Nonlinear Parabolic and Elliptic Equations

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To my wife

Mei-Shan

Preface

The recent development of reaction diffusion systems in biology, ecology and biochemistry, and the traditional importance of these systems in physics, heat-mass transfer, and engineering lead to extensive study in various aspects of nonlinear parabolic and elliptic partial differential equations. A large amount of mathematically rich and physically interesting work, including several excellent books, has been published in the literature since the middle of the 1960s. Because of the fast growth of reaction diffusion type of problems in various different fields it is desirable to have a unified mathematical treatment and practical methods for solving these problems. This book is intended to give a systematic treatment of the basic mathematical theory and constructive methods for a class of nonlinear parabolic and elliptic differential equations as well as their applications to various reaction diffusion problems. The mathematical problems under consideration include scalar boundary-value problems of parabolic and elliptic equations, integroparabolic and integroelliptic boundary-value problems, and coupled systems of parabolic and elliptic equations. The boundary conditions for these equations may be linear or nonlinear, including nonlinear boundary conditions of integral type. The fundamental approach to all of these problems is the method of upper and lower solutions and the associated monotone iterations. This approach leads not only to the basic results of existence, uniqueness, and multiplicity of solutions but also to various qualitative properties of the solution through suitable construction of upper and lower solutions. Moreover, since the book is concerned primarily with classical solutions, the monotone iteration processes for various types of nonlinear problems are adaptable to numerical solutions of the corresponding discrete system. Some of these methods for finite difference parabolic and elliptic equations have already appeared in the current literature.

Extensive discussion is given to the stability analysis and the asymptotic behavior of the time-dependent solution for both scalar boundary-value problems and coupled systems of equations. This includes the stability or instability of a steady-state solution, the asymptotic limit of the time-dependent solution, and decay or growth property of the solution. This qualitative property of the solution leads to an intrinsic relationship between the solutions of a parabolic boundary-value problem and its corresponding elliptic problem, and is a major concern in many physical, ecological, and engineering problems. Attention is given to various model problems in ecology, biochemistry, enzyme kinetics, combustion theory, and chemical and nuclear engineering. A special topic of the analysis is the finite-time blow-up prob-

lem for parabolic equations. Several methods for determining the blow-up behavior of a solution are used for both scalar equations under linear or nonlinear boundary conditions and coupled systems of parabolic equations, including a discussion on the problem of quenching. A chapter is devoted to parabolic and elliptic equations in unbounded domains using the method of upper and lower solutions. This chapter covers the Cauchy problem in $\mathbb{R}^n$, a half-space problem, and problems in the exterior of a bounded domain. Both linear and nonlinear boundary conditions are considered and various qualitative analyses, such as the finite-time blow-up problem, are discussed.

The book consists of twelve chapters; the first seven chapters treat the scalar parabolic and elliptic boundary-value problems and the remaining five chapters are concerned with coupled systems of parabolic and elliptic equations. The problems with unbounded spatial domains are considered in Chapter 7, while all the other chapters involve only bounded domains. Chapter 12 is devoted to applications of coupled systems of parabolic and elliptic equations to various model problems in different fields, including a coupled system of neutron transport and heat equations. These model problems have motivated much of the discussions in the chapters involving coupled systems. An overview of each chapter is given at the beginning of the chapter, and notes and remarks are added at the end of the chapter to provide historical comments and references. Theorems, Lemmas, Equations etc. are ordered in standard form. For example, the first theorem in section 4 of Chapter 2 is referred to as Theorem 4.1 when it appears in Chapter 2 and is referred to as Theorem 2.4.1 when it appears in a different chapter.

Because of the limitation of scope many interesting topics, such as traveling wave solutions, periodic solutions, and Lyapunov method for stability problems, are not discussed in the book. Also omitted is the class of equations where the nonlinear reaction function depends on the gradient of the unknown function. The references given in the book are mostly related to comparison methods and the method of upper and lower solutions, and is not intended to be complete. I apologize for the incompleteness and omissions in the list of references. The book, in part, is an outgrowth of my research on the subject during the past two decades, and much of the development is motivated by some kind of applied problems. Although effort is made to make the presentation self-contained it is necessary to use some basic theory for linear parabolic and elliptic equations. The book can be used as a reference for mathematicians, engineers and scientists, and can also be used as a text for graduate students who are interested in applied partial differential equations or reaction diffusion systems. Portions of the book have been used as the text in a special course on "Nonlinear Reaction Diffusion Equations" at North Carolina State University.

I am grateful to my students and colleagues for their reading the manuscript and helpful comments, and to the editorial staff of Plenum Publishing Corporation for editing the manuscript. I also owe a special debt of gratitude to Ms. Dionne Wilson for her cordial cooperation and excellent typing of the manuscript.

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