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Exponential Functionals of Brownian Motion and Related Processes

Springer

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Preface

This monograph contains:

- ten papers written by the author, and co-authors, between December 1988 and October 1998 about certain exponential functionals of Brownian motion and related processes, which have been, and still are, of interest, during at least the last decade, to researchers in Mathematical finance;

- an introduction to the subject from the view point of Mathematical Finance by H. Geman.

The origin of my interest in the study of exponentials of Brownian motion in relation with mathematical finance is the question, first asked to me by S. Jacka in Warwick in December 1988, and later by M. Chesney in Geneva, and H. Geman in Paris, to compute the price of Asian options, i.e.: to give, as much as possible, an explicit expression for:

$$C^{(\nu)}(t,k) \stackrel{\text{def}}{=} E\left[\left(\frac{1}{t}A_t^{(\nu)} - k\right)^+\right] \tag{1}$$

where $A_t^{(\nu)} = \int_0^t ds \exp 2(B_s + \nu s)$, with $(B_s, s \geq 0)$ a real-valued Brownian motion.

Since the exponential process of Brownian motion with drift, usually called: geometric Brownian motion, may be represented as:

$$\exp(B_t + \nu t) = R^{(\nu)}_{A_t^{(\nu)}}, \qquad t \geq 0, \tag{2}$$

where $(R_u^{(\nu)}, u \geq 0)$ denotes a δ-dimensional Bessel process, with $\delta = 2(\nu+1)$, it seemed clear that, starting from (2) [which is analogous to Feller's representation of a linear diffusion X in terms of Brownian motion, via the scale function and the speed measure of X], it should be possible to compute quantities related to (1), in particular:

$$\int_0^\infty dt\, e^{-\lambda t} E[(A_t^{(\nu)} - k)^+]$$

in hinging on former computations for Bessel processes. This program has been carried out with H. Geman in [5] (a summary is presented in the C.R.A.S. Note [3]).

As a by-product of this approach, the distribution of $A_{T_\lambda}^{(\nu)}$ (i.e.: the process $(A_t^{(\nu)}, t \geq 0)$ taken at an independent exponential time T_λ with parameter λ), was obtained in [2] and [4]: the distribution of $A_{T_\lambda}^{(\nu)}$ is that of the ratio of a beta variable, divided by an independent gamma variable, the parameters of which depend (obviously) on ν and λ. When ν is negative, it is also natural to consider $A_{\infty}^{(\nu)}$ which, as proved originally by D. Dufresne, is distributed as the reciprocal of a gamma variable; again, the representation (2) and known results on Bessel processes give a quick access to this result.

An attempt to understand better the above mentioned ratio representation of $A_{T_\lambda}^{(\nu)}$ is presented in [6], along with some other questions and extensions.

My interest in Bessel processes themselves originated from questions related to the study of the winding number process $(\theta_t, t \geq 0)$ of planar Brownian motion $(Z_t, t \geq 0)$, which may be represented as:

$$\theta_t = \gamma \left(\int_0^t \frac{ds}{|Z_s|^2} \right), \qquad t \geq 0, \tag{3}$$

where $(\gamma(u), u \geq 0)$ is a real-valued Brownian, independent of $(|Z_s|, s \geq 0)$. The interrelations between planar Brownian motion, Bessel processes and exponential functionals are discussed in [7], together with a comparison of computations done partly using excursion theory, with those of De Schepper, Goovaerts, Delbaen and Kaas in vol. 11, n° 4 of *Insurance Mathematics and Economics*, done essentially via the Feynman - Kac formula.

The methodology developed in [2], [3], [4] and [5] to compute the distribution of exponential functionals of Brownian motion adapts easily when Brownian motion is replaced by a certain class of Lévy processes.

This hinges on a bijection, introduced by Lamperti, between exponentials of Lévy processes and semi-stable Markov processes.

A number of computational problems remain in this area; some results about the law of:

$$Z \stackrel{\text{def}}{=} \int_0^\infty dt \exp\left(-\xi \int_0^t ds (R_s^{(\nu)})^\gamma \right) \tag{4}$$

have been obtained in [5] and [9] (see also, in the same volume of Mathematical Finance, the article by F. Delbaen: *Consols in the C.I.R. model*).

It is my hope that the methods developed in this set of papers may prove useful in studying other models in Mathematical Finance.

In particular, models with jumps, involving exponentials of Lévy processes keep being developed intensively, and I should cite here papers by Paulsen, Nilsen and Hove, among many others; see, e.g., the references in [A].

Concerning the different aspects of studies of exponential functionals, D. Dufresne [B] presents a fairly wide panorama.

An effort to present in a unified manner the methodology used in some of the papers in this Monograph is made in [C].

To facilitate the reader's access to the bibliography about exponential functionals of Brownian motion, I have:

a) systematically replaced in the references of each paper/chapter of the volume the references "to appear" by the correct, final reference of the published paper, when this is the case;

b) added at the end of (each) chapter $\#N$, a Postscript $\#N$, which indicates some progress made since the publication of the paper, further references, etc...

Finally, it is a pleasure to thank the coauthors of the papers which are gathered in this book; particular thanks go to H. Geman whose persistence in raising questions about exotic options, and more generally many problems arising in mathematical finance gave me a lot of stimulus.

Last but not least, special thanks to F. Petit for her computational skills and for helping me with the galley proofs.

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