prob. measure. If μ is **ID** and (1.1) (or equivalently (1.2)) holds with $\alpha=2$, then we say μ is *Gaussian*, and, if, in addition, $x(t)=\theta$ (or equivalently $x(r)=\theta$), then we say μ is *centered* (or *symmetric*) *Gaussian*. Finally, we say μ is a **SD** *prob. measure*, if

$$\mu = t\cdot\mu*\nu_t, \qquad \text{for all } 0<t\leqq 1, \tag{1.3}$$

where ν_t is a prob. measure on H.

Let now T be an arbitrary index set and $X\equiv\{X_t\colon t\in T\}$ be a real stochastic process, we say X is an **ID** (resp. a *symmetric* **ID**) *process* if, for every finite set $\{t_1, \ldots, t_n\}$ of T, $\mathscr{L}(X_{t_1}, \ldots, X_{t_n})$, the law of $(X_{t_1}, \ldots, X_{t_n})$, is an **ID** (resp. a symmetric **ID**) prob. measure on $\mathbf{R}^n$, the n-Euclidean space. The definitions of **SD**, $\mathbf{S}(\alpha)$, $\mathbf{S}(r, \alpha)$ and Gaussian processes, of their symmetric counterparts and of centered $\mathbf{S}(\alpha)$ and $\mathbf{S}(r, \alpha)$ processes can be stated in the obvious way.

Now we shall define various **ID** random (r.) measures. Throughout the paper, unless stated otherwise, we denote, by S, an arbitrary non-empty set and, by

$\mathscr{S}$, a δ-ring (i.e., a ring which is closed under countable intersections) of subsets of S with the property:

$$\text{There exists an increasing sequence } \{S_n\} \text{ of sets in } \mathscr{S} \text{ with } \bigcup_n S_n = S. \tag{1.4}$$

Let $\Lambda = \{\Lambda(A)\colon A \in \mathscr{S}\}$ be a real stochastic process defined on some prob. space $(\Omega, \mathscr{F}, P)$. We call Λ to be an *independently scattered r. measure* (or *r. measure*, for short), if, for every sequence $\{A_n\}$ of disjoint sets in $\mathscr{S}$, the r. variables $\Lambda(A_n)$, $n = 1, 2, \ldots$, are independent, and, if $\bigcup_n A_n$ belong to $\mathscr{S}$, then we also have

$$\Lambda(\bigcup_n A_n) = \sum_n \Lambda(A_n) \qquad a.s.,$$

where the series is assumed to converge almost surely. In addition, if $\Lambda(A)$ is a symmetric r. variable, for every $A \in \mathscr{S}$, then we call Λ a *symmetric r. measure*. We call a r. measure Λ to be an **ID** *r. measure* if $\Lambda(A)$ is **ID**; if, in addition, $\Lambda(A)$ is symmetric, then we call Λ to be a *symmetric* **ID** *r. measure*. The definitions of $\mathbf{S}(\alpha)$, $\mathbf{S}(r, \alpha)$, **SD** and Gaussian r. measures, of their symmetric counterparts and of centered $\mathbf{S}(\alpha)$ and $\mathbf{S}(r, \alpha)$ r. measures can be stated analogously.

Before we end this section, we would like to mention a few more conventions and notations: While writing the Lévy representation of the characteristic (ch.) function $\hat{\mu}$ of an **ID** prob. measure μ on H one can use many different centering functions, we found the centering function

$$\tau(z) = \begin{cases} z & \text{if } \|z\| \leqq 1 \\ \dfrac{z}{\|z\|} & \text{if } \|z\| > 1 \end{cases}$$

easier to work with in our calculations. We shall, therefore, use this centering function throughout. By a Lévy measure defined on a Borel subset B of H, we shall always mean any measure M on B satisfying $\int_B \min(1, \|z\|^2)\, dM < \infty$, with $M(\{\theta\}) = 0$, if $\theta \in B$. Whenever it is important that M be defined on the whole of H, we will do so by assigning $M(B^c) = 0$; but will use the same notation for the extended measure.

By the statement "M is a **SD** Lévy measure on B" we would mean that M is a Lévy measure of a **SD** prob. measure on H; we shall adopt a similar convention relative to the Lévy measures of other classes of **ID** prob. measures on H. Finally, for a given topological space X, $\mathscr{B}(X)$ will always denote its Borel σ-algebra.

II. Infinitely Divisible Random Measures and Stochastic Integrals

Throughout this paper $\Lambda = \{\Lambda(A)\colon A \in \mathscr{S}\}$ will denote an **ID** r. measure defined on some prob. space $(\Omega, \mathscr{F}, P)$ (recall that $\mathscr{S}$ stands for a δ-ring of subsets of an arbitrary non-empty set S satisfying (1.4)). Since, for every $A \in \mathscr{S}$, $\Lambda(A)$ is an **ID** r. variable, its ch. function can be written in the Lévy form:

$$\hat{\mathscr{L}}(\Lambda(A))(t) = \exp\{it\nu_0(A) - \tfrac{1}{2}t^2\nu_1(A) + \int_{\mathbf{R}} (e^{itx} - 1 - it\tau(x))\, F_A(dx)\}, \quad (2.1)$$

where $-\infty < \nu_0(A) < \infty$, $0 \leqq \nu_1(A) < \infty$ and F_A is a Lévy measure on $\mathbf{R}$. In this section, we first show (Proposition 2.1) that there is a one to one correspondence between the class of **ID** r. measures on one hand and the class of parameters ν_0, ν_1 and $F_{\boldsymbol{\cdot}}$ on the other. This fact, under various additional assumptions, was "essentially" proved in Prékopa [18, 19] and Urbanik and Woyczynski [30]. We include a proof of this fact here, since this proposition is quite important to us and since our proof is very simple and uses only standard arguments of the classical probability theory. Through this result we also define λ, the control measure of Λ. Next we show (Lemma 2.3) that $F_{\boldsymbol{\cdot}}(\cdot)$ determines a unique measure on $\sigma(\mathbf{S}) \times \mathscr{B}(\mathbf{R})$ which admits a factorization in terms of a family of Lévy measures $\rho(s, \cdot)$, $s \in S$ on $\mathbf{R}$ and the measure λ. This fact plays an important role throughout the paper; in particular, this helps us derive another form of the ch. function of $\mathscr{L}(\Lambda(A))$ in terms of the measures $\rho(s, \cdot)$ and λ (Proposition 2.5). This form of the ch. function plays a crucial role in obtaining the ch. function of the stochastic integral $\int_S f d\Lambda$ (which we also define) (Proposition 2.6) and in the proof of the *main result* of this section (Theorem 2.7) which provides an important characterization of Λ-integrable functions.

Proposition 2.1. (a) *Let Λ be an* **ID** *r. measure with the ch. function given by* (2.1). *Then $\nu_0 : \mathscr{S} \mapsto \mathbf{R}$ is a signed-measure, $\nu_1 : \mathscr{S} \mapsto [0, \infty)$ is a measure, F_A is a Lévy measure on $\mathbf{R}$, for every $A \in \mathscr{S}$, and $\mathscr{S} \ni A \mapsto F_A(B) \in [0, \infty)$ is a measure, for every $B \in \mathscr{B}(\mathbf{R})$, whenever $0 \notin \bar{B}$.*

(b) *Let ν_0, ν_1 and $F_{\boldsymbol{\cdot}}$ satisfy the conditions given in* (a). *Then there exists a unique* (*in the sense of finite-dimensional distributions*) **ID** *r. measure Λ such that* (2.1) *holds.*

(c) *Let ν_0, ν_1 and $F_{\boldsymbol{\cdot}}$ be as in* (a) *and define*

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \int_{\mathbf{R}} \min\{1, x^2\}\, F_A(dx), \qquad A \in \mathscr{S}.$$

Then λ: $\mathscr{S} \mapsto [0, \infty)$ is a measure such that $\lambda(A_n) \mapsto 0$ implies $\Lambda(A_n) \to 0$ in prob. for every $\{A_n\} \subset \mathscr{S}$; further, if $\Lambda(A'_n) \to 0$ in prob. for every sequence $\{A'_n\} \subset \mathscr{S}$ such that $A'_n \subset A_n \in \mathscr{S}$, then $\lambda(A_n) \to 0$.

Proof. (a) Let $\{A_k\}_{k=1}^n$ be pairwise disjoint sets in $\mathscr{S}$. By the uniqueness of Lévy's representation of the ch. function of an **ID** distribution, it follows, using $\hat{\mathscr{L}}\left(\Lambda\left(\bigcup_{k=1}^n A_k\right)\right) = \prod_{k=1}^n \hat{\mathscr{L}}(\Lambda(A_k))$, that all three set functions ν_0, ν_1 and $F_{\boldsymbol{\cdot}}(B)$ are finitely additive. Let now $A_n \in \mathscr{S}$, $A_n \searrow \emptyset$. Since $\Lambda(A_n) \to 0$ in prob., we have that

$\nu_0(A_n) \to 0$, $\nu_1(A_n) \to 0$ and $\int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) \to 0$. By Chebychev's inequality, we get

$$F_{A_n}(\{|x| \geqq \varepsilon\}) \leqq \varepsilon^{-2} \int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) \to 0,$$

for every $\varepsilon \in (0, 1)$, which completes the proof of (a).

(b) The existence of a finitely additive independently scattered r. measure $\Lambda = \{\Lambda(A): A \in \mathscr{S}\}$ follows by a standard application of the Kolmogorov Extension Theorem (see e.g., [11]). To prove that Λ is countably additive, let $A_n \in \mathscr{S}$, $A_n \searrow \emptyset$. Since $F_{A_1} \geqq F_{A_2} \geqq \ldots$, we get

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) &\leqq \overline{\lim_{n\to\infty}} \int_{\{|x|<\varepsilon\}} \min\{1, x^2\} F_{A_n}(dx) + \overline{\lim_{n\to\infty}} F_{A_n}(\{|x| \geqq \varepsilon\}) \\ &\leqq \int_{\{|x|<\varepsilon\}} \min\{1, x^2\} F_{A_1}(dx), \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Letting $\varepsilon \to 0$ we obtain that $\int_{\mathbf{R}} \min\{1, x^2\} F_{A_n}(dx) \to 0$. Since also $\nu_0(A_n) \to 0$ and $\nu_1(A_n) \to 0$, we get $\Lambda(A_n) \to 0$ in prob., proving that Λ is countably additive.

(c) It follows that λ is countably additive by a similar argument as we used for proving the countable additivity of Λ above. For the last part, decompose $A_n = A_n^{(1)} \cup A_n^{(2)}$ such that $\nu_0(A_n^{(1)}) = \nu_0^+(A_n)$ and $\nu_0(A_n^{(2)}) = -\nu_0^-(A_n)$. Since $\Lambda(A_n^{(i)}) \to 0$ in prob. as $n \to \infty$, $i = 1, 2$, we get that $\nu_0(A_n^{(i)}) \to 0$, $\nu_1(A_n^{(i)}) \to 0$ and $\int_{\mathbf{R}} \min\{1, x^2\} F_{A_n^{(i)}}(dx) \to 0$ as $n \to \infty$, $i = 1, 2$. This implies that $\lambda(A_n) \to 0$. □

Definition 2.2. Since $\lambda(S_n) < \infty$, $n = 1, 2, \ldots$ we may (and do) extend λ to a σ-finite measure on $(S, \sigma(\mathscr{S}))$; we call λ, *the control measure* of Λ.

Lemma 2.3. *Let $F_{\cdot}$ be as in Proposition* 2.1(a). *Then there exists a unique σ-finite measure F on $\sigma(\mathscr{S}) \times \mathscr{B}(\mathbf{R})$ such that*

$$F(A \times B) = F_A(B), \qquad \textit{for all } A \in \mathscr{S}, B \in \mathscr{B}(\mathbf{R}).$$

Moreover, there exists a function $\rho: S \times \mathscr{B}(\mathbf{R}) \mapsto [0, \infty]$ such that

(i) $\rho(s, \cdot)$ *is a Lévy measure on* $\mathscr{B}(\mathbf{R})$, *for every* $s \in S$,

(ii) $\rho(\cdot, B)$ *is a Borel measurable function, for every* $B \in \mathscr{B}(\mathbf{R})$,

(iii) $\int_{S \times \mathbf{R}} h(s, x) F(ds, dx) = \int_S [\int_{\mathbf{R}} h(s, x) \rho(s, dx)] \lambda(ds)$, *for every* $\sigma(\mathscr{S}) \times \mathscr{B}(\mathbf{R})$-*measurable function* $h: S \times \mathbf{R} \to [0, \infty]$. *This equality can be extended (with obvious restriction regarding the arithmetic of* $\pm\infty$) *to real and complex-valued functions* h.

The proof of Lemma 2.3 relies on a measure-theoretic fact which says that, under some minimal assumptions every bimeasure can be represented by a measure on the product space. We state this useful fact in the proposition below and sketch its proof for the sake of completeness.

Proposition 2.4. *Let $(X, \mathscr{B})$ be a standard Borel space (i.e., a measurable space such that $\mathscr{B}$ is σ-isomorphic to the Borel σ-algebra of some complete separable* *metric space), and let $(T, \mathscr{A})$ be an arbitrary measurable space. Let $Q_0(A, B)$ be a non-negative function of $A \in \mathscr{A}$, $B \in \mathscr{B}$, satisfying:*

(a) *for every $A \in \mathscr{A}$, $Q_0(A, \cdot)$ is a measure on $(X, \mathscr{B})$,*
(b) *for every $B \in \mathscr{B}$, $Q_0(\cdot, B)$ is a measure on $(T, \mathscr{A})$,*
(c) *the measure λ_0 defined by $\lambda_0(A) = Q_0(A, X)$ is σ-finite on $(T, \mathscr{A})$.*

Then there exists a unique measure Q on the product σ-algebra $\mathscr{A} \times \mathscr{B}$ such that

$$Q(A \times B) = Q_0(A, B) = \int_A q(t, B)\, \lambda_0(dt),$$

for every $A \in \mathscr{A}$, $B \in \mathscr{B}$, where $q\colon T \times \mathscr{B} \to [0, 1]$ fulfills the following conditions:

(d) *for every t, $q(t, \cdot)$ is a probability measure on $\mathscr{B}$,*
(e) *for every B, $q(\cdot, B)$ is $\mathscr{A}$-measurable.*

Further, if $q_1(\cdot, \cdot)$ is some other function satisfying (2.2) below, (d) *and* (e), *then off a set of λ_0-measure zero, $q_1(t, \cdot) = q(t, \cdot)$.*

Sketch of the Proof. It is enough to find a measurable family of probability measures $\{q(t, \cdot)\}_{t \in T}$ such that

$$Q_0(A, B) = \int_A q(t, B)\, \lambda_0(dt) \tag{2.2}$$

for all $A \in \mathscr{A}$, $B \in \mathscr{B}$ (uniqueness of q is obvious). Indeed, if such a family $\{q(t, \cdot)\}_{t \in T}$ is given, then Q defined by

$$Q(C) \equiv \int_T \int_X I_C(t, x)\, q(t, dx)\, \lambda_0(dt),$$

$C \in \mathscr{A} \times B$, is a σ-additive measure (see, e.g., [1] p. 97) and the proof is complete.

To show the existence of $\{q(t, \cdot)\}_{t \in T}$ note that for each fixed $B \in \mathscr{B}$, $Q_0(\cdot, B) \leqq Q_0(\cdot, X) = \lambda_0$; therefore one can define the Radon-Nikodym derivative $q_0(\cdot, B) \equiv dQ_0(\cdot, B)/d\lambda_0$. By the definition of q_0, equality (2.2) is satisfied with q replaced by q_0, further, $0 \leqq q_0(t, B) \leqq q_0(t, x) = 1$ a.e. $[\lambda]$ and $q_0(t, B_1 \cup B_2) = q_0(t, B_1) + q_0(t, B_2)$ a.e. $[\lambda_0]$ for all B, B_1, $B_2 \in \mathscr{B}$, $B_1 \cap B_2 = \emptyset$. Now one can use the method of the construction of regular conditional probabilities (see, e.g. [1], Theorem 6.6.2; since $(X, \mathscr{B})$ is a Borel space, one can assume that $X = \mathbf{R}$) to obtain $\{q(t, \cdot)\}_{t \in T}$ satisfying (d) and (e) and such that $q(\cdot, B) = q_0(\cdot, B)$ a.e. $[\lambda_0]$ for each $B \in \mathscr{B}$. $\square$

Proof of Lemma 2.3. Put

$$G_0(A, B) = \int_B \min\{1, x^2\}\, F_A(dx), \qquad A \in \mathscr{S}, B \in \mathscr{B}(\mathbf{R}).$$

Since for every $B \in \mathscr{B}(\mathbf{R})$, $G_0(\cdot, B)$ is a finite measure on $(S_n, \mathscr{S} \cap S_n)$, $n \geqq 1$, $G_0(\cdot, B)$ has a unique extension to a σ-finite measure on $(S, \sigma(\mathscr{S}))$. Denoting this extension by $Q_0(A, B)$, we see that the assumptions of Proposition 2.4 are satisfied with $(T, \mathscr{A})=(S, \sigma(\mathscr{S}))$ and $(X, \mathscr{B})=(\mathbf{R}, \mathscr{B}(\mathbf{R}))$. Thus there exists a measure Q on the product σ-algebra $\sigma(\mathscr{S})\times\mathscr{B}(\mathbf{R})$ such that

$$Q(A\times B)=G_0(A,B)=\int_A q(s,B)\,\lambda_0(ds),$$

where $\lambda_0(A)=G_0(A,\mathbf{R})$ and q satisfies (d) and (e) of Proposition 2.4. Note that $\lambda_0(A)\leqq\lambda(A)$, for every $A\in\sigma(\mathscr{S})$, which implies that $\lambda_0\ll\lambda$; now define

$$\rho(s,dx)=\frac{d\lambda_0}{d\lambda}(s)(\min\{1,x^2\})^{-1}q(s,dx).$$

Then (ii) is satisfied and

$$\int_{\mathbf{R}}\min\{1,x^2\}\,\rho(s,dx)=\frac{d\lambda_0}{d\lambda}(s)\int_{\mathbf{R}}q(s,dx)=\frac{d\lambda_0}{d\lambda}(s)\leqq 1,$$

which proves (i) $\left(\text{we may always assume that } \frac{d\lambda_0}{d\lambda}(s)\leqq 1 \text{ for all } s\right)$. Define

$$F(C)=\int_S\left[\int_{\mathbf{R}} I_C(s,x)\,\rho(s,dx)\right]\lambda(ds), \tag{2.3}$$

$C\in\sigma(\mathscr{S})\times\mathscr{B}(\mathbf{R})$; then F is a well-defined measure that satisfies, for every $A\in\mathscr{S}$ and $B\in\mathscr{B}(\mathbf{R})$,

$$\begin{aligned}F(A\times B)&=\int_A\left[\int_B\rho(s,dx)\right]\lambda(ds)\\&=\int_A\left[\int_B(\min\{1,x^2\})^{-1}q(s,dx)\right]\lambda_0(ds)\\&=\int_{A\times B}(\min\{1,x^2\})^{-1}Q(ds,dx)\\&=\int_B(\min\{1,x^2\})^{-1}G_0(A,dx)=F_A(B);\end{aligned}$$

(iii) now follows from (2.3) by a standard argument. This completes the proof of Lemma 2.3. □

Using Lemmas 2.1 and 2.3 we obtain a very useful form of the ch. function of $\Lambda(A)$:

Proposition 2.4. *The ch. function* (2.1) *of* $\Lambda(A)$ *can be rewritten in the form*:

$$\hat{\mathscr{L}}(\Lambda(A))(t)=\exp\left\{\int_A K(t,s)\,\lambda(ds)\right\},\qquad t\in\mathbf{R},\ A\in\mathscr{S},$$

where

$$K(t,s)=it\,a(s)-\tfrac{1}{2}t^2\sigma^2(s)+\int_{\mathbf{R}}(e^{itx}-1-it\tau(x))\,\rho(s,dx),$$

$a(s)=\frac{d\nu_0}{d\lambda}(s)$, $\sigma^2(s)=\frac{d\nu_1}{d\lambda}(s)$ *and* ρ *is given by Lemma* 2.3. *Moreover, we have*

$$|a(s)|+\sigma^2(s)+\int_{\mathbf{R}}\min\{1,x^2\}\,\rho(s,dx)=1 \qquad a.e.\ [\lambda]. \tag{2.4}$$

Proof. First part immediately follows from (2.1) and Lemma 2.3. Since, for every $A\in\mathscr{S}$, we have

$$\int_A [|a(s)|+\sigma^2(s)+\int_{\mathbf{R}}\min\{1,x^2\}\,\rho(s,dx)]\,\lambda(ds)$$
$$=|\nu_0|(A)+\nu_1(A)+\int_{A\times\mathbf{R}}\min\{1,x^2\}\,F(ds,dx)=\lambda(A)=\int_A d\lambda,$$

(2.4) follows; which completes the proof. □

The following definition of the stochastic integral, proposed first by Urbanik and Woyczynski [30], is the usual definition of the integrals with respect to a vector measure taking values in the $L_0(\Omega,\mathscr{F},P)$-space (see also [23]).

Definition. (a) Let $f=\sum_{j=1}^{n} x_j I_{A_j}$ be a real simple function on S, where $A_j\in\mathscr{S}$ are disjoint. Then, for every $A\in\sigma(\mathscr{S})$, we define

$$\int_A f\,d\Lambda=\sum_{j=1}^{n} x_j\,\Lambda(A\cap A_j).$$

(b) A measurable function f: $(S,\sigma(\mathscr{S}))\to(\mathbf{R},\mathscr{B}(\mathbf{R}))$ is said to be *Λ-integrable* if there exists a sequence $\{f_n\}$ of simple functions as in (a) such that

(i) $f_n\to f$ a.e. $[\lambda]$,
(ii) for every $A\in\sigma(\mathscr{S})$, the sequence $\{\int_A f_n\,d\Lambda\}$ converges in prob., as $n\to\infty$.

If f is Λ-integrable, then we put

$$\int_A f\,d\Lambda=P-\lim_{n\to\infty}\int_A f_n\,d\Lambda,$$

where $\{f_n\}$ satisfies (i) and (ii).

We note that $\int_A f\,d\Lambda$ is well defined (i.e., it does not depend on the approximating sequence $\{f_n\}$, Urbanik and Woyczynski [30]). Now we proceed to find an expression of the ch. function of $\int_S f\,d\Lambda$:

Proposition 2.6. *If* f *is* Λ*-integrable, then* $\int_S |K(tf(s),s)|\,\lambda(ds)<\infty$, *where* K *is given in Proposition* 2.5, *and*

$$\mathscr{L}(\int_S f\,d\Lambda)(t)=\exp\{\int_S K(tf(s),s)\,\lambda(ds)\},\qquad t\in\mathbf{R}. \tag{2.5}$$

Proof. Note first that (2.5) holds for simple functions. Let $\{f_n\}$ be a sequence of simple functions in the definition of Λ-integral. Define complex measures $\mu_{t,n}$, $t\in\mathbf{R}$, $n\geqq 1$, by

$$\mu_{t,n}(A)=\int_A K(tf_n(s),s)\,\lambda(ds), \qquad A\in\sigma(\mathscr{S}).$$

Since, for every $t\in\mathbf{R}$ and $A\in\sigma(\mathscr{S})$,

$$\lim_{n\to\infty}\mu_{t,n}(A)=\lim_{n\to\infty}\log\hat{\mathscr{L}}\big(\int_A f_n\,d\Lambda\big)(t)=\log\hat{\mathscr{L}}\big(\int_A f\,d\Lambda\big)(t)=\mu_t(A),$$

it follows, by the Hahn-Saks-Vitali Theorem, that μ_t is a countably additive complex measure. Clearly μ_t is absolutely continuous with respect to λ. Therefore, for every $t\in\mathbf{R}$, there exists an $h_t\in L_1(S,\sigma(\mathscr{S}),\lambda;\mathbf{C})$ such that

$$\log\hat{\mathscr{L}}\big(\int_A f\,d\Lambda\big)(t)=\mu_t(A)=\int_A h_t(s)\,\lambda(ds),$$

for every $A\in\sigma(\mathscr{S})$. To end the proof it suffices to show that $h_t(s)=K(tf(s),s)$ a.e. $[\lambda]$, for each $t\in\mathbf{R}$. Let $t\in\mathbf{R}$ be fixed. By the continuity of $K(\cdot,s)$, for each $s\in S$, we obtain

$$K(tf_n(s),s)\to K(tf(s),s) \qquad \text{a.e. } [\lambda], \tag{2.6}$$

as $n\to\infty$. Using Egorov's Theorem, we may decompose S as follows: $S=\bigcup_{j=0}^{\infty}A_j$, where $\lambda(A_0)=0$, $\lambda(A_j)<\infty$, if $j\geqq 1$, and such that (2.6) holds uniformly in $s\in A_j$, $j=1,2,\ldots$. Hence, for every $j\geqq 1$ and $A\in\sigma(\mathscr{S})$,

$$\begin{aligned}\int_{A\cap A_j} h_t(s)\,\lambda(ds)&=\mu_t(A\cap A_j)=\lim_{n\to\infty}\int_{A\cap A_j} K(tf_n(s),s)\,\lambda(ds)\\&=\int_{A\cap A_j} K(tf(s),s)\,\lambda(ds).\end{aligned}$$

It follows that $h_t(s)=K(tf(s),s)$ a.e. $[\lambda]$ on A_j, $j\geqq 1$. Since A_0 is a λ-null set, the last equality holds *a.e.* $[\lambda]$ on S. $\square$

As we noted in the beginning of this section, the following is the main result of this section. It provides a necessary and sufficient condition for the existence of $\int_S f\,d\Lambda$ in terms of the deterministic characteristics of Λ.

Theorem 2.7. *Let $f\colon S\to\mathbf{R}$ be a $\sigma(\mathscr{S})$-measurable function. Then f is Λ-integrable if and only if the following three conditions hold:*

(i) $\int_S |U(f(s),s)|\,\lambda(ds)<\infty$,

(ii) $\int_S |f(s)|^2\sigma^2(s)\,\lambda(ds)<\infty$,

and

(iii) $\int\limits_S V_0(f(s), s)\,\lambda(ds) < \infty$,

where

$$U(u, s) = u\,a(s) + \int\limits_R (\tau(xu) - u\tau(x))\,\rho(s, dx),$$

$$V_0(u, s) = \int\limits_R \min\{1, |xu|^2\}\,\rho(s, dx).$$

Further, if f is Λ-integrable, then the ch. function of $\int\limits_S f\,d\Lambda$ can be written as

(iv) $\hat{\mathscr{L}}(\int\limits_S f\,d\Lambda)(t) = \exp\{it\,a_f - \tfrac{1}{2}t^2\sigma_f^2 + \int\limits_R (e^{itx} - 1 - it\tau(x))\,F_f(dx)\}$,

where

$$a_f = \int\limits_S U(f(s), s)\,\lambda(ds), \qquad \sigma_f^2 = \int\limits_S |f(s)|^2\sigma^2(s)\,\lambda(ds),$$

and

$$F_f(B) = F(\{(s, x) \in S \times \mathbf{R}\colon f(s)\,x \in B \setminus \{0\}\}), \qquad B \in \mathscr{B}(\mathbf{R}).$$

Proof. Assume that f is Λ-integrable. By Proposition 2.6, we have that

$$\begin{aligned}|\hat{\mathscr{L}}(\int\limits_S f\,d\Lambda)(t)|^2 &= \exp\{2\int\limits_S \operatorname{Re} K(tf(s), s)\,\lambda(ds)\}\\ &= \exp\{2\int\limits_S [-\tfrac{1}{2}t^2 f^2(s)\,\sigma^2(s) + \int\limits_R (\cos(tf(s)x) - 1)\,\rho(s, dx)]\,\lambda(ds)\}\\ &= \exp\{-t^2\sigma_f^2 + 2\int\limits_R (\cos tx - 1)\,F_f(dx)\}\end{aligned}$$

is the ch. function of an **ID** distribution. Hence $\sigma_f^2 < \infty$ and $\int\limits_R \min\{1, x^2\}\,F_f(dx) < \infty$. This proves (ii) and (iii). Now, since $|\tau(x) - \sin x| \leqq 2\min\{1, x^2\}$, we get

$$\begin{aligned}|U(u, s)| &\leqq |u\,a(s) + \int\limits_R [\sin xu - u\tau(x)]\,\rho(s, dx)| + |\int\limits_R [\tau(xu) - \sin xu]\,\rho(s, dx)|\\ &\leqq |\operatorname{Im} K(u, s)| + 2V_0(u, s).\end{aligned}$$

Thus (i) follows by Proposition 2.6 and already proven (iii). In view of (i), (ii) and (iii), it is easy to derive (iv) from (2.5).

Conversely, assume that (i), (ii) and (iii) hold. Let $A_n = \{s\colon |f(s)| \leqq n\} \cap S_n$. We have that $A_n \in \mathscr{S}$ and $A_n \nearrow S$. Choose f_n's, simple $\mathscr{S}$-measurable functions, such that $f_n(s) = 0$, if $s \notin A_n$, $|f_n(s) - f(s)| \leqq \frac{1}{n}$, if $s \in A_n$, and $|f_n(s)| \leqq |f(s)|$, for all $s \in S$. Clearly $f_n \to f$ everywhere on S, as $n \to \infty$. Since, for every $A \in \sigma(\mathscr{S})$ and $n, m \geqq 1$,

$$|[f_n(s) - f_m(s)]\,1_A(s)| \leqq 2|f(s)|,$$

by Lemma 2.8, which follows this proof, we get

$$|U([f_n(s) - f_m(s)]\,1_A(s), s)| \leqq 2|U(f(s), s)| + 27\,V_0(f(s), s).$$

Therefore, by the Dominated Convergence Theorem, we obtain that, for every $A \in \sigma(\mathscr{S})$,

$$\lim_{n,m\to\infty} \int_S U([f_n(s)-f_m(s)]\, 1_A(s), s)\, \lambda(ds) = 0,$$

$$\lim_{n,m\to\infty} \int_S [f_n(s)-f_m(s)]^2\, 1_A(s)\, \sigma^2(s)\, \lambda(ds) = 0,$$

and

$$\lim_{n,m\to\infty} \int_S V_0([f_n(s)-f_m(s)]\, 1_A(s), s)\, \lambda(ds) = 0.$$

In view of (iv), $\lim_{n,m\to\infty} \hat{\mathscr{L}}(\int_S [f_n-f_m]\, 1_A\, d\Lambda)(t) \to 1$, for every $t \in \mathbf{R}$ and $A \in \sigma(\mathscr{S})$. Hence the sequence $\{\int_A f_n\, d\Lambda\}_{n=1}^{\infty}$ converges in prob., for every $A \in \sigma(\mathscr{S})$; i.e., f is Λ-integrable. $\square$

Lemma 2.8. *For every $u \in \mathbf{R}$, $s \in S$ and $d > 0$,*

$$\sup\{|U(cu, s)|\colon\ |c| \leqq d\} \leqq d\,|U(u, s)| + (1+d)^3\, V_0(u, s).$$

Proof. Let $|c| \leqq d$. We have

$$\begin{aligned} U(cu, s) &= cu\,a(s) + \int_{\mathbf{R}} [\tau(cux) - cu\,\tau(x)]\, \rho(s, dx) \\ &= cu\,a(s) + c \int_{\mathbf{R}} [\tau(ux) - u\,\tau(x)]\, \rho(s, dx) + \int_{\mathbf{R}} [\tau(cux) - c\,\tau(ux)]\, \rho(s, dx) \\ &= c\,U(u, s) + R(c, u, s), \end{aligned}$$

where $R(c, u, s)$ denotes the last integral. Since $\tau(cux) - c\,\tau(ux) = 0$ if $|ux| \leqq \min\{1, |c|^{-1}\}$ and $|\tau(cux) - c\,\tau(ux)| \leqq 1 + d$ otherwise, we get

$$\begin{aligned} |R(c, u, s)| &\leqq (1+d) \int_{\{|ux| > \min\{1, |c|^{-1}\}\}} \rho(s, dx) \\ &\leqq (1+d)\, \rho(s, \{x\colon\ \min\{1, |ux|\} \geqq \min\{1, |c|^{-1}\}\}) \\ &\leqq \frac{1+d}{\min\{1, |c|^{-2}\}} \int_{\mathbf{R}} \min\{1, |ux|^2\}\, \rho(s, dx), \end{aligned}$$

by Chebyshev's inequality. Since the last quantity is bounded by $(1+d)^3\, V_0(u, s)$, the proof is complete. $\square$

Usually it is easier to verify conditions for the existence of $\int f\, d\Lambda$ when Λ is symmetric. The next position shows how to characterize the Λ-integrable functions f, using $\bar{\Lambda}$-integrability of f, where $\bar{\Lambda}$ is the symmetrization of Λ.

Proposition 2.9. *Let Λ' be an independent copy of Λ and put $\bar{\Lambda}(A) = \Lambda(A) - \Lambda'(A)$, $A \in \mathscr{S}$. Then for an arbitrary function $f\colon S \mapsto \mathbf{R}$, f is Λ-integrable if and only if f is $\bar{\Lambda}$-integrable and the condition (i) of Theorem 2.7 is fulfilled.*

Proof. The Proposition follows immediately from Theorem 2.7 because

$$\mathscr{L}(\bar{\Lambda}(A))(t)=\exp\{\int_A [-t^2\sigma^2(s)+2\int_{\mathbf{R}} (\cos tx-1)\,\bar{\rho}(s,dx)]\,\lambda(ds)\},$$

where $\bar{\rho}(s,B)=\rho(s,B)+\rho(s,-B)$, $B\in\mathscr{B}(\mathbf{R})$. □

III. Continuity of the Stochastic Integral Mapping and Identification of Λ-integrable Functions

In this section we shall identify the set of Λ-integrable functions as a certain Musielak-Orlicz modular space, and shall prove the continuity of the mapping $f\to\int_S f d\Lambda$ from this modular space into $L_p(\Omega, P)$. In addition, under certain conditions on Λ, we shall show that the inverse of this map is also continuous. We also point out that these results on stochastic integrals unify and extend the corresponding results of [23, 29, 30]; further, using these results, we show that one can easily recover, in a unified way, the results concerning stochastic integrals and the space of Λ-integrable functions obtained in [2, 7, 20, 27].

We begin with some preliminaries. Let q be a non-negative number such that

$$(\mathbf{MC})_q \quad E|\Lambda(A)|^q<\infty, \qquad \text{for all } A\in\mathscr{S}.$$

Throughout this section, we shall assume that the above condition is satisfied and $q\in[0,\infty)$ is fixed (note that every Λ satisfies $(\mathbf{MC})_q$ with $q=0$). Hence, using the standard fact which states that for an **ID** distribution μ with Lévy measure G, $\int_{\mathbf{R}} |x|^q\mu(dx)$ is finite if and only if $\int_{\{|x|>1\}} |x|^q G(dx)$ is finite, we have

$$\int_A [\int_{\{|x|>1\}} |x|^q\rho(s,dx)]\,\lambda(ds)=\int_{\{|x|>1\}} |x|^q F_A(dx)<\infty,$$

for every $A\in\mathscr{S}$ (recall F_A is the Lévy measure of $\mathscr{L}(\Lambda(A))$). Hence λ-a.e.

$$\int_{\{|x|>1\}} |x|^q\rho(s,dx)<\infty. \tag{3.1}$$

Thus, without loss of generality, we may (and do) assume that (3.1) holds for all $s\in S$. Define, for $0\leqq p\leqq q$, $u\in\mathbf{R}$ and $s\in S$,

$$\Phi_p(u,s)=U^*(u,s)+u^2\sigma^2(s)+V_p(u,s), \tag{3.2}$$

where

$$U^*(u,s)=\sup_{|c|\leqq 1}|U(cu,s)|$$

and

$$V_p(u,s)=\int_{-\infty}^{\infty} \{|ux|^p I(|ux|>1\}+|ux|^2 I(|ux|\leqq 1)\}\,\rho(s,dx).$$

Next we state and prove two lemmas which will be needed for the identification of the space of Λ-integrable functions as well as for the proof of the continuity of the stochastic integral mapping and its inverse.

Lemma 3.1. *The following are satisfied*:

(i) *for every* $s \in S$, $\Phi_p(\cdot, s)$ *is a continuous non-decreasing function on* $[0, \infty)$ *with* $\Phi_p(0, s)=0$,

(ii) $\lambda(\{s\colon \Phi_p(u, s)=$ *for some* $u=u(s)\neq 0\})=0$,

(iii) *there exists a numerical constant* $C>0$ *such that*

$$\Phi_p(2u, s) \leqq C\Phi_p(u, s),$$

for all $u \geqq 0$ *and* $s \in S$.

Proof. It is easy to prove that $U(\cdot, s)$ is continuous; using this one proves as easily that $U^*(\cdot, s)$ is also continuous. Using this fact and the Dominated Convergence Theorem, we establish the continuity of $\Phi_p(\cdot, s)$. To see that $\Phi_p(\cdot, s)$ is non-decreasing we observe that $U^*(\cdot, s)$ is non-decreasing and, for each fixed u,

$$|ux|^p I(|xu|>1)+|xu|^2 I(|xu| \leqq 1)=\begin{cases}\min\{|xu|^p, |xu|^2\} & \text{if } 0 \leqq p \leqq 2\\ \max\{|xu|^p, |xu|^2\} & \text{if } p>2\end{cases} \tag{3.3}$$

is increasing in $x \geqq 0$. Now we prove (ii). If $\Phi_p(u, s)=0$, for some $u=u(s)\neq 0$, then $\rho(s, \mathbf{R})=0$, $\sigma^2(s)=0$ and $U(u, s)=0$. By the definition of $U(u, s)$, we get $a(s)=0$. Therefore,

$$\begin{aligned} S_0 &\equiv \{s\colon \Phi_p(u, s)=0 \text{ for some } u=u(s)\neq 0\}\\ &=\{s\colon a(s)=\sigma^2(s)=\rho(s, \mathbf{R})=0\}. \end{aligned}$$

(Note that above equality also establishes the measurability of S_0.) Let A be any measurable subset of S_0. Since $\nu_0(A)=\int_A a(s)\,\lambda(ds)=0$, we get $|\nu_0|(S_0)=0$. Thus

$$\lambda(S_0)=|\nu_0|(S_0)+\int_{S_0} \sigma^2(s)\,\lambda(ds)+\int_{S_0} \min\{1, |x|^2\}\,\rho(s, dx)=0.$$

To prove (iii), we use Lemma 2.8 and (3.3), and get

$$\begin{aligned} \Phi_p(2u, s) &\leqq 2|U(u, s)|+27\,V_0(u, s)+4u^2\sigma^2(s)+(2^p+4)\,V_p(u, s)\\ &\leqq (2^p+31)\,\Phi_p(u, s). \quad \square \end{aligned}$$

Lemma 3.2. *Let* $\{\mu_n\}$ *be a sequence of* **ID**. *prob. measures on* $\mathbf{R}$ *with Lévy representation*: $\mu_n \equiv (a_n, \sigma_n^2, G_n)$. *Assume* $\mu_n \xrightarrow{\omega} \delta_0$; *equivalently*, $a_n \to 0$, $\sigma_n^2 \to 0$ *and* $\int_{-\infty}^{\infty} \min\{1, |x|^2\}\, dG_n \to 0$. *Then, for any* $b>0$,

$$\int_{\mathbf{R}} |x|^b \mu_n(dx) \to 0 \Leftrightarrow \int_{\{|x|>1\}} |x|^b G_n(dx) \to 0.$$

(It is, of course, assumed here that $\int_R |x|^b d\mu_n < \infty$ *(and hence* $\int_{\{|x|>1\}} |x|^b G_n(dx) < \infty$*), for all n.)*

Proof. Under the hypotheses of the Lemma, it is easy to prove that

$$\limsup_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b G_n(dx) = 0 \Leftrightarrow \lim_n \int_{\{|x|>1\}} |x|^b G_n(dx) = 0, \tag{3.4}$$

and

$$\limsup_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b \mu_n(dx) = 0 \Leftrightarrow \lim_n \int_{\{|x|>1\}} |x|^b \mu_n(dx) = 0. \tag{3.5}$$

Now assume $\int_{\{|x|>1\}} |x|^b G_n(dx) \to 0$, hence, by (3.4) and Theorem 2 of [10], (note that $\{\mu_n\}$ is compact) $\lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b \mu_n(dx) = 0$. Thus, by (3.5), $\int_{\{|x|>1\}} |x|^b \mu_n(dx) \to 0$. But, as $\mu_n \xrightarrow{\omega} \delta_0$, we have $\int_{\{|x|\leq 1\}} |x|^b \mu_n(dx) \to 0$. This proves $\int_{\mathbf{R}} |x|^b \mu_n(dx) \to 0$. Conversely, if $\int_{\mathbf{R}} |x|^b \mu_n(dx) \to 0$, then, by (3.5), $\lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b \mu_n(dx) = 0$. Thus by [10] again, $\lim_{t\to\infty} \sup_n \int_{\{|x|>t\}} |x|^b G_n(dx) = 0$; which along with (3.4) imply that $\lim_n \int_{\{|x|>1\}} |x|^b G_n(dx) = 0$. □

In order to get ready to state and prove our first main result of this section, we will need a few more notations and definitions:

We define the so-called Musielak-Orlicz space

$$L_{\Phi_p}(S;\lambda) = \{f \in L_0(S;\lambda) \colon \int_S \Phi_p(|f(s)|, s)\,\lambda(ds) < \infty\}.$$

The following properties of $L_{\Phi_p}(S;\lambda)$ (which are well-known for general Musielak-Orlicz spaces generated by functions which satisfy (i), (ii) and (iii) of Lemma 3.3) will be used throughout this paper: The space $L_{\Phi_p}(S;\lambda)$ is a complete linear metric space with the F-norm defined by

$$\|f\|_{\Phi_p} = \inf\{c > 0 \colon \int_S \Phi_p(c^{-1}|f(s)|, s)\,\lambda(ds) \leq c\}.$$

Simple functions are dense in $L_{\Phi_p}(S;\lambda)$ and the natural embedding of $L_{\Phi_p}(S;\lambda)$ into $L_0(S;\lambda)$ is continuous (here $L_0(S;\lambda)$ is equipped with the topology of convergence in λ measure on every set of finite λ-measure). Finally, $\|f_n\|_{\Phi_p} \to 0$ if and only if $\int_S \Phi_p(|f_n(s)|, s)\,\lambda(ds) \to 0$. For these and further facts concerning Musielak-Orlicz spaces, we refer the reader to [16].

Theorem 3.3 *Let* $0 \leq p \leq q$ *and* Φ_p *be as in* (3.2). *Then*

$$\{f \colon f \text{ is } \Lambda\text{-integrable and } E|\int_S f\,d\Lambda|^p < \infty\} = L_{\Phi_p}(S;\lambda),$$

and the linear mapping

$$L_{\Phi_p}(S;\lambda)\ni f\mapsto \int_S f\,d\Lambda\in L_p(\Omega;P)$$

is continuous (note that $p=0$ here signifies that $L_{\Phi_0}(S;\lambda)=\{f: f$ is Λ-integrable$\}$).

Proof. Let $f\in L_{\Phi_p}(S;\lambda)$; i.e. $\int_S \Phi_p(|f(s)|,s)\,\lambda(ds)<\infty$. Then, it is easy to see that the conditions (i), (ii) and (iii) of Theorem 2.7 are satisfied, so, f is Λ-integrable. If F_f denotes the Lévy measure of $\mathscr{L}(\int_S f\,d\Lambda)$ (see Theorem 2.7), then we have

$$\begin{aligned}\int_{\{|u|>1\}} |u|^p F_f(du) &= \int_S \Big[\int_{\{|f(s)x|>1\}} |f(s)\,x|^p \rho(s,dx)\Big]\,\lambda(ds)\\ &\leqq \int_S \Phi_p(|f(s)|,s)\,\lambda(ds)<\infty;\end{aligned}\tag{3.6}$$

and, consequently, $E|\int_S f\,d\Lambda|^p<\infty$.

Conversely, assume that f is Λ-integrable and $E|\int_S f\,d\Lambda|^p<\infty$. By Lemma 2.8 and (i) and (iii) of Theorem 2.7, we get

$$\int_S U^*(|f(s)|,s)\,\lambda(ds)\leqq \int_S |U(f(s),s)|\,\lambda(ds)+8\int_S V_0(f(s),s)\,\lambda(ds)<\infty.$$

Since $E|\int_S f\,d\Lambda|^p<\infty$, we have $\int_{\{|u|>1\}} |x|^p F_f(dx)<\infty$; hence, by (3.6) and (iii) of Theorem 2.7, we get

$$\int_S V_p(f(s),s)\,\lambda(ds)\leqq \int_{\{|u|>1\}} |x|^p F_f(dx)+\int_S V_0(f(s),s)\,\lambda(ds)<\infty.$$

Combining the above and (ii) of Theorem 2.7, we get $f\in L_{\Phi_p}(S;\lambda)$.

Let $f_n\to 0$ in $L_{\Phi_p}(S;\lambda)$; i.e.

$$\int_S \Phi_p(|f_n(s)|,s)\,\lambda(ds)\to 0 \qquad \text{as } n\to\infty. \tag{3.7}$$

Let a_n, σ_n^2 and F_n be, respectively, the centering constant, the variance and the Lévy measure in the canonical representation of the ch. function of $\mathscr{L}(\int_S f_n\,d\Lambda)$ (see (iv) of Theorem 2.7). Then (3.7) implies that $a_n\to 0$, $\sigma_n^2\to 0$ and

$$\int_{\mathbf{R}} \{|x|^p I(|x|>1)+x^2 I(|x|\leqq 1)\}\,F_n(dx)\to 0,$$

as $n\to\infty$. Thus, in view of Lemma 3.2, $E|\int_S f_n\,d\Lambda|^p\to 0$, as $n\to\infty$ if $p>0$; and, if $p=0$, then clearly $\int_S f_n\,d\Lambda\to 0$ in prob. □

We shall study now the conditions under which the mapping $f \to \int_S f\,d\Lambda$ is an isomorphism. First we note that, in general, this mapping is not one-to-one. Indeed, if $\Lambda(ds) = ds$ is the (deterministic) Lebesque measure on $S = [0, 1]$, then obviously $f \mapsto \int_0^1 f(s)\,ds$ is not one-to-one. In view of this, one needs to impose some suitable condition on Λ (or on some of its parameters) which, on one hand, alleviates this difficulty and makes the mapping an isomorphism but, at the same time, is weak enough so that it is satisfied by a large class of **ID** r. measures. We found the following condition quite satisfactory with regard to these criteria; we refer this as **(IC)** (**I** for isomorphism) condition:

$$(\mathbf{IC})_q \equiv (\mathbf{IC}) \left\{ \begin{array}{l} \text{There exists a constant } C = C(p, q),\ 0 \leqq p \leqq q, \\ \text{such that for every } u \geqq 0 \\ |U(u, s)| \leqq C\{u^2 \sigma^2(s) + V_p(u, s)\} \quad \text{a.e. } [\lambda]. \end{array} \right\}$$

The following is our second main result of this section.

Theorem 3.4. *Let* (**IC**) *be satisfied for some* $0 \leqq p \leqq q$. *Then the mapping* $f \to \int_S f\,d\Lambda$ *is an isomorphism from* $L_{\Phi_p}(S; \lambda)$ *into* $L_p(\Omega; P)$. *Moreover,*

$$\{\int_S f\,d\Lambda:\ f \in L_{\Phi_p}(S; \lambda)\} = \overline{\text{lin}}\{\Lambda(A):\ A \in \mathscr{S}\}_{L_p(\Omega; P)}.$$

Proof. By Lemma 2.8 and (**IC**), we get, for every $u \geqq 0$,

$$\begin{aligned} U^*(u, s) &\leqq |U(u, s)| + 8\,V_0(u, s) \\ &\leqq C_1\{u^2 \sigma^2(s) + V_p(u, s)\} \end{aligned} \tag{3.8}$$

a.e. $[\lambda]$, where $C_1 \leqq C + 8$.

Let $E|\int_S f_n\,d\Lambda|^p \to 0$, if $p > 0$ or $\int_S f_n\,d\Lambda \to 0$ in prob. if $p = 0$. By Theorem 2.7(iv) and Lemma 3.2, we have

$$\int_S |f_n(s)|^2 \sigma^2(s)\,\lambda(ds) = \sigma_{f_n}^2 \to 0$$

and

$$\int_S V_p(f_n(s), s)\,\lambda(ds) = \int_{\mathbf{R}} \{|x|^p I(|x| > 1) + |x|^2 I(|x| \leqq 1)\}\,F_{f_n}(dx) \to 0,$$

as $n \to \infty$, where $\sigma_{f_n}^2$ and F_{f_n} are, respectively, the variance and the Lévy measure in the canonical representation of the ch. function of $\mathscr{L}(\int_S f_n\,d\Lambda)$. Thus, by (3.8), we have

$$\int_S U^*(|f_n(s)|, s)\,\lambda(ds) \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\int_S \Phi_p(|f_n(s)|, s)\, \lambda(ds) \to 0$; i.e., $f_n \to 0$ in $L_{\Phi_p}(S; \lambda)$. This proves the invertability of the map $f \to \int_S f d\Lambda$ and the continuity of the inverse map.

Using the fact that simple functions are dense in $L_{\Phi_p}(S; \lambda)$ and that

$$\operatorname{lin}\{\Lambda(A)\colon\ A \in \mathscr{S}\} = \{\int_S f d\Lambda\colon f \text{ is simple}\},$$

the proof of the last statement of the theorem is easy. □

Corollary 3.5. Let **(IC)** be satisfied for some $0 \leqq p \leqq q$ and $\int_S f_n\, d\Lambda \to 0$ in $L_p(\Omega; P)$. Then $f_n \to 0$ in λ on any set of λ-finite measure.

Proof. It follows from Theorem 3.4 and the earlier noted fact that the natural embedding of L_{Φ_p} into $L_0(S; \lambda)$ is continuous. □

The **(IC)** condition is imposed on certain parameters of Λ and not directly on Λ; this limits the usefulness of Theorem 3.4 somewhat. Thus, it is desirable to find sufficient conditions directly in terms of Λ which guarantee **(IC)** and hence also the fact that the integral mapping is an isomorphism. We shall provide such sufficient conditions in Propositions 3.6 and 3.8.

Proposition 3.6. *The condition* **(IC)** *is satisfied under any of the following two hypotheses on the* **ID** *r. measure* Λ *and the real number* p:

(i) Λ *is symmetric and* $0 \leqq p \leqq q$ *arbitrary,*
(ii) $E[\Lambda(A)] = 0$ *for all* A *and* $1 \leqq p \leqq q$.

Proof. That **(IC)** holds under (i) is trivial, since in this case $a(s) = 0$ *and* $\rho(s, \cdot)$ is symmetric, which implies that $U(\cdot, s) \equiv 0$ a.e. $[\lambda]$. Now we prove that **(IC)** holds under (ii). Since $E|\Lambda(A)|^q < \infty$, $q \geqq 1$ and $E\{\Lambda(A)\} = 0$, we have

$$\begin{aligned}\mathscr{L}(\Lambda(A))(t) &= \exp\{-\tfrac{1}{2} t^2 v_1(A) + \int_{\mathbf{R}} (e^{itx} - 1 - itx)\, F_A(dx)\} \\ &= \exp\{it v_0(A) - \tfrac{1}{2} t^2 v_1(A) + \int_{\mathbf{R}} (e^{itx} - 1 - it\tau(x))\, F_A(dx)\}, \end{aligned} \tag{3.9}$$

where $v_0(A) = \int_{\mathbf{R}} [\tau(x) - x]\, F_A(dx)$. Hence, by Proposition 2.5, a.e. $[\lambda]$,

$$a(s) = \int_{\mathbf{R}} (\tau(x) - x)\, \rho(s, dx) \quad \text{and} \quad U(u, s) = \int_{\mathbf{R}} (\tau(ux) - ux)\, \rho(s, dx). \tag{3.10}$$

Thus we get, for every $p \geqq 1$,

$$\begin{aligned}|U(u, s)| &\leqq \int_{\{|ux| > 1\}} |\tau(ux) - ux|\, \rho(s, dx) \\ &\leqq \int_{\{|ux| > 1\}} |ux|\, \rho(s, dx) \leqq V_p(u, s)\end{aligned}$$

a.e. $[\lambda]$, which concludes the proof. □

As we noted in Sect. II, our definition of stochastic integrals is the same as advocated first by Urbanik and Woyczynski [30] and Urbanik [29] and later adopted by Rosinski [23]. Thus our results on stochastic integrals of *real* functions relative to arbitrary **ID** r. measures do unify and extend the pertinent results of these authors. Another approach of defining stochastic integrals relative to symmetric $\mathbf{S}(\alpha)$, and symmetric $\mathbf{S}(r,\alpha)$ and centered $\mathbf{S}(r,\alpha)$, r. measures Λ have been taken in [2, 27] and [20], respectively. In these papers, the integral $\int f d\Lambda$ is defined as L_p-limit, $0<p<\alpha$, of a sequence of integrals of simple functions relative to Λ; and it is shown that the space of Λ-integrable functions is the $L_\alpha(\lambda)$-space and that the integral map $L_\alpha(\lambda)\ni f\mapsto \int f d\Lambda \in L_p(P)$ is a topological and linear isomorphism. The rest of this section is devoted to show that our integrals as well as the space L_{Φ_p} of Λ-integrable function do coincide with those of [2, 27] and [20], when Λ is symmetric $\mathbf{S}(\alpha)$, and symmetric $\mathbf{S}(r,\alpha)$ or centered $\mathbf{S}(r,\alpha)$ r. measures, respectively; and, that the integral map satisfies the above cited property. Thus, we recover all these results of [2, 27, 20] in a unified way. Finally, towards the end of this section we point out certain facts about Λ-integrable functions for certain $\mathbf{S}(r,1)$ r. measures.

If Λ is a centered $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) r. measure where $1<\alpha<2$, then $E|\Lambda(A)|^q<\infty$, for any $q<\alpha$, and $E\Lambda(A)=0$, for every $A\in\mathscr{S}$. Hence the ch. function of $\Lambda(A)$ is of the form (3.9), where $v_1\equiv 0$ and F_A is an $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) Lévy measure.

If Λ is a centered $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) r. measure and $0<\alpha<1$, then

$$\begin{aligned}\hat{\mathscr{L}}(\Lambda(A))(t)&=\exp\Big\{\int_{\mathbf{R}}(e^{itx}-1)\,F_A(dx)\Big\}\\ &=\exp\Big\{itv_0(A)+\int_{\mathbf{R}}(e^{itx}-1-it\tau(x))\,F_A(dx)\Big\},\end{aligned} \tag{3.11}$$

where $v_0(A)=\int_{\mathbf{R}}\tau(x)\,F_A(dx)$ and F_A is an $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) Lévy measure for every $A\in\mathscr{S}$. Therefore, we have (see Proposition 2.5 and Theorem 2.7)

$$a(s)=\int_{\mathbf{R}}\tau(x)\,\rho(s,dx)\quad\text{and}\quad U(u,s)=\int_{\mathbf{R}}\tau(ux)\,\rho(s,dx)\qquad\text{a.e. }[\lambda]. \tag{3.12}$$

Finally, if Λ is a centered $\mathbf{S}(1)$ (resp. $\mathbf{S}(r,1)$) r. measure, then Λ is symmetric and the ch. function of $\Lambda(A)$ is given by (2.1) with $v_0\equiv v_1\equiv 0$ and F_A a symmetric $\mathbf{S}(1)$ (resp. $\mathbf{S}(r,1)$) Lèvy measure, for every $A\in\mathscr{S}$.

In the following lemma, we state the fact that the conditional Lévy measures $\rho(s,\cdot)$ of $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$) r. measure Λ are $\mathbf{S}(\alpha)$ (resp. $\mathbf{S}(r,\alpha)$). The proof of this fact is postponed to the next section mainly for convenience but also because this fact has more relevance there. Formula (3.15) below follow from (3.14) by a standard argument. The proof of (3.14) can be found in [20].

Lemma 3.7. (a) *Let Λ be a $\mathbf{S}(\alpha)$ r. measure. Then a.e. $[\lambda]$*

$$\rho(s,dx)=c_1(s)\,I(x>0)\,x^{-1-\alpha}dx+c_{-1}(s)\,I(x<0)\,|x|^{-1-\alpha}dx, \tag{3.13}$$

where $c_1, c_{-1}: S\mapsto[0,\infty)$ are $\sigma(S)-\mathscr{B}[0,\infty)$ measurable.

(b) *Let Λ be a* $\mathbf{S}(r,\alpha)$ *r. measure. Then, for λ almost all $s\in S$,*

$$\rho(s,B)=\sum_{n=-\infty}^{\infty} r^n \rho(s,(r^{\frac{n}{\alpha}}B)\cap\Delta) \quad \text{for all } B\in\mathscr{B}(\mathbf{R}), \tag{3.14}$$

where $\Delta=\{x\in\mathbf{R}: r^{\frac{1}{\alpha}}<|x|\leqq 1\}$. More generally, for λ-almost all $s\in S$, the following formulas hold

$$\int_{\mathbf{R}} f(x)\,\rho(s,dx)=\sum_{n=-\infty}^{\infty} r^n \int_{\Delta} f(r^{\frac{-n}{\alpha}}x)\,\rho(s,dx),$$

$$\int_{|x|>r^{\frac{k}{\alpha}}} f(x)\,\rho(s,dx)=\sum_{i=1}^{\infty} r^{-k+i}\int_{\Delta} f(r^{\frac{k-1}{\alpha}}x)\,\rho(x,dx), \tag{3.15}$$

$$\int_{|x|\leqq r^{\frac{k}{\alpha}}} f(x)\,\rho(s,dx)=\sum_{i=0}^{\infty} r^{-k-i}\int_{\Delta} f(r^{\frac{k+i}{\alpha}}x)\,\rho(s,dx),$$

for every Borel non-negative function f and an arbitrary integer k.

Proposition 3.8. *Let Λ be a centered* $\mathbf{S}(\alpha)$, *or more generally, a centered* $\mathbf{S}(r,\alpha)$ *r. measure. Then the* (**IC**) *condition holds, for any $0\leqq p<\alpha$, and $L_{\Phi_p}(S;\lambda)=L_\alpha(S;\lambda)$ up to a renorming, for every $0\leqq p<\alpha$. Consequently, there are positive constants C_1 and C_2 depending only on p, r and α such that*

$$C_1\Big(\int_S |f|^\alpha d\lambda\Big)^{\frac{1}{\alpha}}\leqq\Big(E\Big|\int_S f\,d\Lambda\Big|^p\Big)^{\frac{1}{p}}\leqq C_2\Big(\int_S |f|^\alpha d\lambda\Big)^{\frac{1}{\alpha}}, \tag{3.16}$$

for every $f\in L_\alpha(S;\lambda)$.

Proof. Since every centered $\mathbf{S}(\alpha)$ r. variable is also a centered $\mathbf{S}(r,\alpha)$ r. variable for every $0<r<1$, it is enough to prove the proposition for the case when Λ is a centered $\mathbf{S}(r,\alpha)$ r. measure.

First we shall bound $U(u,s)$. If $0<\alpha<1$, then by (3.12), we have

$$|U(u,s)|\leqq\int_{\mathbf{R}}|\tau(ux)|\,\rho(s,dx)=|u|\int_{\{|x|\leqq|u|^{-1}\}}|x|\,\rho(s,dx)+\int_{\{|x|>|u|^{-1}\}}\rho(s,dx) \tag{3.17}$$

(for the sake of brevity we shall omit in this proof the phrase "for λ-almost all s"). Let k be an integer such that $r^{\frac{k}{\alpha}}<|u|^{-1}\leqq r^{\frac{k-1}{\alpha}}$. Using (3.15), we obtain

$$\begin{aligned}\int_{\{|x|\leqq|u|^{-1}\}}|x|\,\rho(s,dx)&\leqq\int_{\{|x|\leqq r^{\frac{k-1}{\alpha}}\}}|x|\,\rho(s,dx)\\&=\sum_{i=0}^{\infty} r^{-k+1-i}\int_{\Delta} r^{\frac{k-1+i}{\alpha}}|x|\,\rho(s,dx)\\&\leqq\sum_{i=0}^{\infty} r^{(\frac{1}{\alpha}-1)(k-1+i)}\rho(s,\Delta)\\&\leqq\frac{r^{1-\frac{1}{\alpha}}}{1-r^{\frac{1}{\alpha}-1}}\rho(s,\Delta)|u|^{\alpha-1};\end{aligned}$$

and, again by (3.15), we get

$$\begin{aligned}\int\limits_{\{|x|>|u|^{-1}\}} \rho(s,dx) &\leqq \int\limits_{\{|x|>r^{\frac{k}{\alpha}}\}} \rho(s,dx) \\ &= \sum_{i=1}^{\infty} r^{-k+i} \int\limits_{\Delta} \rho(s,dx) \leqq \frac{1}{1-r}\,\rho(s,\Delta)|u|^{\alpha}.\end{aligned}$$

By combining the above and (3.17), we obtain

$$|U(u,s)| \leqq D\rho(s,\Delta)|u|^{\alpha}, \tag{3.18}$$

where $D = r^{1-\frac{1}{\alpha}}(1-r^{\frac{1}{\alpha}-1})^{-1} + (1-r)^{-1}$. Let now $1<\alpha<2$. Then, by (3.10), we get

$$|U(u,s)| \leqq \int\limits_{\{|xu|>1\}} |\tau(xu)-xu|\,\rho(s,dx) \leqq |u| \int\limits_{\{|x|>|u|^{-1}\}} |x|\,\rho(s,dx). \tag{3.19}$$

Let k be as above. Utilizing (3.15) again, we obtain

$$\begin{aligned}\int\limits_{\{|x|>|u|^{-1}\}} |x|\,\rho(s,dx) &\leqq \int\limits_{\{|x|>r^{\frac{k}{\alpha}}\}} |x|\,\rho(s,dx) \\ &= \sum_{i=1}^{\infty} r^{-k+i} \int\limits_{\Delta} r^{\frac{k-i}{\alpha}} |x|\,\rho(s,dx) \\ &\leqq \sum_{i=1}^{\infty} r^{\left(1-\frac{1}{\alpha}\right)(i-k)} \rho(s,\Delta) \\ &\leqq (1-r^{1-\frac{1}{\alpha}})^{-1} \rho(s,\Delta)|u|^{\alpha-1},\end{aligned}$$

which, together with (3.19), shows that (3.18) holds for all $1<\alpha<2$ with $D = (1-r^{1-\frac{1}{\alpha}})^{-1}$.

Using (3.15) repeatedly, in a very similar way as above, one can find positive constants D_1 and D_2, depending only on p, r and α, where $0 \leqq p < \alpha$, $0<r<1$ and $0<\alpha<2$, such that

$$\begin{aligned}D_1\,\rho(s,\Delta)|u|^{\alpha} \leqq V_p(u,s) &= u^2 \int\limits_{\{|xu|\leqq 1\}} x^2 \rho(s,dx) + |u|^p \int\limits_{\{|xu|>1\}} |x|^p \rho(s,dx) \\ &\leqq D_2\,\rho(s,\Delta)|u|^{\alpha}.\end{aligned} \tag{3.20}$$

The condition **(IC)** follows now by (3.18) and (3.20) since, if $\alpha \neq 1$,

$$|U(u,s)| \leqq D\rho(s,\Delta)|u|^{\alpha} \leqq DD_1^{-1}\,V_p(u,s).$$

If $\alpha=1$, $\rho(s,\cdot)$ is symmetric and $a(s)=0$; which implies $U(\cdot,s)\equiv 0$ and **(IC)** holds in this case trivially.

Combining (3.18) and (3.20) we get, for every $0 \leqq p < \alpha$ and $0 < \alpha < 2$ (including $\alpha = 1$),

$$\begin{aligned} D_1\, \rho(s, \Delta)|u|^\alpha \leqq \Phi_p(u, s) &= U^*(u, s) + V_p(u, s) \\ &\leqq (D + D_2)\, \rho(s, \Delta)|u|^\alpha, \end{aligned} \tag{3.21}$$

where $D = 0$, if $\alpha = 1$. We shall obtain now bounds for $\rho(s, \Delta)$ utilizing (2.5); which, in view of (3.10) and (3.12), reads

$$|U(1, s)| + V_0(1, s) = 1, \quad \text{if } \alpha \neq 1,$$

and $V_0(1, s) = 1$, if $\alpha = 1$. By (3.18) and (3.20), we get

$$D_1\, \rho(s, \Delta) \leqq |U(1, s)| + V_0(1, s) = 1 \leqq (D + D_2)\, \rho(s, \Delta);$$

hence

$$(D + D_2)^{-1} \leqq \rho(s, \Delta) \leqq D_1^{-1}.$$

Consequently, by (3.21),

$$D_1 (D + D_2)^{-1} |u|^\alpha \leqq \Phi_p(u, s) \leqq D_1^{-1} (D + D_2)|u|^\alpha.$$

This shows that $f \in L_{\Phi_p}$ if and only if $\|f\|_\alpha^\alpha = \int_S |f|^\alpha d\lambda < \infty$ and obviously the F-norms $\|\cdot\|_{\Phi_p}$ and $\|\cdot\|_\alpha^{\min\{1,\alpha\}}$ are comparable. Now, the inequalities (3.16) follow from Theorem 3.4 and the Closed Graph Theorem. □

VI. Spectal Representations of General Discrete and Centered Continuous Parameter ID Processes

Let M be a $\mathbf{S}(\alpha)$ Lévy measure on $l_2 = l_2(N)$; then, as is well known [13], M admits the representation:

$$M = (\rho \cdot v) \circ \Psi^{-1}, \tag{4.1}$$

where v is a *finite* measure on ∂U, the boundary of the unit ball in l_2, ρ is a $\mathbf{S}(\alpha)$ Lévy measure on $\mathbf{R}$ and Ψ is the map: $\partial U \times \mathbf{R}^+ \to l_2 \setminus \{0\}$ defined by $\Psi(u, x) = xu$. It is noted in [20, 21] that a representation similar to (4.1), can be obtained for any $\mathbf{S}(r, \alpha)$ Lévy measure but one must replace ∂U by the annulus $\Delta = \{x\colon r^{\frac{1}{\alpha}} < \|x\| \leqq 1\}$. This fact that M admits the representation like (4.1) plays a crucial role in the proofs of spectral representations of stable and semistable processes obtained in [2, 7, 8, 13, 20, 21, 27, 28]. The basic idea of all these proofs is as follows: Given a stable (resp. semistable) process $X = \{X_n\}$ with paths in l_2, one first represents the Lévy measure M of $\mathscr{L}(X)$ as in (4.1), then one defines a r. measure Λ on ∂U (resp. on Δ) (or via a Borel isomorphism on some other Borel subset of a complete separable metric space) with control measure $F_A(B)=\nu(A)\,\rho(B)$; and, finally by choosing suitable functions f_n, one shows that

$$\{\int_S f_n\,d\Lambda\} \stackrel{d}{=} \{X_n\}. \tag{4.2}$$

Further, using some continuity arguments, one obtains representation like (4.2) for continuous parameter stable and semistable processes.

In order to apply a similar approach to obtain spectral representations of general **ID** processes, it is thus necessary to obtain a *suitable* representation, similar to (4.1), for the Lévy measure M on l_2 of the law of an arbitrary **ID** process $Y=\{Y_n\}$. This representation is given in Theorem 4.2; using this representation of M and using the other two parameters in the Lévy representation of $\mathscr{L}(Y)$, we define several **ID** r. measures Λ which meet the criterion (i) of the Introduction. Using such **ID** r. measures, we obtain spectral representations of all discrete parameters (Theorem 4.9) and "most" centered continuous parameter **ID** processes (Theorem 4.11); these include and extend, to a large degree, all known spectral representations to date of various special **ID** processes. For brevity and convenience of notations, we have obtained our representations on the unit sphere ∂U of l_2; but, using a Borel isomorphism, one can obtain similar representations on any uncountable Borel subset of a complete separable metric space (see Remark 4.12 for more on this point).

We begin by introducing some notations and conventions, which will be used in this section. Given a Lévy measure M on l_2, the finite measure Γ on $\mathscr{B}(\partial U\times\mathbf{R}^+)$, defined by

$$\Gamma=M_0\circ\Psi, \quad \text{where } M_0(dz)=\min(1,\|z\|^2)\,M(dz), \tag{4.3}$$

can be represented (using a theorem on the existence of regular conditional probabilities or by Proposition 2.4) as

$$\Gamma(C)=\int_{\partial U}\Big(\int_{\mathbf{R}^+} I_C(u,x)\,q(u,dx)\Big)\,\nu(du), \tag{4.4}$$

where $q\colon \partial U\times\mathscr{B}(\mathbf{R}^+)\mapsto[0,1]$ satisfies conditions analogous to (d) and (e) of Proposition 2.4 and ν is the finite measure given by

$$\nu(A)=\Gamma(A\times\mathbf{R}^+)=\int_{\left\{z;\,\frac{z}{\|z\|}\in A\right\}}\min\{1,\|z\|^2\}\,M(dz), \tag{4.5}$$

for every Borel set $A\in\mathscr{B}(\partial U)$. Now we define the measures $\rho(u,\cdot)$ on $\mathscr{B}(\mathbf{R}^+)$, F on $\mathscr{B}(\partial U\times\mathbf{R}^+)$ and $F_A(\cdot)$ on $\mathscr{B}(\mathbf{R}^+)$ by

$$\rho(u,dx)=[\min\{1,|x|^2\}]^{-1}\,q(u,dx), \tag{4.6}$$

for every $u\in\partial U$

$$F(C)=\int_{\partial U}\Big(\int_{\mathbf{R}^+} I_C(u,x)\,\rho(u,dx)\Big)\,\nu(dx), \tag{4.7}$$

for every $C \in \mathscr{B}(\partial U \times \mathbf{R}^+)$ and $F_A(\cdot) = F(A \times \cdot)$, for every $A \in \mathscr{B}(\partial U)$. If M is symmetric, then, by (4.3), $\Gamma(A \times B) = \Gamma(-A \times B)$; hence, in particular, $\bar{v}(\equiv v)$ is symmetric (see (4.5)). Using these, (4.4) and (4.6), we choose ρ such that

$$\rho(u, dx) = \rho(-u, dx) \tag{4.8}$$

for all $u \in \partial U$. In the symmetric case, in addition to the measures $\rho(u, \cdot)$, F, $F_\cdot$, we also associate (to M) the measures $\bar{\rho}(u, \cdot)$ on $B(\mathbf{R}_0)$, $\bar{F}$ on $\mathscr{B}(\partial U \times \mathbf{R}_0)$ and $\bar{F}_\cdot$ on $\mathscr{B}(\mathbf{R}_0)$; here and in the following $\mathbf{R}_0$ will be used to denote $\mathbf{R} \setminus \{0\}$. These measures are defined by the following formulas:

$$\bar{\rho}(u, dx) = 2^{-1}[\rho(u, dx) + (-1) \cdot \rho(u, dx)], \tag{4.9}$$

for all $u \in \partial U$,

$$\bar{F}(C) = \int_{\partial U} \left(\int_{\mathbf{R}_0} I_C(u, x)\, \bar{\rho}(u, dx) \right) \bar{v}(du), \tag{4.10}$$

for every $C \in \mathscr{B}(\partial U \times \mathbf{R}_0)$, and

$$\bar{F}_A(\cdot) = \bar{F}(A \times \cdot),$$

for every $A \in \mathscr{B}(\partial U)$. (As we noted in Sect. I, we will assume that $\rho(u, \cdot)$ are naturally extended to $\mathbf{R}_0$ (or to $\mathbf{R}$) and we will use the same notations for the extended measures. Similar remark applies to the measures $\bar{\rho}(s, \cdot)$, and also to the measures $F_A(\cdot)$ and $\bar{F}_A(\cdot)$).

Using the above definitions and Proposition 2.4, one gets the following facts about the measures defined above, these facts are recored here for clarity and ready reference. The proofs of these are straightforward; and use, among other facts, (4.5)–(4.10).

Lemma 4.1. (i) *The functions ρ and $\bar{\rho}$ satisfy analogs of* (*d*) *and* (*e*) *of Proposition 2.4.*

(ii) *The measures $\rho(u, \cdot)$ and $\bar{\rho}(u, \cdot)$ are Lévy measures on* $\mathbf{R}$; *in fact, for all $u \in \partial U$,*

$$\int_{\mathbf{R}^+} \min(1, |x|^2)\, \rho(u, dx) = \int_{\mathbf{R}} \min(1, |x|^2)\, \bar{\rho}(u, dx) = 1;$$

further, for every $u \in \partial U$, the measure $\bar{\rho}(u, dx)$ is symmetric and satisfies $\bar{\rho}(u, dx) = \bar{\rho}(-u, dx)$.

(iii) *The measures $F_A(\cdot)$ and $\bar{F}_A(\cdot)$ are Lévy measures on* $\mathbf{R}$; *in fact,*

$$\int_{\mathbf{R}^+} \min(1, x^2)\, F_A(dx) = v(A) \quad \textit{and} \quad \int_{\mathbf{R}_0} \min(1, x^2)\, \bar{F}_A(dx) = \bar{v}(A),$$

for every $A \in \mathscr{B}(\partial U)$; further, $\bar{F}_A(\cdot)$'s and $\bar{F}$ are symmetric.

(iv) *For every $C \in \mathscr{B}(\partial U \times \mathbf{R}_0)$*

$$\bar{F}(C) = 2^{-1}[F(C \cap (\partial U \times \mathbf{R}^+)) + F(-C \cap (\partial U \times \cdot \mathbf{R}^+))].$$

Now we are ready to state our result providing the useful representation, similar to (4.1), of an arbitrary Lévy measure on l_2. The proof of this follows using the above lemma, (4.4), (4.6), (4.7), (4.10), the standard limiting arguments (e.g. [1] p. 104) and the change of variable formula. We omit the proof for brevity.

Proposition 4.2. (a) *Let M be a Lévy measure on l_2; then F is a unique measure on $\mathscr{B}(\partial U \times \mathbf{R}^+)$ satisfying*

$$M = F \circ \Psi^{-1}; \tag{4.11}$$

(*hence, from* (4.7) *and* (4.11)) *we have the desired representation of M: for every $D \in \mathscr{B}(l_2 \setminus \{0\})$,*

$$M(D) = \int_{\partial U} \Big(\int_{\mathbf{R}^+} I_D(xu)\, \rho(u, dx)\Big)\, \nu(du); \tag{4.12}$$

more generally,

$$\int_{l_2 \setminus \{0\}} f dM = \int_{\partial U} \Big(\int_{\mathbf{R}^+} f(xu)\, \rho(u, dx)\Big)\, \nu(du), \tag{4.13}$$

whenever either $f \geqq 0$ or $\int_{l_2 \setminus \{0\}} |f|\, dM$ is finite, in the second case f can be complex.

(b) *If M is symmetric, then $\bar{F}$ is the unique symmetric measure on $\mathscr{B}(\partial U \times \mathbf{R}_0)$ satisfying*

$$M = \bar{F} \circ \bar{\Psi}^{-1},$$

where $\bar{\Psi}$ is the natural extension of Ψ to $\partial U \times \mathbf{R}_0$; and, in addition to (4.12), *M also admits the representation:*

$$M(D) = \int_{\partial U} \Big(\int_{\mathbf{R}^+} I_D(xu)\, \bar{\rho}(u, dx)\Big)\, \bar{\nu}(du), \tag{4.14}$$

for every $D \in \mathscr{B}(l_2 \setminus \{0\})$; and the analog of (4.13) *also holds.*

We point out here that our polar decompositions of M obtained in the above theorem enjoys similar properties as the decomposition of the stable Lévy measure due to Lévy and Kuelbs as noted in Sect. 1; namely, the measure $\rho(u, \cdot)$, $\bar{\rho}(u, \cdot)$, $F_A(\cdot)$, $\bar{F}_A(\cdot)$ inherit properties of M. We address this point in Proposition 4.4 for three important classes of Lévy measures. This property of our polar decomposition, as noted in the introductory remarks of this section, is very important for us while defining the *right* **ID** r. measures for our spectral representations for **ID** processes. To facilitate the presentation of Proposition 4.4, we first introduce a few more notations and then state a lemma which in needed for the proof of the proposition.

Let H denote a finite or infinite dimensional real separable Hilbert space. Then, we denote, by $\mathscr{M}_1(H)$, the set of all $\mathbf{S}(r, \alpha)$ Lévy measures on H, by $\mathscr{M}_2(H)$, the set of all $\mathbf{S}(\alpha)$ Lévy measures on H and, by $\mathscr{M}_3(H)$, the set of all **SD** Lévy measures on H. We recall that, for a given Lévy measure M on H, the following are well known:

$$M \in \mathscr{M}_1(H) \Leftrightarrow rM = r^{\frac{1}{\alpha}} \cdot M, \tag{4.15}$$

$$M \in \mathscr{M}_2(H) \Leftrightarrow tM = t^{\frac{1}{\alpha}} \cdot M, \quad \text{for all } t \in (0, 1], \tag{4.16}$$

$$M \in \mathscr{M}_3(H) \Leftrightarrow t \cdot M \leqq M, \quad \text{for all } t \in (0, 1]. \tag{4.17}$$

We also recall that if μ is an **ID** measure on H with Lévy measure M; the Lévy measures of μ^s, the s-th roots of μ, and $s\cdot\mu$, $s>0$, are, respectively, sM and $s\cdot M$. Using these facts, the continuity of the semigroup $\{\mu^s\colon s>0\}$ and standard arguments about weak convergence, one gets easily a proof of the following lemma:

Lemma 4.3. *Let M be a Lévy measure on H and T any countable dense subset of* $(0, 1]$, *then* $M\in\mathscr{M}_2(H)$ *(resp.* $M\in\mathscr{M}_3(H))\Leftrightarrow tM=t^{\frac{1}{\alpha}}\cdot M$ *(resp.* $t\cdot M\leqq M$*), for every* $t\in T$.

Proposition 4.4. *Let M be a Lévy measure on l_2; and $\rho(u,\cdot)$, $F_A(\cdot)$ and ν be the measures related to M as defined prior to Proposition 4.1. Then, for any fixed $i=1, 2, 3$, $M\in\mathscr{M}_i(H)\Leftrightarrow$ off a ν-null set, $\rho(u,\cdot)\in\mathscr{M}_i(\mathbf{R})\Leftrightarrow F_A(\cdot)\in\mathscr{M}_i(\mathbf{R})$, for all $A\in\mathscr{B}(\partial U)$.*

Proof. We outline the proof only in the case $i=1$; the other two cases can be proxed with similar methods using Lemma 4.3 and (4.16)–(4.19). Let A, B and D denote the generic elements of $\mathscr{B}(\partial U)$, $\mathscr{B}(R^+)$ and $\mathscr{B}(l_2\setminus\{0\})$, respectively. Observe, form (4.12), that for any $a>0$

$$a\cdot M(D)=\int_{\partial U}\Big(\int_{\mathbf{R}^+} I_D(xu)\,a\cdot\rho(u,dx)\Big)\,\nu(du); \tag{4.18}$$

and, if $D=\Psi(A\times B)$, we get, from (4.18), that

$$a\cdot M(D)=a\cdot F_A(B) \quad\text{and}\quad aM(D)=aF_A(B). \tag{4.19}$$

Now let $M\in\mathscr{M}_1(H)$. Then, by (4.15), $rM=r^{\frac{1}{\alpha}}\cdot M$. Therefore, by (4.19), $rF_A(\cdot)=r^{\frac{1}{\alpha}}\cdot F_A(\cdot)$, for all A; showing $F_A(\cdot)\in\mathscr{M}_1(\mathbf{R})$. Now let $F_A(\cdot)\in\mathscr{M}_1(R)$, for all A; then, from (4.19) again, $r\rho(B)=r^{\frac{1}{\alpha}}\cdot\rho(u,B)$ a.e. $[\nu]$, for every fixed B. But, as $\mathscr{B}(\mathbf{R})$ is countably generated, $r\rho(u,dx)=r^{\frac{1}{\alpha}}\cdot\rho(u,dx)$, of a ν-null set. Showing $\rho(u,\cdot)\in\mathscr{M}_1(\mathbf{R})$, off a ν-null set. Finally, if $\rho(u,\cdot\in\mathscr{M}_1(\mathbf{R})$, off a ν-null set, we have, from (4.18), that $rM=R^{\frac{1}{\alpha}}\cdot M$ or that $M\in\mathscr{M}_1(l_2)$. □

Remark 4.5. If M is symmetric, then exactly the same proofs as above show: For every fixed $i=1, 2, 3$, $M\in\mathscr{M}_i(H)\Leftrightarrow\bar{\rho}(u,\cdot)\in\mathscr{M}_i(\mathbf{R})$, off a $\bar{\nu}$-null set$\Leftrightarrow\bar{F}_A(\cdot)\in\mathscr{M}_i(\mathbf{R})$, for all A. (Here one uses (4.14) instead of (4.12).)

Now we prepare to state and prove our first main result of this section; namely, the spectral representations of general discrete **ID** processes. To obtain these representations, the first important step is to construct a right **ID** r. measure for a given **ID** process. Let $X=\{X_n;\ n=1, 2, \ldots\}$ be an **ID** process; let $b_n>0$ be such that $Y=\{b_n X_n\}\in l_2$ almost surely. Let $\mu=\mathscr{L}(Y)$ be the **ID** law of Y on l_2 with Lévy representation: $\mu\sim[z_0, \mathscr{K}, M]$ where $z_0\in l_2$, $\mathscr{K}$ is the covariance operator and M is the Lévy measure of μ (recall that we always choose our centering function to be τ given in (1.5)). Now, for every $y \in l_2$,

$$\mathscr{K}(y) = \sum_j \beta_j \langle e_j, y \rangle e_j,$$

where $\beta_j \geqq 0$, $\sum_j \beta_j < \infty$ and $\{e_j\}$ is an orthonormal set in l_2.

Define two finite measures on $\mathscr{B}(\partial U)$ by

$$\nu_0 = \|z_0\| \, \delta_{\left\{\frac{z_0}{\|z_0\|}\right\}}, \quad \text{if } z_0 \neq 0, \ =0, \text{ if } z_0 = 0; \quad \text{and} \quad \nu_1 = \sum_j \beta_j \delta_{\{e_j\}}; \tag{4.20}$$

and recall the measures ν and $F.$ (associated to M and defined prior to Lemma 4.1). Now we make the following definition.

Definition 4.6. Let X, ν_0, ν_1 and $F.$ be as above, then the **ID** r. measure on $\mathscr{B}(\partial U)$ with parameters $(\nu_0, \nu_1, F.)$ will be refered to as the associated **ID** r. measure of X (see Proposition 2.1; and note that from Lemma 4(iii) $\nu(A) = \int_{\mathbf{R}^+} \min(1, x^2)\, dF_A(z)$; hence, the control measure λ of Λ is equal to $\nu_0 + \nu_1 + \nu$).

Using this r. measure Λ, we shall obtain the spectral representation of X which meets both criteria (i) and (ii) of the Introduction. Before we can state and prove our representation theorem, however, we will need two lemmas. In the first lemma, we record three integral identities; the proofs of the first two are straightforward and the proof of the last is a direct consequence of (4.13). In the following lemmas and the theorem, we will use above notations and conventions; in addition, we will denote, by π_n, the nth co-ordinate projection in l_2.

Lemma 4.7. *Let $a_1, a_2, \ldots, a_n$ be n-real numbers, then*

$$\int_{\partial U} \left(\sum_{j=1}^n a_j \pi_j(z) \right) \nu_0(dz) = \sum_{j=1}^n a_j \pi_j(z_0), \tag{4.21}$$

$$\int_{\partial U} \left(\sum_{j=1}^n a_j \pi_j(z) \right)^2 \nu_1(dz) = \sum_k \beta_k \left(\sum_{j=1}^n a_j \pi_j(e_k) \right)^2 = \langle \mathscr{K} y, y \rangle, \tag{4.22}$$

where $y = (a_1, \ldots, a_n, 0, 0, \ldots)$ and

$$\int_{\partial U} \left(\int_{\mathbf{R}^+} \min\{1, \pi_n^2(u) x^2\} \rho(u, dx) \right) \nu(du) = \int_{l_2} \min\{1, \pi_n^2(z)\} M(dz). \tag{4.23}$$

Lemma 4.8. *Let $Z \equiv \{Z_n\}$ be an* **ID** *process with almost all sample paths in l_2. Then $\gamma \equiv \mathscr{L}(Z)$ is an* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$*;* **SD***) prob. measure, if Z is an* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$*;* **SD***) process. Further, if Z is centered* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$*) process then γ is a centered* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$*) prob. measure.*

Proof. A proof of the last part in the centered $\mathbf{S}(r, \alpha)$ case is provided in [20]. Similar proof works in the other cases. We outline the proof in the **SD** case.

Denote by $\pi_{1,\ldots,n}$ the natural projection from l_2 onto $\mathbf{R}^n$; and let $0<a<1$ be fixed. First observe $\gamma\circ\pi_{1,\ldots,n}^{-1}=\mathscr{L}(Z_1,\ldots,Z_n)$ and $(a\cdot\gamma)\circ\pi_{1,\ldots,n}^{-1}=a\cdot(\gamma\circ\pi_{1,\ldots,n}^{-1})$ $=a\cdot\mathscr{L}(Z_1,\ldots,Z_n)$. Hence as Z is a **SD** process, there exists a unique prob. measure γ_n on $\mathbf{R}^n$ (recall (1.3)) satisfying

$$\gamma\circ\pi_{1,\ldots,n}^{-1}=(a\cdot\gamma)\circ\pi_{1,\ldots,n}^{-1}*\gamma_n. \tag{4.24}$$

Now, using Kolmogorov's extension theorem, we construct a unique prob. measure γ_0 on $\mathbf{R}^\infty$ with $\gamma_0\circ\pi_{1,\ldots,n}^{-1}=\gamma_n$. Using (4.24) and viewing the measures γ and $a\cdot\gamma$ on $\mathbf{R}^\infty$ and using ch. functions, we find

$$\gamma=a\cdot\gamma*\gamma_0$$

on $\mathbf{R}^\infty$. But, then $1=\gamma_0(l_2)=\int_{l_2}\gamma_0(l_2+x)\,a\cdot\gamma(dx)$; hence $\gamma_0(l_2)=1$. □

Theorem 4.9. *Let $X=\{X_n\}$ be an* **ID** *process and let Λ be its associated* **ID** *r. measure with parameters $(v_0, v_1, F.)$ and control measure λ (see Definition 4.6). Let $f_n=b_n^{-1}\pi_n$; then f_n's are Λ-integrable (equivalently, f_n's belong to $L_{\Phi_0}(\partial U;\lambda)$) and*

$$\{X_n\}\stackrel{d}{=}\{\int_{\partial U} f_n\,d\Lambda\}. \tag{4.25}$$

Further, if X is an $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r,\alpha)$; **SD**) *process, then Λ is an* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r,\alpha)$; **SD**) *r. measure.*

Proof. First we show that π_n's are Λ-integrable (which will trivially imply the Λ-integrability of f_n's). To prove this, we must verify (i)–(iii) of Theorem (2.3). But, in view of (4.22) and (4.23), and the fact that

$$\int_{l_2}\min\{1,\pi_n(z)^2\}\,M(dz)\leqq\int_{l_2}\min\{1,\|z\|^2\}\,M(dz)<\infty,$$

we need only to verify (i). Thus, in view of (4.21), we need to verify that

$$\int_{\partial U}\Big(\int_{\mathbf{R}^+}[\tau(\pi_n(u)\,x)-\pi_n(u)\,\tau(x)]\,\rho(u,dx)\Big)\,v(du)$$

is finite. But this follows since the absolute value of the integrand is no more that $(1+|\pi_n(u)|)\max\{1,\pi_n^2(u)\}$ and since $|\pi_n(u)|\leqq 1$ and v is finite.

Now, recalling that $X_n=b_n^{-1}Y_n$, in order to prove (4.25), it is sufficient (in fact, is equivalent) to prove that $\{Y_n\}\stackrel{d}{=}\{\int_{\partial U}\pi_n\,d\Lambda\}$. To prove this we must show

$$\mathscr{L}\Big(\sum_{j=1}^k a_j Y_j\Big)(1)=\mathscr{L}\Big(\sum_{j=1}^k a_j\int_{\partial U}\pi_j\,d\Lambda\Big)(1), \tag{4.26}$$

for every fixed k, and $a_1, \ldots, a_k$ real. Now the left side of (4.26)

$$
\begin{aligned}
&= E\exp\left(i\sum_{j=1}^{k} a_j Y_j\right) = \int_{l_2} \exp\left(i\sum_{j=1}^{k} a_j \pi_j(z)\right) d\mu = \int_{l_2} e^{i\langle z, y\rangle}\, d\mu \\
&= \exp\{i\langle z_0, y\rangle - \tfrac{1}{2}\langle \mathscr{K} y, y\rangle + \int_{l_2} (e^{i\langle z,y\rangle} - 1 - i\langle \tau(z), y\rangle)\, dM\},
\end{aligned}
$$

where $y = (a_1, \ldots, a_k, 0, 0, \ldots)$; and the right side of (4.26), by (2.5),

$$
\begin{aligned}
&= \exp\left\{ i\int_{\partial U} \left(\sum_{j=1}^{k} a_j \pi_j(u)\right) \nu_0(du) - \tfrac{1}{2}\int_{\partial U} \left(\sum_{j=1}^{k} a_j \pi_j(u)\right)^2 \nu_1(du) \right. \\
&\quad \left. + \int_{\partial U} \left[\int_{\mathbf{R}^+} \left(e^{ix\left(\sum_{j=1}^{k} a_j \pi_j(u)\right)} - 1 - i\left(\sum_{j=1}^{k} a_j \pi_j(u)\, \tau(x)\right)\right) \rho(u, dx)\right] \nu(du) \right\}.
\end{aligned}
$$

Thus, recalling (4.21) and (4.22), we need only to verify that

$$
\begin{aligned}
&\int_{l_2} (e^{i\langle z,y\rangle} - 1 - i\langle \tau(z), y\rangle)\, dM \\
&\quad = \int_{\partial U} \left[\int_{\mathbf{R}^+} \left(e^{ix\left(\sum_{j=1}^{k} a_j \pi_j(u)\right)} - 1 - i\left(\sum_{j=1}^{k} a_j \pi_j(u)\right) \tau(x)\right) \rho(u, dx)\right] \nu(du).
\end{aligned}
$$

But, from (4.13), the left side of this equation

$$
\begin{aligned}
&= \int_{\partial U} \left[\int_{\mathbf{R}^+} (e^{i\langle xu, y\rangle} - 1 - i\langle \tau(xu), y\rangle)\, \rho(u, dx)\right] \nu(du) \\
&= \int_{\partial U} \left[\int_{\mathbf{R}^+} \left(e^{ix\sum_{j=1}^{k} a_j \pi_j(u)} - 1 - i\left(\sum_{j=1}^{k} a_j \pi_j(u)\right) \tau(x)\right) \rho(u, dx)\right] \nu(du),
\end{aligned}
$$

since $\tau(xu) = xu$, if $0 < \|xu\| = x \leqq 1$, $= u$, if $x > 1$, which completes the proof of $\{Y_n\} \stackrel{d}{=} \{\int_{\partial U} \pi_n\, d\Lambda\}$. The last part of the theorem follows immediately from Lemma 4.8 and Proposition 4.4. □

The above theorem yields all known spectral representations for *discrete* parameter stable and semistable processes [2, 7, 13, 20, 27, 28] without having to center or to symmetrize the process; this, in addition, clearly also yields similar spectral representations for **SD** processes. Unlike the discrete case, our methods, unfortunately, do not allow us to obtain spectral representations for arbitrary continuous parameter **ID** processes. However, if the process satisfies some additional conditions then, using Theorem 3.4, we can indeed obtain spectral representations for such a process. These, besides providing spectral representations for new classes of **ID** processes, also yield, in a unified way, all previously known spectral representations for stable and semistable processes. We address these points in the remaining of this section. We begin with some preliminaries which are needed to define associated **ID** r. measures for continuous parameter **ID** processes.

Let $q \geqq 0$ be fixed, and let T be an arbitrary index set. Let $X=\{X_t\colon t\in T\}$ be an **ID** process which satisfies the condition

$$(mc)_q \quad E|X_t|^q<\infty, \qquad \text{for all } t\in T,$$

and which is $L_q(\equiv L_q(\Omega;P))$-separable (i.e., there exists a countable subset $T_0=\{t_n\}$ of T such that, for every $t\in T$, there is a sequence $\{s_m\}\subseteqq T_0$ with $X_{s_m}\to X_t$ in L_q). Recall that if T is a separable metric space and X is L_q-continuous than X is separable in L_q. (Note that if $q=0$, then $(mc)_q$ is vacously satisfied; hence, in this case, it imposes no restriction on X.) If $q=0$, choose $b_n>0$, as prior to Definition 4.1, such that $Y\in l_2$ a.s., where, as before, $Y_n=b_n X_n$ and $X_n=X_{t_n}$, $t_n\in T_0$, for every n. If $q>0$, then we choose $b_n>0$ satisfying, additionally, $E\left(\sum_{n=1}^{\infty} Y_n^2\right)^{q/2}<\infty$. Such a choice of b_n's is always possible; this can be shown, for instance, using the following inequalities:

$$\left(\sum_{n=1}^{\infty}|b_n X_n|^2\right)^{\frac{q}{2}} \leqq \sum_{n=1}^{\infty}\{|b_n X_n|^2\}^{\frac{q}{2}}=\sum b_n^q|X_n|^q,$$

if $0<q\leqq 2$, and if $q>2$, then

$$\begin{aligned}\left(\sum_{n=1}^{\infty}|b_n X_n|^2\right)^{\frac{q}{2}}&=\left(\sum_{n=1}^{\infty}2^{-n}|b_n 2^{\frac{n}{2}} X_n|^2\right)^{\frac{q}{2}}\\ &\leqq \sum_{n=1}^{\infty}2^{-n} b_n^q 2^{q\frac{n}{2}}|X_n|^q=\sum_{n=1}^{\infty}2^{(\frac{q}{2}-1)n} b_n^q|X_n|^q.\end{aligned}$$

As before, let $\mu=\mu_q=\mathscr{L}(Y)$ be the **ID** measure on l_2; denote its convariance operator and Lévy measure by $\mathscr{K}=\mathscr{K}_q$ and $M=M_q$, respectively. If $q>0$, then our choice of b_n's guarantees

$$\int_{l_2}\|z\|^q d\mu<\infty; \qquad \text{hence} \quad \int_{\{\|z\|\geqq 1\}}\|z\|^q dM<\infty.$$

Thus, we have

$$\begin{aligned}\infty &> \int_{\{\|z\|>1\}}\|z\|^q dM\\ &=\int_{\partial U}\int_{\mathbf{R}^+}|x|^q I(|x|>1)\,\rho(u,dx)\,v(du)\geqq \int_{\mathbf{R}^+}|x|^q I(|x|>1)\,F_A(dx),\end{aligned}\tag{4.27}$$

for every $A\in\mathscr{B}(\partial U)$. If X symmetric, then μ (and hence M) is symmetric; in this case the measures $\bar{F}.$ (see Proposition 4.2) are symmetric. Using these preliminaries and notations, we shall now define suitable associated **ID** r. measures for the following three classes of **ID** processes; then we shall state and prove our spectral representations for these classes of processes.

Let $q \geqq 0$ and let $X = \{X_t \colon t \in T\}$ be an L_q-separable **ID** process satisfying $(mc)_q$; we consider the processes which satisfy any one of the following assumptions: (A_1) X symmetric and $q \geqq 0$ arbitrary; (A_2) X arbitrary (as above) $q \geqq 1$ and $E(X_t) = 0$, for all t; and (A_3) X is centered $\mathbf{S}(\alpha)$ or centered $\mathbf{S}(r, \alpha)$ $0 < \alpha < 2$ (so that, in this case, $0 < q < \alpha$).

Definition 4.10. If X satisfies (A_1) (resp. (A_2)) then the r. measure Λ with parameters $(0, \nu_1, \bar{F}.)$ (resp. $(\nu_0, \nu_1, F.)$) will be called the associated **ID** r. measure of the process X satisfying (A_1) (resp. (A_2)), where ν_1 is the measure defined in (4.20) for the covariance $\mathscr{K}$, and ν_0 is as in (3.9) (note that (4.27) is needed here). If X satisfies (A_3) and X is strictly $\mathbf{S}(\alpha)$ process with $1 < \alpha < 2$ and $1 < q < \alpha$, then the r. measure with parameters $(\nu_0, 0, F.)$ will be called the associated **ID** r. measure of X; finally, if X is strictly $\mathbf{S}(\alpha)$ process with $0 < \alpha < 1$ and $0 < q < \alpha$, then the **ID** r. measure with parameters $(\nu_0', 0, F.)$ will be called the associated **ID** r. measure of X, where ν_0' is given by (3.11). Note that in the last two definitions, in order to define ν_0 (resp. ν_0') one must have that $\int_{\{|x| \geqq 1\}} |x| \, dF_A < \infty$ (resp. $\int_{\{|x| \leqq 1\}} |x| \, dF_A < \infty$), for all $A \in \partial U$. That this condition is indeed satisfied follows from the fact that $F.$ is a $\mathbf{S}(\alpha)$ Lévy measure with index $1 < \alpha < 2$ (resp. $0 < \alpha < 1$); (see Proposition 4.2 and Lemma 4.8). The associated **ID** r. measure when X is a strictly $\mathbf{S}(r, \alpha)$ process is defined in an analogous way. Finally, note that, by (4.27), the associated r. measure λ satisfies $(MC)_q$, provided the process X satisfies $(mc)_q$.

Theorem 4.11. *Let $q \geqq 0$ and $X = \{X_t \colon t \in T\}$ be an L_q-separable* **ID** *process satisfying any one of (A_1)–(A_3) assumptions and let Λ be the corresponding associated* **ID** *r. measure with control measure λ. Then, there exist $f_t \in L_{\Phi_q}(\partial U, \lambda)$, $t \in T$, such that*

$$X \stackrel{d}{=} \{ \int_{\partial U} f_t \, d\Lambda \colon t \in T\}; \tag{4.28}$$

and that the map

$$L_q(\Omega; P) \ni \sum_{j=1}^{k} a_j X_{t_j} \mapsto \sum_{j=1}^{k} a_j f_{t_j} \in L_{\Phi_q}(\partial U, \lambda) \tag{4.29}$$

extends to a linear topological isomorphism from the L_q-closure of the span of $\{X_t \colon t \in T\}$ onto the closure of the span of $\{f_t \colon t \in T\}$ in the space $L_{\Phi_q}(\partial U, \lambda)$. Further, under the assumption (A_1) or (A_2), if X is a $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$; **SD**) *process, then Λ is a* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$; **SD**) *r. measure. Finally, under the asumption (A_3), if X is a strictly* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$) *$0 < \alpha < 2$, $\alpha \neq 1$, then Λ is a strictly* $\mathbf{S}(\alpha)$ *(resp.* $\mathbf{S}(r, \alpha)$) *r. measure.*

Proof. The proofs of (4.28) under any one of three assumptions are similar and use Propositions 3.6 and 3.8, the methods of proof the Theorem 4.9, and the L_q-separability of X. To exhibit the ideas of the proof, we outline the proof only under the assumption (A_2). (See Definition 4.10, and notations introduced prior to it.) Also recall the definition of π_n from Lemma 4.9.

The fact that π_n's are Λ-integrable is exactly the same as in Theorem 4.9. We shall now show that

$$\{Y_n\} \stackrel{d}{=} \{\int_{\partial U} \pi_n \, d\Lambda\}. \tag{4.30}$$

Let $a_1, \ldots, a_k$ be k-fixed real numbers; then recalling that $\mathscr{L}(Y)=\mu$ and

$$\int_{l_2} e^{i\langle z,y\rangle} d\mu = \exp\{-\tfrac{1}{2}\langle \mathscr{K} y, y\rangle + \int_{l_2} \{e^{i\langle z,y\rangle} - 1 - i\langle z,y\rangle\} \, dM\},$$

for every $y \in l_2$; and using (4.22) and (4.13), we have

$$\begin{aligned}
&\hat{\mathscr{L}}(\sum_{j=1} a_j Y_j)(1) \\
&= \exp\left[-\tfrac{1}{2}\int_{\partial U} g^2(u)\, v_1(du) + \int_{l_2} \left\{e^{i\sum_{j=1}^{k} a_j \pi_j(z)} - 1 - i\sum_{j=1}^{k} a_j \pi_j(z)\right\} dM\right] \\
&= \exp[-\tfrac{1}{2}\int_{\partial U} g^2(u)\, v_1(du) + \int_{\partial U} \{\int_{\mathbf{R}^+} (e^{ixg(u)} - 1 - ixg(u))\, \rho(u,dx)\}\, v(du)],
\end{aligned}$$

where $g = \sum_{j=1}^{k} a_j \pi_j$. On the other hand, by (2.5),

$$\begin{aligned}
\hat{\mathscr{L}}(\int_{\partial U} g(s)\, d\Lambda)(1) &= \exp[\int_{\partial U} g(u)\, v_0(du) - \tfrac{1}{2}\int_{\partial U} g^2(u)\, v_1(du) \\
&\quad + \int_{\partial U} \{\int_{\mathbf{R}^+} (e^{ixg(u)} - 1 - ig(u)\,\tau(x))\, \rho(u,dx)\}\, v(du)].
\end{aligned} \tag{4.31}$$

The first and last integral on the right side of (4.31) can be combined to see that $\hat{\mathscr{L}}(\int_{\partial U} g(u)\, d\Lambda)(1)$ is equal to $\hat{\mathscr{L}}\left(\sum_{j=1}^{k} a_j Y_j(\cdot)\right)(1)$; proving (4.30). Now recalling that $Y_n = b_n^{-1} X_{t_n}$, we see, from (4.30), that

$$\{X_{t_n}\colon t_n \in T_0\} \stackrel{d}{=} \{\int_{\partial U} f_{t_n}\, d\Lambda\colon t_n \in T_0\},$$

where $f_{t_n} = b_n^{-1}\pi_n$. Now by (4.27) and by the definition of Λ, $E(\Lambda(A))=0$ and $E|\Lambda(A)|^q < \infty$, for every $A \in \mathscr{B}(\partial U)$; hence we have, by Proposition 3.6, that the map

$$L_{\Phi_q}(\partial U, \lambda) \ni f \mapsto \int_{\partial U} f d\Lambda \in L_q(\Omega, P)$$

is an isomorphism. Let $t \in T$; choose a sequence $\{s_m\} \subseteq T_0$ such that $X_{s_m} \to X_t$ in L_q. It follows that $\{\int_{\partial U} f_{s_m} d\Lambda\}$ converges in L_q; hence, from Proposition 3.7, we have that there exists an f_t in $L_{\Phi_q}(\partial U, \lambda)$ and that $\int_{\partial U} f_{s_m}\, d\Lambda \to \int_{\partial U} f_t\, d\Lambda$ in L_q. Now, in order to prove (4.28), we must show

$$\mathscr{L}(X_{l_1}, \ldots\ldots, X_{l_k}) = \mathscr{L}\big(\int_{\partial U} f_{l_1}\, d\Lambda, \ldots, \int_{\partial U} f_{l_k}\, d\Lambda\big)$$

for any fixed $l_1, \ldots, l_k \in T$. But this follows from the usual limiting arguments.

The proof of the last two assertions of the theorem follows easily using the construction of Λ and Proposition 4.4, Remark 4.5 and Lemma 4.8. Finally, the proof of the fact that the map (4.29) extends to a linear topological isomorphism, under any one of the three assumptions, is immediate using Theorem 3.4 and Propositions 3.5 and 3.6 (note that under (A_2), we have already noted that Λ satisfies $(IC)_q$; under (A_1), obviously Λ is symmetric, and, under (A_3), Λ is either strictly $\mathbf{S}(\alpha)$ or $\mathbf{S}(r, \alpha)$ r. measure and $0<\alpha<2$, $\alpha \neq 1$). □

Remark 4.12. (a) As noted in the introductory remarks, the above theorem obviously yields the known spectral representations for stable and semistable representations [2, 7, 13, 20, 27, 28]. For emphasis, we also note again that our r. measure Λ and the functions f_t's, in the above theorem, meet the criteria (i) and (ii) of the Introduction, respectively.

(b) As noted in the introductory remarks of this section, we have obtained the spectral representations in Theorems 4.9 and 4.11 on the space ∂U for simplicity and convenience of notations. However, the space ∂U can be replaced by any other uncountable Borel subset S of a complete separable metric space; we outline this for Theorem 4.9, a similar procedure applies in the case of Theorem 4.11. Let ϕ be a Borel isomorphism from ∂U onto S; and recall the hypotheses and notations used in Theorem 4.9. Set $\tilde{v}_0 = v_0 \circ \phi^{-1}$, $\tilde{v}_1 = v_1 \circ \phi^{-1}$, $\tilde{v} = v \circ \phi^{-1}$ and $\tilde{F}_A(B) = \int_A \int_B \tilde{\rho}(s, dx)\, \tilde{v}(ds)$, where $\tilde{\rho}(s, dx) = \rho(\phi^{-1}(s), dx)$. Let $\tilde{\Lambda}$ be the **ID** r. measure on $(S, \mathscr{B}(S))$ with parameters $(\tilde{v}_0, \tilde{v}_1, \tilde{F}.)$ and let $g_n = (b_n^{-1}\pi_n) \circ \phi^{-1}$; then it follows using Theorem 4.9, that $\{X_n\} \stackrel{d}{=} \{\int_S g_n\, d\tilde{\Lambda}\}$. In particular, one may take $S = [0, 1]$ and replace $\tilde{\Lambda}$ by a process with independent increments.

V. Refinement of Spectral Representations in Distribution to Spectral Representations which Hold Almost Surely

In this section, we shall show that the spectral representations of stochastic processes obtained in the previous section can be modified so that the new representations hold almost surely. This, however, requires that the processes be redefined on a slightly larger prob. space. The possibility of such a refinement, by making use of the randomization lemma (Lemma 1.1 [12]), was suggested to us by O. Kallenberg. It is a great pleasure for both of us to thank Prof. Kallenberg for this suggestion. For our purposes, we shall need a slight generalization of the randomization lemma, which can be proven essentially by the same argument as Lemma 1.1 [12]. We omit this proof.

Lemma 5.1. *Let ξ and η' be random elements defined on the prob. spaces (Ω, P) and (Ω', P'), and taking values in the spaces S and T, respectively, where S is a separable metric space and T is Polish space. Assume that $\xi \overset{d}{=} f(\eta')$ for some Borel measurable function f: $T \to S$. Then there exists a random element $\eta \overset{d}{=} \eta'$ on the ("randomized") prob. space $(\Omega \times [0, 1], P \times Leb)$ such that $\beta = f(\eta)$ a.s. $P \times Leb$.*

Theorem 5.2. *Let $\{X_t\colon t \in T\}$ be an* **ID** *stochastic process defined on a prob. space (Ω, P). Assume*

$$\{X_t\colon\ t \in T\} \overset{d}{=} \{\int_S f_t \, d\Lambda'\colon\ t \in T\},$$

where Λ' is an **ID** *r. measure defined on a prob. space (Ω', P') and S is a Borel subset of a Polish space. Then there exists an* **ID** *r. measure Λ defined on the prob space $(\Omega \times [0, 1], P \times Leb)$ such that*

$$\{\Lambda(A)\colon\ A \in \mathscr{S}\} \overset{d}{=} \{\Lambda'(A)\colon\ A \in \mathscr{S}\}$$

(here $\mathscr{S}$ is the Borel σ-algebra of S) and

$$X_t = \int_S f_t \, d\Lambda \qquad \text{a.s. } P \times Leb,$$

for every $t \in T$.

Proof. We have that $f_t \in L_{\Phi_0}(S; \lambda)$ for every $t \in T$, where λ is the control measure of Λ'. Since $\mathscr{S}$ is countably generated, L_{Φ_0} is separable. Hence there exists a set $T_0 = \{t_n\}_{n=1}^{\infty} \subset T$ such that $\{f_{t_n}\}_{n=1}^{\infty}$ is dense in $\{f_t\}_{t \in T} \subset L_{\Phi_0}$. Define ξ: $\Omega \to \mathbf{R}^{\infty}$ by

$$\xi(\omega) = (X_{t_1}(\omega), X_{t_2}(\omega), \ldots).$$

Choose $\mathscr{S}_0 = \{A_j\}_{j=1}^{\infty}$ to be a countable algebra of sets such $\mathscr{S}_0 \subset \mathscr{S}$ and $\sigma(\mathscr{S}_0) = \mathscr{S}$. Define η': $\Omega \mapsto \mathbf{R}^{\infty}$ by

$$\eta'(\omega') = (\Lambda'(A_1)(\omega'), \Lambda'(A_2)(\omega'), \ldots).$$

Since, for every $f \in L_{\Phi_0}$, there exists a sequence $\{g_k\}$ of simple $\mathscr{S}_0$-measurable functions such that $g_k \to f$ in L_{Φ_0}, we get, by Theorem 3.3, that $\int_S g_k \, d\Lambda' \to \int_S f d\Lambda'$ in prob. as $k \to \infty$. In particular, $\int_S f d\Lambda'$ is equal a.s. $[P']$ to some $\sigma\{\Lambda'(A_j)\colon j \geqq 1\} = \sigma(\eta')$-measurable r. variable. Consequently, for every n, there exists a Borel function $\varphi_n : \mathbf{R}^{\infty} \to \mathbf{R}$ such that

$$\int_S f_{t_n} \, d\Lambda' = \varphi_n(\eta') \qquad \text{a.s. } [P']. \tag{5.1}$$

Then, by the assumption of our theorem, $\{X_{t_n}: n \geqq 1\} \stackrel{d}{=} \{\varphi_n(\eta'): n \geqq 1\}$ or $\xi \stackrel{d}{=} \Phi(\eta')$, where $\Phi: \mathbf{R}^\infty \to \mathbf{R}^\infty$ is the Borel function defined by $\Phi(x) = (\varphi_1(x), \varphi_2(x), \ldots)$, $x \in \mathbf{R}^\infty$. In view of Lemma 5.1, there exists an $\mathbf{R}^\infty$-valued r. element η defined on $(\Omega \times [0, 1], P \times Leb.)$ such that $\eta \stackrel{d}{=} \eta'$ and $\xi = \Phi(\eta)$ a.s. $P \times Leb.$ Put $\Lambda(A_j) = \eta_j$, $A_j \in \mathscr{S}_0$. Since η' is the restriction of the r. measure Λ' to the algebra $\mathscr{S}_0$ and $\eta \stackrel{d}{=} \eta'$, there exists a unique (modulo $P \times Leb.$) extension of Λ to a r. measure on $\sigma(\mathscr{S}_0) = \mathscr{S}$ such that

$$\{\Lambda(A): A \in \mathscr{S}\} \stackrel{d}{=} \{\Lambda'(A): A \in \mathscr{S}\}. \tag{5.2}$$

By (5.1), we get

$$\varphi_n(\eta) = \int_S f_{t_n} \, d\Lambda \qquad \text{a.s. } P \times Leb;$$

which yields

$$X_{t_n} = \int_S f_{t_n} \, d\Lambda \qquad \text{a.s. } P \times Leb, \tag{5.3}$$

for every $n \geqq 1$.

Let now $t \in T$ be arbitrary. We can choose a sequence $\{t_{n(k)}\}_{k=1}^\infty \subset T_0$ such that $f_{t_{n(k)}} \to f_t$ in L_{Φ_0}. By (5.2) and the assumption of our theorem,

$$(X_{t_{n(k)}}, X_t) \stackrel{d}{=} \left(\int_S f_{t_{n(k)}} \, d\Lambda, \int_S f_t \, d\Lambda\right).$$

Since $\int_S f_{t_{n(k)}} \, d\Lambda \to \int_S f_t \, d\Lambda$ in $P \times Leb.$ as $k \to \infty$, we get that $X_{t_{n(k)}} \to X_t$ in $P \times Leb$ as $k \to \infty$. By (5.3), $X_t = \int_S f_t \, d\Lambda$ a.s. $P \times Leb.$ □

Remark 5.3. In the above proof, the fact that the r. measure Λ is **ID** or even independently scattered is not important. In fact, similar methods can be used to prove a version of Theorem 5.2, where Λ is an arbitrary random measure and $\int f d\Lambda$ is defined as a limit, in some appropriate sense, of stochastic integrals of $\mathscr{S}_0$-measurable simple functions.

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