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David Nualart

The Malliavin Calculus and Related Topics

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David Nualart
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
08007 Barcelona, Spain

Series Editors
J. Gani
Stochastic Analysis Group, CMA
Australian National University
Canberra ACT 0200
Australia

C.C. Heyde
Stochastic Analysis Group, CMA
Australian National University
Canberra ACT 0200
Australia

T.E. Kurtz
Departments of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, WI 53706
USA

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To my daughters Eulàlia and Neus

Preface

The origin of this book lies in an invitation to give a series of lectures on Malliavin calculus at the Probability Seminar of Venezuela, in April 1985. The contents of these lectures were published in Spanish in [176]. Later these notes were completed and improved in two courses on Malliavin calculus given at the University of California at Irvine in 1986 and at École Polytechnique Fédérale de Lausanne in 1989. The contents of these courses correspond to the material presented in Chapters 1 and 2 of this book. Chapter 3 deals with the anticipating stochastic calculus and it was developed from our collaboration with Moshe Zakai and Etienne Pardoux. The series of lectures given at the Eighth Chilean Winter School in Probability and Statistics, at Santiago de Chile, in July 1989, allowed us to write a pedagogical approach to the anticipating calculus which is the basis of Chapter 3. Chapter 4 deals with the nonlinear transformations of the Wiener measure and their applications to the study of the Markov property for solutions to stochastic differential equations with boundary conditions. The presentation of this chapter was inspired by the lectures given at the Fourth Workshop on Stochastic Analysis in Oslo, in July 1992. I take the opportunity to thank these institutions for their hospitality, and in particular I would like to thank Enrique Cabaña, Mario Wschebor, Joaquín Ortega, Süleyman Üstünel, Bernt Øksendal, Renzo Cairoli, René Carmona, and Rolando Rebolledo for their invitations to lecture on these topics.

We assume that the reader has some familiarity with the Itô stochastic calculus and martingale theory. In Section 1.1.3 an introduction to the Itô calculus is provided, but we suggest the reader complete this outline of the classical Itô calculus with a review of any of the excellent presentations of this theory that are available (for instance, the books by Revuz and Yor [216] and Karatzas and Shreve [117]).

In the presentation of the stochastic calculus of variations (usually called the Malliavin calculus) we have chosen the framework of an arbitrary centered Gaussian family, and have tried to focus our attention on the notions and results that depend only on the covariance operator (or the associated Hilbert space). We have followed some of the ideas and notations developed by Watanabe in [258] for the case of an abstract Wiener space. In addition to Watanabe's book and the survey on the stochastic calculus of variations written by Ikeda and Watanabe in [102] we would like to mention the book by Denis Bell [14] (which contains a survey of the different approaches to the Malliavin calculus), and the lecture notes by Dan Ocone in [198]. Readers interested in the Malliavin calculus for jump processes can consult the book by Bichteler, Gravereaux, and Jacod [24].

The objective of this book is to introduce the reader to the Sobolev differential calculus for functionals of a Gaussian process. This is called the analysis on the Wiener space, and is developed in Chapter 1. The other chapters are devoted to different applications of this theory to problems such as the smoothness of probability laws (Chapter 2), the anticipating stochastic calculus (Chapter 3), and the shifts of the underlying Gaussian process (Chapter 4). Chapter 1, together with selected parts of the subsequent chapters, might constitute the basis for a graduate course on this subject.

I would like to express my gratitude to the people who have read the several versions of the manuscript, and who have encouraged me to complete the work, particularly I would like to thank John Walsh, Giuseppe Da Prato, Moshe Zakai, and Peter Imkeller. My special thanks go to Michael Röckner for his careful reading of the first two chapters of the manuscript.

March 17, 1995 David Nualart

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