Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift

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Contents

Introduction

The first formulations of linear boundary value problems for analytic functions were due to Riemann (1857). In particular, such problems exhibit as boundary conditions relations among values of the unknown analytic functions which have to be evaluated at *different* points of the boundary. Singular integral equations with a shift are connected with such boundary value problems in a natural way. Subsequent to Riemann's work, D. Hilbert (1905), C. Haseman (1907) and T. Carleman (1932) also considered problems of this type. About 50 years ago, Soviet mathematicians began a systematic study of these topics. The first works were carried out in Tbilisi by D. Kveselava (1946–1948). Afterwards, this theory developed further in Tbilisi as well as in other Soviet scientific centers (Rostov on Don, Kazan, Minsk, Odessa, Kishinev, Dushanbe, Novosibirsk, Baku and others). Beginning in the 1960s, some works on this subject appeared systematically in other countries, e.g., China, Poland, Germany, Vietnam and Korea. In the last decade the geography of investigations on singular integral operators with shift expanded significantly to include such countries as the USA, Portugal and Mexico. It is no longer easy to enumerate the names of the all mathematicians who made contributions to this theory. Beginning in 1957, the author also took part in these developments. Up to the present, more than 600 publications on these topics have appeared. In particular, various applications to the following theories have been developed:

Theory of the limit problems for differential equations with second order partial derivatives of mixed (elliptic-hyperbolic) type,

Theory of the cavity currents in an ideal liquid,

Theory of infinitesimal bonds of surfaces with positive curvature,

Contact theory of elasticity,

Physics of plasma,

Theory of synthesis of signals for linear systems with non-stationary parameters, and so on.

As a result, a vast amout of material has accumulated. This material can be ascribed to one of the following two main directions of investigation:

1) Noether theory of singular integral equations with shift (SIES),

2) Solvability theory of singular integral equations and boundary value problems for analytic functions with a shift.

In the first direction the following two questions are considered:

a) Find a Noetherity criterion for a singular integral operator with shift in terms of the invertibility of its symbol.

b) Calculate the index of a Noether operator through the Cauchy index of its symbol and perhaps through other topological characteristics of the boundary condition.

The second direction includes the calculation of defect numbers of an operator, the construction of bases for defect subspaces, the problems of spectral theory, the determination of exact or approximate solutions for the corresponding equations, and boundary value problems.

All of these developments, in both directions, needed to be gathered and synthesized into a book that would give the topic consistency and cohesion. The first step in this effort was the monograph by G. Litvinchuk "Boundary value problems and singular integral equations with shift", Publishing house "Nauka", Moscow, 1977. First published in Russian, this book became a bibliographical rarity some years ago, even though it had been printed in an edition of 7000 copies, a large number for a monograph in advanced mathematics. Five years after original publication, the book was translated into Chinese and published in Beijing in 1982. However, the book still remained inaccessible for readers not having knowledge of Russian or Chinese. During the 20 years following publication of this book, no less than 350 papers on this topic appeared, mainly by Soviet and former Soviet mathematicians.

During the 1970s–1980s, the first direction (the Noether theory) developed the most intensively. This lively interest prompted V. Kravchenko and G. Litvinchuk to publish in English the monograph "Introduction to the theory of singular integral operators with shift", Kluwer Academic Publisher, Dordrecht, 1994, a book devoted to a systematic description of results of the first direction (the Noether theory of singular integral operators with a shift).

The aim of the present monograph is to provide a systematic description of the second direction (Solvability theory of singular integral equations with a shift and of their corresponding boundary value problems). It builds upon the 1977 monograph of Litvinchuk and covers developments after 1975, including solvability theory of singular integral equations with a Carleman fractional linear shift (V. Kravchenko, A. Schaev, G. Drekova, 1989-1991); solvability theory of the Hasemann boundary value problem on an open contour (A. Aizenstat, Ju. Karlovich, G. Litvinchuk, 1990, 1996); solvability theory of one class of singular integral equations with a non-Carleman shift (V. Kravchenko, A. Baturev, G. Litvinchuk, 1996); calculation of defect numbers of Riemann generalized boundary value problem based on the constructive factorization of a Hermitian matrix-function with a negative determinant (G. Litvinchuk, I. Spitkovsky, 1980-1981), the operator approach for studying boundary value problems in the class of functions analytic in the same domain (Kurtz, Latushkin, Lisovetz, Litvinchuk, Skorohod, Spitkovsky, 1984-1988, 1993); the problem of the spectrum of an operator of singular integration with a Carleman fractional linear shift (V. Kravchenko, A. Lebre, G. Litvinchuk, 1998) and others. A survey of closely related results is given, partly with proofs.

It is necessary to emphasize that, unlike the Noether theory, the solvability theory of SIES has a completed form only for the so-called binomial singular integral operators with a shift. The cause of the difficulties can be explained as follows.

Let's consider in the space $H_\mu(\Gamma)$, $0 < \mu \leq 1$ (or $L_p(\Gamma)$, $1 < p < \infty$), where Γ is a simple closed contour, a singular integral operator, that is an operator of the form

$$K = aP_+ + bP_- \quad , \quad a,\, b \in H_\mu(\Gamma)\ \ (a, b \in C(\Gamma))\ , \tag{1}$$

$P_\pm = 1/2(I \pm S)$, $(S\varphi)(t) = (\pi i)^{-1} \int_\Gamma \varphi(\tau)(\tau - t)^{-1} d\tau$ and I is the identity operator.

The Noether theory of this operator consists of the following facts:

a) The conditions $a(t) \neq 0$, $b(t) \neq 0$, $t \in \Gamma$, are sufficient and necessary for K to be a Noether operator;

b) If $a(t) \neq 0$, $b(t) \neq 0$, $t \in \Gamma$, then

$$\operatorname{ind} K = \dim\ker K - \dim\operatorname{coker} K = \frac{1}{2}\,\{\arg\,(a^{-1}b)\}_\Gamma = \kappa\ .$$

The solvability theory of operator (1) consists of the following facts:

a) $\dim\ker K = \max(0, \kappa)$, $\dim\operatorname{coker} K = \max(0, -\kappa)$;

b) if $\kappa = 0$, $\kappa > 0$ or $\kappa < 0$ there exist, and can be effectively constructed, the corresponding inverse K^{-1}, the right inverse K_r^{-1} or the left inverse K_l^{-1} operators;

c) if $\kappa > 0$ or $\kappa < 0$, bases for the subspaces ker K or coker K can be effectively constructed.

It should be noted that the main tool in constructing the solvability theory of operator (1) is factorization of the function $a^{-1}b$.

So for a general form of operator (1), we have in a complete form both the Noether theory and the solvability theory.

Now let's consider in the space $H_\mu(\Gamma)$ (or $L_p(\Gamma)$) a singular integral operator with shift

$$K = bP_+ + dWP_- \quad , \quad b,\, d \in H_\mu(\Gamma)\ \ (b, d \in C(\Gamma)) \tag{2}$$

where $(W\varphi)(t) = \varphi(\alpha(t))$, $\alpha(t)$ is a diffeomorphism (shift) of Γ onto itself which preserves the orientation on Γ, $\alpha'(t) \in H_\mu(\Gamma)$, $\alpha'(t) \neq 0$, $t \in \Gamma$. The coefficient of P_- in (2) is more complicated than in (1): it is the shift operator which is weighted by the function d. In spite of this, it turns out that the Noether and solvability theories have the same form as in the case of operator (1). The difference consists of the facts that items b) and c) of solvability theory for operator (2) cannot be solved as effectively as for operator (1): the "factorization with a shift" of the function $b^{-1}d$, in contrast to usual factorization, is expressed by means of Cauchy type integrals whose densities are solutions of some Fredholm equations with, generally speaking, non-degenerate kernels. Results of such type can also be obtained for operators of Carleman type

$$K = aP_+ + bWP_+ \quad , \quad K = cP_- + dWP_- \,, \tag{3}$$

where $(W\varphi)(t) = \varphi(\alpha(t))$, $\alpha(t)$ is a diffeomorphism of Γ onto itself changing the orientation on Γ, and the Carleman condition $\alpha(\alpha(t)) \equiv t$ (or, equivalently, $W^2 = I$) holds. So complete results for the solvability theory of operators (1) - (3) can be obtained precisely because operators (1) - (3) are binomial with *scalar* coefficients relative to the projectors P_+ e P_-. For binomial singular integral operators, the theorem of Gakhov-Coburn holds: one of the defect numbers of the binomial operator is equal to zero. If the coefficients of operators (1) - (3) are not scalar, then these operators are not binomial and the Gakhov-Coburn theorem does not appply to them.

Now we consider one of the simplest polynomial operators with shift. Let $\alpha(t)$: $\Gamma \to \Gamma$ be a Carleman shift ($\alpha(\alpha(t)) \equiv t$, $\alpha'(t) \neq 0$, $t \in \Gamma$, $\alpha'(t) \in H_\mu(\Gamma)$). Let's consider operator

$$K = (aI + bW)P_+ + (cI + dW)P_- \tag{4}$$

in $H_\mu(\Gamma)$ (or $L_p(\Gamma)$). If at least three of the four coefficients a, b, c, d are not identically zero on Γ, then operator (4) is polynomial even in the scalar case. Together with operator (4) we also consider the companion operator

$$\tilde{K} = (aI - bW)P_+ + (cI - dW)P_-$$

in $H_\mu(\Gamma)$ (or $L_p(\Gamma)$). Then the following identity of matrices of operators holds

$$\frac{1}{2}\begin{pmatrix} I & I \\ W & -W \end{pmatrix}\begin{pmatrix} K & O \\ O & \tilde{K} \end{pmatrix}\begin{pmatrix} I & W \\ I & -W \end{pmatrix} = \mathcal{A}P_+ + \mathcal{B}P_- + \mathcal{D} \tag{5}$$

where

$$\mathcal{A}(t) = \begin{pmatrix} a(t) & b(t) \\ b(\alpha(t)) & a(\alpha(t)) \end{pmatrix} \quad , \quad \mathcal{B}(t) = \begin{pmatrix} c(t) & d(t) \\ d(\alpha(t)) & c(\alpha(t)) \end{pmatrix} ,$$

if $\alpha(t) = \alpha_+(t)$ preserves the orientation on Γ, and

$$\mathcal{A}(t) = \begin{pmatrix} a(t) & d(t) \\ b(\alpha(t)) & c(\alpha(t)) \end{pmatrix} \quad , \quad \mathcal{B}(t) = \begin{pmatrix} c(t) & b(t) \\ d(\alpha(t)) & a(\alpha(t)) \end{pmatrix} ,$$

if $\alpha(t) = \alpha_-(t)$ changes the orientation on Γ.

The operator $\mathcal{D} = \{d(WSW - \gamma S)\ ,\ c(\alpha(t))(WSW - \gamma S\}$, where $\gamma = \pm 1$ if $\alpha = \alpha_\pm$, is compact because the operator $\mathcal{D}_0 = WSW - \gamma S$ is compact.

The Noether theory of operator (4) is the following:

$$\alpha = \alpha_+ : \Delta_1(t) = c(t)c(\alpha(t)) - d(t)d(\alpha(t)) \neq 0 \ ,\ \Delta_2(t) = a(t)a(\alpha(t)) - b(t)b(\alpha(t)) \neq 0,$$

$$\operatorname{ind} K = \frac{1}{4\pi}\left\{\arg\frac{\Delta_1(t)}{\Delta_2(t)}\right\}_\Gamma \ ;$$

$\alpha = \alpha_- : \Delta(t) = a(t)c(\alpha(t)) - d(t)b(\alpha(t)) \neq 0,$

$$\operatorname{ind} K = -\frac{1}{2\pi} \{\arg \Delta(t)\}_\Gamma .$$

As to the solvability theory of operator (4) in the general case, we only have the assertion

$$\dim \ker K + \dim \ker \tilde{K} = \dim \ker(\mathcal{A}P_+ + \mathcal{B}P_- + \mathcal{D})$$

which follows directly from the identity (5).

So the solvability theory of polynomial operator with shift (4) is reduced to factorization of the matrix operator without shift

$$M = \mathcal{A}P_+ + \mathcal{B}P_- + \mathcal{D}$$

which is also polynomial. So at this point three difficulties arise:

1) It is necessary to estimate the influence of the compact operator $\mathcal{D}$.

2) It is necessary, say, to separate the defect subspaces $\{\ker K\}$ and $\{\ker \tilde{K}\}$ in order to express the defect number $\dim \ker K$ by means of the partial indices of operator M and to construct a basis for $\{\ker K\}$.

3) After solving problems 1) and 2), it is necessary to estimate the signs of the partial indices κ_1 and κ_2 of the operator M and, in case the numbers κ_1 and κ_2 have different signs, to calculate κ_1 and κ_2.

Up to now not one of these three problems has a complete solution. However, there are some interesting results. All the literature relating to the solvability theory of polynomial operators can be subdivided into two groups of results. In the first group,the solvability theory is constructed by means of reducing the polynomial operator with shift (4) to a binomial one (or to a system of binomial operators), at the expense of sufficiently hard restrictions relative to the coefficients a, b, c, d. In the second group, the solvability theory of the polynomial operator (4) can be constructed for arbitrary coefficients a, b, c, d satisfying the Noether conditions, but it is possible to do it only for a Carleman fractional linear shift acting on the circle or on a straight line. In this point it is possible to overcome the difficulties 1) and 2), exactly, to eliminate the compact operator $\mathcal{D}$ and to separate the defect subspaces $\ker K$ and $\ker \tilde{K}$. After this it is possible to overcome even the difficulty 3), but only by means of some additional assumptions.

The main subject of the present monograph is devoted to the following two questions:

I) Construction of a solvability theory for binomial operators with a shift (Chapters 2, 3, 8).

II) Construction of a solvability theory for polynomial operators with a Carleman shift either by means of coefficient restrictions, reducing polynomial operators to binomial ones, or by means of restrictions relative to the Carleman shift, reducing polynomial operators with shift to the characteristic matrix operator without shift (Chapters 5, 6 and, partly, in Chapters 4, 7).

In connection with the latter construction, the reduction of polynomial boundary value problems for functions, analytic in a domain, to the corresponding singular integral equations is not trivial, particularly in a multiply connected domain. We perform this reduction and we obtain theorems about the Noetherity and the index of a boundary value problem of such type (partly in Chapters 7, 8).

At last, Chapter 9 is devoted to a quite new and very difficult question related to the solvability theory of polynomial singular integral operator of type (4) but with a non-Carleman shift. This Chapter is written in the hope of giving a stimulus to further investigations in this direction, prompting applications to problems of mathematical physics.

The author uses the through numbering of sections and the double numbering of all definitions, theorems, equations, etc. However for simplicity the double indices are used only from necessity for references to the objects introduced in other sections. For example, in any section the entry "Theorem 5.2" signifies Theorem 2 from Section 5.

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