Springer Tracts in Natural Philosophy

Volume 18

Jürg T. Marti

Introduction to the Theory of Bases

Springer-Verlag Berlin Heidelberg New York 1969

Jürg T. Marti

Department of Mathematics
University of Illinois, Urbana

ISBN-13: 978-3-642-87142-9 e-ISBN-13: 978-3-642-87140-5
DOI: 10.1007/978-3-642-87140-5

 Library of Congress Catalog Card Number 73-83680

Title No. 6746

Softcover reprint of the hardcover 1st edition 1969

To Rita

Preface

Since the publication of Banach's treatise on the theory of linear operators, the literature on the theory of bases in topological vector spaces has grown enormously. Much of this literature has for its origin a question raised in Banach's book, the question whether every separable Banach space possesses a basis or not. The notion of a basis employed here is a generalization of that of a Hamel basis for a finite dimensional vector space. For a vector space X of infinite dimension, the concept of a basis is closely related to the convergence of the series which uniquely correspond to each point of X. Thus there are different types of bases for X, according to the topology imposed on X and the chosen type of convergence for the series.

Although almost four decades have elapsed since Banach's query, the conjectured existence of a basis for every separable Banach space is not yet proved. On the other hand, no counter examples have been found to show the existence of a special Banach space having no basis. However, as a result of the apparent overconfidence of a group of mathematicians, who it is assumed tried to solve the problem, we have many elegant works which show the tight connection between the theory of bases and structure of linear spaces. In the more general setting of a separable locally convex topological vector space or a complete linear metric space, the basis problem is now solved; there actually are examples of such spaces which have no basis. By the nature of the problem, the methods of proof used in the theory of bases are those of functional analysis.

A few conditions sufficient for a sequence to form a basis of a certain type are now known. Moreover bases have been constructed for most of the separable Banach spaces which are presented as examples in text-books on topological vector spaces. For instance, the trigonometrical system is a basis for the Hilbert space $L_2[0,2\pi]$. On the other hand, if one assumes the existence of a basis for an abstract Banach space X, one obtains valuable indications on the structure of X or of closed linear subspaces of X. The assertions may concern weak sequential completeness, separability, reflexivity, dimension, or weak conditional compactness of bounded sets.

Some generalizations of the concept of a basis for a Banach space have greatly enriched the theory. On the one hand, it is natural to see how results on Banach spaces may be generalized to complete linear metric spaces or locally convex topological vector spaces. The definition of a basis, on the other hand, may be generalized itself. If one replaces the elements of a basis by linear subspaces of a Banach space or an F-space, one obtains decompositions of these spaces, and these decompositions exhibit some properties similar to those of the bases. It is worth noting that decompositions exist for every Banach space of infinite dimension. Carrying the generalization of a basis one step further, one first discards the idea of a series expansion, requiring of a biorthogonal system $\{x_\lambda, f_\lambda\}$ only that $\{x_\lambda\}$ is total in the space X; such a system is called a dual generalized basis. Moreover, if $f_\lambda(x)=0$ for all λ implies that $x=0$ for all x in X, the dual generalized basis is said to be a Markushevich basis. Finally, when the requirement of totalness, common to all definitions of bases and decompositions described above, is dropped along with the requirement of countability, we obtain the concept of a generalized basis for a topological vector space X.

Although there are presently more than two hundred publications on the theory of bases, up to now no text-book has been issued which collects and systematizes the essential results of the subject. This tract is an attempt to meet such a need.

The first chapter contains a short introduction to the working tools from functional analysis. The theorems are given there without proofs. This is justified, since there are now many well-written standard works on this subject. In Chapter II the fundamental theorems on unconditional and absolute convergence of series in Banach spaces are derived. Definitions and properties of the most important types of bases for Banach spaces, together with examples of bases in some well-known spaces of this type are given in Chapter III. The fourth chapter deals with the connections of bases, projections, orthogonality and simple $\mathcal{N}_1$-spaces, as well as with equivalent bases for Banach spaces. The known facts on the structure of Banach spaces with bases are explained in Chapter IV, and the following chapter is concerned with bases for Hilbert spaces. Decompositions are introduced in Chapter VII, and the application of both bases and decompositions in the theory of B-algebras, including compact operators, operators of finite rank, and the theory of proper π-rings, are discussed in Chapter VIII. Some of the most interesting results on generalized bases for topological vector spaces are presented in the final chapter.

I thank the Editor, Professor B. D. Coleman for his friendly co-operation and the Springer-Verlag for the careful preparation of this book.

Urbana, Illinois, February 1969 JÜRG T. MARTI

Contents