

SINGULAR INTEGRAL EQUATIONS

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SINGULAR INTEGRAL EQUATIONS

*Boundary problems of functions theory and their
applications to mathematical physics*

Revised translation from the Russian, edited by

J. R. M. RADOK



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ANNOTATION

This monograph by N. I. Muskhelishvili systematically acquaints the reader with the mathematical apparatus of Cauchy type integrals and singular integral equations, in the study of which the author and his students took active interest. A considerable part of the book is devoted to applications to the solution of numerous problems of potential theory, the theory of elasticity and other sections of mathematical physics.

The book is intended for postgraduates and students of advanced courses of the physico-mathematical faculties, and likewise for research engineers.

EDITOR'S PREFACE

In preparing this translation for publication certain minor modifications and additions have been introduced into the original Russian text, in order to increase its readability and usefulness. Thus, instead of the first person, the third person has been used throughout; wherever possible footnotes have been included with the main text. The chapters and their subsections of the Russian edition have been renamed parts and chapters respectively and the last have been numbered consecutively.

An authors and subject index has been added. In particular, the former has been combined with the list of references of the original text, in order to enable the reader to find quickly all information on any one reference in which he may be especially interested. This has been considered most important with a view to the difficulties experienced outside Russia in obtaining references, published in that country.

Russian names have been printed in Russian letters in the authors index, in order to overcome any possible confusion arising from transliteration.

Zürich.

J. R. M. RADOK

CONTENTS

PART I

FUNDAMENTAL PROPERTIES OF CAUCHY INTEGRALS

Chapter 1

The Hölder Condition

§ 1	Smooth and piecewise smooth lines	7
§ 2	Some properties of smooth lines	9
§ 3	The Hölder Condition (H condition)	11
§ 4	Generalization to the case of several variables	12
§ 5	Two auxiliary inequalities	12
§ 6	Sufficient conditions for the H condition to be satisfied	18
§ 7	Sufficient conditions for the H condition to be satisfied (continued)	16
§ 8	Sufficient conditions for the H condition to be satisfied (continued)	19

Chapter 2

Integrals of the Cauchy type

§ 9	Definitions	22
§ 10	The Cauchy integral	22
§ 11	Connection with logarithmic potential. Historical remarks . . .	23
§ 12	The values of Cauchy integrals on the path of integration . . .	25
§ 13	The tangential derivative of the potential of a simple layer . .	30
§ 14	Sectionally continuous functions	33
§ 15	Sectionally holomorphic functions	35
§ 16	The limiting value of a Cauchy integral	37
§ 17	The Plemelj formulae	42
§ 18	Generalization of the formulae for the difference in limiting values	43
§ 19	The continuity behaviour of the limiting values	45
§ 20	The continuity behaviour of the limiting values (continued) . .	49
§ 21	On the behaviour of the derivative of a Cauchy integral near the boundary	51
§ 22	On the behaviour of a Cauchy integral near the boundary . . .	53

Chapter 3

Some corollaries on Cauchy integrals

§ 23	Poincaré-Bertrand transformation formula	56
§ 24	On analytic continuation of a function given on the boundary of a region	61
§ 25	Generalization of Harnack's theorem	64
§ 26	On sectionally holomorphic functions with discontinuities (case of contours)	65
§ 27	Inversion of the Cauchy integral (case of contours)	66
§ 28	The Hilbert inversion formulae	69

Contents

Chapter 4

Cauchy integrals near ends of the line of integration

§ 29	Statement of the principal results	73
§ 30	An auxiliary estimate.	75
§ 31	Deduction of formula (29.5).	76
§ 32	Deduction of formula (29.8).	78
§ 33	On the behaviour of a Cauchy integral near points of discontinuity	83

PART II

THE HILBERT AND THE RIEMANN-HILBERT PROBLEMS AND SINGULAR INTEGRAL EQUATIONS (CASE OF CONTOURS)

Chapter 5

The Hilbert and Riemann-Hilbert boundary problems

§ 34	The homogeneous Hilbert problem.	86
§ 35	General solution of the homogeneous Hilbert problem. The Index	88
§ 36	Associate homogeneous Hilbert problems	91
§ 37	The non-homogeneous Hilbert problem.	92
§ 38	On the extension to the whole plane of analytic functions given on a circle or half-plane.	94
§ 39	The Riemann-Hilbert problem.	99
§ 40	Solution of the Riemann-Hilbert problem for the circle . . .	100
§ 41	Example. The Dirichlet problem for a circle	107
§ 42	Reduction of the general case to that of a circular region . . .	108
§ 43	The Riemann-Hilbert problem for the half-plane	109

Chapter 6

Singular integral equations with Cauchy type kernels (case of contours)

§ 44	Singular equations and singular operators	118
§ 45	Fundamental properties of singular operators.	118
§ 46	Adjoint operators and adjoint equations	122
§ 47	Solution of the dominant equation.	123
§ 48	Solution of the equation adjoint to the dominant equation. . .	123
§ 49	Some general remarks.	130
§ 50	On the reduction of a singular integral equation	134
§ 51	On the reduction of a singular integral equation (continued). .	135
§ 52	On the resolvent of the Fredholm equation.	137
§ 53	Fundamental theorems	140
§ 54	Real equations.	146
§ 55	I. N. Vekua's theorem of equivalence. An alternative proof of the fundamental theorems.	149

Contents

§ 56	Comparison of a singular integral equation with a Fredholm equation. The Quasi-Fredholm singular equation. Reduction to the canonical form	152
§ 57	Method of reduction, due to T. Carleman and I. N. Vekua	155
§ 58	Introduction of the parameter λ	158
§ 59	Brief remarks on some other results	160

PART III

APPLICATIONS TO SOME BOUNDARY PROBLEMS

Chapter 7

The Dirichlet problem

§ 60	Statement of the Dirichlet and the modified Dirichlet problem. Uniqueness theorems	163
§ 61	Solution of the modified Dirichlet problem by means of the potential of a double layer	167
§ 62	Some corollaries	172
§ 63	Solution of the Dirichlet problem	173
§ 64	Solution of the modified Dirichlet problem, using the modified potential of a simple layer	176
§ 65	Solution of the Dirichlet problem by the potential of a simple layer. Fundamental problem of electrostatics	180

Chapter 8

Various representations of holomorphic functions by Cauchy and analogous integrals

§ 66	General remarks	187
§ 67	Representation by a Cauchy integral with real or imaginary density	188
§ 68	Representation by a Cauchy integral with density of the form $(a + ib)\mu$	190
§ 69	Integral representation by I. N. Vekua.	192

Chapter 9

Solution of the generalized Riemann-Hilbert-Poincaré problem

§ 70	Preliminary remarks	202
§ 71	The generalized Riemann-Hilbert-Poincaré problem (Problem V). Reduction to an integral equation	208
§ 72	Investigation of the solubility of Problem V	207
§ 73	Criteria of solubility of Problem V.	212
§ 74	The Poincaré problem (Problem P)	215
§ 75	Examples	219
§ 76	Some generalizations and applications	223

Contents

PART IV

THE HILBERT PROBLEM IN THE CASE OF ARCS OR DISCONTINUOUS BOUNDARY CONDITIONS AND SOME OF ITS APPLICATIONS

Chapter 10

The Hilbert problem in the case of arcs or discontinuous boundary conditions

§ 77	Definitions.	227
§ 78	Definition of a sectionally holomorphic function for a given discontinuity.	229
§ 79	The homogeneous Hilbert problem for open contours	230
§ 80	The associate homogeneous Hilbert problem. Associate classes	234
§ 81	Solution of the non-homogeneous Hilbert problem for arcs.	235
§ 82	The concept of the class h of functions given on L	238
§ 83	Some generalizations	238
§ 84	Examination of the problem $\Phi^+ + \Phi^- = g$	239
§ 85	The Hilbert problem in the case of discontinuous coefficients	243
§ 86	The Hilbert problem in the case of discontinuous coefficients (continued)	247
§ 87	Connection with the case of arcs	248

Chapter 11

Inversion formulae for arcs

§ 88	The inversion of a Cauchy integral	249
§ 89	Some variations of the inversion problem.	252
§ 90	Some variations of the inversion problem (continued)	257

Chapter 12

Effective solution of some boundary problems of the theory of harmonic functions

§ 91	The Dirichlet and analogous problems for the plane with cuts distributed along a straight line.	261
§ 92	The Dirichlet and analogous problems for the plane with cuts distributed over a circle.	271
§ 93	The Riemann-Hilbert problem for discontinuous coefficients	271
§ 94	Particular cases: The mixed problem of the theory of holomorphic functions	275
§ 95	The mixed problem for the half-plane. Formula of M. V. Keldysh and L. I. Sedov	279

Chapter 13

Effective solution of the principal problems of the static theory of elasticity for the half-plane, circle and analogous regions

§ 96	General formulae of the plane theory of elasticity.	282
§ 97	The first, second and mixed boundary problems for an elastic half-plane	284

Contents

§ 98	The problem of pressure of rigid stamps on the boundary of an elastic half-plane in the absence of friction	292
§ 99	The problem of pressure of rigid stamps on the boundary of an elastic half-plane in the absence of friction (continued)	294
§ 100	Equilibrium of a rigid stamp on the boundary of an elastic half-plane in the presence of friction.	299
§ 101	Another method of solution of the boundary problem for the half-plane	305
§ 102	The problem of contact of two elastic bodies (the generalized plane problem of Hertz)	305
§ 108	The fundamental boundary problems for the plane with straight cuts.	309
§ 104	The boundary problems for circular regions.	316
§ 105	Certain analogous problems. Generalizations.	321

PART V

SINGULAR INTEGRAL EQUATIONS FOR THE CASE OF ARCS OR DISCONTINUOUS COEFFICIENTS AND SOME OF THEIR APPLICATIONS

Chapter 14

Singular integral equations for the case of arcs and continuous coefficients

§ 106	Definitions.	324
§ 107	Solution of the dominant equation.	327
§ 108	Solution of the equation adjoint to the dominant equation.	331
§ 109	Reduction of the singular equation $K\varphi = f$	335
§ 110	Reduction of the singular equation $K'\psi = g$	336
§ 111	Investigation of the equation resulting from the reduction.	338
§ 112	Solution of a singular equation. Fundamental theorems	343
§ 113	Application to the dominant equation of the first kind	350
§ 114	Reduction and solution of an equation of the first kind.	351
§ 115	An alternative method for the investigation of singular equations	358

Chapter 15

Singular integral equations in the case of discontinuous coefficients

§ 116	Definitions.	356
§ 117	Reduction and solution of singular equations in the case of discontinuous coefficients	357

Chapter 16

Application to the Dirichlet problem and similar problems

§ 118	The Dirichlet and similar problems for the plane, cut along arcs of arbitrary shape.	359
§ 119	Reduction to a Fredholm equation. Examples.	365
§ 120	The Dirichlet problem for the plane, cut along a finite number of arcs of arbitrary shape	369

Contents

Chapter 17

Solution of the integro-differential-equation of the theory of aircraft wings of finite span

§ 121	The integro-differential equation of the theory of aircraft wings of finite span	373
§ 122	Reduction to a regular Fredholm equation	374
§ 123	Certain generalizations	379

PART VI

THE HILBERT PROBLEM FOR SEVERAL UNKNOWN FUNCTIONS AND SYSTEMS OF SINGULAR INTEGRAL EQUATIONS

Chapter 18

The Hilbert problem for several unknown functions

§ 124	Definitions.	382
§ 125	Auxiliary theorems.	383
§ 126	The homogeneous Hilbert problem.	384
§ 127	The fundamental system of solutions of the homogeneous Hilbert problem and its general solution.	398
§ 128	The non-homogeneous Hilbert problem.	404
§ 129	Supplement to the solution of a dominant system of singular integral equations and of its associate system	407

Chapter 19

Systems of singular integral equations with Cauchy type kernels and some supplements

§ 130	Definitions. Auxiliary theorems	415
§ 131	Reduction of a system of singular equations. Fundamental theorems	420
§ 132	Other methods of reduction and the investigation of systems of singular equations	421
§ 133	Brief remarks regarding important generalizations and supplements	422

Appendix 1

On smooth and piecewise smooth lines.	424
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Appendix 2

On the behaviour of the Cauchy integral near corner points. .	427
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Appendix 3

An elementary proposition regarding bi-orthogonal systems of functions	433
References and author index	437
Index.	445

PREFACE

This book is intended for an extensive field of readers, in particular for those interested in applications to the theory of elasticity, hydromechanics and other branches of mathematical physics. The book is accessible to those acquainted with the basic theory of functions of a complex variable and the theory of Fredholm integral equations. To facilitate the reading of the book, theorems, the method of proof of which is not of essential independent interest, have been printed in italics, so that the proofs may be omitted without affecting an understanding of the nature of the matter. In addition, wherever possible, the Parts and their chapters devoted to different applications have been made independent of one another. It is hoped that the methods studied in this book may be effectively employed in the solution of many problems of an applied character. Some simple applications to potential theory, the theory of elasticity and hydromechanics are given in this book.

The idea of writing this book resulted from the Author's lectures in a seminar at the Mathematical Institute, Tiflis, in 1940—1942. Under the influence of a series of results obtained by members of the seminar (chiefly due to the excellent work of I. N. Vekua) the range of problems which the Author proposed to examine was essentially altered; the Author may state with much satisfaction that a large proportion of the contents of this book must be considered as the result of the collective work of the young members of the Tiflis Mathematical Institute of the Academy of Sciences of the Georgian S.S.R. together with I. N. Vekua and the Author himself.

Tiflis, Autumn 1944

N. I. MUSKHELISHVILI

In adding the last corrections to the book the Author desires to use this opportunity to express his deep gratitude to the publishing house which always willingly complied with the Author's suggestions. The difficult and responsible task of proof-reading was considerably eased for the Author by the extraordinary kindness of L. I. Bokshitski on the staff of the press, to whom he extends his sincere gratitude just as to all the other members of the staff of the 16th Printing Establishment of the State publishers for their prompt and efficient work.

Moscow, Autumn 1945

N. I. M.

INTRODUCTION

In recent years the theory of singular integral equations has assumed increasing importance in applied problems.

In this book only one-dimensional (i.e., where the range of integration is one-dimensional, i.e., a line) singular equations involving Cauchy principal values will be examined, since the theory of multi-dimensional equations of corresponding form is still far from completion. (Some references to the literature dealing with the latter are given in § 59).

The fundamentals of the theory of one-dimensional singular integral equations of the type described were included in the work of Poincaré and Hilbert, almost directly after the development of the classical theory of integral equations by Fredholm. However, the theory of singular integral equations did not receive the attention of mathematicians for some time. On the other hand many problems of an applied character naturally reduced to singular equations, e.g. problems of the theory of elasticity, etc.; thus often in practice these equations were arrived at by "ordinary methods" and this did not always lead to satisfactory results.

However, during recent years, the theory of one-dimensional singular integral equations has advanced considerably and it can now be presented in a finished form.

This theory appears to be particularly simple and effective, if the solution of a boundary problem of the theory of functions of a complex variable, to be called the Hilbert problem, is considered. Therefore the theory of singular equations is here closely linked with the above boundary problem. The solution of the latter will be used for the development of the theory of singular equations; afterwards this theory will be applied to the solution of other more complicated boundary problems, in particular, to problems encountered in potential theory, the theory of elasticity and in hydromechanics.

Having in mind the implications for different problems of mathematical physics, some restrictions will be imposed upon the unknown and the given functions appearing in the integral equations under consideration or in the boundary conditions of the problems considered, which will largely simplify the investigation, but not affect the final theory.

The fundamental tools of the investigation are Cauchy integrals, the elementary theory of which is given in Part I together with a number of direct simpler applications.

As regards the contents of the remaining Parts nothing will be said here, since a preliminary idea can be gained by means of the sufficiently detailed list of contents at the beginning of the book.