Ergebnisse der Mathematik und ihrer Grenzgebiete

Volume 64

3. Folge

A Series of Modern Surveys in Mathematics

For further volumes:
www.springer.com/series/728

Nihat Ay • Jürgen Jost • Hông Vân Lê •
Lorenz Schwachhöfer

Information Geometry

Springer

Nihat Ay
Information Theory of Cognitive Systems
MPI for Mathematics in the Sciences
Leipzig, Germany
and
Santa Fe Institute, Santa Fe, NM, USA

Jürgen Jost
Geometric Methods and Complex Systems
MPI for Mathematics in the Sciences
Leipzig, Germany
and
Santa Fe Institute, Santa Fe, NM, USA

Hông Vân Lê
Mathematical Institute of ASCR
Czech Academy of Sciences
Praha 1, Czech Republic

Lorenz Schwachhöfer
Department of Mathematics
TU Dortmund University
Dortmund, Germany

ISSN 0071-1136 ISSN 2197-5655 (electronic)
Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics
ISBN 978-3-319-56477-7 ISBN 978-3-319-56478-4 (eBook)
DOI 10.1007/978-3-319-56478-4

Library of Congress Control Number: 2017951855

Mathematics Subject Classification: 60A10, 62B05, 62B10, 62G05, 53B21, 53B05, 46B20, 94A15, 94A17, 94B27

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

Information geometry is the differential geometric treatment of statistical models. It thereby provides the mathematical foundation of statistics. Information geometry therefore is of interest both for its beautiful mathematical structure and for the insight it provides into statistics and its applications. Information geometry currently is a very active field. For instance, Springer will soon launch a new topical journal "Information Geometry". We therefore think that the time is appropriate for a monograph on information geometry that develops the underlying mathematical theory in full generality and rigor, that explores the connections to other mathematical disciplines, and that proves abstract and general versions of the classical results of statistics, like the Cramér–Rao inequality or Chentsov's theorem. These, then, are the purposes of the present book, and we hope that it will become the standard reference for the field.

Parametric statistics as introduced by R. Fisher considers parametrized families of probability measures on some finite or infinite sample space Ω. Typically, one wishes to identify a parameter so that the resulting probability measure best fits the observation among the measures in the family. This naturally leads to quantitative questions, in particular, how sensitively the measures in the family depend on the parameter. For this, a geometric perspective is expedient. There is a natural metric, the Fisher metric introduced by Rao, on the space of probability measures on Ω. This metric is simply the projective or spherical metric obtained when one considers a probability measure as a non-negative measure with a scaling factor to render its total mass equal to unity. The Fisher metric thus is a Riemannian metric that induces a corresponding structure on parametrized families of probability measures as above. Furthermore, moving from one reference measure to another yields an affine structure as discovered by S.I. Amari and N.N. Chentsov. The investigation of these metric and affine structures is therefore called information geometry. Information-theoretical quantities like relative entropies (Kullback–Leibler divergences) then find a natural geometric interpretation.

Information geometry thus provides a way of understanding information-theoretic quantities, statistical models, and corresponding statistical inference methods in geometric terms. In particular, the Fisher metric and the Amari–Chentsov structure are characterized by their invariance under sufficient statistics. Several geometric formalisms have been identified as powerful tools to this end and emphasize respective geometric aspects of probability theory. In this book, we move beyond the applications in statistics and develop both a functional analytic and a geometric theory that are of mathematical interest in their own right. In particular, the theory of dually affine structures turns out to be an analogue of Kähler geometry in a real as opposed to a complex setting.

Also, as the concept of Shannon information can be related to the entropy concepts of Boltzmann and Gibbs, there is also a natural connection between information geometry and statistical mechanics. Finally, information geometry can also be used as a foundation of important parts of mathematical biology, like the theory of replicator equations and mathematical population genetics.

Sample spaces could be finite, but more often than not, they are infinite, for instance, subsets of some (finite- or even infinite-dimensional) Euclidean space. The spaces of measures on such spaces therefore are infinite-dimensional Banach spaces. Consequently, the differential geometric approach needs to be supplemented by functional analytic considerations. One of the purposes of this book therefore is to provide a general framework that integrates the differential geometry into the functional analysis.

Acknowledgements

We would like to thank Shun-ichi Amari for many fruitful discussions. This work was mainly carried out at the Max Planck Institute for Mathematics in the Sciences in Leipzig. It has also been supported by the BSI at RIKEN in Tokyo, the ASSMS, GCU in Lahore-Pakistan, the VNU for Sciences in Hanoi, the Mathematical Institute of the Academy of Sciences of the Czech Republic in Prague, and the Santa Fe Institute. We are grateful for the excellent working conditions and financial support of these institutions during extended visits of some of us. In particular, we should like to thank Antje Vandenberg for her outstanding logistic support.

The research of J.J. leading to his book contribution has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 267087. The research of H.V.L. is supported by RVO: 67985840.

Contents