Introduction to the Theory of Singular Integral Operators with Shift

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Introduction to the Theory of Singular Integral Operators with Shift

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0.1 Introduction

The Riemann boundary value problem is referred to as the Riemann-Hilbert problem, the Hilbert problem and also the problem of the conjugation of analytic functions. It consists of finding a function $\Phi(z)$ analytic everywhere on the complex plane except on the points of the given contour Γ, satisfying the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \ , \ t \in \Gamma \ , \tag{0.1}$$

where $\Phi^{\pm}(t)$ are the limit values of the unknown function $\Phi(z)$, and $G(t), g(t)$ are given functions.

The formulation of the problem is more general than that of problem (0.1), and belongs to Riemann (1, p. 177), and the first investigation of problem (0.1) was carried out by Hilbert (1) in 1904. Using certain additional restrictions Hilbert reduced the problem (0.1) to a Fredholm integral equation. After this (apparently on Hilbert's initiative), in 1907 Haseman (1) carried out an analogous investigation of a problem where the condition of conjugation of the limit values $\Phi^{\pm}(t)$ of the analytic function $\Phi^{\pm}(z)$ was replaced by a more general one,

$$\Phi^+(\alpha(t)) = G(t)\Phi^-(t) + g(t) \ , \ t \in \Gamma \tag{0.2}$$

where $\alpha(t)$ is a diffeomorphism (shift) preserving the orientation of a closed contour Γ onto itself. Historically, the paper by Haseman (1) was the first in which the boundary value problem with a shift (or a displacement) was considered for analytic functions.

The classical theory of singular integral equations (SIE) and boundary value problems for analytic functions (BPAF), at the source of which are to be found the names of Hilbert and Poincaré, have been completed mainly by Soviet mathematicians. The most active founders of the theory, Muskhelishvili (1) and Gakhov (2), have summed up this period in their well-known monographs. These monographs also show that the theory has been mainly applied to problems of the theory of elasticity.

The successful development of the theory of SIE and BPAF naturally stimulated the study of singular integral equations with shift (SIES) and boundary value problems for analytic functions with shift (BPAFS). The investigation of BPAFS was, for the first time, yielding fruit. The papers by Kveselava (1) - (4) (1946-1948) were fundamental in this direction. In these papers, Kveselava, in particular, proved that the solvability picture of problem (0.2) was the same as that of problem (0.1). Incidentally, later on Mandzhavidze and Khvedelidze (1) and Simonenko (1) achieved a direct reduction of problem (0.2) to problem (0.1) with the help of conformal mappings. Apparently, the first paper in which SIES were considered was the paper by Vekua (2) published in 1948. Vekua verified that the equation

$$T\varphi \equiv a_1\varphi + a_2 W\varphi + a_3 S\varphi + a_4 WS\varphi = f \ , \tag{0.3}$$

where $a_i \in C(\Gamma)$, S is the operator of singular integration with a Cauchy kernel $(S\varphi)(t) = (\pi i)^{-1}\int_\Gamma(\tau - t)^{-1}\varphi(\tau)d\tau$, W is the shift operator $(W\varphi)(t) = \varphi(\alpha(t))$, in the case $a_1 = -a_3, a_2 = a_4$, could be reduced to problem (0.2). We note that, in problem (0.2), the shift $\alpha(t)$ need not be a Carleman shift, i.e., it is not necessary that $\alpha_n(t) \equiv t$ for some integer $n \geq 2$, where $\alpha_k(t) = \alpha(\alpha_{k-1}(t))$, $\alpha_0(t) \equiv t$. For the first time, the condition $\alpha_2(t) \equiv t$ appeared in BPAFS theory in connection with the study of the problem

$$\Phi^+(\alpha(t)) = G(t)\Phi^+(t) \tag{0.4}$$

by Carleman (2) who, in particular, showed that problem (0.4) was a natural generalization of the problem on the existence of an automorphic function belonging to a certain group of Fucs. Thus, the paper by Vekua (2) is also the first paper in which a singular integral equation with a non-Carleman shift is considered. The study carried out by Vekua shows, in particular, that the Noether theory of operator T does not depend on the properties of the shift $\alpha(t)$. This is typical of the first period of development of SIES and BPAFS theory which lasted over 20 years. All the results obtained in that period were related either to equations and problems with Carleman shift or to SIES and BPAFS with such restrictions that the shift $\alpha(t)$ could not influence the Noether theory. At the same time, great emphasis was laid on the necessity of investigating SIES and BPAFS that made their Noether theory depend essentially on the properties of the shift $\alpha(t)$. Thus, for example, one of the limit problems for differential equations with second order partial derivatives of mixed (elliptic-hyperbolic) type (this is the problem M according to the terminology of Bitzadze (1) is reduced to SIES in the form

$$\varphi(t) = \int_\Gamma \frac{K(t,\tau)}{\tau - \alpha(t)}\varphi(\tau)d\tau + g(t) \ , \tag{0.5}$$

where $\alpha(t)$ is a non-Carleman shift, which has a finite number of fixed points. The Noether theory of such equations depends essentially on the fact that $\alpha(t)$ is a non-Carleman shift (see Chapter 2, where a class of equations is considered such that equation (0.5) is a particular case, but with the additional assumption that $K(t,\tau)$ does not depend on τ). In the 1950's, there appeared monographs by N. P. Vekua (1) (first edition) and I. N.

Vekua, where BPAFS and SIES and their applications to mechanics and geometry were considered. Undoubtedly, all this actively stimulated further development of the theory. The informative burst in these subjects in the early 1940's with papers by Kveselava and N. P. Vekua led to an extensive accumulation of factual material up to the early 1970's. Naturally, it became necessary to systematize this material and make it available to a wider circle of readers in the form of a monograph. This work was carried out by Litvinchuk (1). The contents of this book can be of interest regarding any of the three following topics:

(1) Solvability theory of BPAFS;

(2) The reduction of BPAFS to SIES;

(3) The Noether theory of SIES.

Fifteen years after the publication of Litvinchuk's book, mentioned above, BPAFS theory has not undergone important changes. Two pages (see Chapter 4, Section 4.1) is all we require to show (certainly without proofs) the main new results arrived at in connection with the solvability theory of BPFAS. One can say that this important part of BPAFS and SIES theory is still awaiting its investigators. At present, the material of Topic (2) above would require adding a number of new classes of BPAFS equivalent to the SIES considered in this book. This would, however, unduly increase its volume. Rather than dwell upon this subject, we refer the interested reader to some papers, e.g.: Isakhanov (2), Latushkin, Litvinchuk and Spitkovskii (1), N.I. Lisovets (1), (2), Litvinchuk (4), Scorokhod (1).

Changes of the utmost importance have, however, taken place in SIES theory. During the 15-year period referred to, no less than 300 papers were published on the Noether theory of SIES and also on a closely related topic, the Noether theory of integral operators of convolution type (with periodic coefficients) and their discrete analogues, on the theory of pseudo-differential operators with shift, etc. This large current of information has not, however, been systematized and described in monograph literature. The Noether theory of singular integral operators (SIO) with shift which have a non-empty set of periodic points has been particularly developed . In this respect we point out that in the above-mentioned monograph of Litvinchuk only one case is somewhat incompletely discussed: that's where a set of periodic points of the shift coincides with a contour-bearer of the operator (the case of a Carleman shift). In this case, the symbol of the SIO with shift as well as the symbol of the classical SIO without shift have a finite-dimensional representation (in the form of a matrix function). This has mainly allowed to reduce an SIO with a Carleman shift to an SIO without shift. But if the set of periodic points is non-empty and does not coincide with the contour-bearer, then there appears a new factor in the Noether theory of SIO with shift. Just now the symbol of SIO with shift has essentially an infinite dimensional representation (in the form of an operator-valued function). This led to the necessity of creating new methods of investigation as compared with the case of a Carleman shift. In particular, local methods have been created. These methods essentially generalize the local principle of Simonenko-Gohberg-Krupnik-Douglas-Allan developed, at the time, for the creation of the Noether theory of an SIO without shift. Moreover, it was necessary to solve the problem of functional (non-integral) operators (FO) invertibility, itself of great importance. Incidently, this problem is trivial in the Carleman case because it is reduced to the solution of a finite system of linear algebraic equations. In the non-Carleman case, the problem of FO invertibility turned out to be complicated enough and actually closely related to certain problems of the theory of dynamic systems. In this respect, the investigation of the problem of the invertibility of functional operators by "Singularing people" has already led to new results in the area of dynamic systems (see Chapter 3, Section 6.1). Finally, the development of the SIO theory with non-Carleman shift has also enabled a new approach to the classical Noether theory of SIO without shift, thus providing a statement of the theory which is at once complete, clear and concise. It should also be pointed out that, during the last fifteen years, there have been essential developments in the theory of an SIO with shift, which has no periodic points, and in that of an SIO with a non-invertible shift; the theory of an SIO with an amenable discrete group of shifts has also started to develop. Such results have matured to the point of making it possible to attempt a fresh review of the theory.

It has therefore become necessary to generalize and to order the material on the Noether theory of SIO with shift accumulated during the last fifteen years, which is a purpose this monograph tries to serve. First of all, we have sought to describe the methods to investigate the Noetherity of SIO with shift. This seems to us to be of particular importance because, as already pointed out, the methods developed by SIO theory with shift also make it possible to organize a scheme, both new and simple, to study the Noetherity of classical objects (SIO with a Cauchy kernel, Wiener-Hopf operators, etc.).

The authors have a twofold purpose which consists in presenting a statement of the theory that can be easily understood, accessible to a wider circle of readers, but that which is also sufficiently condensed. In order to achieve this goal, only the simplest model situations are discussed, with detailed proofs, in this monograph. Cases considered to be more complex are reviewed in the final parts of the corresponding chapters with citation of the original literature. In this respect, the authors have not aimed at supplying an exhaustive bibliography but, at the same time, have tried not to omit publications directly concerned with the material under consideration. For example, we leave out questions dealing with the normalization of non-Noether SIO with shift (see, for example, Mikhlin and Prössdorf (1), Kravchenko (6)), the theory of which has just started to be studied in its fundamental points. We do not dwell upon the Noether theory of non-bounded SIO and SIO with shift (see, for example, Kravchenko (7, 8)), etc. The reading of the final paragraphs of chapters is not generally necessary for the understanding of the main material of the book.

The authors have tried to favour clarity and simplicity rather than pursue questions of generality and completeness from a purely mathematical point of view. For instance, the authors consider the simplest bearer of an operator, i.e., a simple closed Lyapunov contour bounding a simply connected domain instead of a composite contour formed by simple rectifiable lines on which the operator of singular integration is bounded ; the segment $[0,1]$ with fixed points of the shift only at the ends 0 and 1, instead of a collection of nonclosed curves with an arbitrary disposition of periodic points of the shift; or assume a Holderness of the derivative of the shift where a piecewise continuity of this derivative would be considered, etc.

As a rule, we do not begin a particular discussion by presenting the most general aspect of the problem but, rather, its simplest case. Thus, in Chapter 2, we study the case where the "coefficients" of the SIO with shift are binomial FO of type $aI+bU$, where U is the shift operator, I is the identity operator and aI, bI are operators of multiplication by functions continuous on the closed contour-bearer. We call such operators singular integral functional operators (SIFO) of the $1^{\underline{st}}$-order. In Chapter 3, these "coefficients" are already polynomials $\sum_{k=0}^{m} a_k U^k$ relative to the shift operator U (SIFO of $m^{\underline{th}}$ order) but here the contour-bearer is a closed one, and the functions a_k are continuous as before. In Chapter 4, we first consider binomial "coefficients" and we assume that the functions a_k are continuous as before but the contour $\Gamma(\Gamma=[0,1])$ is already nonclosed. Towards the end of this chapter, we consider polynomial "coefficients" . Only Chapter 5 is devoted to the study of an SIFO of the algebra generated by operators of multiplication by continuous functions, the shift operator U and the operator of singular integration S; the contour Γ is first assumed as being closed and then nonclosed. In general, we may say that Chapters 1-4 of this monograph serve as a prologue to Chapter 5 which is, in turn, a brief introduction to the theory of FO and SIFO as shown in the conclusions of this chapter (see §4 of Chapter 5).

Since, in this monograph, a new methodical approach to the Noether theory of classical SIO with Cauchy kernel is presented, this book can also be recommended to those readers who are interested only in the SIO without shift. To these, we would suggest reading only §§1 and 2 of Chapter 1, Sections 1.1 and 1.2 of §1 and Section 2.1 of §2 of Chapter 2, §1 of Chapter 3, §1 of Chapter 4, §1 of Chapter 5. All of this is so presented that it can be read independently of the rest of the material and will thus be of use to such readers.

Finally, the reader interested only in functional operators (and we hope there will also be such readers) may read only §§3 and 4 of Chapter 1, Sections 1.3 - 1.6 of §1 of Chapter 2, §§2, 4 and 6 of Chapter 3, Sections 2.1 and 2.2 of 2 and Sections 3.1 and 3.3 of 3 of Chapter 5. All of this material, mainly dealing with FO invertibility, is also presented independently of the rest.

This book was written so that most of it can be studied as part of the fundamentals of University courses. The knowledge necessary for reading this book, which is not provided by University courses, is offered together with complete proofs in Chapter 1. The above mentioned book by Litvinchuk does not have to be read before this monograph, which is autonomous with regard to that book.

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