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COMPLETENESS, SIMILAR REGIONS, AND UNBIASED ESTIMATION—PART II[1]

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In Part I of this study (Lehmann and Scheffé, 1950), to which we shall refer as LSI, we defined and illustrated the concept of completeness of a family of measures and developed some of its statistical applications. In this part we shall state in section 7 some general results about complete families, showing how from given families one can construct others, and a specific result about the completeness of a certain parametric family. Uniformly most powerful unbiased tests for certain statistical hypotheses about this parametric family are considered in section 8. Some examples are given in section 9. As mentioned[2] in LSI, these results on testing are extensions and simplifications of earlier ones by Neyman (1941), Scheffé (1942), Lehmann (1947), Ghosh (1948), and Hoel (1948). Since then further papers have appeared on the subject by Nandi (1951) and Sverdrup (1953); however, the present version is in a number of ways more general than previous ones. The notation throughout will be that of LSI.

7. SOME FURTHER RESULTS ON COMPLETENESS

In this and the next section we shall restrict ourselves to families of measures $\mathfrak{M} = \{M_\theta \mid \theta \epsilon \omega\}$ for which the "label set" ω is a Borel set in Euclidean space and for which certain measurability assumptions with respect to θ are satisfied. Thus, if $\mathfrak{M}$ is a family $\mathfrak{M}^x = \{M_\theta^x \mid \theta \epsilon \omega\}$ of measures M_θ^x on the additive family $\mathfrak{F}^x$ of sets in the space W^x of points x, we shall denote by L^θ Lebesgue measure on the family $\mathfrak{L}^\theta$ of Borel subsets of the Borel set ω in a Euclidean space. We shall also assume that all the

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[2] We shall not give the derivation of the solutions of the differential equations of Neyman and Scheffé for the parametric family of probability densities, as announced in LSI, because this is lengthy, tedious, and now mainly of historical interest.

measures M^x for $\theta\epsilon\omega$ are absolutely continuous with respect to a common σ-finite measure μ^x on $\mathfrak{F}^x$. The measure μ^x is said to be σ-finite on $\mathfrak{F}^x$ if W^x is the union of a countable number of disjoint sets in $\mathfrak{F}^x$ on each of which μ^x is finite. Then there exists a function $g_\theta(x)$, measurable ($\mathfrak{F}^x$), which may be written $g_\theta(x) = dM^x_\theta/d\mu^x$, such that

$$M^x_\theta(A) = \int_A g_\theta(x) d\mu^x$$

for all $A\epsilon\mathfrak{F}^x$ and $\theta\epsilon\omega$; we shall assume further that $g_\theta(x)$ can be taken to be measurable $(\mathfrak{L}^\theta \times \mathfrak{F}^x)$.

We shall define the family $\mathfrak{M}^x$ to be *strongly complete* if it satisfies all the assumptions of the preceding paragraph and

$$\int_{W^x} f(x) dM^x_\theta = 0 \qquad \text{(a.e. } L^\theta)$$

implies $f(x) = 0$ (a.e. $\mathfrak{M}^x$). The relation between the three kinds of completeness we have now defined is that strong completeness implies completeness implies bounded completeness. The reader may now wish to proceed directly to Theorem 7.3, which is the basis for the following sections of this paper. The developments up to that point are of some general interest but may be avoided.

The concept of strong completeness permits us to formulate the following theorem concerning families of product measures.

Theorem 7.1: *Let* $x = (x', x'')$, $W^x = W^{x'} \times W^{x''}$, $\mathfrak{F}^x = \mathfrak{F}^{x'} \times \mathfrak{F}^{x''}$, $\theta = (\theta', \theta'')$, $\omega = \omega' \times \omega''$, $\mathfrak{L}^\theta = \mathfrak{L}^{\theta'} \times \mathfrak{L}^{\theta''}$, $L^\theta = L^{\theta'} \times L^{\theta''}$. *Suppose* $\mathfrak{M}^x = \{M^x_\theta | \theta\epsilon\omega\}$, *where* $dM^x_\theta = g_\theta(x) d\mu^x$, μ^x *is* σ*-finite,*

$$g_\theta(x) = g'_{\theta'}(x') g''_{\theta''}(x''),$$

$g'_{\theta'}(x')$ *is measurable* $(\mathfrak{L}^{\theta'} \times \mathfrak{F}^{x'})$, *and* $g''_{\theta''}(x'')$ *is measurable* $(\mathfrak{L}^{\theta''} \times \mathfrak{F}^{x''})$. *Suppose further there exists a measure* $\mu^{x''}$ *on* $\mathfrak{F}^{x''}$ *and a "conditional measure"* $\mu^{x'|x''}$ *on* $\mathfrak{F}^{x'}$ *depending on* x'', *such that whenever the integral*

$$\int_{W^x} f(x', x'') \, g_\theta(x) d\mu^x \qquad \ldots \quad (7.1)$$

is defined[3] *it equals the iterated integral*

$$\int_{W^{x''}} F(\theta', x'') g''_{\theta''}(x'') d\mu^{x''}, \qquad \ldots \quad (7.2)$$

where

$$F(\theta', x'') = \int_{W^{x'}} f(x', x'') g'_{\theta'}(x') d\mu^{x'|x''} \qquad \ldots \quad (7.3)$$

[3] We allow the values $+\infty$ and $-\infty$.

is measurable $(\mathfrak{L}^{\theta'} \times \mathfrak{F}^{x''})$. *Then* $\mathfrak{M}^x$ *is strongly complete providing the same is true of* $\mathfrak{M}^{x''} = \{M^{x''}_{\theta''} | \theta'' \epsilon w''\}$ *and (a.e.* $\mu^{x''}$*) of* $\mathfrak{M}^{x'|x''} = \{M^{x'|x''}_{\theta'} | \theta' \epsilon w'\}$, *where* $dM^{x''}_{\theta''} = g''_{\theta''}(x'')\ d\mu^{x''}$ *and* $dM^{x'|x''}_{\theta'} = g'_{\theta'}(x') d\mu^{x'|x''}$.

To prove the theorem we need to show that if the integral (7.1) vanishes (a.e. L^θ) then $f(x', x'') = 0$ (a.e. $\mathfrak{M}^x$). Let N be the set in ω where the integral (7.1) does not vanish, so $N \epsilon \mathfrak{L}^\theta$ and $L^\theta(N) = 0$. Then by the hypothesis of the theorem the iterated integral (7.2) exists and equals zero for $\theta \notin N$. Let S be the set of points (θ', x'') where $F(\theta', x'')$ defined in (7.3) does not vanish; $S \epsilon \mathfrak{L}^{\theta'} \times \mathfrak{F}^{x''}$. We show first that the product measure $L^{\theta'} \times M^{x''}_{\theta''}$ of S is zero for all $\theta'' \epsilon \omega''$: Denote by $N_{\theta'}$ the section of N by $\theta' =$ constant, i.e., $N_{\theta'} = \{\theta'' | (\theta', \theta'') \epsilon N\}$. Since $(L^{\theta'} \times L^{\theta''})(N) = 0$, $L^{\theta''}(N_{\theta'}) = 0$ except for $\theta' \epsilon N'$, where $N' \epsilon \mathfrak{L}^{\theta'}$ and $L^{\theta'}(N') = 0$. For $\theta' \notin N'$ we now have

$$\int_{W^{x'}} F(\theta', x'') dM^{x''}_{\theta''} = 0 \qquad \text{(a.e. } L^{\theta''}\text{)},$$

and so by strong completeness of $\mathfrak{M}^{x''}$, $F(\theta', x'') = 0$ (a.e. $\mathfrak{M}^{x''}$). Thus all sections of S by $\theta' =$ constant with $\theta' \notin N'$ have $M^{x''}_{\theta''}$ measure zero for all $\theta'' \epsilon \omega''$. Hence $(L^{\theta'} \times M^{x''}_{\theta''})(S) = 0$ for all $\theta'' \epsilon \omega''$.

We may now conclude that if $S_{x''}$ denotes a section of S by $x'' =$ constant, then the set N'' in $W^{x''}$ for which $L^{\theta'}(S_{x''}) \neq 0$ has $M^{x''}_{\theta''}$ measure zero for all $\theta'' \epsilon \omega''$. In other words, except for $x'' \epsilon N''$, a null set of $\mathfrak{M}^{x''}$,

$$F(\theta', x'') = \int_{W^{x'}} f(x', x'') dM^{x'|x''}_{\theta'} = 0 \qquad \text{(a.e. } L^{\theta'}\text{)}.$$

Hence by strong completeness of $\mathfrak{M}^{x'|x''}$, we have for $x'' \notin N''$, $f(x', x'') = 0$ (a.e. $\mathfrak{M}^{x'|x''}$), and from this we will now conclude that $f(x', x'') = 0$ (a.e. $\mathfrak{M}^x$): We may calculate the measure M^x_θ of the set Q where $f(x', x'') \neq 0$ as the integral of the characteristic function $\chi_Q(x', x'')$ of Q,

$$M^x_\theta(Q) = \int_{W^x} \chi_Q(x', x'') dM^x_\theta,$$

and then express this as an iterated integral

$$M^x_\theta(Q) = \int_{W^{x''}} \left[\int_{W^{x'}} \chi_Q(x', x'') dM^{x'|x''}_{\theta'} \right] dM^{x''}_{\theta''},$$

as allowed by the hypotheses of the theorem. This gives

$$M^x_\theta(Q) = \int_{W^{x''}} [M^{x'|x''}_{\theta'}(Q_{x''})] dM^{x''}_{\theta''},$$

where $Q_{x''}$ is the section of Q by $x'' =$ constant, and for $x'' \notin N''$ the integrand vanishes.

If μ^x is itself a product of σ-finite measures $\mu^{x'}$ and $\mu^{x''}$, the conditional measure $\mu^{x'|x''}$ of Theorem 7.1 obviously exists and may be taken to be $\mu^{x'}$ by the Fubini theorem. This gives the following important corollary for product measures:

Corollary 7.1: *Suppose* $\mathfrak{M}^{x'} = \{M^{x'}_{\theta'} | \theta' \epsilon \omega'\}$ *and* $\mathfrak{M}^{x''} = \{M^{x''}_{\theta''} | \theta'' \epsilon \omega''\}$, *where* $dM^{x'}_{\theta'} = g'_{\theta'}(x')d\mu^{x'}$ *and* $dM^{x''}_{\theta''} = g''_{\theta''}(x'')d\mu^{x''}$, *are strongly complete families of measures. Let* $x = (x', x'')$, *etc. as in the first sentence of Theorem* 7.1 *and let* $\mathfrak{M}^x = \{M^x_\theta | \theta \epsilon \omega\}$, *where* $M^x_\theta = M^{x'}_{\theta'} \times M^{x''}_{\theta''}$. *Then the family* $\mathfrak{M}^x$ *of product measures is strongly complete.*

The following theorem is also useful.

Theorem 7.2: *If* $\mathfrak{M}^x = \{M^x_\theta | \theta \epsilon \omega\}$ *is strongly complete,*[4] *and if* $\mathfrak{K}^x = \{K^x_\theta | \theta \epsilon \omega\}$ *is defined by* $dK^x_\theta = u(\theta)v(x)dM^x_\theta$, *where* $u(\theta)$ *and* $v(x)$ *are non-negative,* $u(\theta)$ *is measurable* $(\mathfrak{L}^\theta)$, $u(\theta) \neq 0$ (a.e. L^θ), *and* $v(x)$ *is measurable* $(\mathfrak{F}^x)$, *then* $\mathfrak{K}^x$ *is strongly complete.*

The absolute continuity of the family $\mathfrak{K}^x$ of measures with respect to a σ-finite measure μ^x and the measurability $(\mathfrak{L}^\theta \times \mathfrak{F}^x)$ of $dK^x_\theta/d\mu^x$ follow from the corresponding properties of the family $\mathfrak{M}^x$ and the relation $dK^x_\theta/d\mu^x = u(\theta)v(x)dM^x_\theta/d\mu^x$. It will thus suffice to prove that if

$$\int_{W^x} f(x)dK^x_\theta = 0 \qquad \text{(a.e. } L^\theta), \qquad \ldots \quad (7.4)$$

then the set $H_f = \{x | f(x) \neq 0\}$ is a null set of $\mathfrak{K}^x$. We may write (7.4) in the form

$$\int_{W^x} f(x)v(x)dM^x_\theta = 0 \qquad \text{(a.e. } L^\theta). \qquad \ldots \quad (7.5)$$

Let $H_v = \{x | v(x) \neq 0\}$, $H_{fv} = \{x | f(x)v(x) \neq 0\}$, so $H_f \subset H_{fv} \cup (W^x - H_v)$. Since $\mathfrak{M}^x$ is complete we see from (7.5) that H_{fv} is a null set for $\mathfrak{M}^x$, and hence for $\mathfrak{K}^x$. Also, $W^x - H_v$ is a null set for $\mathfrak{K}^x$ since $v(x) = 0$ in $W^x - H_v$. Hence H_f is a null set for $\mathfrak{K}^x$.

A completeness result for a class of families of probability distributions of exponential form which has received much attention in statistical literature may be obtained by applying the above theorems to a result from the theory of Laplace transforms which we shall now state.

Lemma 7.1: *If* $\mathfrak{M}^x = \{M^x_\theta | \theta \epsilon \omega\}$ *is a family of measures on the real line* W^x *with a real parameter* θ *in a nondegenerate interval* ω, *and if the family has a density of the form*

$$dM^x_\theta / d\mu^x = C(\theta)e^{\theta x}h(x) \qquad \ldots \quad (7.6)$$

with respect to a σ-*finite measure* μ^x *on the class* $\mathfrak{F}^x$ *of Borel sets in* W^x, *then* $\mathfrak{M}^x$ *is strongly complete.*

[4] The proof will show that Theorem 7.2 remains true if "strongly complete" is replaced by "complete" in hypothesis and conclusion and the hypotheses $u(\theta) \geqslant 0$, $u(\theta) \neq 0$ (a.e. L^θ), $u(\theta)$ is measurable $(\mathfrak{L}^\theta)$ are replaced by the hypothesis $u(\theta) > 0$.

Since $C(\theta)e^{\theta x}h(x)$ is measurable ($\mathfrak{F}^x$), so is $h(x)$. Since

$$C(\theta) = 1\Big/\int_W e^{\theta x}h(x)d\mu^x,$$

it is measurable ($\mathfrak{L}^\theta$). Strong completeness of $\mathfrak{M}^x$ will then follow from Theorem 7.2 if we can prove $\mathfrak{N}^x = \{N_\theta^x \mid \theta\epsilon\omega\}$ strongly complete, where $dN_\theta^x = e^{\theta x}d\mu^x$. But this is an immediate consequence of applying to the relation

$$\int_{W^x} f(x)e^{\theta x}d\mu^x = 0 \qquad (\text{a.e.} \quad L^\theta) \qquad \ldots \quad (7.7)$$

the uniqueness theorem for Laplace transforms, as given for example in Widder (1941, Ch. 6, Th. 6a). The theorem in Widder requires that $\alpha(t) = \int_0^t f(x)d\mu^x$ be of bounded variation in every finite interval. Let θ_0 be a value of θ for which

$$\int_{W^x} f(x)e^{\theta_0 x}d\mu^x = 0. \qquad \ldots \quad (7.8)$$

Since the Lebesgue integral is an absolutely convergent integral, (7.8) implies

$$\int_{W^x} |f(x)e^{\theta_0 x}|\, d\mu^x < \infty.$$

Let $I^t = \{t \mid a \leqslant t \leqslant b\}$ be any finite interval. The total variation of $\alpha(t)$ on I^t is

$$\int_{I^x} |f(x)|\, d\mu^x \leqslant a^{-1} \int_{I^x} |f(x)e^{\theta_0 x}|\, d\mu^x < \infty,$$

where $I^x = \{x \mid a \leqslant x \leqslant b\}$ and a is the minimum of $e^{\theta_0 x}$ on I^x. The theorem in Widder now tells us $\alpha(t) = 0$; thus $f(x)$ vanishes (a.e. μ^x), hence (a.e. $\mathfrak{M}^x$).

We remark that in the following theorems where a family of measures is a family of probability distributions (in which case we agreed in LSI to write $\mathfrak{P}^x$ instead of $\mathfrak{M}^x$), $\mathfrak{P}^x = \{P_\theta^x \mid \theta\epsilon\omega\}$ with $dP_\theta^x = p_\theta(x)d\mu^x$, it is not necessary to assume μ^x to be σ-finite if there exists a set W_+^x in $\mathfrak{F}^x$ such that for all $\theta\epsilon\omega$, $p_\theta(x) \neq 0$ (a.e. μ^x) in W_+^x and $p_\theta(x) = 0$ (a.e. μ^x) in $W^x - W_+^x$; for in this case μ^x is σ-finite in $W_+^x(\theta) = \{x \mid p_\theta(x) \neq 0\}$ (Halmos, 1950, sec. 25, Th. F), and hence in W_+^x, and we may assume $\mu^x(W^x - W_+^x) = 0$ without affecting the family $\mathfrak{P}^x$. It is also understood that the density $p_\theta(x)$ is measurable ($\mathfrak{F}^x$) since it is integrable ($\mathfrak{F}^x$, μ^x).

Theorem 7.3: *If* $\mathfrak{P}^x = \{P_\theta^x \mid \theta\epsilon\omega\}$, $x = (x_1, \ldots, x_r)$, $\theta = (\theta_1, \ldots, \theta_r)$, *where* x_i *and* θ_i *are real* ($i = 1, \ldots, r$), *if* ω *contains a nondegenerate r-dimensional interval, and if* P_θ^x *has a density of the form*

$$dP_\theta^x/d\mu^x = C(\theta_1, \ldots, \theta_r)h(x_1, \ldots, x_r) \exp\left(\sum_{i=1}^r \theta_i x_i\right) \qquad \ldots \quad (7.9)$$

with respect to a measure μ^x on the class $\mathfrak{F}^x$ of Borel sets in the r-dimensional Euclidean space W^x, then $\mathfrak{P}^x$ is strongly complete.

This theorem may be proved with the aid of Theorems 7.1 and 7.2 and Lemma 7.1. A shorter proof may be made along different lines by a method used by Sverdrup (1953, Th. 1) to obtain a similar result.

Theorem 7.3 contains as special cases the completeness results of Examples 3.2, 3.3, 3.5, 3.8, 3.9 of LSI, as well as many multidimensional examples that can be obtained from these. The example at the top of p. 322 of LSI, in which the sufficient statistic $T = (T_1, T_2)$ consists of the mean T_1 of a random sample from a normal population and the sum of squares T_2 of the deviations from T_1 does not fall directly under Theorem 7.3; however, if we let $U = (U_1, U_2)$, $U_1 = T_1$, $U_2 = T_2+nT_1^2$, then the completeness of $\mathfrak{P}^u$ is given immediately by Theorem 7.3, and it is obvious in general that if $t = t(u)$ and $\mathfrak{P}^u$ is complete, so is $\mathfrak{P}^t$.

8. Uniformly most powerful unbiased tests

Throughout this section we shall be concerned with the case where the sample point $X = (X_1, \ldots, X_n)$ in an n-dimensional Euclidean space W^x has a density

$$dP^x_{\vartheta,\theta}/d\mu^x = p_{\vartheta,\theta}(x) = C(\vartheta, \theta)h(x) \exp\left[\vartheta s(x) + \sum_{i=1}^{r} \theta_i t_i(x)\right] \qquad \ldots (8.1)$$

with respect to a measure μ^x in the class $\mathfrak{F}^x$ of Borel sets in W^x. The statistical hypotheses considered will be statements about the real parameter ϑ of the form

H_1: $\vartheta \leqslant \vartheta_0$ against alternatives $\vartheta > \vartheta_0$,

H_2: $\vartheta = \vartheta_0$ against alternatives $\vartheta \neq \vartheta_0$,

H_3: $\vartheta_0 \leqslant \vartheta \leqslant \vartheta_1$ against alternatives $\vartheta < \vartheta_0$ or $\vartheta > \vartheta_1$,

H_4: $\vartheta \leqslant \vartheta_0$ or $\vartheta \geqslant \vartheta_1$ against alternatives $\vartheta_0 < \vartheta < \vartheta_1$.

The set of real parameters $(\theta_1, \ldots, \theta_r) = \theta$ are nuisance parameters $(r = 1, 2, \ldots)$. Denote by Ω the domain of (ϑ, θ). We shall assume the parameter space Ω is convex[5]. In the problem of testing H_i we shall assume there are points of Ω with $\vartheta > \vartheta_0$ for $i = 1, 2$; $\vartheta < \vartheta_0$ for $i = 2, 3$; $\vartheta > \vartheta_1$ for $i = 3$; $\vartheta \leqslant \vartheta_0$ and $\vartheta \geqslant \vartheta_0$ for $i = 4$. Let Ω_j denote the section of Ω by $\vartheta = \text{constant} = \vartheta_j (j = 0 \text{ or } 1)$, that is $\Omega_j = \{(\vartheta, \theta) \mid \vartheta = \vartheta_j, (\vartheta, \theta) \epsilon \Omega\}$, and let

$$\mathfrak{P}^x_j = \{P^x_{\vartheta,\theta} \mid (\vartheta, \theta) \epsilon \Omega_j\}. \qquad \ldots (8.2)$$

It will be understood that statements about Ω_j, $\mathfrak{P}^x_j$, and $\mathfrak{P}^t_j$ (defined below) are made for $j = 0$ in the problem of testing hypotheses H_1 and H_2, for $j = 0$ and 1 in the case of H_3 and H_4. It is assumed that Ω_j contains a non-degenerate r-dimensional interval.

[5] The assumption that Ω is convex is made because it is brief to state and does not seem to exclude any interesting applications. The proof of Theorem 8.2 is valid with less restrictive assumptions on Ω.

The real-valued functions $h(x)$, $s(x)$, $t_i(x)$ are understood to be measurable ($\mathfrak{F}^x$). Let $t(x) = (t_1(x), \ldots, t_r(x))$; then clearly $T = t(X)$ is a sufficient statistic for $\mathfrak{P}_j^x$. We shall take for $\mathfrak{F}^t$ the family of Borel sets in the Euclidean space W^t, instead of the more inclusive additive family induced in W^t by $t(x)$. Let $P^t_{\vartheta,\theta}$ be the probability measure induced on $\mathfrak{F}^t$ by $P^x_{\vartheta,\theta}$ and t(x), and write

$$\mathfrak{P}_j^t = \{P^t_{\vartheta,\theta} \mid (\vartheta, \theta)\epsilon\Omega_j\}. \qquad \ldots \quad (8.3)$$

In order to prove $\mathfrak{P}_j^t$ strongly complete we need

Lemma 8.1: *Let $t = t(x)$ be a measurable*[6] *transformation from* $(W^x, \mathfrak{F}^x)$ *into* $(W^t, \mathfrak{F}^t)$. *If the probability distribution of X is given by a density of the form*

$$dP^x_\theta / d\mu^x = q_\theta(t(x))h(x)$$

with respect to a measure μ^x on $\mathfrak{F}^x$, then there exists a measure ν^t on $\mathfrak{F}^t$ such that the distribution of $T = t(X)$ has the density

$$dP^t_\theta / d\nu^t = q_\theta(t)$$

with respect to ν^t.

This lemma may be proved by defining the measure ν^x on $\mathfrak{F}^x$ by $d\nu^x = h(x)d\mu^x$, and the measure ν^t on $\mathfrak{F}^t$ by $\nu^t(B) = \nu^x(t^{-1}(B))$ for all $B\epsilon\mathfrak{F}^t$. Then for all $B\epsilon\mathfrak{F}^t$ we have

$$P^t_\theta(B) = P^x_\theta(t^{-1}(B)) = \int_{t^{-1}(B)} q_\theta(t(x))h(x)d\mu^x = \int_{t^{-1}(B)} q_\theta(t(x))d\nu^x = \int_B q_\theta(t)d\nu^t,$$

which is the desired result.

From Lemma 8.1 and Theorem 7.3 we now have

Theorem 8.1: *If the sample point X has a probability density* (8.1) *and satisfies all the conditions stated in connection with* (8.1). *then, in the notation there defined, T is a sufficient statistic for $\mathfrak{P}_j^x$ with a probability density*

$$C(\vartheta_j, \theta) \exp\left(\sum_{i=1}^{r} \theta_i t_i\right)$$

with respect to a measure ν^t on the family $\mathfrak{F}^t$ of Borel sets in W^t, and the family $\mathfrak{P}_j^t$ is strongly complete.

In problems of testing statistical hypotheses about a parameter ϑ in the presence of nuisance parameters $(\theta_1, \ldots, \theta_r) = \theta$, we will denote the sample space by W^x, the parameter point by (ϑ, θ) the parameter space by Ω, the hypothesis by H: $\theta\epsilon\omega$,

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[6] For any given additive classes $\mathfrak{F}^x$, $\mathfrak{F}^t$, $t = t(x)$ is said to be a *measurable transformation* if for every $A\epsilon\mathfrak{F}^t$, $t^{-1}(A)\epsilon\mathfrak{F}^x$. In the application of this lemma W^x, W^t will be Euclidean spaces and $\mathfrak{F}^x$, $\mathfrak{F}^t$ the classes of Borel sets in each.

where ω is a given subset of Ω. Any function $\varphi(x)$ measurable $(\mathfrak{F}^x)$, such that $0 \leqslant \varphi(x) \leqslant 1$, may be considered as a *critical function* for testing H: H is to be rejected with probability $\varphi(x)$ when $X = x$, so that the power of the test is $E_{\vartheta,\theta}(\Phi)$ with $\Phi = \varphi(X)$, as indicated in LSI, pp. 317-318, where the terminology of *similar* critical function, and *Neyman structure* is also defined. By the *size* of the test based on $\varphi(x)$ we mean sup $E_{\vartheta,\theta}(\Phi)$ for $(\vartheta, \theta)\epsilon\omega$. The test is said to be *unbiased* if for every pair of parameter points $(\vartheta', \theta')\epsilon\omega$ and $(\vartheta'', \theta'')\epsilon\Omega-\omega$ we have $E_{\vartheta',\theta'}(\Phi) \leqslant E_{\vartheta'',\theta''}(\Phi)$. Since for any critical function $\varphi(x)$, the power $E_{\vartheta,\theta}(\Phi)$ is continuous (in fact, analytic) in ϑ for the family of distributions defined by (8.1) under the assumptions stated there, it follows that a critical function is similar for the family $\mathfrak{P}_0^x$ if it gives an unbiased test of H_1 or H_2, and similar for $\mathfrak{P}_0^x \cup \mathfrak{P}_1^x$ if it gives an unbiased test of H_3 or H_4. We can thus dispense with the classificatory terminology of "type B_1," "best one-sided similar," etc., and yet apply the results of LSI about similar critical functions to get the following theorem about uniformly most powerful unbiased tests.

Theorem 8.2: *If the sample point X has a probability density of the form* (8.1), *then for testing any of the hypotheses H_1, H_2, H_3, H_4 defined below* (8.1), *and for each $\alpha(0<\alpha<1)$, there exists a uniformly most powerful unbiased test of size α if the assumptions stated in connection with* (8.1) *are satisfied. The corresponding critical function $\varphi_i(x)$ for testing H_i may be constructed as follows: The function $\varphi_i(x)$ is defined by means of four functions of t, $v_i(t)$, $w_i(t)$ $\delta_i(t)$, $\gamma_i(t)$, satisfying the inequalities $v_i(t)<w_i(t)$, $0 \leqslant \delta_i(t) \leqslant 1$, $0 \leqslant \gamma_i(t) \leqslant 1$. Let $I_i(t)$ denote the interval from $v_i(t)$ to $w_i(t)$. For $i = 1, 2, 3$ we define $\varphi_i(x) = 0$ if $s(x)$ is inside the interval $I_i(t(x))$, $= 1$ if outside the interval; if $s(x)$ falls on an end point of $I_i(t(x))$ we define $\varphi_i(x) = \delta_i(t(x))$ or $\gamma_i(t(x))$ according as $s(x) = v_i(t(x))$ or $w_i(t(x))$. For $i = 4$ the definition of $\varphi_i(x)$ is similar except that we put $\varphi_i(x) = 1$ inside the interval, 0 outside. The four functions $v_i, w_i, \delta_i, \gamma_i$ are determined by the conditions $\mathfrak{C}_i$ below where Φ_i denotes $\varphi_i(X)$, S denotes $s(X)$, and $\mathfrak{P}_j^t$ is defined by* (8.3):

$$\mathfrak{C}_1: \quad v_1 = -\infty \text{ (hence, } \delta_1 \text{ need not be defined)},$$

$$E_{\vartheta_0\theta}(\Phi_1|t) = \alpha \qquad (a.e.\ \mathfrak{P}_0^t).$$

$$\mathfrak{C}_2: \quad E_{\vartheta_0,\theta}(\Phi_2|t) = \alpha \qquad (a.e.\ \mathfrak{P}_0^t),$$

$$E_{\vartheta_0,\theta}(S\Phi_2|t) = \alpha E_{\vartheta_0,\theta}(S|t) \qquad (a.e.\ \mathfrak{P}_0^t).$$

$$\mathfrak{C}_3,\ \mathfrak{C}_4: \quad E_{\vartheta_0,\theta}(\Phi_i|t) = \alpha \qquad (a.e.\ \mathfrak{P}_0^t).$$

$$E_{\vartheta_1,\theta}(\Phi_i|t) = \alpha \qquad (a.e.\ \mathfrak{P}_1^t).$$

The proof of the theorem will be facilitated by the following modified form of the Neyman-Pearson fundamental lemma.

Lemma 8.2: *Suppose $\mathfrak{P}^x = \{P_\eta^x | \eta\epsilon\Lambda\}$ is a family of probability measures on the additive family $\mathfrak{F}^x$ of sets in W^x with $dP_\eta^x = P_\eta(x)d\mu^x$, and $\mathfrak{P}_i^x = \{P_\eta^x | \eta\epsilon\Lambda_i\}$, where $\Lambda_i \subset \Lambda (i = 1, \ldots, k)$. Let $U_i = u_i(X)$ be a sufficient statistic for $\mathfrak{P}_i^x$. Suppose the following are given: $\eta^*\epsilon\Lambda$, $\eta_i\epsilon\Lambda_i$, constants α_i, and measurable $(\mathfrak{F}^x)$ functions $f_i(x) \geqslant 0$.*

Denote by $\mathfrak{P}_i^{u_i}$ *the family of measures on the additive family* $\mathfrak{F}_i^{u_i}$ *of sets in the range* W^{u_i} *of* $u_i(x)$, *where* $\mathfrak{F}_i^{u_i}$ *is such that* $u_i(x)$ *is a measurable transformation. If there exists a critical function* $\varphi^*(x)$ *and Borel-measurable functions* $g_i(u_i) \geqslant 0$ *such that*

$$E_{\eta_i}(f_i(X)\varphi^*(X)\,|\,u_i) = \alpha_i \quad (a.e.\ \mathfrak{P}_i^{u_i})(i = 1, \ldots, k) \qquad \ldots \quad (8.4)$$

and

$$\varphi^*(x) = \begin{cases} 1 \ \textit{where}\ p_{\eta^*}(x) > \sum_{i=1}^{k} g_i(u_i(x))\,f_i(x)\,p_{\eta_i}(x), \\ 0 \ \textit{where}\ p_{\eta^*}(x) < \sum_{i=1}^{k} g_i(u_i(x))\,f_i(x)\,p_{\eta_i}(x), \end{cases}$$

then, among all critical functions $\varphi(x)$ *satisfying* (8.4), $\varphi^*(x)$ *maximizes* $E_{\eta^*}(\varphi(X))$.

To prove the lemma suppose that $g_i(u_i)$ and $\varphi^*(x)$ satisfy the conditions of Lemma 8.2. Then β_i defined as

$$\beta_i = E_{\eta_i}(g_i(U_i)\,f_i(X)\varphi^*(X))$$

is finite since $$g_i(u_i(x))f_i(x)\varphi^*(x)p_{\eta_i}(x) \leqslant p_{\eta^*}(x).$$

Hence $$\beta_i = E_{\eta_i}[g_i(U_i)E_{\eta_i}(f_i(X)\varphi(X)\,|\,u_i)] = \alpha_i E_{\eta_i}(g_i(U_i)).$$

Now consider the wider class of critical functions $\varphi(x)$ satisfying

$$\int_{W^x} g_i(u_i(x))\,f_i(x)\varphi(x)p_{\eta_i}(x)d\mu^x = \beta_i \qquad (i = 1, \ldots, k). \qquad \ldots \quad (8.5)$$

It follows from the Neyman-Pearson lemma that in this wider class $\varphi^*(x)$ maximizes $E_{\eta^*}(\Phi(X))$.

We now prove Theorem 8.2 for testing hypothesis H_1. By the remark preceding the theorem, if $\varphi(x)$ gives an unbiased test then it is a similar critical function for the family $\mathfrak{P}_0^x$. By Theorem 8.1, the family $\mathfrak{P}_0^t$ generated by the sufficient statistic T is strongly complete. Hence by Theorem 4.1 of LSI, $\varphi(x)$ has the Neyman structure[6a] with respect to T, i.e.,

$$E_{\vartheta_0,\theta}(\varphi(X)\,|\,t) = \alpha \qquad (\text{a.e. } \mathfrak{P}_0^t). \qquad \ldots \quad (8.6)$$

Since T is sufficient for $\mathfrak{P}_0^x$, the left side is independent of θ for all $\varphi(x)$. Let (ϑ^*, θ^*) be any alternative with $\vartheta^* > \vartheta_0$, and let (ϑ_0, θ^0) be any point in Ω_0. We shall now apply Lemma 8.2 with $\Lambda = \Omega$, $k = 1$, $\Lambda_1 = \Omega_0$, $\eta^* = (\vartheta^*, \theta^*)$, $\eta_1 = (\vartheta_0, \theta^0)$, $\alpha_1 = \alpha$, $f_1 \equiv 1$, and $U_1 = T$. Then $\varphi(x)$ will maximize $E_{\vartheta^*, \theta^*}(\varphi(X))$ subject to (8.6) if it satisfies (8.6) and there exists a $g(t)$ such that $\varphi(x) = 1$ or 0 according as the ratio $p_{\vartheta^*, \theta^*}(x)/p_{\vartheta_0, \theta^0}(x)$ is $>$ or $< g(t(x))$. This condition reduces to $\varphi(x) = 1$ or 0 according as $s(x) >$ or $< w_1(t(x))$, where the function $w_1(t)$ may depend on ϑ^*, ϑ_0, θ^*, and θ^0.

[6a] We have only recently become aware of Bartlett's (1937) paper published at about the same time as Neyman's, in which he uses the same construction.

We must determine $w_1(t)$, and a suitable value $\gamma_1(t)$ for φ when $s = w_1(t)$, so that (8.6) is satisfied. This can be done so that $w_1(t)$ and $\gamma_1(t)$ do not depend on ϑ^*, θ^*, or θ^0, as follows: Write $F_t(s)$ for the conditional distribution function $P_{\vartheta_0,\theta}(S \leqslant s \mid t)$, which does not depend on θ, so that $F_t(s) = F_t(s+0)$ and $F_t(s) - F_t(s-0) = P_{\vartheta_0,\theta}(S = s \mid t)$. Then $w_1(t)$ and $\gamma_1(t)$ are to be determined so that

$$F_t(w_1-0) \leqslant 1-\alpha \leqslant F_t(w_1), \qquad \ldots \quad (8.7)$$

$$F_t(w_1-0)+(1-\gamma_1)[F_t(w_1)-F_t(w_1-0)] = 1-\alpha. \qquad \ldots \quad (8.8)$$

This is clearly possible; in particular if $F_t(s)$ is continuous at w_1 defined by (8.7) then for such t the definition of $\gamma_1(t)$ is immaterial. Since the resulting most powerful test against (ϑ^*, θ^*) does not depend on $\vartheta^* > \vartheta_0$ or on θ^*, it is uniformly most powerful among those satisfying (8.6).

It remains to verify that the above test has the proper size, i.e.

$$E_{\vartheta,\theta}(\Phi) \leqslant \alpha \ \text{ for } \vartheta < \vartheta_0, \qquad \ldots \quad (8.9)$$

and is unbiased, i.e.

$$E_{\vartheta,\theta}(\Phi) \geqslant \alpha \ \text{ for } \vartheta > \vartheta_0. \qquad \ldots \quad (8.10)$$

The inequality (8.10) may be deduced from the fact that $\varphi(x) \equiv \alpha$ is in the class satisfying (8.6) of which the φ defined above is uniformly most powerful for $\vartheta > \vartheta_0$. The inequality (8.9) can be demonstrated by noting that the above φ minimizes $E_{\vartheta,\theta}(\Phi)$ for $\vartheta < \vartheta_0$ subject to (8.6) and again comparing it with $\varphi(x) \equiv \alpha$.

The proof that φ_1 thus constructed is Borel measurable, as are the other φ_i, is given in the appendix at the end of the paper.

We next prove Theorem 8.2 for hypothesis H_2. As before, unbiasedness implies (8.6) . This time it also implies that

$$E_{\vartheta,\theta}(\Phi) = \int_{W^x} \varphi(x) p_{\vartheta,\theta}(x) d\mu^x \qquad \ldots \quad (8.11)$$

has a minimum at $\vartheta = \vartheta_0$. From well known properties of Laplace transforms, (8.11) has a derivative with respect to ϑ, which may be calculated by differentiation under the integral sign. Hence

$$\partial E_{\vartheta,\theta}(\Phi)/\partial\vartheta \,|_{\vartheta=\vartheta_0} = 0 \qquad \ldots \quad (8.12)$$

for all $(\vartheta_0, \theta) \epsilon \Omega_0$, and thus, differentiating (8.11) under the integral sign and using (8.1), we get from (8.12)

$$E_{\vartheta_0,\theta}(S\Phi) = -\alpha C^{-1}(\vartheta_0, \theta) \partial C(\vartheta, \theta)/\partial\vartheta \,|_{\vartheta=\vartheta_0}. \qquad \ldots \quad (8.13)$$

Since
$$C(\vartheta, \theta) = \left\{ \int_{W^x} h(x) \exp[\vartheta s(x) + \sum_{i=1}^{r} \theta_i t_i(x)] d\mu^x \right\}^{-1},$$

therefore, $\partial C(\vartheta, \theta)/\partial\vartheta = -C(\vartheta, \theta) E_{\vartheta,\theta}(S)$, and so (8.13) may be written

$$E_{\vartheta_0, \theta}(S\Phi) = \alpha E_{\vartheta_0, \theta}(S),$$

or,
$$E_{\vartheta_0, \theta}[E_{\vartheta_0, \theta}(S\Phi - \alpha S \mid t)] = 0$$

for all θ such that $(\vartheta_0, \theta) \epsilon \Omega_0$. The last condition implies

$$E_{\vartheta_0, \theta}(S\Phi - \alpha S \mid t) = 0 \qquad (\text{a.e. } \mathfrak{P}_0^t) \qquad \dots \quad (8.14)$$

since $\mathfrak{P}_0^t$ is strongly complete. Conditions (8.6) and (8.14) may be written

$$E_{\vartheta_0, \theta}(S^{i-1}\Phi \mid t) = \alpha_i \qquad (\text{a.e. } \mathfrak{P}_0^t) \quad (i = 1, 2), \qquad \dots \quad (8.15)$$

where $\alpha_i = \alpha E_{\vartheta_0, \theta}(S^{i-1} \mid t)$, which does not depend on θ for $(\vartheta_0, \theta)\epsilon\Omega_0$ since T is sufficient for $\mathfrak{P}_0^t$.

Let (ϑ^*, θ^*) be an alternative in Ω with $\vartheta^* \neq \vartheta_0$, and let $(\vartheta_0, \theta^0)\epsilon\Omega_0$. We now apply Lemma 8.2 with $\Lambda = \Omega$, $k = 2$, $\Lambda_1 = \Lambda_2 = \Omega_0$, $\eta^* = (\vartheta^*, \theta^*)$, $\eta_1 = \eta_2 = (\vartheta_0, \theta^0)$, $f_i(x) = s^{i-1}(x)$, and $U_1 = U_2 = T$: $\varphi(x)$ will maximize $E_{\vartheta^*, \theta^*}(\varphi(X))$ subject to (8.14) if it satisfies (8.15) and there exist $g_1(t)$ and $g_2(t)$ such that $\varphi(x) = 1$ or 0 according as the ratio $p_{\vartheta^*, \theta^*}(x)/p_{\vartheta_0, \theta^0}(x)$ is $>$ or $<$ than $g_1(t(x)) + g_2(t(x))s(x)$. Since the ratio is a convex function of s and depends on x only through $s(x)$ and $t(x)$, the existence of such $g_1(t)$ and $g_2(t)$ is equivalent to that of $v_2(t)$ and $w_2(t)$ with $v_2(t) < w_2(t)$ such that $\varphi(x) = 1$ or 0 according as $s(x)$ lies outside or inside the interval from $v_2(t)$ to $w_2(t)$. At this stage of the argument $v_2(t)$ and $w_2(t)$ may depend on ϑ^*, ϑ_0, θ^*, and θ^0. It remains to show that the functions $v_2(t)$ and $w_2(t)$, and values $\delta_2(t)$ and $\gamma_2(t)$ for $\varphi(x)$ when $s(x) = v_2(t)$ or $w_2(t)$, can be determined so that they do not depend on ϑ^*, θ^*, or θ^0, and (8.15) is satisfied.

In terms of $v_2(t)$, $w_2(t)$, $\delta_2(t)$, $\gamma_2(t)$, and the conditional distribution function $F_t(s)$ introduced above (8.7), the conditions (8.15) become

$$F_t(v_2-0) + 1 - F_t(w_2) + \delta_2[F_t(v_2) - F_t(v_2-0)] + \gamma_2[F_t(w_2) - F_t(w_2-0)] = \alpha, \quad \dots \quad (8.16)$$

$$\int_{-\infty}^{v_2-0} s\,dF_t + \int_{w_2+0}^{\infty} s\,dF_t + v_2\delta_2[F_t(v_2) - F_t(v_2-0)] + w_2\gamma_2[F_t(w_2) - F_t(w_2-0)] = \alpha_2, \dots \quad (8.17)$$

where (8.16) and (8.17) are to be satisfied (a.e. $\mathfrak{P}_0^t$). Since (8.16) and (8.17) do not depend on ϑ^*, θ^*, or θ^0 (F_t depends on ϑ_0), neither will solutions $v_2(t)$, $w_2(t)$, $\delta_2(t)$, and $\gamma_2(t)$, whose existence we shall now establish by a continuity argument: For each fixed t, let y be a variable with range $0 \leqslant y \leqslant \alpha$ and define v and δ as functions of y to satisfy

$$F_t(v-0) \leqslant y \leqslant F_t(v),$$

$$y = F_t(v-0) + \delta[F_t(v) - F_t(v-0)],$$

so that v is unique unless there is an interval where F_t is constant and equal to y, and δ is unique if $F_t(s)$ is discontinuous at $s = v$. Next determine w and γ similarly so that

$$F_t(w-0) \leqslant 1-\alpha+ y \leqslant F_t(w),$$

$$1-\alpha+y = F_t(w-0)+(1-\gamma)[F_t(w)-F_t(w-0)].$$

For any y $(0 \leqslant y \leqslant \alpha)$ the v, w, δ, γ thus defined will satisfy (8.16). Let $G_t(y)$ denote the value obtained when the v, w, δ, γ are substituted for the $v_2, w_2, \delta_2, \gamma_2$ in the left member of (8.17). Then $G_t(y)$ is a unique function of y even though v, w, δ, γ are in general not, and is continuous in y. If we can show (a.e. $\mathfrak{P}_0^t$) that $G_t(0) \geqslant \alpha_2$ and $G_t(\alpha) \leqslant \alpha_2$, then there must exist a y_t for which $G(y_t) = \alpha_2$. If we then define $v_2, w_2, \delta_2, \gamma_2$ as the v, w, δ, γ corresponding to $y = y_t$, both (8.16) and (8.17) will be satisfied.

To fill the above gap in the proof for H_2 we note that the test found above for for H_1 corresponds to $y = 0$, and this maximizes $E_{\vartheta,\theta}(\Phi)$ for every $\vartheta > \vartheta_0$ subject to (8.6) , hence it also maximizes the derivative

$$\partial E_{\vartheta,\theta}(\Phi)/\partial\vartheta\Big|_{\vartheta=\vartheta_0} = E_{\vartheta_0,\theta}(S\Phi-\alpha S).$$

It must therefore maximize $E_{\vartheta_0,\theta}(S\Phi-\alpha S\,|\,t)$ (a.e. $\mathfrak{P}_0^t$). But this has the value 0 for $\Phi \equiv \alpha$. Consequently the Φ corresponding to $y = 0$ makes $E_{\vartheta_0,\theta}(S\Phi\,|\,t) \geqslant \alpha E_{\vartheta_0,\theta}(S\,|\,t)$, or $G_t(0) \geqslant \alpha_2$ (a.e. $\mathfrak{P}_0^t$). By similar reasoning about the test corresponding to $y = \alpha$ it can be shown $G_t(\alpha) \leqslant \alpha_2$ (a.e. $\mathfrak{P}_0^t$). Unbiasedness of the test follows as before from comparison with $\Phi \equiv \alpha$.

We shall treat together the cases of the theorem for hypotheses H_3 and H_4. By the remark preceding the theorem unbiasedness implies similarity for $\mathfrak{P}_0^\omega \cup \mathfrak{P}_1^\omega$, while sufficiency of T for $\mathfrak{P}_0^\omega$ and for $\mathfrak{P}_1^\omega$ and strong completeness of $\mathfrak{P}_0^t$ and of $\mathfrak{P}_1^t$ imply

$$E_{\vartheta_j,\theta}(\Phi\,|\,t) = \alpha \quad \text{(a.e. } \mathfrak{P}_j^t) \quad (j = 0, 1). \qquad \ldots \quad (8.18)$$

Let (ϑ^*, θ^*) be a point in Ω, $\vartheta^* \neq \vartheta_0$ or ϑ_1. We choose arbitrary θ^0 and θ^1 such that $(\vartheta_j, \theta^j)\epsilon\Omega_j$ $(j = 0, 1)$, and we again apply Lemma 8.2, this time with $k = 2$, $\Lambda_i = \Omega_{i-1}$, $\eta^* = (\vartheta^*, \theta^*)$, $\eta_i = (\vartheta_{i-1}, \theta^{i-1})$, $f_i \equiv 1$, $U_i = T$ $(i = 1, 2)$, to find that $\varphi(x)$ will maximize $E_{\vartheta^*,\theta^*}(\varphi(X))$ subject to (8.18) if it satisfies (8.18) and there exist $g_0(t), g_1(t)$ such that $\varphi(x) = 1$ or 0 according as

$$p_{\vartheta^*,\theta^*}(x) > \text{ or } < g_0(t(x))\, p_{\vartheta_0,\theta^0}(x)+g_1(t(x))\, p_{\vartheta_1,\theta^1}(x). \qquad \ldots \quad (8.19)$$

Substituting from (8.1) we find (8.19) equivalent to the existence of h_0 and h_1 such that $\varphi(x) = 1$ or 0 according as $H(s) <$ or > 1, where

$$H(s) = h_0 e^{\Delta_0 s}+h_1 e^{\Delta_1 s}, \qquad \ldots \quad (8.20)$$

$\Delta_i = \vartheta_i - \vartheta^*$, and where h_0 and h_1 may depend on $\vartheta^*, \vartheta_0, \vartheta_1, \theta^*, \theta^0, \theta^1$, as well as on t.

For any v, w with $-\infty \leqslant v < w \leqslant +\infty$ we shall define h_0, h_1 to be the unique solutions of the equations $H(v) = H(w) = 1$, namely

$$h_0 = (e^{\Delta_1 w} - e^{\Delta_1 v})/D, \quad h_1 = (e^{\Delta_0 v} - e^{\Delta_0 w})/D,$$

where
$$D = e^{\Delta_0 v + \Delta_1 w} - e^{\Delta_0 w + \Delta_1 v} > 0.$$

Suppose first $\vartheta_0 < \vartheta^* < \vartheta_1$, so $\Delta_0 < 0 < \Delta_1$, hence $h_0 > 0$, $h_1 > 0$. Then $H''(s) > 0$ for all s, and for this determination of h_0 and h_1, $H(s) < 1$ or > 1 according as s is inside or outside the interval (v, w). Suppose next $\vartheta_0 < \vartheta_1 < \vartheta^*$ so $\Delta_0 < \Delta_1 < 0$, hence $h_0 < 0$, $h_1 > 0$. Now $H(s) <$ or > 1 according as $\tilde{H}(s) >$ or < 1, where

$$\tilde{H}(s) = \tilde{h}_0 e^{\tilde{\Delta}_0 s} + \tilde{h}_1 e^{\tilde{\Delta}_1 s},$$

$$\tilde{h}_0 = -g_0/g_1 > 0, \quad \tilde{h}_1 = 1/g_1 > 0, \quad \tilde{\Delta}_0 = \Delta_0 - \Delta_1 < 0, \quad \tilde{\Delta}_1 = -\Delta_1 > 0.$$

Applying the previous reasoning about $H(s)$ to $\tilde{H}(s)$ we see that in the present case according as s is inside or outside the interval (v, w), $\tilde{H}(s)$ is < 1 or > 1, hence $H(s)$ is > 1 or < 1. The same result is found in the case $\vartheta^* < \vartheta_0 < \vartheta_1$. At this point we see that (8.18) is equivalent to $\varphi(x) = 1$ or 0 according as $s(x)$ is inside or outside an interval (v, w) if $\vartheta_0 < \vartheta^* < \vartheta_1$, the opposite if $\vartheta^* < \vartheta_0$ or $\vartheta^* > \vartheta_1$, where $v = v(t)$ and $w = w(t)$ may depend also on $\vartheta^*, \vartheta_0, \vartheta_1, \theta^*, \theta^0, \theta^1$.

Next we must show that $v(t)$, $w(t)$, $\delta(t)$, $\gamma(t)$ may be chosen so that they do not depend on $\vartheta^*, \theta^*, \theta^0$, or θ^1 and (8.18) is satisfied, and that the resulting test has size α and is unbiased. Since this part of the proof is very similar for H_3 and H_4 we indicate it only for H_3. Let $F_{t,j}(s) = P_{\vartheta_j, \theta}(S \leqslant s \mid t)$, which does not depend on θ $(j = 0, 1)$. The conditions (8.18) are then equivalent to two equations similar to (8.16), namely,

$$F_{t,j}(v_3 - 0) + 1 - F_{t,j}(w_3) + \delta_3[F_{t,j}(v_3) - F_{t,j}(v_3 - 0)]$$
$$+ \gamma_3[F_{t,j}(w_3) - F_{t,j}(w_3 - 0)] = \alpha \; (j = 0, 1). \quad \ldots \quad (8.21_j)$$

First consider the solution (v, w, δ, γ) of (8.21_0) obtained by setting $v = -\infty$, which gives the uniformly most powerful unbiased test of H_1: $\vartheta \leqslant \vartheta_0$ against $\vartheta > \vartheta_0$ at size α. For this solution the left member of (8.21_1) is the conditional power of this test against $\vartheta = \vartheta_1$ and is therefore $\geqslant \alpha$ (a. e. $\mathfrak{P}_1^t$). Next consider the solution (v, w, δ, γ) of (8.21_0) obtained by setting $w = +\infty$, which gives the uniformly most powerful unbiased test of H_1': $\vartheta \geqslant \vartheta_0$ against $\vartheta < \vartheta_0$ at size α. For this (v, w) the left member of (8.21_1) is the conditional probability that this test will reject H_1' when $\vartheta = \vartheta_1$ and is therefore $\leqslant \alpha$ (a. e. $\mathfrak{P}_1^t$). If we now think of varying the solution (v, w, δ, γ) of (8.21_0) between these two extremes, it follows again from continuity considerations that there exists a solution of (8.21_0) for which (8.21_1) is also satisfied, and that the resulting $v = v_3(t)$, $w = w_3(t)$, $\delta = \delta_3(t)$, $\gamma = \gamma_3(t)$ do not depend on $\vartheta^*, \theta^*, \theta^0$, or θ^1. Verification of correct size and unbiasedness follow again from comparison with the test corresponding to $\Phi \equiv \alpha$.

9. Some examples

In applications of Theorem 8.2 the $\vartheta, \theta_1, \ldots, \theta_r$ of the density (8.1) are usually not the original parameters of the statistical problem but functions of these. Suppose that originally the parameters are $(\psi_1, \ldots, \psi_{r+1}) = \psi$ and that they appear in a density of the form

$$D(\psi)h(x) \exp \sum_{i=1}^{r+1} f_i(\psi)u_i(x).$$

If there exists a 1:1 transformation from $\psi_1, \ldots, \psi_{r+1}$ to $\vartheta, \theta_1, \ldots, \theta_r$ such that

$$\sum_{i=1}^{r+1} f_i(\psi)u_i(x) \equiv \vartheta s(x) + \sum_{i=1}^{r} \theta_i t_i(x) \qquad \ldots \quad (9.1)$$

identically in ψ and x, so that the density becomes of the form (8.1) and if the assumptions stated in connection with (8.1) are satisfied, then there exist uniformly most powerful unbiased tests of hypotheses of the form H_1, H_2, H_3, H_4 defined at the beginning of section 8, and Theorem 8.2 tells us how to construct them.

We are especially interested in determining the functions $s(x)$, $t_1(x), \ldots, t_r(x)$, since Theorem 8.2 gives the test in terms of these functions. It is obvious that if we put $(\vartheta, \theta_1, \ldots, \theta_r)$ equal to a nonsingular linear transformation[7] of $(f_1(\psi), \ldots, f_{r+1}(\psi))$, then (9.1) will be satisfied. We may choose any constants $a_{11}, \ldots, a_{1,r+1}$ not all zero, and constants $\vartheta_0, \vartheta_1 (\vartheta_0 < \vartheta_1)$, and consider the hypotheses $H_i (i = 1, \ldots, 4)$ about

$$\vartheta \equiv \sum_{j=1}^{r+1} a_{1j} f_j(\psi). \qquad \ldots \quad (9.2)$$

Without affecting these hypotheses we can assume $\Sigma_{j=1}^{r+1} a_{ij}^2 = 1$, by redefinition of ϑ_0 and ϑ_1. We define the transformation from $(\psi_1, \ldots, \psi_{r+1})$ to $(\vartheta, \theta_1, \ldots, \theta_r)$ by the equations

$$a_{1i}\, \vartheta + \sum_{j=2}^{r+1} a_{ji}\theta_{j-1} = f_i(\psi) \quad (i = 1, \ldots, r+1), \qquad \ldots \quad (9.3)$$

where (a_{ij}) is any orthogonal matrix whose first row is $(a_{11}, \ldots, a_{1,r+1})$. Then (9.2) is satisfied. If we substitute (9.3) in (9.1) and collect coefficients of ϑ and of $\theta_1, \ldots, \theta_r$, we find that

$$s(x) = \sum_{j=1}^{r+1} a_{1j} u_j(x), \qquad \ldots \quad (9.4)$$

that is, $s(x)$ is the same linear form in the $u_j(x)$ as ϑ is in the $f_j(\psi)$, while $t_1(x), \ldots, t_r(x)$ are r linearly independent forms orthogonal to (9.4).

[7] It can be shown, under mild restrictions, that this is the most general transformation preserving (9.1).

Example 9.1: Suppose X is a random sample of n from a normal population with unknown mean μ and unknown variance σ^2. The density may be written $D\ \exp\ (f_1u_1+f_2u_2)$, where $f_1 = 1/\sigma^2$, $f_2 = \mu/\sigma^2$, $u_1 = -\frac{1}{2}\Sigma_1^n x_i^2$, $u_2 = \Sigma_1^n x_i$, and D is a function of the parameters. Set

$$\vartheta = A(1/\sigma^2)+B(\mu/\sigma^2),$$

where A and B are known constants. By specializing A, B, ϑ_0, ϑ_1 we get from Theorem 8.2 UMPU (uniformly most powerful unbiased) tests of any of the following hypotheses: $\mu = \mu_0$, $\mu \leqslant \mu_0$, $\sigma^2 = \sigma_0^2$, $\sigma^2 \geqslant \sigma_0^2$, σ^2 inside (or outside) an interval (σ_1^2, σ_0^2). We do not get a test of $\mu_0 \leqslant \mu \leqslant \mu_1$ by this method. It is curious that UMPU tests fall out for such apparently "unnatural" hypotheses as that the point (μ,σ) lies between two parabolas $\mu = \mu_0+c_1\sigma^2$ and $\mu = \mu_0+c_2\sigma^2$, tangent to each other at $(\mu_0, 0)$.

Example 9.2: For independent random samples from k normal populations with equal variance σ^2, the ith sample of size n_i from a population with mean μ_i, the density is of the form $D \exp\ [u_0(1/\sigma^2)+\Sigma_1^k u_i(\mu_i/\sigma^2)]$, and so we may take

$$\vartheta = A(1/\sigma^2)+\Sigma_1^k B_i(\mu_i/\sigma^2).$$

By specializing $A, B_1, \ldots, B_k, \vartheta_0, \vartheta_1$ we get UMPU tests of the hypotheses $\Sigma_1^k c_i\mu_i = c_0$, $\Sigma_1^k c_i\mu_i \leqslant c_0$, $\sigma^2 = \sigma_0^2$, $\sigma^2 \geqslant \sigma_0^2$, σ^2 inside (or outside) an interval (σ_1^2, σ_0^2).

Example 9.3: Suppose Example 9.2 is changed so that the variance of the ith population is σ_i^2 $(i = 1, \ldots, k)$. Then the density is $D\ \exp\ [\Sigma_1^k u_i'(1/\sigma_i^2) + \Sigma_1^k u_i(\mu_i/\sigma_i^2)]$, and we may take

$$\vartheta = \sum_1^k A_i(1/\sigma_i^2)+ \sum_1^k B_i(\mu_i/\sigma_i^2),$$

yielding UMPU tests of the hypotheses that $\Sigma_1^k\ A_i(1/\sigma_i^2)$, a linear form in the precisions $1/\sigma_i^2$, is $= A$, $\leqslant A$, inside (or outside) an interval. For $k = 2$ this includes the hypotheses that σ_1^2/σ_2^2 is $= A$, $\leqslant A$, but not the hypothesis that σ_1^2/σ_2^2 is in a given interval.

Example 9.4: For independent random samples of sizes n_1 and n_2 from binomial populations with parameters p_1 and p_2 it is found that the density is of the form $Dh(x)\ \exp\ (f_1u_1+f_2u_2)$ with $f_i = \log\ [p_i/(1-p_i)]$, leading to

$$\vartheta = \log\left[\left(\frac{p_1}{1-p_1}\right)^{A_1}\left(\frac{p_2}{1-p_2}\right)^{A_2}\right].$$

Specializing $A_1 = -A_2 = 1$ yields UMPU tests of the hypotheses that the "odds ratio" $[p_1/(1-p_1)]/[p_2/(1-p_2)]$ is $= A$, $\leqslant A$, inside (outside) an interval. With $A = 1$ we get UMPU tests of the hypotheses $p_1 = p_2$, $p_1 \leqslant p_2$.

Example 9.5: For independent observations from two Poisson populations with parameters λ_1 and λ_2 the density is $Dh(x)\exp[x_1 \log \lambda_1 + x_2 \log \lambda_2]$, leading to

$$\vartheta = \log\left(\lambda_1^{A_1}\lambda_2^{A_2}\right).$$

With $A_1 = -A_2 = 1$, we get UMPU tests of the hypotheses that the ratio λ_1/λ_2 is $= A, \leqslant A$, inside (outside) an interval.

APPENDIX

MEASURABILITY OF PROPOSED CRITICAL FUNCTIONS

It is unfortunately necessary to show that our construction of the critical functions $\varphi_i(x)$ of Theorem 8.2 leads to B.m. (Borel measurable) functions. Since these are functions of the B.m. functions $s(x)$ and $t(x)$ it suffices to prove they are B.m. functions of (s, t). Following the above construction, the definitions of the φ_i are made as functions of s for each fixed t in a set which may differ from the t-space by a null set of $\mathfrak{P}^t$. In this connection, we note that the qualification "(a.e. $P^t_{\vartheta,\theta}$)" for some $(\vartheta, \theta) \in \Omega$ is equivalent to "(a.e. $\mathfrak{P}^t$)" since by (8.1) and the proof of Lemma 8.1, a null set of $P^t_{\vartheta,\theta}$ is of measure ν^t zero, where $d\nu^t = h(x)d\mu^x$, and is hence a null set of $\mathfrak{P}^t$.

Consider the critical function φ_1 for H_1. The conditional distribution function $F_t(s)$ introduced above (8.7) may be chosen to be B.m. in t (for each s) and non-decreasing and continuous from the right in s (for each t). It follows from the identity[8]

$$\{(s, t) \| F_t(s) \geqslant c\} = \bigcap_n \bigcup_i \{(s, t) \mid 0 \leqslant r_i - s < 1/n,\ F_t(r_i) \geqslant c\},$$

where (r_i) denotes the rationals, that $F_t(s)$ is then B.m. in (s, t). Likewise, $F_t(s-0)$ is also B.m. in (s, t). Define $F_t^{-1}(y)$ as inf s for which $F_t(s) \geqslant y$; then $F_t^{-1}(y)$ and $F_t^{-1}(y-0)$ are B.m. in (y, t). The functions $w_1(t)$ and $\gamma_1(t)$ of (8.7) and (8.8) may be uniquely defined as $w_1(t) = F_t^{-1}(1-\alpha)$, and

$$\gamma_1(t) = 1 - \frac{F_t(w_1(t)-0)}{F_t(w_1(t)) - F_t(w_1(t)-0)},$$

where, here and below, if a denominator is zero the fraction is defined to be zero. Then $w_1(t)$ and hence $\gamma_1(t)$ are B.m. in t. Since $\varphi_1 = 1$ for $s > w_1(t)$, $\varphi_1 = 0$ for $s < w_1(t)$, and $\varphi_1 = \gamma_1(t)$ for $s = w_1(t)$, it follows that φ_1 is B.m. in (s, t).

In the proof for φ_2 we shall need that

$$g(x, t) = \int_{-\infty}^{x-0} s\,dF_t(s)$$

is B.m. in (x, t). This integral converges since $E_{\vartheta_0,\theta}$ is finite because of the exponential form of the density (8.1), hence the integral $g(\infty, t) = E_{\vartheta_0,\theta}(s|t)$ is absolutely

[8] This proof was pointed out to us by Professor L. M. LeCam.

convergent (a.e.$\mathfrak{P}^t$). Now suppose $x<+\infty$. We may express $g(x, t)$ as the limit of approximating sums $g_n(x, t)$ in which "subdivision of the ordinate" leads to

$$g_n(x, t) = \sum_{k=1}^{\infty}\left(x-\frac{k}{2^n}\right)\left[F_t\left(x-\frac{k-1}{2^n}-0\right)-F_t\left(x-\frac{k}{2^n}-0\right)\right]$$

which is B.m. in (x, t), hence $g(x, t)$ is also. That $g(\infty, t) = \alpha_2(t)/\alpha$ is B.m. in t follows from the fact that it is a conditional expectation.

For $0\leqslant y\leqslant\alpha$ define $v(y, t) = F_t^{-1}(y)$, $w(y, t) = F_t^{-1}(1-\alpha+y)$, and

$$\delta(y, t) = \frac{y-F_t(v(y, t)-0)}{F_t(v(y, t))-F_t(v(y, t)-0)},$$

$$\gamma(y, t) = 1-\frac{1-\alpha+y-F_t(w(y, t)-0)}{F_t(w(y, t))-F_t(w(y, t)-0)}.$$

These four functions are then B.m. in (y, t). Let $H(y, t)+\alpha_2(t)$ denote the left member of (8.17) with v_2, w_2, δ_2, γ_2 replaced by $v(y, t)$, $w(y, t)$, $\delta(y, t)$, $\gamma(y, t)$, respectively, i.e., $H(y, t)$ equals the $G_t(y)$ introduced above, diminished by $\alpha_2(t)$. Thus $H(0, t)\geqslant 0$, $H(\alpha, t)\leqslant 0$, and $H(y, t)$ is B.m. in (y, t) and continuous in y. Now define y_t as α on the set where $H(\alpha, t) = 0$ and otherwise as the inf of y for which $H(y, t)<0$. Because $H(y, t)$ is continuous in y, y_t satisfies the desired condition $H(y_t, t) = 0$, equivalent to (8.17). That y_t is B.m. in t may be seen as follows: $\{t| y_t<c, H(\alpha, t)<0\}$ is the set of t for which $H(\alpha, t)<0$ and $H(y,t)<0$ for some $y<c$, hence for some rational $y = r_n<c$, because of the continuity of $H(y, t)$ in y, and this is the union of the denumerable number of B.m. sets $\{t|H(r_n, t)<0, H(\alpha, t)<0\}$ for rational $r_n<c$. If $v_2(t)$, $w_2(t)$, $\delta_2(t)$, $\gamma_2(t)$ are defined as $v(y, t)$, $w(y, t)$, $\delta(y, t)$, $\gamma(y, t)$, which are B.m. in (y,t), with y replaced by y_t, which is B.m. in t, then it is clear these are then B.m. in t, and hence φ_2 is B.m. in (s, t).

The proof for φ_3 is rather similar to that for φ_2. We again define functions $v(y, t)$, $w(y, t)$, $\delta(y, t)$, $\gamma(y, t)$ by replacing the F_t in the previous definitions by $F_{t,1}$, where $F_{t,j}$ is defined above (8.21_j). We then define $H(y, t)$ as the left member of (8.21_1) diminished by α and with v_3, etc. replaced by $v(y, t)$, etc. The continuity of $H(y, t)$ in y now depends on the fact that for each t except a null set of $\mathfrak{P}^t$, the countable set of discontinuities in s of $F_{t,j}(s)$ is the same for $j = 0$ as for $j = 1$, and the set of intervals where $F_{t,j}$ is constant is the same for $j = 0$ and 1. To show this define A_j to be the set of points (s, t) where $F_{t,j}(s)-F_{t,j}(s-0) = 0$ and $F_{t,1-j}(s)-F_{t,1-j}(s-0) > 0$, so A_j is a B.m. set in (s, t). Let $f_{jk}(t)$ be the probability assigned by $F_{t,k}(s)$ to the cross-section of A_j on which $t =$ constant ($k = 0, 1$); so $f_{jj}(t) = 0$ and $f_{j,1-j}(t)>0$ for all non-empty cross-sections of A_j. We have

$$P^{s,t}_{\vartheta,\theta}(A_j) = \int_{W^t} f_{jk}(t)dP^t_{\vartheta,\theta}$$

for all $(\vartheta, \theta)\epsilon\Omega_k$. If B_j is the pre-image of A_j in W^x, so $P^{s,t}_{\vartheta,\theta}(A_j) = P^x_{\vartheta,\theta}(B_j)$, we see from (8.1) that if $P^x_{\vartheta,\theta}(B_j) = 0$ for some $(\vartheta, \theta)\epsilon\Omega$, then $h(x) = 0$ (a.e.μ^x) on B_j and hence $P^{\mathfrak{F}}_{\vartheta,\theta}(B_j) = 0$ for all $(\vartheta,\theta)\epsilon\Omega$. But $P^{s,t}_{\vartheta,\theta}(A_j) = 0$ for $(\vartheta,\theta)\,\epsilon\Omega_j$, therefore for $(\vartheta,\theta)\,\epsilon\Omega_{1-j}$, hence (a.e.$\mathfrak{P}^t$) $f_{j,1-j}(t) = 0$ and the cross-section of A_j for $t =$ constant is empty.

This proves the above assertion about the discontinuities of $F_{t,1}$ and $F_{t,2}$; to prove it for the intervals of constancy, denote by $F^{-1}_{t,j}(y)$ the inf of s for which $F_t(s) \geqslant y$, and by $\bar{F}^{-1}_{t,j}(y)$ the sup for $F_{t,j}(s) \leqslant y$. Write $K_j(s,t) = \bar{F}^{-1}_{t,j}(F_{t,j}(s) - F^{-1}_{t,j}(F_{t,j}(s))$, so $K_j(s,t)$ is B.m. in (s,t), and s_0 will be in an interval where $F_{t,j}(s)$ is constant if and only if $K_j(s_0,t) > 0$. Now let D_j be the set of points (s,t) where $K_j(s,t) > 0$ and $K_{1-j}(s,t) = 0$. Applying to D_j an argument similar to the above for A_j we can conclude the proof.

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