Applied and Numerical Harmonic Analysis

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David F. Walnut

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To my parents

and to Megan

Unless the LORD *builds the house,*
its builders labor in vain.

– Psalm 127:1a (NIV)

Contents

Preface

These days there are dozens of wavelet books on the market, some of which are destined to be classics in the field. So a natural question to ask is: Why another one? In short, I wrote this book to supply the particular needs of students in a graduate course on wavelets that I have taught several times since 1991 at George Mason University. As is typical with such offerings, the course drew an audience with widely varying backgrounds and widely varying expectations. The difficult if not impossible task for me, the instructor, was to present the beauty, usefulness, and mathematical depth of the subject to such an audience.

It would be insane to claim that I have been entirely successful in this task. However, through much trial and error, I have arrived at some basic principles that are reflected in the structure of this book. I believe that this makes this book distinct from existing texts, and I hope that others may find the book useful.

(1) Consistent assumptions of mathematical preparation. In some ways, the subject of wavelets is deceptively easy. It is not difficult to understand and implement a discrete wavelet transform and from there to analyze and process signals and images with great success. However, the underlying ideas and connections that make wavelets such a fascinating subject require some considerable mathematical sophistication. There have been some excellent books written on wavelets emphasizing their elementary nature (e.g., Kaiser, *A Friendly Guide to Wavelets*; Strang and Nguyen, *Wavelets and Filter Banks*; Walker, *Primer on Wavelets and their Scientific Applications*; Frazier, *Introduction to Wavelets through Linear Algebra*; Nievergelt, *Wavelets Made Easy*; Meyer, *Wavelets: Algorithms and Applications*). For my own purposes, such texts required quite a bit of "filling in the gaps" in order to make some connections and to prepare the student for more advanced books and research articles in wavelet theory.

This book assumes an upper-level undergraduate semester of advanced calculus. Sufficient preparation would come from, for example, Chapters 1–5 of Buck, *Advanced Calculus*. I have tried very hard not to depart from this assumption at any point in the book. This has required at times sacrificing elegance and generality for accessibility. However, all proofs are completely rigorous and contain the gist of the more general argument. In this way, it is hoped that the reader will be prepared to tackle more sophisticated books and articles on wavelet theory.

(2) Proceeding from the continuous to the discrete. I have always found it more meaningful and ultimately easier to start with a presentation of wavelets and wavelet bases in the continuous domain and use this to motivate the discrete theory, even though the discrete theory hangs together in its own right and is easy to understand. This can be frustrating for the student whose primary interest is in applications, but I believe that a better understanding of applications can ultimately be achieved by doing things in this order.

(3) Prepare readers to explore wavelet theory on their own. Wavelets is too broad a subject to cover in a single book and is most interesting to study when the students have a particular interest in what they are studying. In choosing what to include in the book, I have tried to ensure that students are equipped to pursue more advanced topics on their own. I have included an appendix called *Excursions in Wavelet Theory* (Appendix B) that gives some guidance toward what I consider to be the most readable articles on some selected topics. The suggested topics in this appendix can also be used as the basis of semester projects for the students.

Structure of the Book

The book is divided into five parts: *Preliminaries*, *The Haar System*, *Multiresolution Analysis and Orthonormal Wavelet Bases*, *Other Wavelet Constructions*, and *Applications.*

Preliminaries

Wavelet theory is really very hard to appreciate outside the context of the language and ideas of Fourier Analysis. Chapters 1-4 of the book provide a background in some of these ideas and include everything that is subsequently used in the text. These chapters are designed to be more than just a reference but less than a "book-within-a-book" on Fourier analysis. Depending on the background of the reader or of the class in which this book is being used, these chapters are intended to be dipped into either superficially or in detail as appropriate.

Naturally there are a great many books on Fourier analysis that cover the same material better and more thoroughly than do Chapters 1-4 and at the same level (more or less) of mathematical sophistication. I will list some of my favorites below. Walker, *Fourier Analysis*; Kammler, *A First Course in Fourier Analysis*; Churchill and Brown, *Fourier Series and Boundary Value Problems*; Dym and McKean, *Fourier Series and Integrals*; Körner, *Fourier Analysis*; and Benedetto, *Harmonic Analysis and Applications.*

The Haar System

Chapters 5 and 6 provide a self-contained exposition of the Haar system, the earliest example of an orthonormal wavelet basis. These chapters could be presented as is in a course on advanced calculus, or an undergraduate Fourier analysis course. In the context of the rest of the book, these chapters are designed to motivate the search for more general wavelet bases with different properties, and also to illustrate some of the more advanced concepts such as multiresolution analysis that are used throughout the rest of the book. Chapter 5 contains a description of the Haar basis on $[0, 1]$ and on $\mathbf{R}$, and Chapter 6 shows how to implement a discrete version of the Haar basis in one and two dimensions. Some examples of images analyzed with the Haar wavelet are also included.

Multiresolution Analysis and Orthonormal Wavelet Bases

Chapters 7-9 represent the heart of the book. Chapter 7 contains an exposition of the general notion of a multiresolution analysis (MRA) together with several examples. Next, we describe the recipe that gives the construction of a wavelet basis from an MRA, and then construct corresponding examples of wavelet orthonormal bases. Chapter 8 describes the passage from the continuous domain to the discrete domain. First, properties of MRA are then used to motivate and define the quadrature mirror filter (QMF) conditions that any orthonormal wavelet filter must satisfy. Then the discrete wavelet transform (DWT) is defined for infinite signals, periodic signals, and for finite sets of data. Finally the techniques used to pass from discrete filters satisfying the QMF conditions to continuously defined wavelet functions are described. Chapter 9 presents the construction of compactly supported orthornomal wavelet bases due to Daubechies. Daubechies's approach is motivated by a lengthy discussion of the importance of vanishing moments in the design of wavelet filters.

Other Wavelet Constructions

Chapters 10 and 11 contain a discussion of two important variations on the theme of the construction of orthonormal wavelet bases. The first, in Chapter 10, shows what happens when you allow yourself to consider nonorthogonal wavelet systems. This chapter contains a discussion of Riesz bases, and describes the semi-orthogonal wavelets of Chui and Wang, as well as the notion of dual MRA and the fully biorthogonal wavelets of Daubechies, Cohen, and Feauveau. Chapter 11 discusses wavelet packets, another natural variation on orthonormal wavelet bases. The motivation here is to consider what happens to the DWT when the "full wavelet tree" is computed. Waveletpacket functions are described, their time and frequency localization properties are discussed, and necessary and sufficient conditions are given under which a collection of scaled and shifted waveletpackets constitutes an orthonormal basis on $\mathbf{R}$. Finally, the notion of a best basis is described, and the so-called best basis algorithm (due to Coifman and Wickerhauser) is given.

Applications

Many wavelet books have been written emphasizing applications of the theory, most notably, Strang and Nguyen, *Wavelets and Filter Banks*, and Mallat's comprehensive, *A Wavelet Tour of Signal Processing*. The book by Wickerhauser, *Applied Wavelet Analysis from Theory to Software*, also contains descriptions of several applications. The reader is encouraged to consult these texts and the references therein to learn more about wavelet applications.

The description of applications in this book is limited to a brief description of two fundamental examples of wavelet applications. The first, described in Chapter 12, is to image compression. The basic components of a transform image coder as well as how wavelets fit into this picture are described. Chapter 13 describes the Beylkin-Coifman-Rokhlin (BCR) algorithm, which is useful for numerically estimating certain integral operators known as singular integral operators. The algorithm is very effective and uses the same basic properties of wavelets that make them useful for image compression. Several examples of singular integral operators arising in ordinary differential equations, complex variable theory, and image processing are given before the BCR algorithm is described.

Acknowledgments

I want to express my thanks to the many folks who made this book possible. First and foremost, I want to thank my advisor and friend John Benedetto for encouraging me to take on this project and for graciously agreeing to publish it in his book series. Thanks also to Wayne Yuhasz, Lauren Schultz, Louise Farkas, and Shoshanna Grossman at Birkhäuser for their advice and support. I want to thank Margaret Mitchell for LaTeX advice and Jim Houston and Clovis L. Tondo for modifying some of the figures to make them more readable. All of the figures in this book were created by me using MATLAB and the Wavelet ToolBox. Thanks to the MathWorks for creating such superior products.

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I want give special thanks to my Dad, with whom I had many conversations about book-writing. He passed away suddenly while this book was in production and never saw the finished product. He was pleased and proud to have another published author in the family. He is greatly missed.

Finally, I want to thank my wife Megan for her constant love and support, and my delightful children John and Genna who will someday read their names here and wonder how their old man actually did it.

Fairfax, Virginia *David F. Walnut*

Albrecht Dürer (1471-1528), *Melencholia I* (engraving). Courtesy of the Fogg Art Museum, Harvard University Art Museums, Gift of William Gray from the collection of Francis Calley Gray. Photograph by Rick Stafford, ©President and Fellows of Harvard College. A detail of this engraving, a portion of the magic square, is used as the sample image in 22 figures in this book. The file processed is a portion of the image file detail.mat packaged with MATLAB version 5.0.