

ON THE EQUIVALENCE OF EULERIAN AND LAGRANGIAN VARIABLES FOR THE TWO-COMPONENT CAMASSA–HOLM SYSTEM

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ABSTRACT. The Camassa–Holm equation and its two-component Camassa–Holm system generalization both experience wave breaking in finite time. To analyze this, and to obtain solutions past wave breaking, it is common to reformulate the original equation given in Eulerian coordinates, into a system of ordinary differential equations in Lagrangian coordinates. It is of considerable interest to study the stability of solutions and how this is manifested in Eulerian and Lagrangian variables. We identify criteria of convergence, such that convergence in Eulerian coordinates is equivalent to convergence in Lagrangian coordinates. In addition, we show how one can approximate global conservative solutions of the scalar Camassa–Holm equation by smooth solutions of the two-component Camassa–Holm system that do not experience wave breaking.

1. INTRODUCTION

The prevalent way to analyze the ubiquitous wave breaking for the Camassa–Holm (CH) equation, is to transform the original equation from its Eulerian variables into a new coordinate system, e.g. in Lagrangian variables. The reason for the transformation is that while the solution develops singularities in Eulerian coordinates, the solution remains smooth in the Lagrangian framework. This invites the question of a closer analysis of the transformation between the Eulerian and the Lagrangian variables. That is the goal of the present paper.

A two-component generalization of the CH equation was introduced in [30, Eq. (43)], and we will study the above question in this setting. It turns out that this system, denoted the two-component Camassa–Holm (2CH) system, has a regularizing effect on the original CH equation as long as the density ρ remains positive. To set the stage, we recall that the 2CH system can be written as

$$(1.1a) \quad u_t + uu_x + P_x = 0,$$

$$(1.1b) \quad \rho_t + (u\rho)_x = 0,$$

where P is implicitly defined by

$$(1.2) \quad P - P_{xx} = u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2.$$

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The original CH equation [4, 5] is the special case where ρ vanishes identically. The CH equation possesses many intriguing properties, and the main challenge when one considers the Cauchy problem, is that the solution develops singularities in finite time, independent of the smoothness initially. This singularity is characterized by the H^1 -norm of the function u remaining finite, while the spatial derivative u_x goes to negative infinity at a specific point at the time of wave breaking. The structure of the points of wave breaking may be intricate [13]. The behavior in the proximity of the point of wave breaking, and, in particular, the prolongation of the solution past wave breaking, has been extensively studied. See, e.g., [2, 3, 6, 7, 8, 10, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29] and references therein. The key point here is that past wave breaking uniqueness fails, and there is a continuum of distinct solutions [19], with two extreme solutions called dissipative and conservation solutions, respectively. The various solutions can be characterized by the behavior of the total energy, as measured by the local H^1 density of the solution u . As mentioned above, the density ρ has a regularizing effect on the solution: If ρ is positive on the line initially, then the solution will not develop singularities [9, 18]. A local result, saying that if ρ initially is smooth on an interval, then the solution will remain smooth on the interval determined by the characteristics emanating from the endpoints of the original interval, can be found in [18, Thm. 6.1]. This is surprising, as the 2CH system has infinite speed of propagation [26].

In this paper we study in detail the relation between the Eulerian and the Lagrangian variables, and, in particular, the stability of solutions in the two coordinate systems. Two aspects are considered. First one may ask if the solution of the 2CH system will converge to a solution of the CH equation in the limit when the density ρ vanishes, and if so, to which of the plethora of solutions. This problem has also been studied in [18]. We show that the limit is the so-called conservative solution of the CH equation where the energy is preserved, see Theorem 6.2. The second question addresses the relation between stability in Eulerian variables and stability in Lagrangian variables in general. The short answer is that the two notions are equivalent. This result can hardly be considered surprising. However, as each of the norms for the variables is rather intricate, and the relation between them is highly nonlinear, the actual proofs are considerably more technical than we expected. In part, this is due to the fact that the solution does develop singularities in Eulerian coordinates, while it remains smooth in the Lagrangian framework. We have chosen to give rather detailed proofs, as we find that eases the understanding. Each proof is broken down into shorter technical arguments for the benefit of the reader.

Let us describe more precisely the content of this paper. A key role is played by the non-negative Radon measure μ with absolutely continuous part $\mu_{ac} = (u_x^2 + \bar{\rho}^2)dx$. Here $\rho - \bar{\rho}$ is a real constant, and $\bar{\rho}$ is square integrable. The dynamics between the singular and absolutely continuous part of the measure encode the wave breaking. In Section 2 we consider the Cauchy problem for the CH equation with initial data (u, μ) . We mollify these data to obtain a sequence (u_n, ρ_n, μ_n) with positive density ρ_n . The main result in this section, Theorem 2.2, shows that indeed $u_n \rightarrow u$ in H^1 while $\rho_n \rightarrow 0$, and $\mu_n((-\infty, x]) \rightarrow \mu((-\infty, x])$ at points of continuity of the limit. In Theorem 6.2 we prove that the same result applies to the solution of the initial value problem. More specifically, we show (in obvious notation) that the solution $(u_n(t), \rho_n(t), \mu_n(t))$ of (1.1) with initial data

$(u_{0,n}, \rho_{0,n}, \mu_{0,n})$ will converge to the conservative solution $(u(t), \mu(t))$ with initial data (u_0, μ_0) . In Section 3 we study how this approximation by a mollification procedure carries over in Lagrangian coordinates. To detail this, we first need to recall the transformation between Eulerian and Lagrangian variables. We are given the pair of functions $(u, \rho) \in H^1 \times L^2$ (Eulerian variables). For simplicity we let $\rho = \bar{\rho}$. In addition, we need the energy density in the form of a positive Radon measure μ , that was introduced above, such that the absolutely continuous part equals $\mu_{ac} = (u_x^2 + \rho^2) dx$. The characteristic is given by $y(\xi) = \sup\{y \mid \mu((-\infty, y)) + y < \xi\}$. The Lagrangian velocity, energy density, and density read $U = u(y)$, $h = 1 - y_\xi$, and $r = \rho(y)y_\xi$, respectively. The full set of Lagrangian variables is then $X = (y, U, h, r)$. We write $X = L((u, \rho, \mu))$, and $(u, \rho, \mu) = M(X)$. There is a lack of uniqueness in this transformation, corresponding to the fact that a particle trajectory can be parametrized in several distinct ways. In our context we denote this by *relabeling*. Thus $M \circ L = \text{Id}$, while $L \circ M$ is only the identity on the equivalence classes of Lagrangian functions that correspond to one and the same Eulerian solution, see [27, Thm. 3.12]. We prove that the convergence $(u_n, \rho_n, \mu_n) \rightarrow (u, 0, \mu)$ implies that $X_n \rightarrow X$ (in obvious notation) in the appropriate norm, see Theorem 3.4. The proof is surprisingly intricate and applies the notion of relabeling.

The situation is turned around in Section 4, where we consider an arbitrary sequence of Lagrangian coordinates X_n that converges to X , thus $X_n \rightarrow X$ in an appropriate norm. It is then shown that the corresponding Eulerian variables (u_n, ρ_n, μ_n) converge to (u, ρ, μ) , see Theorem 4.3. In Section 5 we study how general convergence in Eulerian coordinates carries over to Lagrangian variables. To be more specific, consider a sequence (u_n, ρ_n, μ_n) that converges to (u, ρ, μ) . Then we show in Theorem 5.1 that the corresponding Lagrangian coordinates converge. Here it is not assumed that the sequence (u_n, ρ_n, μ_n) is a mollification of (u, ρ, μ) . Finally, in Section 6 we consider the time-dependent case. Consider a sequence of initial data $(u_{n,0}, \rho_{n,0}, \mu_{n,0})$ that converges to (u_0, ρ_0, μ_0) in \mathcal{D} . In Theorem 6.1 it is shown that the corresponding solutions converge for each fixed positive time. The proof transfers the convergence issue from Eulerian variables to Lagrangian coordinates, analyzes it in these variables, and finally translates the result back to the original variables.

2. APPROXIMATION IN EULERIAN COORDINATES

The aim of this section is to show that any initial data $(u, 0, \mu)$ of the CH equation can be approximated by a sequence of smooth initial data (u_n, ρ_n, μ_n) of the 2CH system. We start by introducing the Banach spaces needed in this context, before recalling the definition of the set of Eulerian coordinates for the 2CH system (and hence also for the CH equation). Thereafter we state and prove the approximation theorem.

Let

$$(2.1) \quad L_{\text{const}}^2(\mathbb{R}) = \{\rho \in L_{\text{loc}}^1(\mathbb{R}) \mid \rho(x) = \bar{\rho}(x) + k, \bar{\rho} \in L^2(\mathbb{R}), k \in \mathbb{R}\}.$$

Then we can associate to any $\rho \in L_{\text{const}}^2(\mathbb{R})$ the unique pair $(\bar{\rho}, k) \in L^2(\mathbb{R}) \times \mathbb{R}$. Thus, if we equip $L_{\text{const}}^2(\mathbb{R})$ with the norm

$$(2.2) \quad \|\rho\|_{L_{\text{const}}^2} = \|\bar{\rho}\|_{L^2} + |k|,$$

then $L_{\text{const}}^2(\mathbb{R})$ is a Banach space.

We are now ready to introduce the set of Eulerian coordinates of the 2CH system (and hence also of the CH equation). The case of the CH equation corresponds to $\rho(x) = 0$ for all $x \in \mathbb{R}$.

Definition 2.1 (Eulerian coordinates). *The set \mathcal{D} is composed of all triples (u, ρ, μ) such that $u \in H^1(\mathbb{R})$, $\rho \in L^2_{\text{const}}(\mathbb{R})$, and μ is a positive finite Radon measure whose absolutely continuous part μ_{ac} satisfies*

$$\mu_{\text{ac}} = (u_x^2 + \bar{\rho}^2)dx.$$

We write $F(x) = \mu((-\infty, x])$.

We will need a standard *Friedrichs mollifier* $\phi \in C_c^\infty(\mathbb{R})$, chosen in such a way that $\phi(x) = \phi(-x) \geq 0$, $\|\phi\|_{L^1} = 1$, $\phi'(x) > 0$ for $x \in (-1, 0)$, and $\text{supp}(\phi) = [-1, 1]$.

Theorem 2.2. *Given $(u, 0, \mu) \in \mathcal{D}$, let (u_n, ρ_n, μ_n) be given through*

$$(2.3a) \quad u_n(x) = \int_{\mathbb{R}} n\phi(n(x-y))u(y)dy,$$

$$(2.3b) \quad \rho_n(x) = \left(\frac{1}{n^2} + \int_{\mathbb{R}} n\phi'(y)F(x - \frac{y}{n})dy - \left(\int_{\mathbb{R}} \phi(y)u_x(x - \frac{y}{n})dy \right)^2 \right)^{1/2} \\ = \frac{1}{n} + \bar{\rho}_n(x),$$

$$(2.3c) \quad \mu_n(x) = u_{n,x}^2(x) + \bar{\rho}_n^2(x).$$

Define moreover

$$F_n(x) = \mu_n((-\infty, x]).$$

Then $(u_n, \rho_n, \mu_n) \in \mathcal{D}$ is a sequence of smooth functions, which approximates $(u, 0, \mu)$ in the following sense:

$$u_n \rightarrow u \quad \text{in } H^1(\mathbb{R}),$$

$$F_n(x) \rightarrow F(x) \quad \text{for every } x \text{ at which } F \text{ is continuous.}$$

Proof. We split the proof into several steps.

Step 1. Approximation of u by smooth functions u_n . By assumption we have $u \in H^1(\mathbb{R})$. Thus, application of Minkowski's inequality for integrals and the dominated convergence theorem yield that u_n defined in (2.3a) converges to u in $H^1(\mathbb{R})$. Moreover, the smoothness of ϕ implies that $u_n \in C^\infty(\mathbb{R})$.

Step 2. Construction of some auxiliary functions and measures.

We start by defining the auxiliary function

$$(2.4) \quad \hat{F}_n(x) = \int_{\mathbb{R}} n\phi(n(x-y))F(y)dy.$$

Then \hat{F}_n is smooth and converges pointwise to F at every point x at which F is continuous. Now recall that $F(x) = \mu((-\infty, x])$ and denote by μ_d the purely discrete part of the finite Radon measure μ . Then μ_d can be written as an at most countable sum of Dirac measures, the positions of which coincide with the set of discontinuities of F . In particular, F is continuous almost everywhere, and thus \hat{F}_n converges to F pointwise almost everywhere. Define moreover

$$(2.5) \quad \hat{\mu}_n(x) = \int_{\mathbb{R}} n^2\phi'(n(x-y))F(y)dy = \int_{\mathbb{R}} n\phi(n(x-y))d\mu(y) \geq 0.$$

Then we obtain by Fubini's theorem that $\|\hat{\mu}_n\|_{L^1} = \|\mu\|$ for all $n \in \mathbb{N}$.

As a next step, we will associate a sequence of densities $\hat{\rho}_n$ to $(u_n, \hat{\mu}_n)$. To that end, we note, using the Cauchy–Schwarz inequality and the fact that $\|\phi\|_{L^1} = 1$, that

$$\begin{aligned} & \int_{\mathbb{R}} \phi(z) u_x^2 \left(x - \frac{z}{n} \right) dz - \left(\int_{\mathbb{R}} \phi(z) u_x \left(x - \frac{z}{n} \right) dz \right)^2 \\ & \geq \int_{\mathbb{R}} \phi(z) u_x^2 \left(x - \frac{z}{n} \right) dz - \left(\int_{\mathbb{R}} \phi(z) dz \right) \left(\int_{\mathbb{R}} \phi(z) u_x^2 \left(x - \frac{z}{n} \right) dz \right) = 0. \end{aligned}$$

As a consequence, as μ is a positive Radon measure and $\mu_{ac} = u_x^2 dx$, we see that

$$\begin{aligned} \hat{\mu}_n(x) &= \int_{\mathbb{R}} n\phi(n(x-y)) d\mu(y) \geq \int_{\mathbb{R}} n\phi(n(x-y)) d\mu_{ac}(y) \\ &= \int_{\mathbb{R}} n\phi(n(x-y)) u_x^2(x) dx \geq \left(\int_{\mathbb{R}} n\phi(n(x-y)) u_x(x) dx \right)^2, \end{aligned}$$

and we may define $\hat{\rho}_n$ to be the non-negative root of

$$(2.6) \quad \hat{\rho}_n^2(x) = \hat{\mu}_n(x) - u_{n,x}^2(x).$$

Note that by construction $\hat{\rho}_n^2 \in L^1(\mathbb{R})$ and $\hat{\rho}_n^2 \in C^\infty(\mathbb{R})$. The function $\hat{\rho}_n$ itself need not be smooth, though.

Step 3. Smooth, approximating sequences ρ_n and μ_n .

Let ρ_n be defined by (2.3b), then

$$(2.7) \quad \rho_n^2(x) = \hat{\rho}_n^2(x) + \frac{1}{n^2}.$$

In particular, ρ_n is well-defined, since the term within the square root is always positive. Furthermore, we can decompose ρ_n as

$$\rho_n(x) = \frac{1}{n} + \bar{\rho}_n(x).$$

Then

$$\bar{\rho}_n(x) = -\frac{1}{n} + \left(\frac{1}{n^2} + \hat{\rho}_n^2(x) \right)^{1/2} \geq 0,$$

where we always take the positive root on the right-hand side. Since $\hat{\rho}_n^2$ is smooth and the term within the square root is bounded away from zero, it follows that $\bar{\rho}_n \in C^\infty(\mathbb{R})$ and consequently also $\mu_n \in C^\infty(\mathbb{R})$. Note also that this implies that

$$(2.8) \quad \rho_n(x) \geq \frac{1}{n} \quad \text{for all } x \in \mathbb{R}.$$

Moreover, we have that

$$(2.9) \quad \bar{\rho}_n^2(x) + \frac{2}{n} \bar{\rho}_n(x) = \hat{\rho}_n^2(x),$$

which in particular implies that

$$(2.10) \quad \bar{\rho}_n^2(x) \leq \hat{\rho}_n^2(x) \quad \text{for all } x \in \mathbb{R}.$$

Next, we see, using the definition of μ_n in (2.3c) and the equations (2.10) and (2.6), that

$$(2.11) \quad \mu_n(x) = u_{n,x}^2(x) + \bar{\rho}_n^2(x) \leq u_{n,x}^2(x) + \hat{\rho}_n^2(x) = \hat{\mu}_n(x)$$

for all $x \in \mathbb{R}$. As a consequence,

$$(2.12) \quad \|u_{n,x}\|_{L^2}^2 + \|\bar{\rho}_n\|_{L^2}^2 = \|\mu_n\| \leq \|\hat{\mu}_n\| = \|\mu\|,$$

which in particular shows that μ_n is a finite Radon measure, but also that $\bar{\rho}_n \in L^2(\mathbb{R})$ and therefore $\rho_n \in L^2_{\text{const}}(\mathbb{R})$.

So far, we have shown that (u_n, ρ_n, μ_n) is a sequence of smooth functions contained in \mathcal{D} , and that $u_n \rightarrow u$ in $H^1(\mathbb{R})$. It now remains to show that $F_n(x) \rightarrow F(x)$ at every point x at which F is continuous, which is in this case equivalent to $\mu_n \rightarrow \mu$ weakly, cf. [11, Props. 7.19 and 8.17]. This means we have to prove that

$$(2.13) \quad \int_{\mathbb{R}} \psi(x) d\mu_n(x) \rightarrow \int_{\mathbb{R}} \psi(x) d\mu \quad \text{as } n \rightarrow \infty$$

for all $\psi \in C_c^\infty(\mathbb{R})$. To that end observe first that, due to (2.3c), (2.6), and (2.9), we have

$$(2.14) \quad \int_{\mathbb{R}} \psi(x) d\mu_n(x) dx = \int_{\mathbb{R}} \psi(x) d\hat{\mu}_n(x) - \frac{2}{n} \int_{\mathbb{R}} \psi(x) \bar{\rho}_n(x) dx.$$

We already know that $\hat{\mu}_n \rightarrow \mu$ weakly, that is,

$$(2.15) \quad \int_{\mathbb{R}} \psi(x) d\hat{\mu}_n(x) \rightarrow \int_{\mathbb{R}} \psi(x) d\mu(x) \quad \text{as } n \rightarrow \infty,$$

for all $\psi \in C_c^\infty(\mathbb{R})$. Moreover we obtain from (2.12) and the Cauchy–Schwarz inequality that

$$\left| \frac{2}{n} \int_{\mathbb{R}} \psi(x) \bar{\rho}_n(x) dx \right| \leq \frac{2}{n} \|\psi\|_{L^2} \|\bar{\rho}_n\|_{L^2} \leq \frac{2}{n} \|\psi\|_{L^2} \|\mu\|^{1/2} \rightarrow 0$$

for all $\psi \in C_c^\infty(\mathbb{R})$, which concludes the proof. \square

Remark 2.3. Note that one can show that the function $\frac{1}{n} \bar{\rho}_n$ converges pointwise to 0. Indeed, according to (2.6) and (2.9), we have

$$(2.16) \quad 0 \leq \bar{\rho}_n^2(x) \leq \hat{\rho}_n^2(x) = \hat{\mu}_n(x) - u_{n,x}^2(x) \leq \hat{\mu}_n(x).$$

Moreover, from (2.5) we get

$$(2.17) \quad \frac{1}{n^2} \hat{\mu}_n(x) = \frac{1}{n} \int_{-1}^0 \phi'(z) \left(F\left(x - \frac{z}{n}\right) - F\left(x + \frac{z}{n}\right) \right) dz \leq \frac{1}{n} \|\mu\| \phi(0).$$

Thus combining (2.16) and (2.17) yields that the sequence $\frac{1}{n} \bar{\rho}_n$ is uniformly bounded and that

$$\frac{1}{n} \bar{\rho}_n \rightarrow 0 \quad \text{pointwise as } n \rightarrow \infty.$$

Remark 2.4. In the next section we are not only going to use the splitting of $\hat{\mu}_n(x)$ into $u_{n,x}^2(x)$ and $\hat{\rho}_n^2(x)$ as introduced in (2.6), but also a second one, which we are introducing next. Namely, let $F_s(x) = \mu_s((-\infty, x])$, where μ_s denotes the singular part of the measure μ , and let ϕ be the Friedrichs mollifier. Define

$$(2.18) \quad \tilde{u}_{n,x}^2(x) = \int_{\mathbb{R}} \phi(z) u_x^2\left(x - \frac{z}{n}\right) dz$$

and

$$(2.19) \quad \tilde{\rho}_n^2(x) = \int_{\mathbb{R}} n \phi'(z) F_s\left(x - \frac{z}{n}\right) dz.$$

Then

$$(2.20) \quad \begin{aligned} \tilde{u}_{n,x}^2(x) + \tilde{\rho}_n^2(x) &= \int_{\mathbb{R}} n\phi'(z)F_s\left(x - \frac{z}{n}\right)dz + \int_{\mathbb{R}} \phi(z)u_x^2\left(x - \frac{z}{n}\right)dz \\ &= \int_{\mathbb{R}} n\phi'(z)F\left(x - \frac{z}{n}\right)dz = \hat{\mu}_n(x). \end{aligned}$$

Remark 2.5. Let $(u, 0, \mu) \in \mathcal{D}$, $n \in \mathbb{N}$, and $(u_n, \rho_n, \mu_n) \in \mathcal{D}$ be defined as in Theorem 2.2. By construction we then have that $u_n, \rho_n \in C^\infty(\mathbb{R})$, μ_n is absolutely continuous, and, according to (2.8), that $\rho_n(x) \geq \frac{1}{n}$ for all $x \in \mathbb{R}$. Hence [18, Cor. 6.2] implies that the corresponding solution $(u_n(t), \rho_n(t), \mu_n(t))$ has the same regularity for all times t , and, in particular, no wave breaking occurs.

3. CONVERGENCE IN LAGRANGIAN COORDINATES

The aim of this section is to show that the smooth approximating sequence constructed in Theorem 2.2 not only converges in the set of Eulerian coordinates \mathcal{D} but also in the set of Lagrangian coordinates \mathcal{F} . Hence, we are first going to introduce the set of Lagrangian coordinates \mathcal{F} and the mapping L from \mathcal{D} to \mathcal{F} , before stating and proving the outlined convergence theorem.

Let V be the Banach space defined by

$$V = \{f \in C_b(\mathbb{R}) \mid f_\xi \in L^2(\mathbb{R})\},$$

where $C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and the norm is given by $\|f\|_V = \|f\|_{L^\infty} + \|f_\xi\|_{L^2}$. Let moreover

$$E = V \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{R},$$

then E equipped with the norm

$$(3.1) \quad \|(\zeta, U, h, \bar{r}, k)\|_E = \|\zeta\|_V + \|U\|_{H^1} + \|h\|_{L^2} + \|\bar{r}\|_{L^2} + |k|$$

is a Banach space. Note that we can associate to each $(\zeta, U, h, \bar{r}, k) \in E$ the tuple (y, U, h, r) by setting

$$(3.2) \quad y = \zeta + \text{Id} \quad \text{and} \quad r = \bar{r} + ky_\xi.$$

Conversely, for any pair (y, r) such that $y - \text{Id} \in V$ and $r \in L^2_{\text{const}}(\mathbb{R})$ there exists a unique triplet $(\zeta, \bar{r}, k) \in [L^2(\mathbb{R})]^2 \times \mathbb{R}$ such that (3.2) holds. For more details we refer to [18, Sect. 3]. In what follows we will slightly abuse the notation by writing $(y, U, h, r) \in E$ instead of $(\zeta, U, h, \bar{r}, k) \in E$.

In addition we have to introduce the set of relabeling functions, which are not only needed for identifying equivalence classes in Lagrangian coordinates, but also for determining the set of Lagrangian coordinates.

Definition 3.1 (Relabeling functions). *We denote by G the subgroup of the group of homeomorphisms f of \mathbb{R} such that*

$$(3.3a) \quad f - \text{Id} \text{ and } f^{-1} - \text{Id} \text{ both belong to } W^{1,\infty}(\mathbb{R}),$$

$$(3.3b) \quad f_\xi - 1 \text{ belongs to } L^2(\mathbb{R}),$$

where Id denotes the identity function.

Given $\kappa \geq 0$, we denote by G_κ the subset of G defined by

$$G_\kappa = \{f \in G \mid \|f - \text{Id}\|_{W^{1,\infty}} + \|f^{-1} - \text{Id}\|_{W^{1,\infty}} \leq \kappa\}.$$

We are now ready to introduce the set of Lagrangian coordinates of the 2CH system (and hence also of the CH equation). The case of the CH equation corresponds to $r(\xi) = 0$ for all $\xi \in \mathbb{R}$.

Definition 3.2 (Lagrangian coordinates). *The set \mathcal{F} is composed of all tuples $X = (y, U, h, r) \in E$, such that*

$$(3.4a) \quad (\zeta, U, h, r) \in [W^{1,\infty}(\mathbb{R})]^2 \times [L^\infty(\mathbb{R})]^2,$$

$$(3.4b) \quad y_\xi \geq 0, \quad h \geq 0, \quad y_\xi + h > 0 \text{ almost everywhere,}$$

$$(3.4c) \quad y_\xi h = U_\xi^2 + \bar{r}^2 \text{ almost everywhere,}$$

$$(3.4d) \quad y + H \in G,$$

where we denote $y(\xi) = \zeta(\xi) + \xi$ and $H(\xi) = \int_{-\infty}^{\xi} h(\eta) d\eta$.

Moreover, we set

$$\mathcal{F}_\kappa = \{X \in \mathcal{F} \mid y + H \in G_\kappa\}.$$

Observe that

$$(3.5) \quad \mathcal{F}_0 = \{X \in \mathcal{F} \mid y(\xi) + H(\xi) = \xi \text{ for all } \xi \in \mathbb{R}\}.$$

We note that the group G acts on \mathcal{F} by means of right composition of the form

$$X = (y, U, h, r) \mapsto X \circ g := (y \circ g, U \circ g, (h \circ g)g_\xi, (r \circ g)g_\xi).$$

This group action then allows us to define equivalence classes of Lagrangian coordinates, where we say that two coordinates X and \hat{X} are equivalent, if there exists some $g \in G$ such that $\hat{X} = X \circ g$.

Given an arbitrary $X = (y, U, h, r)$, we note that $y + H \in G$ and hence also $(y + H)^{-1} \in G$. In particular, if we introduce

$$(3.6) \quad \Gamma(X) = X \circ (y + H)^{-1},$$

then a short computation yields that $\Gamma(X) \in \mathcal{F}_0$. This shows that every equivalence class $X \circ G$ of Lagrangian coordinates has a unique canonical representative $\Gamma(X)$ in \mathcal{F}_0 . Moreover, it has been shown in [18, Lem. 4.6] that the mapping $\Gamma|_{\mathcal{F}_\kappa} : \mathcal{F}_\kappa \rightarrow \mathcal{F}_0$ is continuous for each $\kappa > 0$.

Finally we can introduce the mapping L from Eulerian to Lagrangian coordinates.

Theorem 3.3 ([18, Thm. 4.9]). *For any (u, ρ, μ) in \mathcal{D} , let*

$$(3.7a) \quad y(\xi) = \sup \{y \mid \mu((-\infty, y)) + y < \xi\},$$

$$(3.7b) \quad h(\xi) = 1 - y_\xi(\xi),$$

$$(3.7c) \quad U(\xi) = u \circ y(\xi),$$

$$(3.7d) \quad r(\xi) = \rho \circ y(\xi) y_\xi(\xi).$$

Then $(y, U, h, r) \in \mathcal{F}_0$. We denote by $L : \mathcal{D} \rightarrow \mathcal{F}_0$ the mapping which to any element $(u, \rho, \mu) \in \mathcal{D}$ associates $X = (y, U, h, r) \in \mathcal{F}_0$ given by (3.7).

In the case of the CH equation, we have $r(\xi) = 0$ for all $\xi \in \mathbb{R}$.

Theorem 3.4. *Let $(u, 0, \mu) \in \mathcal{D}$, and let $(u_n, \rho_n, \mu_n) \in \mathcal{D}$ be the corresponding approximating sequence defined in Theorem 2.2. Moreover, let $(y, U, h, 0) = L((u, 0, \mu))$ and $(y_n, U_n, h_n, r_n) = L((u_n, \rho_n, \mu_n))$. Then*

$$(y_n, U_n, h_n, r_n) \rightarrow (y, U, h, 0) \text{ in } E.$$

$$\begin{array}{ccccc}
(u_n, \rho_n, \mu_n) & \rightarrow & (u, 0, \mu) & \xrightarrow{\text{Step 1}} & \hat{X}_n = X_n \circ g_n \\
\text{Theorem 3.4} \downarrow & & & & \downarrow \text{Steps 2-9} \\
(y_n, U_n, h_n, r_n) & \rightarrow & (y, U, h, r) & \xleftarrow{\text{Step 10}} & (\hat{y}_n, \hat{U}_n, \hat{h}_n, \hat{r}_n) \rightarrow (y, U, h, r)
\end{array}$$

FIGURE 1. Outline of the proof of Theorem 3.4.

Proof. Let $(u, 0, \mu) \in \mathcal{D}$ and $(u_n, \rho_n, \mu_n) \in \mathcal{D}$ be the approximating series defined in Theorem 2.2. Furthermore, let $X = (y, U, h, 0) = L((u, 0, \mu))$ and $X_n = (y_n, U_n, h_n, r_n) = L((u_n, \rho_n, \mu_n))$, which yields a smooth sequence in Lagrangian coordinates, cf. [18, Proof of Thm. 6.1]. However, due to the construction of our approximating sequence (u_n, ρ_n, μ_n) , it turns out that in order to prove that $X_n \rightarrow X$ in Lagrangian coordinates, it is better to introduce another sequence $\hat{X}_n = (\hat{u}_n, \hat{\rho}_n, \hat{\mu}_n)$ which is linked to the sequence X_n via relabeling. For better understanding, we split the proof into several steps. After first defining the new sequence \hat{X}_n , we show that for every $n \in \mathbb{N}$ there exists $g_n \in G$ such that $\hat{X}_n = X_n \circ g_n$ (Step 1). Thereafter, we establish that $\hat{X}_n \rightarrow X$ in E (Steps 2–9). Finally, we show that $\hat{X}_n \rightarrow X$ implies $X_n \rightarrow X$ in E (Step 10). The situation is also depicted in Figure 1.

Step 1. Definition of the sequence \hat{X}_n and proof that $\hat{X}_n = X_n \circ g_n$. Define¹ $\hat{F}_n(x)$ by

$$(3.8) \quad \hat{F}_n(x) = \int_{\mathbb{R}} n\phi(n(x-y))F(y)dy = \hat{\mu}_n((-\infty, x]),$$

such that

$$\hat{\mu}_n(x) = \int_{-1}^0 n\phi'(z) \left(F\left(x - \frac{z}{n}\right) - F\left(x + \frac{z}{n}\right) \right) dz.$$

Introduce $\hat{\rho}_n^2 = \hat{\mu}_n - u_{n,x}^2$. Then $\hat{\mu}_n = u_{n,x}^2 + \bar{\rho}_n^2 + \frac{2}{n}\bar{\rho}_n = u_{n,x}^2 + \hat{\rho}_n^2$, as in (2.6)–(2.9). Let now $\hat{X}_n = (\hat{y}_n, \hat{U}_n, \hat{h}_n, \hat{r}_n) \in \mathcal{F}$, where

$$(3.9a) \quad \hat{y}_n(\xi) = \sup \{y \mid \hat{\mu}_n((-\infty, y)) + y < \xi\},$$

$$(3.9b) \quad \hat{h}_n(\xi) = 1 - \frac{2}{n}\bar{r}_n(\xi) - \hat{y}_{n,\xi}(\xi),$$

$$(3.9c) \quad \hat{U}_n(\xi) = u_n \circ \hat{y}_n(\xi),$$

$$(3.9d) \quad \bar{r}_n(\xi) = \bar{\rho}_n \circ \hat{y}_n(\xi)\hat{y}_{n,\xi}(\xi),$$

$$(3.9e) \quad \hat{r}_n(\xi) = \rho_n \circ \hat{y}_n(\xi)\hat{y}_{n,\xi}(\xi).$$

We are going to show that we can write $\hat{X}_n = X_n \circ g_n$ for some $g_n \in G$, that is,²

$$\begin{aligned}
(3.10) \quad \hat{X}_n(\xi) &= (\hat{y}_n(\xi), \hat{U}_n(\xi), \hat{h}_n(\xi), \hat{r}_n(\xi)) \\
&= (y_n(g_n(\xi)), U_n(g_n(\xi)), h_n(g_n(\xi))g_{n,\xi}(\xi), r_n(g_n(\xi))g_{n,\xi}(\xi)) \\
&= X_n \circ g_n(\xi),
\end{aligned}$$

¹This construction resembles the one used in Step 2 of the proof of Theorem 2.2. However, here we perform the construction in Lagrangian variables.

²Note the factors $g_{n,\xi}(\xi)$.

which implies immediately that $\hat{X}_n \in \mathcal{F}$ and that it belongs to the same equivalence class as X_n . Additionally, we will show that there exists some κ independent of n such that $g_n \in G_\kappa$ for all $n \in \mathbb{N}$.

Since both μ_n and $\hat{\mu}_n$ are smooth and purely absolutely continuous, we have that

$$(3.11) \quad y_n(\xi) + F_n(y_n(\xi)) = \xi,$$

and

$$(3.12) \quad \hat{y}_n(\xi) + \hat{F}_n(\hat{y}_n(\xi)) = \xi,$$

for all $\xi \in \mathbb{R}$. Moreover, recall that

$$\mu_n(x) + \frac{2}{n}\bar{\rho}_n(x) = \hat{\mu}_n(x) \quad \text{for all } x \in \mathbb{R}$$

according to (2.9), (2.6) and (2.3c), and $\frac{2}{n}\bar{\rho}_n \in L^1(\mathbb{R})$. Hence we can rewrite (3.12) as

$$(3.13) \quad \hat{y}_n(\xi) + \hat{H}_n(\xi) = \hat{y}_n(\xi) + F_n(\hat{y}_n(\xi)) = \xi - \frac{2}{n} \int_{-\infty}^{\hat{y}_n(\xi)} \bar{\rho}_n(x) dx = g_n(\xi),$$

which defines $g_n(\xi)$. Here $\hat{H}_n(\xi) = \int_{-\infty}^{\xi} \hat{h}_n(\eta) d\eta$. Moreover, using (3.11) we have

$$y_n(g_n(\xi)) + F_n(y_n(g_n(\xi))) = g_n(\xi) \quad \text{for all } \xi \in \mathbb{R},$$

and, since $\text{Id} + F_n$ is strictly increasing, we conclude that

$$(3.14) \quad \hat{y}_n(\xi) = y_n(g_n(\xi)) \quad \text{for all } \xi \in \mathbb{R},$$

which immediately implies that

$$\hat{U}_n(\xi) = u_n(\hat{y}_n(\xi)) = u_n(y_n(g_n(\xi))) = U_n(g_n(\xi)) \quad \text{for all } \xi \in \mathbb{R}.$$

Using (3.9b), (3.9d), (3.13), (3.14), and (3.7d) we infer that

$$(3.15) \quad \begin{aligned} \hat{h}_n(\xi) &= 1 - \frac{2}{n} \bar{r}_n(\xi) - \hat{y}_{n,\xi}(\xi) = g_{n,\xi}(\xi) - \hat{y}_{n,\xi}(\xi) \\ &= (1 - y_{n,\xi}(g_n(\xi))) g_{n,\xi}(\xi) = h_n(g_n(\xi)) g_{n,\xi}(\xi). \end{aligned}$$

In addition, we see that

$$\hat{r}_n(\xi) = \rho_n(\hat{y}_n(\xi)) \hat{y}_{n,\xi}(\xi) = r_n(g_n(\xi)) g_{n,\xi}(\xi).$$

Thus we conclude that $\hat{X}_n = X_n \circ g_n$, and it remains to show that $g_n \in G_\kappa$ for some κ independent of n .

Instead of checking that g_n satisfies all the properties listed in Definition 3.1, we are going to apply [27, Lem. 3.2]. Namely, if g_n is absolutely continuous, $g_{n,\xi} - 1 \in L^2(\mathbb{R})$, and there exist $c_1 \geq 1$ and $c_2 > 0$ such that $\frac{1}{c_1} \leq g_{n,\xi}(\xi) \leq c_1$ almost everywhere and $\|g_n - \text{Id}\|_{L^\infty} \leq c_2$, then $g_n \in G_\kappa$ for some $\kappa > 0$ depending only on c_1 and c_2 . By construction, \hat{X}_n is smooth and therefore g_n is smooth and, in particular, absolutely continuous. Since $\frac{2}{n}\bar{\rho}_n \in L^1(\mathbb{R})$ and $\bar{\rho}_n$ is strictly positive, we get from (3.13) that $g_n - \text{Id} \in L^\infty(\mathbb{R})$, and from (2.6), (2.9), and (2.12) that $\|g_n - \text{Id}\|_{L^\infty} \leq \|\mu\|$. Moreover, using the notation in the proof of Theorem 2.2, we obtain from (3.12) and (2.6) that

$$(3.16) \quad \hat{y}_{n,\xi}(\xi) = \frac{1}{1 + u_{n,x}^2(\hat{y}_n(\xi)) + \hat{\rho}_n^2(\hat{y}_n(\xi))} \leq 1.$$

Thus

$$\begin{aligned}
\frac{1}{2} &\leq \frac{1 + u_{n,x}^2(\hat{y}_n(\xi)) + \bar{\rho}_n^2(\hat{y}_n(\xi))}{1 + \frac{1}{n^2} + u_{n,x}^2(\hat{y}_n(\xi)) + 2\bar{\rho}_n^2(\hat{y}_n(\xi))} \\
&\leq \frac{1 + u_{n,x}^2(\hat{y}_n(\xi)) + \bar{\rho}_n^2(\hat{y}_n(\xi))}{1 + u_{n,x}^2(\hat{y}_n(\xi)) + \bar{\rho}_n^2(\hat{y}_n(\xi)) + \frac{2}{n}\bar{\rho}_n(\hat{y}_n(\xi))} \\
&= 1 - \frac{\frac{2}{n}\bar{\rho}_n(\hat{y}_n(\xi))}{1 + u_{n,x}^2(\hat{y}_n(\xi)) + \bar{\rho}_n^2(\hat{y}_n(\xi))} \\
(3.17) \quad &= g_{n,\xi}(\xi) \leq 1,
\end{aligned}$$

due to (2.9) and (3.13). Finally, we have to check that $g_{n,\xi} - 1 \in L^2(\mathbb{R})$. Direct computations, using (3.13) and (3.16), yield

$$\int_{\mathbb{R}} (g_{n,\xi}(\xi) - 1)^2 d\xi = \frac{4}{n^2} \int_{\mathbb{R}} \bar{\rho}_n^2(\hat{y}_n(\xi)) \hat{y}_{n,\xi}^2(\xi) d\xi \leq \frac{4}{n^2} \int_{\mathbb{R}} \bar{\rho}_n^2(x) dx.$$

Thus $g_{n,\xi} - 1 \in L^2(\mathbb{R})$, since $\bar{\rho}_n \in L^2(\mathbb{R})$. Thus [27, Lem. 3.2] implies that g_n is a relabeling function and that there exists $\kappa > 0$ independent of n such that $g_n \in G_\kappa$ for all $n \in \mathbb{N}$.

Step 2: The sequence $\hat{y}_n - \text{Id}$ converges to $y - \text{Id}$ in $L^\infty(\mathbb{R})$. Recall that we have by definition that

$$(3.18) \quad y(\xi) + F(y(\xi)-) \leq \xi \leq y(\xi) + F(y(\xi)) \quad \text{for all } \xi \in \mathbb{R},$$

where $F(x) = \mu((-\infty, x])$. Moreover, since $\hat{\mu}_n(x)$ is smooth and purely absolutely continuous, we have

$$(3.19) \quad \hat{y}_n(\xi) + \hat{F}_n(\hat{y}_n(\xi)) = \xi \quad \text{for all } \xi \in \mathbb{R},$$

where $\hat{F}_n(x) = \hat{\mu}_n((-\infty, x])$. Introducing

$$(3.20) \quad G(x) := x + F(x) \quad \text{and} \quad \hat{G}_n(x) := x + \hat{F}_n(x),$$

we conclude that

$$\begin{aligned}
\hat{G}_n(\hat{y}_n(\xi)) &= \hat{y}_n(\xi) + \hat{F}_n(\hat{y}_n(\xi)) \\
&= \int_{-1}^1 \phi(z) \left(\hat{y}_n(\xi) - \frac{z}{n} + F\left(\hat{y}_n(\xi) - \frac{z}{n}\right) \right) dz \\
&= \int_{-1}^1 \phi(z) G\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz,
\end{aligned}$$

where we used (3.8). Moreover, since $G(x)$ is strictly increasing and due to (3.19) and (3.20), one has that

$$(3.21) \quad G\left(\hat{y}_n(\xi) - \frac{1}{n}\right) < \hat{G}_n(\hat{y}_n(\xi)) = \xi < G\left(\hat{y}_n(\xi) + \frac{1}{n}\right),$$

and by (3.18) that

$$(3.22) \quad G(y(\xi)-) \leq \xi \leq G(y(\xi)).$$

Combining (3.21) and (3.22) yields on the one hand that

$$G\left(\hat{y}_n(\xi) - \frac{1}{n}\right) < \xi \leq G(y(\xi))$$

and on the other hand that

$$G(y(\xi)-) \leq \xi < G\left(\hat{y}_n(\xi) + \frac{1}{n}-\right).$$

Recalling that $G(x)$, and hence also $G(x-)$, is strictly increasing, we obtain that

$$\hat{y}_n(\xi) - \frac{1}{n} < y(\xi) < \hat{y}_n(\xi) + \frac{1}{n}$$

or, equivalently,

$$-\frac{1}{n} < y(\xi) - \hat{y}_n(\xi) < \frac{1}{n} \quad \text{for all } \xi \in \mathbb{R}.$$

In particular, this shows that $\|\hat{y}_n - y\|_{L^\infty} \rightarrow 0$ as $n \rightarrow \infty$.

Step 3. Convergence of \tilde{h}_n to h in $L^1(\mathbb{R})$. Let

$$(3.23) \quad \tilde{h}_n = 1 - \hat{y}_{n,\xi}.$$

To show that $\tilde{h}_n \rightarrow h$ in $L^1(\mathbb{R})$ is the main (and most difficult) step.

Due to our change from Eulerian to Lagrangian coordinates, it is not clear at first sight that \tilde{h}_n and h belong to $L^1(\mathbb{R})$. We know that $hy_\xi = U_\xi^2$, or, equivalently, $h = U_\xi^2 + h^2$, because $y_\xi + h = 1$. However, since U_ξ and h both belong to $L^2(\mathbb{R})$, it follows that $h \in L^1(\mathbb{R})$. Combining (3.23) and (3.9b), and recalling that $\hat{X} \in \mathcal{F}$ which implies that (3.4c) is satisfied, one obtains

$$(3.24) \quad \tilde{h}_n = \tilde{h}_n \hat{y}_{n,\xi} + \tilde{h}_n^2 = \hat{h}_n \hat{y}_{n,\xi} + \frac{2}{n} \tilde{r}_n \hat{y}_{n,\xi} + \tilde{h}_n^2 = \hat{U}_{n,\xi}^2 + \tilde{r}_n^2 + \frac{2}{n} \tilde{r}_n \hat{y}_{n,\xi} + \tilde{h}_n^2,$$

where $\tilde{r}_n, \hat{h}_n, \hat{U}_{n,\xi} \in L^2(\mathbb{R})$, $\frac{2}{n} \tilde{r}_n \in L^1(\mathbb{R})$ and $\hat{y}_{n,\xi} \in L^\infty(\mathbb{R})$. Thus also $\tilde{h}_n \in L^1(\mathbb{R})$. Define

$$(3.25) \quad \tilde{H}_n(\xi) = \int_{-\infty}^{\xi} \tilde{h}_n(\eta) d\eta \quad \text{and} \quad H(\xi) = \int_{-\infty}^{\xi} h(\eta) d\eta.$$

Then the identities $\tilde{H}_n + \hat{y}_n = \text{Id}$ and $H + y = \text{Id}$ together with the pointwise convergence of $\hat{y}_n \rightarrow y$ imply that \tilde{H}_n converges pointwise to H and $\tilde{H}_n(\infty) = \|\hat{\mu}_n\|_{L^1}$ and $H(\infty) = \|\mu\|$. In particular, this means that $\tilde{H}_n(\infty) = H(\infty)$ for all $n \in \mathbb{N}$ and hence

$$(3.26) \quad \|\tilde{h}_n\|_{L^1} = \|h\|_{L^1}, \quad n \in \mathbb{N}.$$

Next we will prove that \tilde{h}_n converges to h pointwise almost everywhere, which will imply that $\tilde{h}_n \rightarrow h$ in $L^1(\mathbb{R})$, see [1, Prop. 1.33]. To that end, observe first that $\tilde{h}_n(\xi) - h(\xi) = y_\xi(\xi) - \hat{y}_{n,\xi}(\xi)$ for all $\xi \in \mathbb{R}$. Thus it suffices to show that $\hat{y}_{n,\xi}$ converges pointwise to y_ξ almost everywhere. Recalling (3.19), (3.20), and that \hat{y}_n is smooth, we see that this is equivalent to showing that

$$(3.27) \quad \hat{G}'_n(\hat{y}_n(\xi)) \rightarrow \frac{1}{y_\xi(\xi)} \quad \text{for almost every } \xi \in \mathbb{R}.$$

Moreover, note that

$$\begin{aligned} \hat{G}'_n(\hat{y}_n(\xi)) &= n \int_{-1}^1 \phi'(z) G\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz \\ &= n \int_{-1}^1 \phi'(z) \left(G\left(\hat{y}_n(\xi) - \frac{z}{n}\right) - \xi\right) dz \end{aligned}$$

$$(3.28) \quad = n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z))(G(z) - \xi) dz.$$

Introducing the strictly increasing function $\tilde{G}(z) = G(z) - \xi$, it follows that we have to show that

$$(3.29) \quad \hat{G}'_n(\hat{y}_n(\xi)) = n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \rightarrow \frac{1}{y_\xi(\xi)}$$

for almost every $\xi \in \mathbb{R}$.

In fact, we will show below that (3.29) holds at every $\xi \in \mathbb{R}$ where the function y is differentiable. Since y is Lipschitz continuous and therefore differentiable almost everywhere, this will prove the convergence of \tilde{h}_n to h in $L^1(\mathbb{R})$. In the proof of (3.29), we will consider separately the cases where the derivative of y is zero, and where it is strictly positive.

(a) The case $y_\xi(\xi) = 0$. We have to show that $\hat{G}'_n(\hat{y}_n(\xi)) \rightarrow \infty$ as $n \rightarrow \infty$. By assumption $y_\xi(\xi) = 0$ and hence for every $\varepsilon > 0$ there exists some $\delta_\varepsilon > 0$ such that

$$(3.30) \quad 0 \leq \frac{y(\eta) - y(\xi)}{\eta - \xi} < \varepsilon \quad \text{whenever } |\xi - \eta| < \delta_\varepsilon.$$

Define $\gamma_\varepsilon := \varepsilon \delta_\varepsilon$ and let $z \in \mathbb{R}$ such that $|z - y(\xi)| < \gamma_\varepsilon$. In addition, recall (3.18) and (3.20), and observe that $y(G(z)) = z$ for all $z \in \mathbb{R}$. If $|G(z) - \xi| < \delta_\varepsilon$, we have by (3.30) that

$$\frac{G(z) - \xi}{z - y(\xi)} = \frac{G(z) - \xi}{y(G(z)) - y(\xi)} > \frac{1}{\varepsilon}.$$

On the other hand, if $|G(z) - \xi| \geq \delta_\varepsilon$, then

$$\frac{G(z) - \xi}{z - y(\xi)} = \frac{|G(z) - \xi|}{|z - y(\xi)|} > \frac{\delta_\varepsilon}{\gamma_\varepsilon} = \frac{1}{\varepsilon}.$$

Thus

$$(3.31) \quad \frac{G(z) - \xi}{z - y(\xi)} > \frac{1}{\varepsilon} \quad \text{whenever } |z - y(\xi)| < \gamma_\varepsilon.$$

In the remainder of this subsection we are going to show that there exists a constant $C > 0$ independent of n and ε such that

$$(3.32) \quad \hat{G}'_n(\hat{y}_n(\xi)) > \frac{C}{\varepsilon} \quad \text{for all } n \text{ such that } \frac{2}{n} < \gamma_\varepsilon,$$

which will prove the claim. Let

$$(3.33) \quad I_0 := \{z \mid |z - \hat{y}_n(\xi)| \leq |y(\xi) - \hat{y}_n(\xi)|\} = \hat{y}_n(\xi) + \{t \mid |t| \leq |y(\xi) - \hat{y}_n(\xi)|\}.$$

Direct computations show that for all $z \in \mathbb{R} \setminus I_0$

$$(3.34) \quad \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) \geq 0$$

and that

$$(3.35) \quad \begin{aligned} & \int_{I_0} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\ &= \int_{-|\hat{y}_n(\xi) - y(\xi)|}^{|\hat{y}_n(\xi) - y(\xi)|} \phi'(nz) \tilde{G}(\hat{y}_n(\xi) - z) dz \\ &= \int_0^{|\hat{y}_n(\xi) - y(\xi)|} \phi'(nz) (\tilde{G}(\hat{y}_n(\xi) - z) - \tilde{G}(\hat{y}_n(\xi) + z)) dz \geq 0, \end{aligned}$$

since both terms in the last integral are non-positive on the interval of integration.

Again, we have to consider two situations separately depending on the difference of $y(\xi)$ and $\hat{y}_n(\xi)$.

(a.I) *The case* $|\hat{y}_n(\xi) - y(\xi)| \leq \frac{1}{2n}$. We only prove (3.32) in the case $y(\xi) \leq \hat{y}_n(\xi) \leq y(\xi) + \frac{1}{2n}$ and leave the other case, which follows the same lines, to the interested reader. Using (3.28), (3.31), (3.34), and (3.35) we have

$$\begin{aligned}
\hat{G}'_n(\hat{y}_n(\xi)) &= n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\quad + n^2 \int_{I_0} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\quad + n^2 \int_{2\hat{y}_n(\xi) - y(\xi)}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\geq n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\geq n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi'(n(\hat{y}_n(\xi) - z)) \frac{z - y(\xi)}{\varepsilon} dz \\
&= \frac{n}{\varepsilon} \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi(n(\hat{y}_n(\xi) - z)) dz \\
&\geq \frac{n}{\varepsilon} \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) - \frac{1}{2n}} \phi(n(\hat{y}_n(\xi) - z)) dz \\
&= \frac{1}{\varepsilon} \int_{\frac{1}{2}}^1 \phi(z) dz = \frac{C}{\varepsilon}
\end{aligned}$$

with $C = \int_{1/2}^1 \phi(z) dz$. Here we applied (3.31) to $\tilde{G}(z) = G(z) - \xi$, which is satisfied since we assume that $\frac{2}{n} < \gamma_\varepsilon$.

(a.II) *The case* $\frac{1}{2n} < |\hat{y}_n(\xi) - y(\xi)| < \frac{1}{n}$. We only prove (3.32) in the case $y(\xi) + \frac{1}{2n} < \hat{y}_n(\xi) < y(\xi) + \frac{1}{n}$ and leave the other case, which follows the same lines, to the interested reader. Due to (3.34) and (3.35) we have

$$\begin{aligned}
\hat{G}'_n(\hat{y}_n(\xi)) &= n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\quad + n^2 \int_{I_0} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\quad + n^2 \int_{2\hat{y}_n(\xi) - y(\xi)}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\geq n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&= n \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi'(z) \tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz.
\end{aligned}$$

Let us turn our attention to the last integral

$$n \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi'(z) \tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz,$$

where $n(\hat{y}_n(\xi) - y(\xi)) \in (\frac{1}{2}, 1)$. Since $\tilde{G}(\hat{y}_n(\xi) - \frac{z}{n})$ is strictly decreasing and $\tilde{G}(\hat{y}_n(\xi) - \frac{z}{n}) \leq 0$ for all $z \in [n(\hat{y}_n(\xi) - y(\xi)), 1]$, we have

$$\begin{aligned} 0 &\leq n \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi'(z) \tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz \\ &= -n \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{z}{n})}^0 \phi'(z) dp dz. \end{aligned}$$

Since the area of integration has finite measure and the integrand is uniformly bounded, we can interchange the order of integration and get

$$\begin{aligned} -n \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{z}{n})}^0 \phi'(z) dp dz \\ = -n \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{1}{n})}^0 \int_{\max(n(\hat{y}_n(\xi) - y(\xi)), n(\hat{y}_n(\xi) - \tilde{G}^{-1}(p))}^1 \phi'(z) dz dp. \end{aligned}$$

Evaluating the inner integral and using that $\phi(z)$ is decreasing on $[0, 1]$, we end up with

$$\begin{aligned} -n \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{1}{n})}^0 \int_{\max(n(\hat{y}_n(\xi) - y(\xi)), n(\hat{y}_n(\xi) - \tilde{G}^{-1}(p))}^1 \phi'(z) dz dp \\ = n \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{1}{n})}^0 \phi(\max(n(\hat{y}_n(\xi) - y(\xi)), n(\hat{y}_n(\xi) - \tilde{G}^{-1}(p))) dp \\ \geq n \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{1}{n})}^0 \int_{\max(n(\hat{y}_n(\xi) - y(\xi)), n(\hat{y}_n(\xi) - \tilde{G}^{-1}(p))}^1 \frac{\phi(z)}{1 - \max(n(\hat{y}_n(\xi) - y(\xi)), n(\hat{y}_n(\xi) - \tilde{G}^{-1}(p)))} dz dp \\ \geq \frac{n}{1 - n(\hat{y}_n(\xi) - y(\xi))} \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{1}{n})}^0 \int_{\max(n(\hat{y}_n(\xi) - y(\xi)), n(\hat{y}_n(\xi) - \tilde{G}^{-1}(p))}^1 \phi(z) dz dp \\ = \frac{n}{1 - n(\hat{y}_n(\xi) - y(\xi))} \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \int_{\tilde{G}(\hat{y}_n(\xi) - \frac{z}{n})}^0 \phi(z) dp dz \\ = -\frac{n}{1 - n(\hat{y}_n(\xi) - y(\xi))} \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi(z) \tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz. \end{aligned}$$

In the last step we used once more that both the area of integration and the integrand are bounded, which justifies once more the interchange of the order of integration. Thus we showed, so far, that

$$\begin{aligned} \hat{G}'_n(\hat{y}_n(\xi)) &\geq n \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi'(z) \tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz \\ &\geq -\frac{n}{1 - n(\hat{y}_n(\xi) - y(\xi))} \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi(z) \tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right) dz. \end{aligned}$$

The last step towards (3.32) is to replace the interval of integration $[n(\hat{y}_n(\xi) - y(\xi)), 1]$ by $[-1, n(\hat{y}_n(\xi) - y(\xi))]$ and to use (3.31). To that end observe that we have

$$(3.36) \quad \int_{-1}^1 \phi(z)G\left(\hat{y}_n(\xi) - \frac{z}{n}\right)dz = \hat{G}_n(\hat{y}_n(\xi)) = \xi = \int_{-1}^1 \phi(z)\xi dz \quad \text{for all } \xi \in \mathbb{R}.$$

Since $\tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right) = G\left(\hat{y}_n(\xi) - \frac{z}{n}\right) - \xi$, we have

$$\int_{-1}^{n(\hat{y}_n(\xi) - y(\xi))} \phi(z)\tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right)dz = - \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi(z)\tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right)dz,$$

and, accordingly,

$$\begin{aligned} \hat{G}'_n(\hat{y}_n(\xi)) &\geq -\frac{n}{1 - n(\hat{y}_n(\xi) - y(\xi))} \int_{n(\hat{y}_n(\xi) - y(\xi))}^1 \phi(z)\tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right)dz \\ &= \frac{n}{1 - n(\hat{y}_n(\xi) - y(\xi))} \int_{-1}^{n(\hat{y}_n(\xi) - y(\xi))} \phi(z)\tilde{G}\left(\hat{y}_n(\xi) - \frac{z}{n}\right)dz \\ &\geq \frac{n}{1 - n(\hat{y}_n(\xi) - y(\xi))} \int_{-1}^0 \phi(z) \frac{\hat{y}_n(\xi) - \frac{z}{n} - y(\xi)}{\varepsilon} dz \\ &\geq \frac{n(\hat{y}_n(\xi) - y(\xi))}{1 - n(\hat{y}_n(\xi) - y(\xi))} \frac{1}{\varepsilon} \int_{-1}^0 \phi(z) dz \\ &\geq \frac{1}{2\varepsilon}, \end{aligned}$$

where we used in the last step that $\frac{1}{2} \leq n(\hat{y}_n(\xi) - y(\xi)) \leq 1$. This finishes the proof of (3.32).

(b) The case $y_\xi(\xi) = c > 0$. By assumption $y_\xi(\xi) = c > 0$ and hence for every $\varepsilon > 0$ there exists some $\delta_\varepsilon > 0$ such that

$$(3.37) \quad (1 - \varepsilon)c < \frac{y(\eta) - y(\xi)}{\eta - \xi} < (1 + \varepsilon)c \quad \text{whenever } |\eta - \xi| \leq \delta_\varepsilon.$$

Let $0 < \varepsilon < 1$ be fixed and define $\gamma_\varepsilon := (1 - \varepsilon)c\delta_\varepsilon$. In addition, let $z \in \mathbb{R}$ be such that $|y(\xi) - z| < \gamma_\varepsilon$. We will first show that $|G(z) - \xi| < \delta_\varepsilon$. Indeed, assume the opposite. Then, due to (3.37), if $G(z) \geq \xi + \delta_\varepsilon$, we have

$$z = y(G(z)) \geq y(\xi + \delta_\varepsilon) \geq y(\xi) + \delta_\varepsilon(1 - \varepsilon)c = y(\xi) + \gamma_\varepsilon,$$

and, if $G(z) \leq \xi - \delta_\varepsilon$, then

$$z = y(G(z)) \leq y(\xi - \delta_\varepsilon) \leq y(\xi) - \delta_\varepsilon(1 - \varepsilon)c = y(\xi) - \gamma_\varepsilon.$$

Together, these estimates contradict $|y(\xi) - z| < \gamma_\varepsilon$, and hence prove that $|G(z) - \xi| < \delta_\varepsilon$.

As an immediate consequence, we obtain

$$(1 - \varepsilon)c < \frac{y(G(z)) - y(\xi)}{G(z) - \xi} < (1 + \varepsilon)c \quad \text{whenever } |z - y(\xi)| < \gamma_\varepsilon,$$

and thus, as $z = y(G(z))$ for all $z \in \mathbb{R}$,

$$(3.38) \quad \frac{1}{(1 + \varepsilon)c} < \frac{G(z) - \xi}{z - y(\xi)} < \frac{1}{(1 - \varepsilon)c} \quad \text{whenever } |z - y(\xi)| < \gamma_\varepsilon.$$

In view of the above inequality (3.38), which will play a key role, we assume without loss of generality that $\frac{2}{n} < \gamma_\varepsilon$ for the rest of this subsection.

The other main ingredient is to establish that $\lim_{n \rightarrow \infty} |n(\hat{y}_n(\xi) - y(\xi))| = 0$. We note here that this fast convergence of $\hat{y}_n(\xi)$ to $y(\xi)$ need not necessarily hold in points ξ where $y_\xi(\xi) = 0$, cf. the remark after this proof. We will only consider the case $\hat{y}_n(\xi) \leq y(\xi)$ and leave the other case, which follows the same lines, to the interested reader. From (3.36), we can deduce that

$$\begin{aligned}
0 &= \int_{-1}^1 \phi(z) \tilde{G} \left(\hat{y}_n(\xi) - \frac{z}{n} \right) dz \\
&= n \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&= n \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\quad + n \int_{y(\xi)}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&\leq n \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{y(\xi)} \phi(n(\hat{y}_n(\xi) - z)) \frac{z - y(\xi)}{(1 + \varepsilon)c} dz \\
&\quad + n \int_{y(\xi)}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi(n(\hat{y}_n(\xi) - z)) \frac{z - y(\xi)}{(1 - \varepsilon)c} dz \\
&= n \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi(n(\hat{y}_n(\xi) - z)) \frac{z - y(\xi)}{(1 + \varepsilon)c} dz \\
&\quad + n \int_{y(\xi)}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi(n(\hat{y}_n(\xi) - z)) \frac{z - y(\xi)}{c} \left[\frac{1}{1 - \varepsilon} - \frac{1}{1 + \varepsilon} \right] dz \\
&= \frac{\hat{y}_n(\xi) - y(\xi)}{(1 + \varepsilon)c} + n \frac{2\varepsilon}{(1 - \varepsilon^2)c} \int_{y(\xi)}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi(n(\hat{y}_n(\xi) - z)) (z - y(\xi)) dz \\
&\leq \frac{\hat{y}_n(\xi) - y(\xi)}{(1 + \varepsilon)c} + n \frac{2\varepsilon}{(1 - \varepsilon^2)c} \int_{\hat{y}_n(\xi)}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi(n(\hat{y}_n(\xi) - z)) \frac{2}{n} dz \\
&\leq \frac{\hat{y}_n(\xi) - y(\xi)}{(1 + \varepsilon)c} + \frac{2\varepsilon}{n(1 - \varepsilon^2)c},
\end{aligned}$$

where we used (3.38). Thus

$$(3.39) \quad 0 \leq y(\xi) - \hat{y}_n(\xi) \leq \frac{2\varepsilon}{n(1 - \varepsilon)},$$

which implies that $\lim_{n \rightarrow \infty} n(y(\xi) - \hat{y}_n(\xi)) = 0$.

Let us return to the term $\hat{G}'_n(\hat{y}_n(\xi))$. We have from (3.28) that

$$\begin{aligned}
\hat{G}'_n(\hat{y}_n(\xi)) &= n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \tilde{G}(z) dz \\
&= n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \frac{z - y(\xi)}{c} dz \\
&\quad + n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \left[\tilde{G}(z) - \frac{z - y(\xi)}{c} \right] dz
\end{aligned}$$

$$= \frac{1}{c} + n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \left[\tilde{G}(z) - \frac{z - y(\xi)}{c} \right] dz.$$

Thus (3.27) will follow if we can show that the last term on the right-hand side tends to 0 as $n \rightarrow \infty$. Now observe that (3.38) implies that

$$\left| \tilde{G}(z) - \frac{z - y(\xi)}{c} \right| = \frac{|z - y(\xi)|}{c} \left| 1 - \frac{c\tilde{G}(z)}{z - y(\xi)} \right| \leq \frac{\varepsilon}{(1 - \varepsilon)c} |z - y(\xi)|,$$

and hence

$$\begin{aligned} & \left| n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \left[\tilde{G}(z) - \frac{z - y(\xi)}{c} \right] dz \right| \\ & \leq \frac{n^2}{c} \frac{\varepsilon}{1 - \varepsilon} \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} |\phi'(n(\hat{y}_n(\xi) - z))| |z - y(\xi)| dz \\ & \leq n^2 \frac{\varepsilon}{(1 - \varepsilon)c} \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) (z - y(\xi)) dz \\ & \quad + 2n^2 \frac{\varepsilon}{(1 - \varepsilon)c} \int_{I_0} |\phi'(n(\hat{y}_n(\xi) - z))| |z - y(\xi)| dz \\ & = \frac{\varepsilon}{(1 - \varepsilon)c} \left(1 + 2n^2 \int_{I_0} |\phi'(n(\hat{y}_n(\xi) - z))| |z - y(\xi)| dz \right), \end{aligned}$$

where we used (3.34).

Recall from (3.33) that for $z \in I_0$ one has

$$\hat{y}_n(\xi) - |\hat{y}_n(\xi) - y(\xi)| \leq z \leq \hat{y}_n(\xi) + |\hat{y}_n(\xi) - y(\xi)|$$

and therefore

$$|z - y(\xi)| \leq 2|\hat{y}_n(\xi) - y(\xi)| \leq \frac{4\varepsilon}{n(1 - \varepsilon)},$$

where we used (3.39).

This implies that

$$\begin{aligned} \left| \hat{G}'_n(\hat{y}_n(\xi)) - \frac{1}{c} \right| & \leq \left| n^2 \int_{\hat{y}_n(\xi) - \frac{1}{n}}^{\hat{y}_n(\xi) + \frac{1}{n}} \phi'(n(\hat{y}_n(\xi) - z)) \left[\tilde{G}(z) - \frac{z - y(\xi)}{c} \right] dz \right| \\ & \leq \frac{\varepsilon}{(1 - \varepsilon)c} \left(1 + \frac{8\varepsilon}{(1 - \varepsilon)} n \int_{I_0} |\phi'(n(\hat{y}_n(\xi) - z))| dz \right) \\ & \leq \frac{\varepsilon}{(1 - \varepsilon)c} \left(1 + \frac{16\varepsilon}{1 - \varepsilon} \phi(0) \right). \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this implies that $\hat{G}'_n(\hat{y}_n(\xi)) \rightarrow \frac{1}{c}$ as $n \rightarrow \infty$.

To summarise, we have in this step shown that $\hat{G}'_n(\hat{y}_n(\xi))$ converges to $1/y_\xi(\xi)$ in every point $\xi \in \mathbb{R}$ where y is differentiable. Thus also $\tilde{h}_n(\xi)$ converges to $h(\xi)$ in all of these points. Together with the fact that $\|\tilde{h}_n\|_{L^1} = \|h\|_{L^1}$ for all n (see (3.26)), this shows that $\|\tilde{h}_n - h\|_{L^1} \rightarrow 0$.

Step 4. Convergence of \tilde{h}_n to h in $L^2(\mathbb{R})$. Recall that

$$(3.40) \quad \tilde{h}_n = 1 - \hat{y}_{n,\xi} \quad \text{and} \quad h = 1 - y_\xi.$$

Since \tilde{h}_n , $\hat{y}_{n,\xi}$, h , and y_ξ all are non-negative, it follows that

$$0 \leq \tilde{h}_n(\xi) \leq 1 \quad \text{and} \quad 0 \leq h(\xi) \leq 1 \quad \text{for all } \xi \in \mathbb{R}.$$

Thus we have

$$\|\tilde{h}_n - h\|_{L^2}^2 \leq \|\tilde{h}_n - h\|_{L^1}.$$

Since we already know that $\tilde{h}_n \rightarrow h$ in $L^1(\mathbb{R})$, the claim follows.

Step 5. Convergence of $\hat{y}_{n,\xi} - 1$ to $y_\xi - 1$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. By definition we have

$$\tilde{h}_n = 1 - \hat{y}_{n,\xi} \quad \text{and} \quad h = 1 - y_\xi.$$

Since $\tilde{h}_n \rightarrow h$ both in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ the claim follows.

Step 6. Convergence of \hat{U}_n to U in $L^2(\mathbb{R})$. A proof can be found in [27, Prop. 5.1].

Step 7. Convergence of $\hat{U}_{n,\xi}$ to U_ξ in $L^2(\mathbb{R})$.

Let $S = \{\xi \in \mathbb{R} \mid y_\xi(\xi) = 0\}$. Then $U_\xi(\xi) = 0$ for almost all $\xi \in S$, since $U_\xi^2 = hy_\xi$ almost everywhere. Thus we have

$$(3.41) \quad \|\hat{U}_{n,\xi} - U_\xi\|_{L^2}^2 = \int_S \hat{U}_{n,\xi}^2(\xi) d\xi + \int_{S^c} (\hat{U}_{n,\xi} - U_\xi)^2(\xi) d\xi.$$

From (3.24) and the fact that $\tilde{r}_n \geq 0$, it follows that we have for almost every $\xi \in S$ that

$$\hat{U}_{n,\xi}^2(\xi) \leq \tilde{h}_n(\xi) \hat{y}_{n,\xi}(\xi) = \tilde{h}_n(\xi) (\hat{y}_{n,\xi} - y_\xi)(\xi) = \tilde{h}_n(\xi) (h - \tilde{h}_n)(\xi),$$

and, therefore,

$$\int_S \hat{U}_{n,\xi}^2(\xi) d\xi \leq \|h - \tilde{h}_n\|_{L^1},$$

since $\|\tilde{h}_n\|_{L^\infty} \leq 1$. Thus the first integral in (3.41) tends to 0 as $n \rightarrow \infty$.

As far as the integral over S^c is concerned, the proof of the convergence follows closely the one of $\tilde{r}_n \rightarrow \tilde{r}$ in $L^2(\mathbb{R})$ as $n \rightarrow \infty$ in [18, Lemma 6.4], which we reproduce here for completeness. Note that by definition we have $\hat{U}_{n,\xi}(\xi) = u_{n,x}(\hat{y}_n(\xi)) \hat{y}_{n,\xi}(\xi)$ and $U_\xi(\xi) = u_x(y(\xi)) y_\xi(\xi)$ for almost every $\xi \in S^c$, so that

$$(3.42) \quad \begin{aligned} \|U_{n,\xi} - U_\xi\|_{L^2(S^c)}^2 &= \|(u_{n,x} \circ \hat{y}_n) \hat{y}_{n,\xi} - (u_x \circ y) y_\xi\|_{L^2(S^c)}^2 \\ &= \int_{S^c} (u_{n,x} \circ \hat{y}_n)^2 \hat{y}_{n,\xi} (\hat{y}_{n,\xi} - y_\xi) d\xi \\ &\quad + \int_{S^c} (u_{n,x} \circ \hat{y}_n) \hat{y}_{n,\xi} (u_{n,x} \circ \hat{y}_n - u_{n,x} \circ y) y_\xi d\xi \\ &\quad + \int_{S^c} (u_{n,x} \circ \hat{y}_n) \hat{y}_{n,\xi} (u_{n,x} \circ y - u_x \circ y) y_\xi d\xi \\ &\quad + \int_{S^c} (u_x \circ y)^2 y_\xi (y_\xi - \hat{y}_{n,\xi}) d\xi \\ &\quad + \int_{S^c} (u_x \circ y) y_\xi (u_x \circ y - u_x \circ \hat{y}_n) \hat{y}_{n,\xi} d\xi \\ &\quad + \int_{S^c} (u_x \circ y) y_\xi (u_x \circ \hat{y}_n - u_{n,x} \circ \hat{y}_n) \hat{y}_{n,\xi} d\xi. \end{aligned}$$

The first and the fourth term have the same structure, and we therefore only treat the first one. Since $(u_{n,x} \circ \hat{y}_n)^2 \hat{y}_{n,\xi} \leq \tilde{h}_n \leq 1$, we have

$$\|(u_{n,x} \circ \hat{y}_n)^2 \hat{y}_{n,\xi} (\hat{y}_{n,\xi} - y_\xi)\|_{L^1(S^c)} \leq \|y_\xi - \hat{y}_{n,\xi}\|_{L^1}$$

and thus this term tends to 0 as $n \rightarrow \infty$. In order to investigate the fifth term we will use that $u_x \in L^2(\mathbb{R})$ and therefore there exists for any $\varepsilon > 0$ a continuous function \tilde{l} with compact support such that $\|u_x - \tilde{l}\|_{L^2} \leq \varepsilon / (3 \max(1, \|u_x\|_{L^2}))$. Thus we can write

$$\begin{aligned}
(3.43) \quad & \|(u_x \circ y)y_\xi(u_x \circ y - u_x \circ \hat{y}_n)\hat{y}_{n,\xi}\|_{L^1(S^c)} \\
& \leq \|(u_x \circ y)y_\xi(u_x \circ y - \tilde{l} \circ y)\hat{y}_{n,\xi}\|_{L^1(S^c)} \\
& \quad + \|(u_x \circ y)y_\xi(\tilde{l} \circ y - \tilde{l} \circ \hat{y}_n)\hat{y}_{n,\xi}\|_{L^1(S^c)} \\
& \quad + \|(u_x \circ y)y_\xi(\tilde{l} \circ \hat{y}_n - u_x \circ \hat{y}_n)\hat{y}_{n,\xi}\|_{L^1(S^c)} \\
& \leq \|u_x\|_{L^2} \left(2\|u_x - \tilde{l}\|_{L^2} + \|\tilde{l} \circ \hat{y}_n - \tilde{l} \circ y\|_{L^2} \right).
\end{aligned}$$

Here we have used in the last inequality that both y_ξ and $\hat{y}_{n,\xi}$ are non-negative and bounded above by 1. Since $\hat{y}_n - y \rightarrow 0$ in $L^\infty(\mathbb{R})$ and \tilde{l} is continuous with compact support, we obtain by Lebesgue's dominated convergence theorem that $\tilde{l} \circ \hat{y}_n \rightarrow \tilde{l} \circ y$ in $L^2(\mathbb{R})$. In particular, we can choose n large enough so that $\|(u_x \circ y)y_\xi(u_x \circ y - u_x \circ \hat{y}_n)\hat{y}_{n,\xi}\|_{L^1(S^c)} \leq \varepsilon$. Since ε can be chosen arbitrarily small, we obtain in particular that

$$(3.44) \quad \lim_{n \rightarrow \infty} \|(u_x \circ y)y_\xi(u_x \circ y - u_x \circ \hat{y}_n)\hat{y}_{n,\xi}\|_{L^1(S^c)} = 0.$$

For the convergence of the second term, we estimate, using again that y_ξ is bounded by 1,

$$\begin{aligned}
& \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}(u_{n,x} \circ \hat{y}_n - u_{n,x} \circ y)y_\xi\|_{L^1(S^c)} \\
& \leq \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}(u_{n,x} \circ \hat{y}_n - u_x \circ \hat{y}_n)y_\xi\|_{L^1(S^c)} \\
& \quad + \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}(u_x \circ \hat{y}_n - u_x \circ y)y_\xi\|_{L^1(S^c)} \\
& \quad + \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}(u_x \circ y - u_{n,x} \circ y)y_\xi\|_{L^1(S^c)} \\
& \leq \left(\int_{S^c} (u_{n,x} \circ \hat{y}_n)^2 \hat{y}_{n,\xi} d\xi \right)^{1/2} \left(\int_{S^c} (u_{n,x} \circ \hat{y}_n - u_x \circ \hat{y}_n)^2 \hat{y}_{n,\xi} d\xi \right)^{1/2} \\
& \quad + \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}(u_x \circ \hat{y}_n - u_x \circ y)y_\xi\|_{L^1(S^c)} \\
& \quad + \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}\|_{L^2} \|(u_x \circ y - u_{n,x} \circ y)y_\xi\|_{L^2}.
\end{aligned}$$

The first and third term in this last estimate tend to zero because $u_{n,x} \rightarrow u_x \in L^2(\mathbb{R})$ and both y_ξ and $\hat{y}_{n,\xi}$ are uniformly bounded, and for the convergence of the second term we can use the same method as in (3.43). Thus also the second term in (3.42) tends to zero. As far as the third (and, similarly, the last) term in (3.42) is concerned, we have that

$$\begin{aligned}
& \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}(u_{n,x} \circ y - u_x \circ y)y_\xi\|_{L^1(S^c)} \\
& \leq \|(u_{n,x} \circ \hat{y}_n)\hat{y}_{n,\xi}\|_{L^2(S^c)} \|(u_{n,x} \circ y - u_x \circ y)y_\xi\|_{L^2(S^c)} \\
& \leq \|u_{n,x}\|_{L^2} \|u_{n,x} - u_x\|_{L^2},
\end{aligned}$$

which again tends to zero since by assumption $u_{n,x} \rightarrow u_x \in L^2(\mathbb{R})$. Hence all terms in (3.42) tend to 0 as $n \rightarrow \infty$ and therefore $\hat{U}_{n,\xi} \rightarrow U_\xi$ in $L^2(\mathbb{R})$.

Step 8. Convergence of \hat{r}_n to zero in $L^2(\mathbb{R})$. By construction, we have $\bar{r}_n \geq 0$ since $\bar{\rho}_n \geq 0$ and $\hat{y}_{n,\xi} \geq 0$. Hence, by (3.4c), (3.7b), (3.24), and (3.40), we

have

$$\begin{aligned}
\|\tilde{r}_n\|_{L^2}^2 &\leq \left\| \tilde{r}_n^2 + \frac{2}{n} \tilde{r}_n \hat{y}_{n,\xi} \right\|_{L^1} \\
&= \|\tilde{h}_n \hat{y}_{n,\xi} - h y_\xi - \hat{U}_{n,\xi}^2 + U_\xi^2\|_{L^1} \\
&= \|\tilde{h}_n - h - \tilde{h}_n^2 + h^2 - \hat{U}_{n,\xi}^2 + U_\xi^2\|_{L^1} \\
&\leq 3\|h - \tilde{h}_n\|_{L^1} + (\|\hat{U}_{n,\xi}\|_{L^2} + \|U_\xi\|_{L^2}) \|\hat{U}_{n,\xi} - U_\xi\|_{L^2} \\
&\leq 3\|h - \tilde{h}_n\|_{L^1} + 2\|\mu\| \|\hat{U}_{n,\xi} - U_\xi\|_{L^2}.
\end{aligned}$$

Since $\tilde{h}_n \rightarrow h$ in $L^1(\mathbb{R})$ and $\hat{U}_{n,\xi} \rightarrow U_\xi$ in $L^2(\mathbb{R})$, the above estimate implies that $\tilde{r}_n \rightarrow 0$ in $L^2(\mathbb{R})$ as $n \rightarrow \infty$.

Step 9. Convergence of \hat{h}_n to h in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. According to (3.7b) and (3.9b), we have

$$\|\hat{h}_n - h\|_{L^2} \leq \frac{2}{n} \|\tilde{r}_n\|_{L^2} + \|\hat{y}_{n,\xi} - y_\xi\|_{L^2}.$$

Since $\tilde{r}_n \rightarrow 0$ and $\hat{y}_{n,\xi} - y_\xi \rightarrow 0$ in $L^2(\mathbb{R})$, this inequality implies that $\hat{h}_n \rightarrow h$ in $L^2(\mathbb{R})$. As far as the $L^1(\mathbb{R})$ convergence is concerned, observe that

$$\begin{aligned}
\hat{h}_n - h &= \hat{h}_n(\hat{y}_{n,\xi} + \tilde{h}_n) - h(y_\xi + h) \\
&= \hat{h}_n \hat{y}_{n,\xi} - h y_\xi + \hat{h}_n \tilde{h}_n - h^2 \\
&= \hat{U}_{n,\xi}^2 - U_\xi^2 + \tilde{r}_n^2 + \hat{h}_n \tilde{h}_n - h^2.
\end{aligned}$$

Here the first equality follows from (3.7b) and (3.23), and the last equality follows from (3.4c) and (3.24). Thus

$$\begin{aligned}
\|\hat{h}_n - h\|_{L^1} &\leq \|\hat{U}_{n,\xi} + U_\xi\|_{L^2} \|\hat{U}_{n,\xi} - U_\xi\|_{L^2} + \|\tilde{r}_n\|_{L^2}^2 \\
&\quad + \|\tilde{h}_n\|_{L^2} \|\hat{h}_n - h\|_{L^2} + \|h\|_{L^2} \|\tilde{h}_n - h\|_{L^2},
\end{aligned}$$

which implies that $\hat{h}_n \rightarrow h$ in $L^1(\mathbb{R})$.

Step 10. Convergence of X_n to X in E . So far we have shown that $\hat{X}_n \rightarrow X$ in E . In addition we showed in Step 1 for all $n \in \mathbb{N}$ that we can write $\hat{X}_n = X_n \circ g_n$ with $g_n \in G_\kappa$ for some $\kappa > 0$ independent of n , or, equivalently that $\hat{X}_n \in \mathcal{F}_\kappa$ for all $n \in \mathbb{N}$. Moreover, it is known, see, e.g., [18, Lemma 4.6], that the mapping $\Gamma: \mathcal{F}_\kappa \rightarrow \mathcal{F}_0$ defined in (3.6) is continuous. Thus we also have that $X_n = \Gamma(\hat{X}_n) \rightarrow X$ in E , which completes the proof. \square

Remark 3.5. A closer look at the proof of Theorem 3.4 reveals that we showed that for every $\xi \in \mathbb{R}$ where y is differentiable and $y_\xi > 0$, we have

$$\lim_{n \rightarrow \infty} n(\hat{y}_n(\xi) - y(\xi)) = 0.$$

As the following example illustrates, we cannot expect this convergence to hold for almost every $\xi \in \mathbb{R}$ such that $y_\xi(\xi) = 0$.

Consider the following initial data for the CH equation which corresponds to a symmetric/antisymmetric peakon-antipeakon solution, which vanishes at breaking time $t = 0$, i.e.,

$$u(x) = 0 \text{ for all } x \in \mathbb{R} \quad \text{and} \quad F(x) = \mu((-\infty, x)) = \begin{cases} 0, & x < 0, \\ \alpha, & 0 \leq x, \end{cases}$$

where $\alpha > 0$. Then

$$y(\xi) = \begin{cases} \xi, & \xi < 0, \\ 0, & 0 \leq \xi \leq \alpha, \\ \xi - \alpha, & \alpha < \xi. \end{cases}$$

and especially $y_\xi(\xi) = 0$ for all $\xi \in (0, \alpha)$. For the approximating sequence we know that

$$\hat{y}_n(\xi) + \hat{F}_n(\hat{y}_n(\xi)) = \xi \quad \text{for all } \xi \in \mathbb{R},$$

where

$$\hat{F}_n(x) = \int_{\mathbb{R}} n\phi(n(x-y))F(y)dy.$$

We are going to show that $\lim_{n \rightarrow \infty} n(\hat{y}_n(\xi) - y(\xi)) \neq 0$ for any $\xi \in (0, \alpha)$ except $\xi = \frac{\alpha}{2}$.

Indeed, if we denote

$$\Phi(x) := \int_{-\infty}^x \phi(y) dy,$$

then we see that

$$\hat{F}_n(x) = \int_{\mathbb{R}} n\phi(n(x-y))F(y)dy = \alpha \int_0^\infty n\phi(n(x-y))dy = \alpha\Phi(nx)$$

for all $x \in \mathbb{R}$. Now assume that $0 < \xi < \alpha$. Then $y(\xi) = 0$ and thus

$$\xi = \hat{y}_n(\xi) + \hat{F}_n(\hat{y}_n(\xi)) = \hat{y}_n(\xi) + \alpha\Phi(n\hat{y}_n(\xi)) = \hat{y}_n(\xi) - y(\xi) + \alpha\Phi(n(\hat{y}_n(\xi) - y(\xi))).$$

In Step 2 in the proof of Theorem 3.4 we have shown that $\hat{y}_n(\xi) \rightarrow y(\xi)$. Taking the limit $n \rightarrow \infty$ in the previous equation therefore implies that

$$\frac{\xi}{\alpha} = \lim_{n \rightarrow \infty} \Phi(n(\hat{y}_n(\xi) - y(\xi))).$$

Using that $|n(\hat{y}_n(\xi) - y(\xi))| < 1$ for all n and that Φ is continuously invertible on $(-1, 1)$, it follows that

$$\lim_{n \rightarrow \infty} n(\hat{y}_n(\xi) - y(\xi)) = \Phi^{-1}(\xi/\alpha).$$

Since $\Phi(0) = 1/2$ and therefore $\Phi^{-1}(1/2) = 0$, this shows in particular that the sequence $n(\hat{y}_n(\xi) - y(\xi))$ only converges to 0 for $\xi = \alpha/2$.

4. CONVERGENCE IN LAGRANGIAN COORDINATES IMPLIES CONVERGENCE IN EULERIAN COORDINATES

In the previous two sections, we saw that we can approximate any given initial data (u, μ) for the CH equation by a sequence of smooth initial data (u_n, ρ_n, μ_n) for the 2CH system where the measures μ_n are purely absolutely continuous. Afterwards we saw that this convergence in Eulerian coordinates is transported, via the mapping L , to convergence in Lagrangian coordinates.

In this section we consider the case when we are given a sequence $X_n \in \mathcal{F}_0$ and $X \in \mathcal{F}_0$, such that $X_n \rightarrow X$ in E . Does $M(X_n) \rightarrow M(X)$ in some sense in Eulerian coordinates? Here $M: \mathcal{F}_0 \rightarrow \mathcal{D}$ denotes the mapping from Lagrangian to Eulerian coordinates, which is defined as follows.

Definition 4.1 ([18, Thm. 4.10]). *Given any element $X = (y, U, h, r) \in \mathcal{F}_0$, we define (u, ρ, μ) as follows³*

$$(4.1a) \quad u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi),$$

$$(4.1b) \quad \mu = y_{\#}(h(\xi) d\xi),$$

$$(4.1c) \quad \bar{\rho}(x) dx = y_{\#}(\bar{r}(\xi) d\xi),$$

$$(4.1d) \quad \rho(x) = k + \bar{\rho}(x),$$

where k is implicitly given through the relation $r(\xi) = \bar{r}(\xi) + ky_{\xi}(\xi)$ for all $\xi \in \mathbb{R}$. We have that (u, ρ, μ) belongs to \mathcal{D} and, in particular, that the measure $y_{\#}(\bar{r}(\xi) d\xi)$ is absolutely continuous. We denote by $M: \mathcal{F}_0 \rightarrow \mathcal{D}$ the mapping which to any X in \mathcal{F}_0 associates the element $(u, \rho, \mu) \in \mathcal{D}$ as given by (4.1).

We recall from Definition 3.2 that for any $(y, U, h, r) \in \mathcal{F}_0$ we have that $y_{\xi} \geq 0$ and $U_{\xi} = 0$ whenever $y_{\xi} = 0$, or, in other words, that U is constant whenever the increasing function y is constant. As a consequence, the value $U(\xi)$ is uniquely determined by $y(\xi)$, which means that the definition of the function u in (4.1a) is independent of the choice of ξ satisfying $x = y(\xi)$. Also, the fact that y is Lipschitz continuous (see (3.4a)) implies that the push-forward of the absolutely continuous measure $\bar{r}(\xi) d\xi$ under y is again absolutely continuous, cf. [18, Thm. 4.10].

Moreover, we consider the following notion of sequential convergence on \mathcal{D} .

Definition 4.2. *We say that a sequence $(u_n, \rho_n, \mu_n) \in \mathcal{D}$ converges to $(u, \rho, \mu) \in \mathcal{D}$ as $n \rightarrow \infty$ if⁴*

$$(4.2a) \quad u_n \rightarrow u \in L^2(\mathbb{R}) \cap L^{\infty}(\mathbb{R}),$$

$$(4.2b) \quad u_{n,x} \rightharpoonup u_x,$$

$$(4.2c) \quad \bar{\rho}_n \rightarrow \bar{\rho},$$

$$(4.2d) \quad k_n \rightarrow k,$$

$$(4.2e) \quad \int_{\mathbb{R}} \frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx \rightarrow \int_{\mathbb{R}} \frac{u_x^2(x)}{1 + u_x^2(x) + \bar{\rho}^2(x)} dx,$$

$$(4.2f) \quad \int_{\mathbb{R}} \frac{\bar{\rho}_n^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx \rightarrow \int_{\mathbb{R}} \frac{\bar{\rho}^2(x)}{1 + u_x^2(x) + \bar{\rho}^2(x)} dx,$$

$$(4.2g) \quad F_n(x) \rightarrow F(x) \text{ for every } x \text{ at which } F \text{ is continuous,}$$

$$(4.2h) \quad F_n(\infty) \rightarrow F(\infty),$$

where $F_n(x) = \mu_n((-\infty, x])$ for all $n \in \mathbb{N}$ and $F(x) = \mu((-\infty, x])$.

With this definition, the convergence result can be stated as follows.

Theorem 4.3. *Given a sequence $X_n = (y_n, U_n, h_n, r_n) \in \mathcal{F}_0$ and $X = (y, U, h, r) \in \mathcal{F}_0$ such that $X_n \rightarrow X$ in E as $n \rightarrow \infty$, then $(u_n, \rho_n, \mu_n) = M(X_n)$ converges to $(u, \rho, \mu) = M(X)$ in the sense of Definition 4.2.*

Proof. The proof is divided into 8 steps for convenience.

Step 1. Convergence of u_n to u in $L^{\infty}(\mathbb{R})$. For a proof we refer the interested reader to [18, Thm. 6.5].

³Here we denote by $y_{\#}(h d\xi)$ the push-forward of the measure $h d\xi$ by y , defined by $y_{\#}(h d\xi)(A) = \int_{y^{-1}(A)} h(\xi) d\xi$ for all Borel sets $A \subset \mathbb{R}$.

⁴We say that $f_n \rightharpoonup f$ if $\int_{\mathbb{R}} f_n(x)g(x)dx \rightarrow \int_{\mathbb{R}} f(x)g(x)dx$ for every $g \in L^2(\mathbb{R})$.

Step 2. Convergence of u_n to u in $L^2(\mathbb{R})$. If we can show that the assumptions of the Radon–Riesz theorem are fulfilled, see, e.g., [1, Thm. 1.37], the claim follows. Thus we have to show that $\|u_n\|_{L^2} \rightarrow \|u\|_{L^2}$ and that u_n converges weakly to u as $n \rightarrow \infty$.

A straightforward computation using (4.1a) yields

$$(4.3) \quad \|u_n\|_{L^2}^2 = \int_{\mathbb{R}} (U_n^2 - U^2)y_{n,\xi}(\xi)d\xi + \int_{\mathbb{R}} U^2(y_{n,\xi} - y_\xi)(\xi)d\xi + \|u\|_{L^2}^2,$$

where we have used that $U^2 y_\xi(\xi) = 0$ whenever $y_\xi(\xi) = 0$, and similarly that $U_n^2 y_{n,\xi}(\xi) = 0$ whenever $y_{n,\xi}(\xi) = 0$. Applying the Cauchy–Schwarz inequality to the first and second term on the right-hand side of (4.3) yields that

$$\begin{aligned} & \left| \int_{\mathbb{R}} (U_n^2 - U^2)y_{n,\xi}(\xi)d\xi \right| + \left| \int_{\mathbb{R}} U^2(y_{n,\xi} - y_\xi)(\xi)d\xi \right| \\ & \leq (\|U_n - U\|_{L^2} + 2\|U\|_{L^2}) \|U_n - U\|_{L^2} + \|U\|_{L^\infty} \|U\|_{L^2} \|y_{n,\xi} - y_\xi\|_{L^2}, \end{aligned}$$

where we used that $0 \leq y_{n,\xi} \leq 1$. Since $U_n \rightarrow U$ in $L^2(\mathbb{R})$ and $\zeta_{n,\xi} \rightarrow \zeta_\xi$ in $L^2(\mathbb{R})$, we obtain from (4.3) that $\|u_n\|_{L^2} \rightarrow \|u\|_{L^2}$ as $n \rightarrow \infty$.

Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and $\|u_n\|_{L^2} \rightarrow \|u\|_{L^2}$, it suffices to show that

$$(4.4) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} u_n \psi(x) dx = \int_{\mathbb{R}} u \psi(x) dx$$

for all $\psi \in C_c^\infty(\mathbb{R})$. This however follows immediately, as

$$\left| \int_{\mathbb{R}} (u_n - u) \psi(x) dx \right| \leq \|u_n - u\|_{L^\infty} \|\psi\|_{L^1} \rightarrow 0$$

according to Step 1.

Step 3. Convergence of $F_n(x)$ to $F(x)$ for all x at which $F(x)$ is continuous. According to [11, Props. 7.19 and 8.17], this is equivalent to showing that

$$(4.5) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \psi(x) d\mu_n(x) = \int_{\mathbb{R}} \psi(x) d\mu(x)$$

for all $\psi \in C_c^\infty(\mathbb{R})$. It follows from (4.1b) that

$$(4.6) \quad \int_{\mathbb{R}} \psi d\mu_n(x) = \int_{\mathbb{R}} \psi(y_n(\xi)) h_n(\xi) d\xi,$$

and

$$(4.7) \quad \int_{\mathbb{R}} \psi d\mu(x) = \int_{\mathbb{R}} \psi(y(\xi)) h(\xi) d\xi.$$

Since $y_n - \text{Id} \rightarrow y - \text{Id}$ in $L^\infty(\mathbb{R})$, the support of $\psi \circ y_n$ is contained in some compact set which can be chosen independently of n , and, from Lebesgue’s dominated convergence theorem, we have that $\psi \circ y_n \rightarrow \psi \circ y$ in $L^2(\mathbb{R})$. Hence, since $h_n \rightarrow h$ in $L^2(\mathbb{R})$,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \psi(y_n(\xi)) h_n(\xi) d\xi = \int_{\mathbb{R}} \psi(y(\xi)) h(\xi) d\xi,$$

and (4.5) follows from (4.6) and (4.7).

Note that, in particular, $\mu_n(\mathbb{R}) \rightarrow \mu(\mathbb{R})$ as $n \rightarrow \infty$, since $\|h_n\|_{L^1} \rightarrow \|h\|_{L^1}$ by assumption. Moreover,

$$\mu_{n,\text{ac}} = (u_{n,x}^2 + \bar{\rho}_n^2) dx \quad \text{and} \quad \mu_{\text{ac}} = (u_x^2 + \bar{\rho}^2) dx,$$

which implies

$$(4.8) \quad \|u_{n,x}\|_{L^2}^2 + \|\bar{\rho}_n\|_{L^2}^2 \leq \mu_n(\mathbb{R}) \quad \text{and} \quad \|u_x\|_{L^2}^2 + \|\bar{\rho}\|_{L^2}^2 \leq \mu(\mathbb{R}),$$

and hence $u_{n,x}$, u_x , $\bar{\rho}_n$ and $\bar{\rho}$ belong to $L^2(\mathbb{R})$.

Step 4. Weak convergence of $u_{n,x}$ to u_x . Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and $\|u_{n,x}\|_{L^2}$ and $\|u_x\|_{L^2}$ can be uniformly bounded according to (4.8), it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} u_{n,x}(x) \psi(x) dx = \int_{\mathbb{R}} u_x(x) \psi(x) dx$$

for all $\psi \in C_c^\infty(\mathbb{R})$. To that end, observe that

$$\int_{\mathbb{R}} u_{n,x}(x) \psi(x) dx = \int_{\mathbb{R}} U_{n,\xi}(\xi) \psi(y_n(\xi)) d\xi,$$

since $U_{n,\xi}(\xi) = 0$ for all $\xi \in \mathbb{R}$ such that $y_n(\xi) = 0$, and

$$\int_{\mathbb{R}} u_x(x) \psi(x) dx = \int_{\mathbb{R}} U_\xi(\xi) \psi(y(\xi)) d\xi.$$

Thus it suffices to show that

$$(4.9) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} U_{n,\xi}(\xi) \psi(y_n(\xi)) d\xi = \int_{\mathbb{R}} U_\xi(\xi) \psi(y(\xi)) d\xi,$$

for all $\psi \in C_c^\infty(\mathbb{R})$. By assumption we have that $U_{n,\xi} \rightarrow U_\xi$ in $L^2(\mathbb{R})$ and $y_n - \text{Id} \rightarrow y - \text{Id}$ in $L^\infty(\mathbb{R})$ and hence the support of $\psi(y_n(\xi))$ and $\psi(y(\xi))$ is contained in some compact set that can be chosen independent of n . Thus $\psi(y_n(\xi)) \rightarrow \psi(y(\xi))$ in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, and Lebesgue's dominated convergence theorem implies (4.9).

Step 5. Weak convergence of $\bar{\rho}_n$ to $\bar{\rho}$. The argument closely follows the one of $u_{n,x}$ convergences weakly to u_x in Step 4.

Step 6. Convergence of $\int_{\mathbb{R}} \frac{u_{n,x}^2(x)}{1+u_{n,x}^2(x)+\bar{\rho}_n^2(x)} dx$ to $\int_{\mathbb{R}} \frac{u_x^2(x)}{1+u_x^2(x)+\bar{\rho}^2(x)} dx$.

Let $S = \{\xi \in \mathbb{R} \mid y_\xi(\xi) = 0\}$ and $S_n = \{\xi \in \mathbb{R} \mid y_{n,\xi}(\xi) = 0\}$. Furthermore, let $B_n = y_n(S)$. Then we claim that $\text{meas}(B_n) \rightarrow 0$ as $n \rightarrow \infty$.

By definition, we have that $S = \{\xi \in \mathbb{R} \mid y_\xi(\xi) = 0\} = \{\xi \in \mathbb{R} \mid h(\xi) = 1\}$, which implies that $\text{meas}(S) \leq \|h\|_{L^1}$. Thus

$$\begin{aligned} \text{meas}(B_n) &= \text{meas}(y_n(S)) = \int_{y_n(S)} dx \\ &= \int_S y_{n,\xi}(\xi) d\xi = \int_S (y_{n,\xi}(\xi) - y_\xi(\xi)) d\xi \\ &\leq \text{meas}(S)^{1/2} \|y_{n,\xi} - y_\xi\|_{L^2} \\ &\leq \|h\|_{L^1}^{1/2} \|y_{n,\xi} - y_\xi\|_{L^2}. \end{aligned}$$

By assumption $y_{n,\xi} - y_\xi \rightarrow 0$ in $L^2(\mathbb{R})$, and hence $\text{meas}(B_n)$ tends to 0 as $n \rightarrow \infty$. Moreover,

$$U_{n,\xi}(\xi) = 0 \text{ for } \xi \in S_n \quad \text{and} \quad U_\xi(\xi) = 0 \text{ for } \xi \in S.$$

As far as $y_{n,\xi}(\xi)$ and $y_\xi(\xi)$ are concerned, we have the representations

$$y_{n,\xi}(\xi) = \frac{1}{1 + u_{n,x}^2(y_n(\xi)) + \bar{\rho}_n^2(y_n(\xi))} \quad \text{for almost every } \xi \in S_n^c$$

and

$$y_\xi(\xi) = \frac{1}{1 + u_x^2(y(\xi)) + \bar{\rho}^2(y(\xi))} \quad \text{for almost every } \xi \in S^c.$$

This means, in particular, that

$$\begin{aligned}
\int_{S^c} U_{n,\xi}^2(\xi) d\xi &= \int_{S^c \cap S_n^c} U_{n,\xi}^2(\xi) d\xi \\
&= \int_{S^c \cap S_n^c} u_{n,x}^2(y_n(\xi)) y_{n,\xi}^2(\xi) d\xi \\
&= \int_{S^c \cap S_n^c} \frac{u_{n,x}^2(y_n(\xi))}{1 + u_{n,x}^2(y_n(\xi)) + \bar{\rho}_n^2(y_n(\xi))} y_{n,\xi}^2(\xi) d\xi \\
&= \int_{B_n^c \cap y_n(S_n^c)} \frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx \\
&= \int_{B_n^c} \frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx,
\end{aligned}$$

since y_n is Lipschitz continuous and therefore $\text{meas}(y_n(S_n)) = 0$ for all $n \in \mathbb{N}$. Similarly, one obtains

$$\int_{S^c} U_\xi^2(\xi) d\xi = \int_{\mathbb{R}} \frac{u_x^2(x)}{1 + u_x^2(x) + \bar{\rho}^2(x)} dx,$$

as $\text{meas}(y(S)) = 0$. Since $U_{n,\xi} \rightarrow U_\xi$ in $L^2(\mathbb{R})$ we find that

$$\lim_{n \rightarrow \infty} \int_{B_n^c} \frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx = \int_{\mathbb{R}} \frac{u_x^2(x)}{1 + u_x^2(x) + \bar{\rho}^2(x)} dx.$$

Furthermore, note that $\frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)}$ is uniformly bounded by 1 for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. This means, in particular, that

$$\int_{B_n} \frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx \leq \text{meas}(B_n),$$

and thus the term on the left-hand side converges to 0 as $n \rightarrow \infty$ since $\text{meas}(B_n) \rightarrow 0$. Thus we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx = \int_{\mathbb{R}} \frac{u_x^2(x)}{1 + u_x^2(x) + \bar{\rho}^2(x)} dx.$$

Step 7. Convergence of $\int_{\mathbb{R}} \frac{\bar{\rho}_n^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx$ to $\int_{\mathbb{R}} \frac{\bar{\rho}^2(x)}{1 + u_x^2(x) + \bar{\rho}^2(x)} dx$. The argument is similar to the one in Step 6.

Step 8. Convergence of k_n to k . By definition we have

$$r(\xi) = \bar{r}(\xi) + k y_\xi(\xi)$$

and

$$r_n(\xi) = \bar{r}_n(\xi) + k_n y_{n,\xi}(\xi).$$

By assumption X_n converges to X in E , and thus according to (3.1), we infer that $k_n \rightarrow k$. □

Remark 4.4. Note that the convergence in Lagrangian coordinates implies in particular that $\bar{\rho}_n$ converges to $\bar{\rho}$ weakly. Thus, in the special case $k = 0$ and $r(\xi) = \bar{r}(\xi) = 0$ for all $\xi \in \mathbb{R}$, we infer that $\bar{\rho}_n$ converges weakly to zero and $k_n \rightarrow 0$. Thus $\rho(x) = 0$ for all $x \in \mathbb{R}$, and hence $(u, \rho, \mu) = (u, 0, \mu)$ belongs to the set of Eulerian coordinates for the CH equation. Thus the sequence $\bar{\rho}_n$ in Theorem 2.2

converges to zero in the weak sense, since all the assumptions in Theorem 4.3 are satisfied due to Theorem 3.4.

5. CONVERGENCE IN EULERIAN COORDINATES IMPLIES CONVERGENCE IN LAGRANGIAN COORDINATES

In this section we want to show that convergence in Eulerian coordinates implies convergence in Lagrangian coordinates. Due to the definition of Eulerian coordinates, one might guess that it is natural to impose only the condition $u_n \rightarrow u$ in $H^1(\mathbb{R})$. However, due to Theorem 4.3 we will require a somewhat stronger mode of convergence for $u_{n,x}$ to u_x , which in the end yields an equivalence between convergence in Eulerian and Lagrangian coordinates.

Theorem 5.1. *Given a sequence $(u_n, \rho_n, \mu_n) \in \mathcal{D}$ and $(u, \rho, \mu) \in \mathcal{D}$ such that (u_n, ρ_n, μ_n) converges to (u, ρ, μ) as $n \rightarrow \infty$ in the sense of Definition 4.2, let $X_n = (y_n, U_n, h_n, r_n) = L((u_n, \rho_n, \mu_n)) \in \mathcal{F}_0$ and $X = (y, U, h, r) = L((u, \rho, \mu)) \in \mathcal{F}_0$. Then $X_n \rightarrow X$ in E as $n \rightarrow \infty$.*

Proof. The proof is divided into 7 steps.

Step 1. The sequence y_n converges pointwise to y . Denote $D = \{\xi \in \mathbb{R} \mid F \text{ is discontinuous at } y(\xi)\}$. By construction we have for all $\xi \in D^c$ that

$$(5.1) \quad F(y(\xi)) + y(\xi) = \xi,$$

and, in particular,

$$(5.2) \quad F(y(\xi)) = \mu((-\infty, y(\xi))) = \mu((-\infty, y(\xi))).$$

As far as $y_n(\xi)$ is concerned, we have by (3.7a) that

$$(5.3) \quad \mu_n((-\infty, y_n(\xi))) + y_n(\xi) \leq \xi \leq \mu_n((-\infty, y_n(\xi))) + y_n(\xi) = F_n(y_n(\xi)) + y_n(\xi).$$

To show the pointwise convergence of $y_n(\xi)$ to $y(\xi)$ for $\xi \in D^c$, we have to distinguish two cases.

For $y_n(\xi) \leq y(\xi)$, combining (5.1)–(5.3) yields

$$\begin{aligned} F(y(\xi)) + y(\xi) = \xi &\leq F_n(y_n(\xi)) + y_n(\xi) \\ &= F_n(y(\xi)) - \mu_n((y_n(\xi), y(\xi))) + y_n(\xi). \end{aligned}$$

Thus

$$0 \leq y(\xi) - y_n(\xi) + \mu_n((y_n(\xi), y(\xi))) \leq F_n(y(\xi)) - F(y(\xi)).$$

For $y(\xi) < y_n(\xi)$, combining again (5.1)–(5.3) yields

$$\begin{aligned} \mu_n((-\infty, y_n(\xi))) + y_n(\xi) &= F_n(y(\xi)) + \mu_n((y(\xi), y_n(\xi))) + y_n(\xi) \\ &\leq \xi = F(y(\xi)) + y(\xi). \end{aligned}$$

Hence

$$0 \leq y_n(\xi) - y(\xi) + \mu_n((y(\xi), y_n(\xi))) \leq F(y(\xi)) - F_n(y(\xi)).$$

Since μ_n and μ are positive finite Radon measures for all $n \in \mathbb{N}$, we get that

$$(5.4) \quad |y_n(\xi) - y(\xi)| \leq |F_n(y(\xi)) - F(y(\xi))|, \quad \xi \in D^c.$$

Since by assumption $\xi \in D^c$, we have that F is continuous at the point $y(\xi)$, which in turn implies that $|F_n(y(\xi)) - F(y(\xi))|$ converges to zero. Thus $y_n(\xi) \rightarrow y(\xi)$ for every $\xi \in D^c$.

For $\xi \in D$, we argue as follows. Any point x at which F is discontinuous in Eulerian coordinates, corresponds to a maximal interval $[\xi_1, \xi_2]$ in Lagrangian

coordinates such that $y(\xi) = x$ for all $\xi \in [\xi_1, \xi_2]$ and $\xi_2 - \xi_1 = \mu(\{x\})$. In particular, there exists an increasing sequence $\xi_i \in D^c$ such that ξ_i converges to ξ_1 . We may write

$$y_n(\xi_1) - y(\xi_1) = (y_n(\xi_1) - y_n(\xi_i)) + (y_n(\xi_i) - y(\xi_i)) + (y(\xi_i) - y(\xi_1)).$$

Because y_n and y are Lipschitz continuous with Lipschitz constant at most 1 due to (3.7a), we can thus estimate

$$|y_n(\xi_1) - y(\xi_1)| \leq 2|\xi_i - \xi_1| + |y_n(\xi_i) - y(\xi_i)|.$$

Since y is continuous at ξ_i (cf. (5.4)), the second term on the right-hand side tends to 0 as $n \rightarrow \infty$, which shows that $|y_n(\xi_1) - y(\xi_1)|$ can be made arbitrarily small and thus $y_n(\xi_1) \rightarrow y(\xi_1)$. A similar argument shows that $y_n(\xi_2) \rightarrow y(\xi_2)$ by taking a decreasing sequence $\xi_i \in D^c$ such that ξ_i converges to ξ_2 .

We can now show that $y_n(\xi) \rightarrow y(\xi)$ for all $\xi \in [\xi_1, \xi_2]$. By definition y_n is an increasing function, and $y(\xi)$ is constant on $[\xi_1, \xi_2]$. Thus

$$|y_n(\xi) - y(\xi)| \leq \max(|y_n(\xi_1) - y(\xi_1)|, |y_n(\xi_2) - y(\xi_2)|) \quad \text{for all } \xi \in [\xi_1, \xi_2].$$

Since both $|y_n(\xi_1) - y(\xi_1)|$ and $|y_n(\xi_2) - y(\xi_2)|$ tend to zero as $n \rightarrow \infty$, it follows immediately that $y_n(\xi) \rightarrow y(\xi)$ for all $\xi \in [\xi_1, \xi_2]$. Thus $y_n \rightarrow y$ pointwise.

Step 2. Convergence of h_n to h and $\zeta_{n,\xi}$ to ζ_ξ in $L^2(\mathbb{R})$. By definition, we have that $X_n \in \mathcal{F}_0$ for all $n \in \mathbb{N}$ and $X \in \mathcal{F}_0$. Thus $H_n(\xi) = \xi - y_n(\xi)$, $n \in \mathbb{N}$, and $H(\xi) = \xi - y(\xi)$ for almost every $\xi \in \mathbb{R}$. As $y_n(\xi)$ converges pointwise to $y(\xi)$, we infer that $H_n(\xi) \rightarrow H(\xi)$ pointwise almost everywhere as $n \rightarrow \infty$. Moreover, $H_n(\xi)$, $n \in \mathbb{N}$, and $H(\xi)$ are all continuous, and hence we conclude that, actually, we have pointwise convergence of $H_n(\xi) \rightarrow H(\xi)$ for every $\xi \in \mathbb{R}$. Moreover, since

$$H_n(\xi) = \int_{-\infty}^{\xi} h_n(\eta) d\eta \quad \text{and} \quad H(\xi) = \int_{-\infty}^{\xi} h(\eta) d\eta,$$

h_n and h can be seen as positive finite Radon measures, and hence

$$(5.5) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} h_n(\xi) \psi(\xi) d\xi = \int_{\mathbb{R}} h(\xi) \psi(\xi) d\xi$$

for all $\psi \in C_c^\infty(\mathbb{R})$ according to [11, Props. 7.19 and 8.17]. If we can show that $\|h_n\|_{L^2} \rightarrow \|h\|_{L^2}$, (5.5) will remain true for all $\psi \in L^2(\mathbb{R})$ by a density argument and hence all assumptions of the Radon–Riesz theorem are satisfied. Thus $h_n \rightarrow h$ in $L^2(\mathbb{R})$, provided $\|h_n\|_{L^2} \rightarrow \|h\|_{L^2}$.

In order to show this convergence, observe that

$$(5.6) \quad \|h_n\|_{L^1} = \|H_n\|_{L^\infty} = \mu_n(\mathbb{R}) \rightarrow \mu(\mathbb{R}) = \|H\|_{L^\infty} = \|h\|_{L^1} \quad \text{as } n \rightarrow \infty.$$

Since $X_n \in \mathcal{F}_0$ and $X \in \mathcal{F}_0$, we have because of (3.4c) that

$$(5.7) \quad h_n^2(\xi) = h_n(\xi) - U_{n,\xi}^2(\xi) - \bar{r}_n^2(\xi) \quad \text{and} \quad h^2(\xi) = h(\xi) - U_\xi^2(\xi) - \bar{r}^2(\xi),$$

respectively. Moreover, let $S = \{\xi \in \mathbb{R} \mid y_\xi(\xi) = 0\}$ and $S_n = \{\xi \in \mathbb{R} \mid y_{n,\xi}(\xi) = 0\}$. Then

$$y_{n,\xi}(\xi) = \frac{1}{1 + u_{n,x}^2(y_n(\xi)) + \bar{\rho}_n^2(y_n(\xi))} \quad \text{for almost every } \xi \in S_n^c,$$

and

$$y_\xi(\xi) = \frac{1}{1 + u_x^2(y(\xi)) + \bar{\rho}^2(y(\xi))} \quad \text{for almost every } \xi \in S^c.$$

Hence we get

$$\begin{aligned} \int_{\mathbb{R}} U_{n,\xi}^2(\xi) d\xi &= \int_{S_n^c} U_{n,\xi}^2(\xi) d\xi = \int_{S_n^c} \frac{u_{n,x}^2(y_n(\xi))}{1 + u_{n,x}^2(y_n(\xi)) + \bar{\rho}_n^2(y_n(\xi))} y_{n,\xi}(\xi) d\xi \\ &= \int_{\mathbb{R}} \frac{u_{n,x}^2(x)}{1 + u_{n,x}^2(x) + \bar{\rho}_n^2(x)} dx, \end{aligned}$$

where we used that $U_{n,\xi}(\xi) = y_{n,\xi}(\xi) = 0$ for all $\xi \in S_n$ and that $\text{meas}(y_n(S_n)) = 0$. Similar arguments yield

$$\int_{\mathbb{R}} U_{\xi}^2(\xi) d\xi = \int_{\mathbb{R}} \frac{u_x^2(x)}{1 + u_x^2(x) + \bar{\rho}^2(x)} dx.$$

Thus, according to (4.2e),

$$(5.8) \quad \lim_{n \rightarrow \infty} \|U_{n,\xi}\|_{L^2}^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} U_{n,\xi}^2(\xi) d\xi = \int_{\mathbb{R}} U_{\xi}^2(\xi) d\xi = \|U_{\xi}\|_{L^2}^2.$$

Following the same argument, this time using (4.2f), we obtain

$$(5.9) \quad \lim_{n \rightarrow \infty} \|\bar{r}_n\|_{L^2}^2 = \|\bar{r}\|_{L^2}^2.$$

Hence combining (5.6)–(5.7) and (5.8)–(5.9) yields that $\|h_n\|_{L^2} \rightarrow \|h\|_{L^2}$, and, in particular, $h_n \rightarrow h$ in $L^2(\mathbb{R})$ and $y_{n,\xi} \rightarrow y_{\xi}$ in $L^2(\mathbb{R})$, since both X_n and X belong to \mathcal{F}_0 .

Step 3. Convergence of $U_{n,\xi}$ to U_{ξ} in $L^2(\mathbb{R})$. In order to conclude that $U_{n,\xi} \rightarrow U_{\xi}$ in $L^2(\mathbb{R})$, it suffices to show, according to the Radon–Riesz theorem, that $U_{n,\xi}(\xi) \rightarrow U_{\xi}(\xi)$ since we have convergence of the corresponding norms, cf. (5.8). Due to the fact that $C_c^{\infty}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} U_{n,\xi}(\xi) \psi(\xi) d\xi = \int_{\mathbb{R}} U_{\xi}(\xi) \psi(\xi) d\xi$$

for all $\psi \in C_c^{\infty}(\mathbb{R})$. Observe that we have for any $\psi \in C_c^{\infty}(\mathbb{R})$

$$\int_{\mathbb{R}} U_{n,\xi}(\xi) \psi(\xi) d\xi = \int_{S_n^c} U_{n,\xi}(\xi) \psi(\xi) d\xi = \int_{\mathbb{R}} u_{n,x}(x) \psi(y_n^{-1}(x)) dx.$$

Here $S_n = \{\xi \in \mathbb{R} \mid y_{n,\xi}(\xi) = 0\}$ and hence, according to (3.4c), $U_{n,\xi}(\xi) = 0$ for almost every $\xi \in S_n$, and $y_n^{-1}(x)$ denotes the pseudo inverse to $y_n(\xi)$ defined as

$$y_n^{-1}(x) = \inf\{\xi \in \mathbb{R} \mid y_n(\xi) > x\}.$$

Similarly,

$$\int_{\mathbb{R}} U_{\xi}(\xi) \psi(\xi) d\xi = \int_{\mathbb{R}} u_x(x) \psi(y^{-1}(x)) dx,$$

where $y^{-1}(x)$ denotes the pseudo inverse to $y(\xi)$, i.e.,

$$y^{-1}(x) = \inf\{\xi \in \mathbb{R} \mid y(\xi) > x\}.$$

Thus it suffices to show that

$$(5.10) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} u_{n,x}(x) \psi(y_n^{-1}(x)) dx = \int_{\mathbb{R}} u_x(x) \psi(y^{-1}(x)) dx$$

for all $\psi \in C_c^{\infty}(\mathbb{R})$.

Let $\psi \in C_c^{\infty}(\mathbb{R})$. By assumption, there exist $c, d \in \mathbb{R}$ such that $\text{supp}(\psi) \subset [c, d]$. Then $\text{supp}(\psi \circ y^{-1}) \subset [\xi_1, \xi_2]$, where

$$\xi_1 = \min\{\xi \in \mathbb{R} \mid y(\xi) \geq c\} \quad \text{and} \quad \xi_2 = \max\{\xi \in \mathbb{R} \mid y(\xi) \leq d\}.$$

Since $y(\xi) + H(\xi) = \xi$ for all $\xi \in \mathbb{R}$ and $H(\xi) \leq \|\mu\|$, we have

$$c - \|\mu\| \leq \xi_1 < \xi_2 \leq d + \|\mu\|$$

or, in other words, $\text{supp}(\psi \circ y^{-1}) \subset [c - \|\mu\|, d + \|\mu\|]$. Similarly one obtains that $\text{supp}(\psi \circ y_n^{-1}) \subset [c - \|\mu_n\|, d + \|\mu_n\|]$. In particular, cf. (4.2g), there exists $N \in \mathbb{N}$ such that

$$\text{supp}(\psi \circ y_n^{-1}) \subset [c - 2\|\mu\|, d + 2\|\mu\|], \quad \text{for all } n \geq N.$$

Moreover, to any $x \in \mathbb{R}$ we can assign a unique $y^{-1}(x)$ and $y_n^{-1}(x)$ using the pseudo inverse to $y(\xi)$ and $y_n(\xi)$, respectively. Thus we have from (3.7a)

$$y^{-1}(x) = x + \mu((-\infty, x]) = x + F(x).$$

and

$$y_n^{-1}(x) = x + \mu_n((-\infty, x]) = x + F_n(x).$$

In particular,

$$|y_n^{-1}(x) - y^{-1}(x)| = |F_n(x) - F(x)|.$$

Thus $y_n^{-1}(x)$ converges to $y^{-1}(x)$ for any $x \in \mathbb{R}$ at which $F(x)$ is continuous. In particular, $y_n^{-1}(x)$ converges to $y^{-1}(x)$ for almost every $x \in \mathbb{R}$, since $F(x)$ has at most countably many discontinuities. Hence, after using Lebesgue's dominated convergence theorem, we obtain that $\psi \circ y_n^{-1} \rightarrow \psi \circ y^{-1}$ in $L^2(\mathbb{R})$. Moreover, we have by assumption that $u_{n,x}$ converges weakly to u_x . Thus $u_{n,x} \psi \circ y_n^{-1}$ is the product of a weakly convergent sequence and a strongly convergent sequence, which implies that its integral converges to the integral of the limit $u_x \psi \circ y^{-1}$, which in turn proves (5.10).

Step 4. Convergence of \bar{r}_n to \bar{r} in $L^2(\mathbb{R})$. The proof follows exactly the same lines as the one of $U_{n,\xi} \rightarrow U_\xi$ in $L^2(\mathbb{R})$ in Step 3.

Step 5. Convergence of ζ_n to ζ in $L^\infty(\mathbb{R})$. Since both X_n and X belong to \mathcal{F}_0 , we have

$$(5.11) \quad |y(\xi) - y_n(\xi)| = |H(\xi) - H_n(\xi)| \leq \|h_n - h\|_{L^1}.$$

Moreover, by (5.7),

$$h_n(\xi) - h(\xi) = h_n^2(\xi) - h^2(\xi) + U_{n,\xi}^2(\xi) - U_\xi^2(\xi) + \bar{r}_n^2(\xi) - \bar{r}^2(\xi),$$

which together with (5.11) implies that $\|y_n - y\|_{L^\infty} \rightarrow 0$ as $n \rightarrow \infty$, since $h_n \rightarrow h$, $U_{n,\xi} \rightarrow U_\xi$ and $\bar{r}_n \rightarrow \bar{r}$ in $L^2(\mathbb{R})$.

Step 6. Convergence of U_n to U in $L^2(\mathbb{R})$. For a proof we refer the interested reader to [27, Prop. 5.1].

Step 7. Convergence of k_n to k . By definition we have

$$r(\xi) = \bar{r}(\xi) + ky_\xi(\xi) = \bar{\rho}(y(\xi))y_\xi(\xi) + ky_\xi(\xi) = \rho(y(\xi))y_\xi$$

and

$$r_n(\xi) = \bar{r}_n(\xi) + k_n y_{n,\xi}(\xi) + \bar{\rho}_n(y_n(\xi))y_{n,\xi}(\xi) + k_n y_{n,\xi}(\xi) = \rho_n(y_n(\xi))y_{n,\xi}(\xi).$$

Thus, the constants k_n in Eulerian and Lagrangian coordinates coincide and the same is true for k , and the claim is an immediate consequence of (4.2d). \square

6. CONVERGENCE FOR THE INITIAL DATA IMPLIES CONVERGENCE FOR THE SOLUTION OF THE 2CH SYSTEM

Finally, we would like to turn our attention to the 2CH system. In particular, we are going to study the consequences of the results derived so far in the context of the global weak conservative solutions of the 2CH system.

Theorem 6.1. *Given (u_0, ρ_0, μ_0) in \mathcal{D} , let $(u_{n,0}, \rho_{n,0}, \mu_{n,0})$ in \mathcal{D} be such that $(u_{n,0}, \rho_{n,0}, \mu_{n,0}) \rightarrow (u_0, \rho_0, \mu_0)$ in the sense of Definition 4.2. Consider the weak conservative solutions $(u(t, \cdot), \rho(t, \cdot), \mu(t, \cdot))$ and $(u_n(t, \cdot), \rho_n(t, \cdot), \mu_n(t, \cdot))$ of the 2CH system with initial data $(u, \rho, \mu)|_{t=0} = (u_0, \rho_0, \mu_0)$ and $(u_n, \rho_n, \mu_n)|_{t=0} = (u_{n,0}, \rho_{n,0}, \mu_{n,0})$, respectively. Then we have, for all $t \in [0, \infty)$,*

$$\begin{aligned} u_n(t, \cdot) &\rightarrow u(t, \cdot) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ u_{n,x}(t, \cdot) &\rightarrow u_x(t, \cdot), \\ \bar{\rho}_n(t, \cdot) &\rightarrow \bar{\rho}(t, \cdot), \\ k_n &\rightarrow k, \\ \int_{\mathbb{R}} \frac{u_{n,x}^2(t, x)}{1 + u_{n,x}^2(t, x) + \bar{\rho}_n^2(t, x)} dx &\rightarrow \int_{\mathbb{R}} \frac{u_x^2(t, x)}{1 + u_x^2(t, x) + \bar{\rho}^2(t, x)} dx, \\ \int_{\mathbb{R}} \frac{\bar{\rho}_n^2(t, x)}{1 + u_{n,x}^2(t, x) + \bar{\rho}_n^2(t, x)} dx &\rightarrow \int_{\mathbb{R}} \frac{\bar{\rho}^2(t, x)}{1 + u_x^2(t, x) + \bar{\rho}^2(t, x)} dx, \\ F_n(t, x) &\rightarrow F(t, x) \text{ for every } x \text{ at which } F(t, x) \text{ is continuous,} \\ F_n(t, \infty) &\rightarrow F(t, \infty), \end{aligned}$$

where $F_n(t, x) = \mu_n(t, (-\infty, x])$ for all $n \in \mathbb{N}$ and $F(t, x) = \mu(t, (-\infty, x])$. That is, for every $t \geq 0$ we have that the sequence $(u_n(t, \cdot), \rho_n(t, \cdot), \mu_n(t, \cdot))$ converges to $(u(t, \cdot), \rho(t, \cdot), \mu(t, \cdot))$ in the sense of Definition 4.2.

Proof. Again, we are going to split the proof into several steps.

Step 1. Convergence in Eulerian coordinates implies convergence in Lagrangian coordinates for the initial data. Let $X_0 = (y_0, U_0, h_0, r_0) = L((u_0, \rho_0, \mu_0))$ and $X_{n,0} = (y_{n,0}, U_{n,0}, h_{n,0}, r_{n,0}) = L((u_{n,0}, \rho_{n,0}, \mu_{n,0}))$. Then according to Theorem 5.1, $X_{n,0} \rightarrow X_0$ in E .

Step 2. Convergence at initial time implies convergence at any later time for the solution in Lagrangian coordinates. Consider the following semi-linear system of ordinary differential equations, which describes weak conservative solutions of the 2CH system in Lagrangian coordinates, cf. [18],

$$\begin{aligned} (6.1a) \quad &\zeta_t = U, \\ (6.1b) \quad &U_t = -Q(X), \\ (6.1c) \quad &h_t = 2(U^2 + \frac{1}{2}k^2 - P)U_\xi, \\ (6.1d) \quad &\bar{r}_t = -kU_\xi, \\ (6.1e) \quad &k_t = 0, \end{aligned}$$

where $y(t, \xi) = \zeta(t, \xi) + \xi$, and

$$P(X)(t, \xi) = \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \eta)|} (2U^2 y_\xi + 2k\bar{r} + h)(t, \eta) d\eta + \frac{1}{2} k^2,$$

and

$$Q(X)(t, \xi) = -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(\xi - \eta) e^{-|y(t, \xi) - y(t, \eta)|} (2U^2 y_\xi + 2k\bar{r} + h)(t, \eta) d\eta.$$

Then to $X(0) = X_0$ and $X_n(0) = X_{n,0} \in \mathcal{F}$, there exists a unique global solution to (6.1) in \mathcal{F} , which we denote $X(t)$ and $X_n(t)$, respectively. Moreover, the mappings $X \mapsto P(X) - \frac{1}{2}k^2$ and $X \mapsto Q(X)$ are Lipschitz continuous on bounded sets as mappings from E to $H^1(\mathbb{R})$. In particular, one has, cf. [27, Lemma 2.1], that

$$(6.2) \quad \|P(X_n(t)) - P(X(t))\|_{L^2 \cap L^\infty} \leq C_{t,n} \|X_n(t) - X(t)\|_E,$$

where $C_{t,n}$ is dependent on $\|X_n(t)\|_E, \|X(t)\|_E$. Similarly, we have

$$(6.3) \quad \|Q(X_n(t)) - Q(X(t))\|_{L^2 \cap L^\infty} \leq C_{t,n} \|X_n(t) - X(t)\|_E.$$

Furthermore, following closely the proof of [17, Thm. 3.5] and [27, Thm. 2.8], we get from (6.1) and (3.5) that

$$(6.4) \quad \begin{aligned} \Sigma(X(t)) &= \int_{\mathbb{R}} (U^2 y_\xi + h)(t, \eta) d\eta = \int_{\mathbb{R}} (U^2 y_\xi + h)(0, \eta) d\eta \\ &= \int_{\mathbb{R}} (U^2 y_\xi + h^2 + U_\xi^2 + \bar{r}^2)(0, \eta) d\eta = \Sigma(X(0)), \end{aligned}$$

and, in particular,

$$\|X(t)\|_E \leq D(t, \Sigma(X(0))) \|X(0)\|_E,$$

where $D(t, \Sigma(X(0)))$ depends on t and $\Sigma(X(0))$. A similar estimate holds for $X_n(t)$ with $n \in \mathbb{N}$. Thus $C_{t,n}$ in (6.2) and (6.3) only depends on $t, \|X_n(0)\|_E$, and $\|X(0)\|_E$, due to (6.4). Furthermore, since $X_n(0) \rightarrow X(0)$ in E , there exists $M > 0$ such that $\max(\|X_n(0)\|_E, \|X(0)\|_E) \leq M$. Thus, (6.2) and (6.3) imply that the right-hand side of (6.1) is Lipschitz continuous on bounded sets, and, in particular, applying Gronwall's inequality yields

$$(6.5) \quad \|X_n(t) - X(t)\|_E \leq C_t \|X_n(0) - X(0)\|_E,$$

where C_t only depends on M and t .

Step 3. Convergence independent of relabeling in \mathcal{F} . As in [27, Lemma 3.3], one can show, given $T \geq 0$ and $\bar{X}(0) \in \mathcal{F}_0$, one has

$$\bar{X}(t) \in \mathcal{F}_\alpha$$

for all $t \in [0, T]$, where α only depends on t and $\|\bar{X}(0)\|_E$. In our case, since $X_n(0) \rightarrow X(0)$ in E , there exists $M > 0$ such that $M \geq \|X(0)\|_E$ and $M \geq \|X_n(0)\|_E$ for all $n \in \mathbb{N}$. Thus there exists $\beta(t) > 0$ independent of n , such that

$$X(t) \in \mathcal{F}_{\beta(t)} \quad \text{and} \quad X_n(t) \in \mathcal{F}_{\beta(t)} \quad \text{for all } n \in \mathbb{N}.$$

Moreover, it is known, see, e.g., [18, Lemma 4.6], that for $\beta(t) \geq 0$, the mapping $\Gamma: \mathcal{F}_{\beta(t)} \rightarrow \mathcal{F}_0$ with $X \mapsto \Gamma(X) = X \circ (y + H)^{-1}$ is continuous. Let $\tilde{X}_n(t) = \Gamma(X_n(t))$ and $\tilde{X}(t) = \Gamma(X(t))$. Then for each $t \geq 0$ the convergence $X_n(t) \rightarrow X(t)$ in E implies $\tilde{X}_n(t) \rightarrow \tilde{X}(t)$ in E .

Step 4. Convergence of the solutions in Eulerian coordinates. Since $\tilde{X}_n(t) \rightarrow \tilde{X}(t)$ in E for all $t \geq 0$ and $\tilde{X}_n(t), \tilde{X}(t) \in \mathcal{F}_0$ for all $t \geq 0$, applying Theorem 4.3 finishes the proof. \square

The next result gives the corresponding result in the case where the approximation is constructed using the mollifier.

Theorem 6.2. *Given $(u_0, 0, \mu_0)$ in \mathcal{D} and let $(u_{n,0}, \rho_{n,0}, \mu_{n,0})$ in \mathcal{D} be the smooth approximation given by (2.3) in Theorem 2.2. Consider the weak, conservative solutions $(u(t, \cdot), 0, \mu(t, \cdot))$ and $(u_n(t, \cdot), \rho_n(t, \cdot), \mu_n(t, \cdot))$ of the 2CH system with initial data $(u, 0, \mu)|_{t=0} = (u_0, 0, \mu_0)$ and $(u_n, \rho_n, \mu_n)|_{t=0} = (u_{n,0}, \rho_{n,0}, \mu_{n,0})$, respectively. Then we have, for all $t \in [0, \infty)$,*

$$\begin{aligned} u_n(t, \cdot) &\rightharpoonup u(t, \cdot) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \\ u_{n,x}(t, \cdot) &\rightharpoonup u_x(t, \cdot), \\ \bar{\rho}_n(t, \cdot) &\rightharpoonup \bar{\rho}(t, \cdot), \\ k_n &\rightarrow 0, \\ \int_{\mathbb{R}} \frac{u_{n,x}^2(t, x)}{1 + u_{n,x}^2(t, x) + \bar{\rho}_n^2(t, x)} dx &\rightarrow \int_{\mathbb{R}} \frac{u_x^2(t, x)}{1 + u_x^2(t, x)} dx, \\ \int_{\mathbb{R}} \frac{\bar{\rho}_n^2(t, x)}{1 + u_{n,x}^2(t, x) + \bar{\rho}_n^2(t, x)} dx &\rightarrow 0, \\ F_n(t, x) &\rightarrow F(t, x) \text{ for every } x \text{ at which } F(t, x) \text{ is continuous,} \\ F_n(t, \infty) &\rightarrow F(t, \infty), \end{aligned}$$

where $F_n(t, x) = \mu_n(t, (-\infty, x])$ for all $n \in \mathbb{N}$ and $F(t, x) = \mu(t, (-\infty, x])$.

Moreover, for each $n \in \mathbb{N}$, $(u_n(t, \cdot), \rho_n(t, \cdot), \mu_n(t, \cdot))$ is a smooth solution to the 2CH system, that is, $u_n(t, x)$ and $\rho_n(t, x)$ belong to $C^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R})$ and $\mu_n(t, x) = \mu_{n,ac}(t, x) = (u_x^2(t, x) + \bar{\rho}^2(t, x))dx$ for all $t \geq 0$, and, in particular, no wave breaking occurs.

Proof. Since we showed in Theorem 3.4 that $X_n = (y_n, U_n, h_n, r_n)$ converges to $X = (y, U, h, 0)$ in E , and hence according to Theorem 4.3, the sequence $(u_{n,0}, \rho_{n,0}, \mu_{n,0})$ converges to $(u_0, 0, \mu_0)$ in the sense of Definition 4.2, the first part of the theorem is an immediate consequence of Theorem 6.1.

As far as the smoothness of the solution $(u_n(t, \cdot), \rho_n(t, \cdot), \mu_n(t, \cdot))$ for any $t \geq 0$ and $n \in \mathbb{N}$ is concerned, we refer the interested reader to [18, Sect. 6]. \square

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