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Andreas E. Kyprianou

Introductory Lectures on Fluctuations of Lévy Processes with Applications

With 22 Figures

 Springer

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The background text on the front cover is written in old (pre-1941) Mongolian scripture. It is a translation of the words ‘stochastic processes with stationary and independent increments’ and should be read from the top left hand corner of the back cover to the bottom right hand corner of the front cover.

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Preface

In 2003 I began teaching a course entitled *Lévy processes* on the Amsterdam-Utrecht masters programme in stochastics and financial mathematics. Quite naturally, I wanted to expose my students to my own interests in Lévy processes; that is, the role that certain subtle behaviour concerning their fluctuations play in explaining different types of phenomena appearing in a number of classical models of applied probability. Indeed, recent developments in the theory of Lévy processes, in particular concerning path fluctuation, have offered the clarity required to revisit classical applied probability models and improve on well established and fundamental results.

Whilst teaching the course I wrote some lecture notes which have now matured into this text. Given the audience of students, who were either engaged in their ‘afstudeerfase’¹ or just starting a Ph.D., these lecture notes were originally written with the restriction that the mathematics used would not surpass the level that they should in principle have reached. Roughly speaking that means the following: experience to the level of third year or fourth year university courses delivered by a mathematics department on

- foundational real and complex analysis,
- basic facts about L^p spaces,
- measure theory, integration theory and measure theoretic probability theory,
- elements of the classical theory of Markov processes, stopping times and the Strong Markov Property.
- Poisson processes and renewal processes,
- Brownian motion as a Markov process and elementary martingale theory in continuous time.

For the most part this affected the way in which the material was handled compared to the classical texts and research papers from which almost all of the results and arguments in this text originate. A good example of this is

¹The afstudeerfase is equivalent to at least a European masters-level programme.

VIII Preface

the conscious exclusion of calculations involving the master formula for the Poisson point process of excursions of a Lévy process from its maximum.

There are approximately 80 exercises and likewise these are pitched at a level appropriate to the aforementioned audience. Indeed several of the exercises have been included in response to some of the questions that have been asked by students themselves concerning curiosities of the arguments given in class. Arguably the exercises are at times quite long. Such exercises reflect some of the other ways in which I have used preliminary versions of this text. A small number of students in Utrecht also used the text as an individual reading/self-study programme contributing to their ‘kleinescriptie’ (extended mathematical essay) or ‘onderzoekopdracht’ (research option); in addition, some exercises were used as (take-home) examination questions. The exercises in the first chapter in particular are designed to show the reader that the basics of the material presented thereafter is already accessible assuming basic knowledge of Poisson processes and Brownian motion.

There can be no doubt, particularly to the more experienced reader, that the current text has been heavily influenced by the outstanding books of Bertoin (1996) and Sato (1999), and especially the former which also takes a predominantly pathwise approach to its content. It should be reiterated however that, unlike the latter two books, this text is *not* intended as a research monograph nor as a reference manual for the researcher.

Writing of this text began whilst I was employed at Utrecht University, The Netherlands. In early 2005 I moved to a new position at Heriot Watt University in Edinburgh, Scotland, and in the final stages of completion of the book to The University of Bath. Over a period of several months my presence in Utrecht was phased out and my presence in Edinburgh was phased in. Along the way I passed through the Technical University of Munich and The University of Manchester. I should like to thank these four institutes and my hosts for giving me the facilities necessary to write this text (mostly time and a warm, dry, quiet room with an ethernet connection). I would especially like to thank my colleagues at Utrecht for giving me the opportunity and environment in which to develop this course, Ron Doney during his two-month absence for lending me the key to his office and book collection whilst mine was in storage and Andrew Cairns for arranging to push my teaching duties into 2006 allowing me the focus to finalise this text.

Let me now thank the many, including several of the students who took the course, who have made a number of remarks, corrections and suggestions (minor and major) which have helped to shape this text. In alphabetical order these are: Larbi Alili, David Applebaum, Johnathan Bagley, Erik Baurdoux, M.S. Bratiychuk, Catriona Byrne, Zhen-Qing Chen, Gunther Cornelissen, Irmingard Erder, Abdelghafour Es-Saghouani, Serguei Foss, Uwe Franz, Shota Gugushvili, Thorsten Kleinow, Paweł Kliber, Claudia Klüppelberg, V.S. Korolyuk, Ronnie Loeffen, Alexander Novikov, Zbigniew Palmowski, Goran Peskir, Kees van Schaik, Sonja Scheer, Wim Schoutens, Budhi Arta Surya, Enno Veerman, Maaïke Verloop, Zoran Vondraček. In particular I would also

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Finally, the deepest gratitude of all goes to Jagaa, Sophia and Sanaa for whom the special inscription is written.

Edinburgh
June 2006

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Contents

1	Lévy Processes and Applications	1
1.1	Lévy Processes and Infinite Divisibility	1
1.2	Some Examples of Lévy Processes	5
1.3	Lévy Processes and Some Applied Probability Models	14
	Exercises	26
2	The Lévy–Itô Decomposition and Path Structure	33
2.1	The Lévy–Itô Decomposition	33
2.2	Poisson Random Measures	35
2.3	Functionals of Poisson Random Measures	41
2.4	Square Integrable Martingales	44
2.5	Proof of the Lévy–Itô Decomposition	51
2.6	Lévy Processes Distinguished by Their Path Type	53
2.7	Interpretations of the Lévy–Itô Decomposition	56
	Exercises	62
3	More Distributional and Path-Related Properties	67
3.1	The Strong Markov Property	67
3.2	Duality	73
3.3	Exponential Moments and Martingales	75
	Exercises	83
4	General Storage Models and Paths of Bounded Variation	87
4.1	General Storage Models	87
4.2	Idle Times	88
4.3	Change of Variable and Compensation Formulae	90
4.4	The Kella–Whitt Martingale	97
4.5	Stationary Distribution of the Workload	100
4.6	Small-Time Behaviour and the Pollaczek–Khintchine Formula	102
	Exercises	105

5	Subordinators at First Passage and Renewal Measures	111
5.1	Killed Subordinators and Renewal Measures	111
5.2	Overshoots and Undershoots	119
5.3	Creeping	121
5.4	Regular Variation and Tauberian Theorems	126
5.5	Dynkin–Lamperti Asymptotics	130
	Exercises	133
6	The Wiener–Hopf Factorisation	139
6.1	Local Time at the Maximum	140
6.2	The Ladder Process	147
6.3	Excursions	154
6.4	The Wiener–Hopf Factorisation	157
6.5	Examples of the Wiener–Hopf Factorisation	168
6.6	Brief Remarks on the Term “Wiener–Hopf”	174
	Exercises	174
7	Lévy Processes at First Passage and Insurance Risk	179
7.1	Drifting and Oscillating	179
7.2	Cramér’s Estimate of Ruin	185
7.3	A Quintuple Law at First Passage	189
7.4	The Jump Measure of the Ascending Ladder Height Process	195
7.5	Creeping	197
7.6	Regular Variation and Infinite Divisibility	200
7.7	Asymptotic Ruinous Behaviour with Regular Variation	203
	Exercises	206
8	Exit Problems for Spectrally Negative Processes	211
8.1	Basic Properties Reviewed	211
8.2	The One-Sided and Two-Sided Exit Problems	214
8.3	The Scale Functions $W^{(q)}$ and $Z^{(q)}$	220
8.4	Potential Measures	223
8.5	Identities for Reflected Processes	227
8.6	Brief Remarks on Spectrally Negative GOUs	231
	Exercises	233
9	Applications to Optimal Stopping Problems	239
9.1	Sufficient Conditions for Optimality	240
9.2	The McKean Optimal Stopping Problem	241
9.3	Smooth Fit versus Continuous Fit	245
9.4	The Novikov–Shiryaev Optimal Stopping Problem	249
9.5	The Shepp–Shiryaev Optimal Stopping Problem	255
9.6	Stochastic Games	260
	Exercises	269

10 Continuous-State Branching Processes	271
10.1 The Lamperti Transform	271
10.2 Long-term Behaviour	274
10.3 Conditioned Processes and Immigration	280
10.4 Concluding Remarks	291
Exercises	293
Epilogue	295
Solutions	299
References	361
Index	371