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# Limit Operators and Their Applications in Operator Theory

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# Preface

This text has two goals. It describes a topic: band and band-dominated operators and their Fredholm theory, and it introduces a method to study this topic: limit operators.

**Band-dominated operators.** Let $H = l^2(\mathbb{Z})$ be the Hilbert space of all squared summable functions $x : \mathbb{Z} \to \mathbb{C}$, $i \mapsto x_i$ provided with the norm

$$\|x\|^2 := \sum_{i \in \mathbb{Z}} |x_i|^2.$$

It is often convenient to think of the elements $x$ of $l^2(\mathbb{Z})$ as two-sided infinite sequences $(x_i)_{i \in \mathbb{Z}}$.

The standard basis of $l^2(\mathbb{Z})$ is the family of sequences

$$(e_i)_{i \in \mathbb{Z}} \quad \text{where} \quad e_i = (\ldots, 0, 0, 1, 0, 0, \ldots)$$

with the 1 standing at the $i$th place. Every bounded linear operator $A$ on $H$ can be described by a two-sided infinite matrix $(a_{ij})_{i,j \in \mathbb{Z}}$ with respect to this basis, where $a_{ij} = \langle Ae_j, e_i \rangle$. The band operators on $H$ are just the operators with a matrix representation of finite band-width, i.e., the operators for which $a_{ij} = 0$ whenever $|i - j| > k$ for some $k$. Operators which are in the norm closure of the algebra of all band operators are called band-dominated. Needless to say that band and band-dominated operators appear in numerous branches of mathematics. Archetypal examples come from discretizations of partial differential operators.

It is easy to check that every band operator can be uniquely written as a finite sum $\sum d_k V_k$ where the $d_k$ are multiplication operators (i.e., they are given by a diagonal matrix with respect to the standard basis), and where the $V_k$ are the shift operators $e_j \mapsto e_{j+k}$. Conversely, every finite sum of this form is a band operator. This equivalence allows us to think of band operators as being composed of two kinds of generators – multiplication operators and shift operators.

**Fredholmness.** We will be mainly concerned with the Fredholm properties of band-dominated operators. A bounded linear operator $A$ on $H$ is called a Fredholm operator if both its kernel $\{x \in H : Ax = 0\}$ and its cokernel $H/(AH)$ are finite-dimensional linear spaces. Equivalently, an operator $A$ is Fredholm if its coset $A + K(H)$ is invertible in the Calkin algebra $L(H)/K(H)$ where $L(H)$ stands for

the algebra of all bounded linear operators on $H$ and $K(H)$ for the ideal of $L(H)$ consisting of the compact operators. In particular, the property of being Fredholm is invariant with respect to compact perturbations. Thus, no finite part of the matrix representation $(a_{ij})_{i,j\in\mathbb{Z}}$ of $A \in L(H)$ is responsible for the Fredholmness of $A$, and the whole information on the Fredholm properties of $A$ is hidden at infinity, i.e., in the asymptotic behavior of the entries $a_{ij}$ .

**Limit operators.** How can one draw information from infinity? A convenient way is to fix a basis vector $e_k$, to shift the operator $A \mapsto V_{-n}AV_n$, and to observe the evolution of the vectors $V_{-n}AV_ne_k$ as $n$ tends to $\pm\infty$. This has to be done for each basis vector $e_k$, which amounts to considering the behavior of the sequence $(V_{-n}AV_n)$ for large $n$ with respect to the strong convergence of operators.

Assume for a moment that we are in the lucky case where the entries of each diagonal of $A$ stabilize at infinity (i.e., where the limits $\lim_{i\to\pm\infty} a_{i+k,i}$ exist for every integer $k$). Then the strong limits of the sequence $(V_{-n}AV_n)$ as $n \to +\infty$ and $n \to -\infty$ exist, and these limits tell us exactly how the operator looks at infinity. What happens in the general situation where the entries of each diagonal are allowed to form an arbitrary bounded sequence? Then we cannot expect that the sequence $(V_{-n}AV_n)$ converges strongly but, hopefully, certain subsequences will still do. Indeed, using a Cantor diagonal argument, we will even get that if $A$ is band-dominated, then *every sequence $h : \mathbb{N} \to \mathbb{Z}$ which tends to (plus or minus) infinity possesses a subsequence $g : \mathbb{N} \to \mathbb{Z}$ such that the sequence of the operators*

$$V_{-g(n)}AV_{g(n)}$$

*converges strongly as $n \to \infty$.* The strong limit of this sequence is called the limit operator of $A$ with respect to the sequence $g$, and the set of all limit operators of a given operator $A$ is called the operator spectrum of $A$. The crucial and surprising point is that the operator spectrum of a band-dominated operator contains exactly the information from infinity which is needed to decide whether the operator is Fredholm or not. The precise statement is: *a band-dominated operator is Fredholm if, and only if, its limit operators are invertible and if the norms of their inverses are uniformly bounded.*

**Contents of the book.** We will not restrict our attention to band-dominated operators on $l^2(\mathbb{Z})$; rather we consider band-dominated operators on $l^p(\mathbb{Z}^N, X)$ where $N$ is a positive integer, $1 \le p \le \infty$, and where $X$ is a complex Banach space. The main reason for this is that, after a suitable discretization, functions in $L^p(\mathbb{R}^N)$ become sequences in $l^p(\mathbb{Z}^N, X)$ with $X = L^p([0, 1]^N)$, and that a related discretization, applied to (wide classes of) convolution and pseudodifferential operators on $L^p(\mathbb{R}^N)$, indeed produces band-dominated operators on the discrete space $l^p(\mathbb{Z}^N, X)$. Thus, the theory of band-dominated operators which will be developed in the first two chapters can immediately be applied to convolution and pseudodifferential operators to reproduce some known facts and to uncover some new properties of these and other operators.

The inclusion of the 'exotic' case $p = \infty$ and the consideration of sequences with values in infinite-dimensional Banach spaces involve some subtleties. For example, we are limited when working with strong convergence of operators, because the projections $P_n : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$ which replace all entries of a sequence $(x_k)$ with $|k| > n$ by zero converge strongly to the identity operator if and only if $p \neq \infty$. Another point is that the notions of Fredholmness and of invertibility at infinity (which are synonym for operators on spaces of sequences which take values in a finite-dimensional Banach space) become basically different for operators on spaces of sequences with values in infinite-dimensional Banach spaces. The applications we have in mind suggest to give preference to the aspect of invertibility at infinity over Fredholmness.

So we start in the first chapter with modifying the standard concepts of strong convergence, compactness and Fredholmness by introducing the notions of $\mathcal{P}$-strong convergence, $\mathcal{P}$-compactness and $\mathcal{P}$-Fredholmness. Here, $\mathcal{P}$ is a given approximate identity, for instance, the sequence $(P_n)$ we encountered in the preceding paragraph. Based on these $\mathcal{P}$-notions, we introduce the general concept of a limit operator in Section 1.2.

In case $1 < p < \infty$ and $\dim X < \infty$, all $\mathcal{P}$-notions reduce to their usual meanings. Thus, readers who are exclusively interested in band-dominated operators on scalar-valued sequences can skip the first part of this chapter. It is perhaps also a good advice for a first reading of Chapters 1 and 2 to skip Section 1.1, to ignore the $\mathcal{P}$, and to set $X = \mathbb{C}$ in what follows.

The second chapter is the heart of the book. Here we introduce band-dominated operators on the spaces $l^p(\mathbb{Z}^N, X)$ and prove that they possess sufficiently many limit operators in the sense that a band-dominated operator is $\mathcal{P}$-Fredholm if and only if all of its limit operators are invertible and if the norms of their inverses are uniformly bounded.

The status of the uniform invertibility hypothesis is not really evident at this moment. On the one hand, it might be quite hard to check the uniform boundedness of the norms of the inverses of a family of operators explicitly; so the condition is very unpleasant from this practical point of view. On the other hand, we do not know any example of a band-dominated operator, all limit operators of which are invertible, but which fails to be $\mathcal{P}$-Fredholm. Moreover, it turns out that there are large and important classes of band-dominated operators for which we can prove that the uniformity of the invertibility is indeed redundant. In particular, this happens for band-dominated operators in the Wiener algebra (which includes all band operators) as well as for band-dominated operators with slowly oscillating coefficients. These results will also be presented in Chapter 2. Partially, these results are based on the compatibility of the limit operators method with another local principle, which is due to Allan and Douglas. Thus, a large part of the second chapter is devoted to the study of the relations between these local theories.

The last part of Chapter 2 deals with the problem of calculating the index of a Fredholm band-dominated operator $A$ in terms of its limit operators. Here we restrict ourselves to band-dominated operators on $l^2(\mathbb{Z})$ with scalar-valued coeffi-

cients. Under these assumptions, we get an index formula of astonishing simplicity. For, choose an arbitrary limit operator $B_+$ of $A$ with respect to a sequence tending to $+\infty$ as well as an arbitrary limit operator $B_-$ of $A$ with respect to a sequence tending to $-\infty$. Then the index of $A$ is the sum of the local indices of $B_\pm$ at $\pm\infty$,

$$\operatorname{ind} A = \operatorname{ind}_+ B_+ + \operatorname{ind}_- B_-.$$

In Chapters 3 and 4, we are going to specify the results on Fredholmness obtained for general band-dominated operators to convolution operators and pseudodifferential operators on $L^p(\mathbb{R}^N)$, respectively. The key steps are to embed these operators into suitable operator algebras of Wiener type and to discretize in an appropriate manner these operators to get band-dominated operators on a discrete $l^p$-space of vector-valued sequences. A similar approach is chosen in Chapter 5 in order to illustrate the applicability of the limit operators method to study the Fredholmness of pseudodifference operators on $l^p(\mathbb{Z}^N)$-spaces with weight. Particular attention is paid to Phragmen-Lindelöf type theorems on the exponential decay at infinity of solutions to pseudodifference equations, to the description of the essential spectrum of discrete Schrödinger operators, and to the decay of their eigenfunctions at infinity.

Chapter 6 shifts the attention from analysis to numerical analysis. We consider the finite section method for the approximate solution of equations with band-dominated system matrices. The basic observation here is that the sequence of the finite sections of a band-dominated operator on $\mathbb{Z}^N$ can be interpreted as a band-dominated operator on $\mathbb{Z}^{N+1}$. Moreover, the sequence is stable if and only if the corresponding operator is $\mathcal{P}$-Fredholm. Thus, the results from Chapter 2 apply immediately to yield stability results for the finite section method. In the Hilbert space case $p = 2$, these stability results will be further used to derive characterizations of the asymptotic behavior of the norms, condition numbers, eigenvalues, pseudo-eigenvalues, and Rayleigh quotients of the finite section matrices.

So far we have only discussed the application of the limit operators method to the Fredholm theory of band-dominated (and related) operators. The goal of the final Chapter 7 is to indicate that the range of the applicability of the limit operators method is much larger. So we will develop an axiomatic scheme which covers most applications of the limit operators method. As concrete examples we consider the Fredholmness of convolution operators as well as of convolutions combined with (non-Carleman) shifts on the Heisenberg group.

**Preliminaries.** We assume that the reader has basic knowledge in linear Functional Analysis. For Chapters 4 and 7, a first acquaintance with pseudodifferential operators and with non-commutative harmonic analysis would be helpful.

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