Lecture Notes in Computational Science and Engineering

6

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Library of Congress Cataloging-in-Publication Data

Turek, Stefan.
Efficient solvers for incompressible flow problems / an algorithmic and computational approach / Stefan Turek.
p. cm. -- (Lecture notes in computational science and engineering ; 6)
ISBN 978-3-642-63573-1 ISBN 978-3-642-58393-3 (eBook)
DOI 10.1007/978-3-642-58393-3
1. Fluid dynamics. I. Title. II. Series.
TA357.T895 1999
620.1'064--dc21 99-21146
CIP

Mathematics Subject Classification (1991): 76Mxx, 65Mxx, 65Nxx

Additional material to this book can be downloaded from http://extras.springer.com.

Originally published by Springer-Verlag Berlin Heidelberg New York in 1999
Softcover reprint of the hardcover 1st edition 1999

Cover Design: Friedhelm Steinen-Broo, Estudio Calamar, Spain
Cover production: *design & production* GmbH, Heidelberg
Typeset by the editors using a Springer TEX macro package

SPIN 10555489 46/3143 – 5 4 3 2 1 0 – Printed on acid-free paper

Stefan Turek

Efficient Solvers for Incompressible Flow Problems

An Algorithmic and Computational Approach

Springer

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Turek, Stefan.
Efficient solvers for incompressible flow problems / an algorithmic and computational approach / Stefan Turek.
p. cm. -- (Lecture notes in computational science and engineering ; 6)
ISBN 3-540-65433-X (hc. : alk. paper)
1. Fluid dynamics. I. Title. II. Series.
TA357.T895 1999
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Preface

The scope of this book is to discuss recent numerical and algorithmic tools for the solution of certain flow problems arising in *Computational Fluid Dynamics* (CFD). Here, we mainly restrict ourselves to the case of the incompressible Navier–Stokes equations,

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad , \quad \nabla\cdot\mathbf{u} = 0\,. \tag{1}$$

These *basic* equations already play an important role in CFD, both for mathematicians as well as for more practical scientists: Physically important facts with "real life" character can be described by them, including also economical aspects in industrial applications. On the other hand, the equations in (1) provide the complete spectrum of numerical problems nowadays concerning the mathematical treatment of partial differential equations.

Although this field of research may appear to be a small part only inside of CFD, it was and still is of great interest for mathematicians as well as engineers, physicists, computer scientists and many more: a fact which can be easily checked by counting the numerous publications. Nevertheless, our contribution has some unique characteristics since it contains a few of the latest results for the numerical solution of (complex) flow problems on modern computer platforms. In this book, our particular emphasis lies on the **solution process** of the resulting high dimensional discrete systems of equations which is often neglected in other works. Together with the included CDROM, which contains the 'FEATFLOW 1.1' software and parts of the 'Virtual Album of Fluid Motion', the interested reader may find a lot of suggestions for improving his own computational simulations.

Organisation:

Chapter 1 contains the motivation for our work. We provide the reader with detailed results from several recent Benchmark configurations for incompressible flow solvers. These are discussed in view of the numerical and also computational problems of the existing mathematical methodology and CFD software.

The mathematical description of a large variety of Navier–Stokes solvers is the subject of Chapter 2. The essential point in our approach is a strict splitting of tasks, namely the **outer control part** which is responsible for the global convergence and accuracy of the overall problem, and the **inner solver engine** which has to provide approximate solutions with respect to a given (discrete) framework. The aim is to demonstrate how classical schemes, as for instance introduced by Chorin, Van Kan or by Vanka, can be generalized and essentially improved, as "pure solvers" and also as powerful ingredients in the modern mathematical discretization context. We concentrate on the algorithmic aspects concerning the solution process while the other part related to discretization techniques is discussed more in detail in Chapter 3.

The tool for a better understanding of the existing solver methodology is a *Navier–Stokes tree* which contains most of the employed CFD techniques as *subtrees*. The basic assumption is that we can reduce the various solution schemes for the incompressible Navier–Stokes equations to - among others - the treatment of discrete nonlinear saddle point problems,

$$S\mathbf{u} + kBp = \mathbf{g} \quad , \quad B^T\mathbf{u} = 0\,, \tag{2}$$

with matrices B and B^T as discrete analogues of the operators ∇ and $\nabla\cdot$, time step k and the velocity matrix S coming from the discretized momentum equations. Then, the various approaches can be characterized through differences in the

- treatment of the nonlinearity,
- treatment of the incompressibility,
- complete outer control.

Having reached well-known tested ground for numerical analysts, namely the **solution of discrete systems of equations**, we can continue with standard techniques derived from Numerical Linear Algebra. We may treat the nonlinearity using some *fixed point defect correction* techniques or other quasi–Newton variants, and apply the general *pressure Schur complement* (PSC) approach which formally transforms the original coupled system of equations into an equivalent scalar equation for the pressure only,

$$B^T S^{-1} Bp = \frac{1}{k} B^T S^{-1} \mathbf{g}\,. \tag{3}$$

Then, the velocity $\mathbf{u}$ can be derived from p once calculated. As it is well-known for scalar linear systems, arising for instance from Poisson or transport–diffusion problems, we apply the simple *preconditioned Richardson* iteration with certain preconditioners C^{-1},

$$p^l = p^{l-1} - C^{-1}(B^T S^{-1} B p^{l-1} - \frac{1}{k} B^T S^{-1} \mathbf{g})\,. \tag{4}$$

This general *defect correction* approach is our basic iteration for the following, and all derived techniques concentrate on the "numerical linear algebraic" task of accelerating this simple iteration scheme. As usual, the first step to increase efficiency is to derive better preconditioners C^{-1}. Two different approaches are proposed:

1. We construct – on discrete and/or continuous level – globally defined operators of the type $C_i := B^T \tilde{S}_i^{-1} B$ and use them in an additive way. These are the **global pressure Schur complement** methods (*global PSC*) which contain *projection-like schemes* (or *fractional step*, *pressure correction*) as proposed by Chorin [22] or Van Kan [115].

2. We construct local preconditioners $C_i^{-1} := B^T_{|\Omega_i} S^{-1}_{|\Omega_i} B_{|\Omega_i}$ on certain patches Ω_i and combine them in the typical way related to block Jabobi- or Gauß–Seidel schemes. These are **local pressure Schur complement** methods (*local PSC*) and include schemes as for instance the *Vanka* smoother [114].

The next step is to accelerate these simple schemes as preconditioners in Krylov space methods or, often significantly better, as *smoothers* in the standard multigrid context. We explain this multilevel approach more in detail since this technique is the crucial step towards very efficient and robust CFD solvers.

In Chapter 3, we discuss other important tools which are necessary in our framework of numerical solution techniques for incompressible flow problems. While we have mainly concentrated so far on the solution process of discretized Navier–Stokes problems, we examine supplementary issues as discretization techniques, error control mechanisms, adaptivity and other necessary numerical ingredients:

1. **Finite element** spaces (including approximation and stability properties of Stokes elements, the nonconforming $\tilde{Q}1/Q0$ finite elements, stabilization techniques for convective terms via upwind or streamline–diffusion techniques, explicit construction of discretely divergence–free subspaces, a posteriori error control mechanisms).

2. **Time discretization** techniques (including the Fractional–step–θ–scheme and other One–step θ–schemes, adaptive time step control).

3. **Nonlinear iteration** schemes (including adaptive fixed point defect correction techniques, quasi–Newton schemes, stopping criterions, linearization techniques for nonstationary problems, least square CG methods).

4. **Multigrid** tools (including properties of simple basic iterations as smoothers or as preconditioners in Krylov–space methods, construction of grid transfer operators and coarse grid matrices, adaptive step-length control of the correction).

5. **Boundary conditions** (including natural *do nothing* conditions, pressure drop and flux settings, iterative implementation techniques, treatment of moving boundaries).

Having derived all necessary conmponents, one can arrange the 'Navier-Stokes solvers' in a scale which is mainly oriented at their stability and robustness. However, the probably more important question arising in the numerical treatment of the Navier–Stokes equations is:

> *'What are the total numerical cost to obtain a certain accuracy? The answer involves the measurement of number of time steps, nonlinear iteration steps and linear multigrid sweeps, but the final measure is the elapsed CPU time to achieve a desired accuracy!'*

We have to remark explicitely that all tests and resulting conclusions are for **fixed spatial meshes**. These are systematically varied in order to simulate the most usual instances which may appear in fully adaptive approaches. However, up to now, no adaptive configuration has been included. It is obvious that our "optimality" statements are not complete since they neglect the "optimal" spatial mesh. Nevertheless, all tests in this book can be viewed as special exercises with the aim to derive the optimal solver for "any" fixed discretization. Since we examine explicitely the case of complex triangulations including also highly-stretched meshes, all conclusions are relevant for the future fully adaptive framework, too.

We present most of these numerical tests in Chapter 4, for several characteristic flow situations. The quality of the applied solution schemes is examined with respect to:

- the complexity of the domain, resp., the shape of the mesh (large aspect ratios!),

- the size of the viscosity parameter ν,
- the size of the performed time step k.

Based on this knowledge we can finally show that there are indeed "discrete Black Box" solution approaches - at least for fixed but arbitrary discrete frameworks - which work likewise robust and efficient for all examined flow configurations.

Acknowledgments:

Let me start with mentioning my fruitful work together with Rolf Rannacher in Heidelberg who mainly formed my mathematical background over the last 10 years. Let me also thank Heribert Blum in Dortmund who was my mentor in mathematical software engineering. Both, Blum and Rannacher, are mainly responsible for my understanding of today's numerical work.

Let me also emphasize Owen Walsh and John Heywood from UBC Vancouver, Friedhelm Schieweck and Lutz Tobiska from Magdeburg and Phil Gresho from the LLNL who are interlocutors since years, giving me a lot of inspiration and new ideas. Especially the work of Phil has to be pointed out, and this not only because of his important "link" function – as one of the editors of the 'International Journal for Numerical Methods in Fluids' – between the mathematical and the engineering CFD community.

Furthermore, let me mention Michael Schäfer from Darmstadt for the common work over the last years concerning the development and evaluation of the *1995 DFG Benchmark*, and also the IWR at Heidelberg, especially Bernhard Przywara and Stefan Schnadt, for providing the "infinite" compute power which is necessary for such numerical studies. And not to forget, my thanks go to all members of our *FEAST group*, particularly Peter Schreiber, Susanne Kilian, Hubertus Oswald, Rainer Schmachtel, Guohui Zhou, Andreas Prohl, Christian Becker, John Wallis, Ludmilla Rivkind and the many others at our institute.

Finally, this book is dedicated to Monika, my children and my parents who have to live with a *numerical computing scientist* for many years.

Heidelberg, October 1998 *Stefan Turek*

Contents

Notation

This book is sometimes written in a very technical style and it is mainly directed to the professional CFD specialist who is familiar with key words as *finite elements*, *multigrid* or *projection schemes* and some other special issues from mathematical and computational sciences. Nevertheless, a short explanation for many of these main topics in Computational Fluid Dynamics will be given in this book, in particular in the Chapter 'Other mathematical components'.

Furthermore, it is very natural that many abbreviations will be utilized. To give the reader a better chance that one can easier find the meaning of such technical terms, we list some of the most important items in the following list.

Notations for PDE's (Partial Differential Equations):

Ω	domain $\Omega \subset \mathbf{R}^d$ with space dimension $d = 2$ or $d = 3$
$\partial\Omega$	boundary of Ω
p	scalar pressure $p(x,y,t)$ in 2D, resp., $p(x,y,z,t)$ in 3D
$\mathbf{u}$	velocity field $\mathbf{u}(x,y,z,t)$ with components $(u_1,\ldots,u_d)$
$\mathbf{u}_t$	time derivative operator $\frac{\partial \mathbf{u}}{\partial t}$
Δu_i	Laplacian operator $\frac{\partial^2 u_i}{\partial^2 x} + \frac{\partial^2 u_i}{\partial^2 y} + \frac{\partial^2 u_i}{\partial^2 z}$
∇p	gradient operator $(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z})^T$
$\nabla \cdot \mathbf{u}$	divergence operator $\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$
$\mathbf{v} \cdot \nabla u_i$	transport operator $v_1 \cdot \frac{\partial u_i}{\partial x} + v_2 \cdot \frac{\partial u_i}{\partial y} + v_3 \cdot \frac{\partial u_i}{\partial z}$
ν	viscosity parameter
Re	Reynolds number, $Re = \frac{U \cdot L}{\nu}$, with U, L *characteristic* velocity and length

Notations for discretizations and finite element constructs:

h	mesh width parameter
$k, \Delta t$	time step
$\mathrm{L}^2(\Omega), \mathbf{H}_0^1(\Omega)$	usual Lebesgue and Sobolev spaces
$a(\cdot,\cdot)$	bilinear form, mostly $a(\mathbf{u},\mathbf{v}) := (\nabla\mathbf{u}, \nabla\mathbf{v})$
$b(\cdot,\cdot)$	bilinear form, mostly $b(p,\mathbf{v}) := -(p, \nabla\cdot\mathbf{v})$
$a_h(\cdot,\cdot), b_h(\cdot,\cdot)$	discrete counterparts
$(\cdot,\cdot), \|\cdot\|$	inner product, resp., norm of $\mathrm{L}^2(\Omega)$
$\langle\cdot,\cdot\rangle_E$	euclidian scalar product
$\|\|\|\cdot\|\|\|$	discrete norms
$\mathbf{T}_h$	(regular) decomposition $\mathbf{T}_h = \bigcup\{T\}$ with simple elements T
H_h, L_h	discrete spaces for velocity and pressure ansatz functions
I_{2h}^h, I_{k-1}^k	prolongation operator
I_h^{2h}, I_k^{k-1}	restriction operator
NEL,NMT,NVT	number of elements, midpoints and vertices
FE,FV,FD	finite element, finite volume, finite difference
UPW	Upwind
SD	Streamline–diffusion
$\tilde{Q}1/Q0$	nonconforming velocity/piecewise constant pressure ansatz
$Q1/Q0$	conforming bilinear velocity/piecewise constant pressure ansatz
$Q1/Q1$	conforming bilinear velocity/conforming bilinear pressure ansatz
CN	Crank–Nicolson scheme
FS	Fractional–step–θ scheme
IE,BE	Implicit Euler/Backward Euler scheme

Notations for matrices and Numerical Linear Algebra:

M	velocity mass matrix, in the finite element context arising from (φ_i, φ_j)
M_l	*lumped* – that means diagonalized – velocity mass matrix
M_p	pressure mass matrix, analogous to M, with pressure ansatz functions
L, Δ_h	Laplacian matrix according to the Δ–operator
K	transport matrix according to the $(\mathbf{v}\cdot\nabla)$–operator
S	velocity matrix, typically $S := \alpha M + \theta_1 \nu k L + \theta_2 k K$
B	gradient matrix according to the ∇–operator, $B = (B_1, \ldots, B_d)^T$

B^T	divergence matrix, equals transposed gradient matrix B
$B^T S^{-1} B$	discrete *pressure Schur complement* operator
P	reactive preconditioner $P := B^T M_l^{-1} B$
D,L,U	diagonal, lower, upper part of a given matrix
JAC	Jacobi scheme
GS	Gauß–Seidel scheme
SOR	SOR scheme
SSOR	SSOR scheme
ILU	ILU scheme
CG	conjugate gradient scheme
PCG	preconditioned conjugate gradient scheme
BiCGSTAB	BiCGSTAB scheme
GMRES	GMRES scheme
MG	multigrid/multilevel scheme

Other notations:

AR	aspect ratio
VR	volume ratio
PSC	*pressure Schur complement*
MPSC	*multilevel pressure Schur complement*
SPSC	*single (grid) pressure Schur complement*
c_a	lift coefficient ('*Auftrieb*')
c_w	drag coefficient ('*Widerstand*')
PP2D	*Discrete projection scheme* in the FEATFLOW package (∼ global MPSC)
CC2D	*coupled Galerkin scheme* in the FEATFLOW package (∼ local MPSC)