# Entanglement Entropy of Nontrivial States ${ }^{\square}$ 

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#### Abstract

We study the entanglement entropy arising from coherent states and oneparticle states. We show that it is possible to define a finite entanglement entropy by subtracting the vacuum entropy from that of the considered states, when the unobserved region is the same.


[^0]Proposed running head: Entanglement Entropy

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## 1 Introduction

Recently, quantum corrections to the black hole entropy from matter fields have been studied extensively [1-11]. One source of the quantum corrections may be understood as entropy of entanglement, which arises when the density matrix of a pure quantum field theoretic state is reduced because the quantum field is not observed in some region of space. It is hoped that this concept provides a description for some of the contributions to the black hole entropy, where the unobserved quantum field lies within the black hole horizon.

In quantum field theoretic calculations, ultraviolet divergences appear due to the infinite number of degrees of freedom at short distances. Such infinities arise in the entanglement entropy because there is an infinite number of states near the boundary between the observed and unobserved regions. This implies that there is a conflict between the entropy defined by the counting of quantum states and the finite Bekenstein-Hawking thermodynamic entropy of a black hole.

In this paper we examine the possiblity of defining a finite entanglement entropy of a nontrivial state by subtracting the one associated with the vacuum. We study the entanglement entropy associated with coherent states and one-particle states in a massless scalar field theory in $(1+1)$-dimensional Minkowski space-time. We find that the entanglement entropy for the coherent states is the same as that for the vacuum, a result that can be generalized to a space-time of arbitrary dimension, with an unobserved region of arbitrary shape. For a restricted class of one-particle states we calculate the entanglement entropy explicitly and show that, once the vacuum expression is subtracted, the remainder is finite. We discuss possible divergences in the entanglement entropy for more general states.

We begin by presenting a brief review of the entanglement entropy associated with the vacuum state, which arises when we trace over the fields in the negative $x$ region by considering an imaginary boundary at $x=0$. The Hamiltonian of the system is

$$
\begin{equation*}
H=\frac{1}{2} \int d x \Pi^{2}(x)+\frac{1}{2} \int d x d y \Phi(x) \Omega^{2}(x, y) \Phi(y) \tag{1.1}
\end{equation*}
$$

where $\Pi(x)$ is the canonical momentum of $\Phi(x)$ and $\Omega^{2}(x, y)=-\nabla^{2} \delta(x-y)$. In the functional Schrödinger representation, the vacuum wave functional has the form

$$
\begin{equation*}
\langle\phi \mid 0\rangle_{M}=\Psi_{0}[\phi]=\operatorname{det}^{\frac{1}{4}}\left(\frac{\Omega}{\pi}\right) \exp \left\{-\frac{1}{2} \int d x d y \phi(x) \Omega(x, y) \phi(y)\right\} \tag{1.2}
\end{equation*}
$$

where $\phi(x)$ is a $c$-number field at a fixed time (the label $M$ indicates that this is the Minkowski vacuum). By constructing the pure state density matrix and tracing over the field in the negative $x$ region, we obtain a reduced density matrix,

$$
\begin{align*}
\rho_{0}\left(\phi_{+}^{1}, \phi_{+}^{2}\right) & =\int \mathcal{D} \phi_{-} \Psi_{0}\left(\phi_{+}^{1}, \phi_{-}\right) \Psi_{0}^{*}\left(\phi_{+}^{2}, \phi_{-}\right) \\
& =\left[\frac{\operatorname{det} \Omega}{\operatorname{det} \Omega_{--}}\right]^{\frac{1}{2}} e^{-\frac{1}{2} \int\left(\phi_{+}^{1} A_{+} \phi_{+}^{1}+\phi_{+}^{2} A_{+}+\phi_{+}^{2}+2 \phi_{+}^{1} B_{+}+\phi_{+}^{2}\right)} \tag{1.3}
\end{align*}
$$

where $A_{++}$and $B_{++}$are functions of the kernel entering in Eq. (11.2),

$$
\begin{equation*}
A_{++}=\Omega_{++}+B_{++} \quad ; \quad B_{++}=-\frac{1}{2} \Omega_{+-} \Omega_{--}^{-1} \Omega_{-+} \tag{1.4}
\end{equation*}
$$

[Throughout we use a self-evident functional/matrix notation, with $\phi_{-} \equiv \phi(x<0)$, $\phi_{+} \equiv \phi(x>0), \Omega_{+-} \equiv \Omega(x>0, y<0)$ etc., and $\left.\int \phi A \phi \equiv \iint d x d y \phi(x) A(x, y) \phi(y).\right]$ In Eq. (1.4) $\Omega_{--}^{-1}$ is the inverse of the restricted kernel $\Omega_{--}$. It has been shown first by Bombelli et al. [2], that $\rho_{0}$ may be diagonalized by solving the eigenvalue problem

$$
\begin{equation*}
\int_{0}^{\infty} d z \Lambda_{++}(x, z) \psi(z)=\lambda \psi(x) \tag{1.5}
\end{equation*}
$$

where

$$
\begin{equation*}
\Lambda_{++}(x, z) \equiv-\int_{-\infty}^{0} d y\left[\Omega^{-1}\right]_{+-}(x, y) \Omega_{-+}(y, z) \tag{1.6}
\end{equation*}
$$

In this expression $\left[\Omega^{-1}\right]_{+-}$is the inverse of the full kernel with argument restricted. The eigenfunctions and eigenvalues of $\Lambda_{++}$are

$$
\begin{align*}
\psi(x) & \propto \exp (i k \ln x)  \tag{1.7a}\\
\lambda(k) & =\frac{1}{\sinh ^{2} \pi k} \tag{1.7~b}
\end{align*}
$$

To calculate the entropy we must discretize the spectrum. We adopt the procedure used in Ref [5]: we introduce an infrared cutoff $L$ and an ultraviolet cutoff $\epsilon$, i.e. $\epsilon \leq x \leq L$, and demand that $\psi(x)$ vanish at $x=L$ and $x=\epsilon$,

$$
\begin{align*}
\psi(x) & =\sin \left(k_{n} \ln x / \epsilon\right)  \tag{1.8a}\\
k_{n} & =\frac{\pi n}{\ln (L / \epsilon)} \quad, \quad n \text { an integer. } \tag{1.8b}
\end{align*}
$$

The vacuum entanglement entropy can be approximated as an integral,

$$
\begin{equation*}
S_{0}=\sum_{n} \mathcal{S}_{0}\left(k_{n}\right) \approx \frac{2}{\pi} \ln \frac{L}{\epsilon} \int_{0}^{\infty} d \omega \mathcal{S}_{0}(\omega) \tag{1.9}
\end{equation*}
$$

where $\mathcal{S}_{0}(\omega)$ is the contribution of the eigenmode $\omega \equiv|k|$,

$$
\begin{equation*}
\mathcal{S}_{0}(\omega)=-\ln (1-\mu)-\frac{\mu}{1-\mu} \ln \mu \quad ; \quad \mu=e^{-2 \pi \omega} \tag{1.10}
\end{equation*}
$$

This contribution is finite as $\omega \rightarrow \infty$, but diverges as $\omega \rightarrow 0$. The divergence is integrable, however, and the integral in Eq. (1.9) is finite. As can be seen from Eq. (1.9) $S_{0}$ is infinite as $\epsilon \rightarrow 0$ due to the infinite density of states near the boundary at $x=0$.

## 2 Entanglement Entropy of a Coherent State

We consider a coherent state $\left|\Psi_{\alpha}\right\rangle$, which is an eigenstate of the annihilation operator,

$$
\begin{equation*}
a(x)=\frac{1}{\sqrt{2}}\left[\int_{-\infty}^{\infty} d y \Omega^{\frac{1}{2}}(x, y) \Phi(y)+i \int_{-\infty}^{\infty} d y \Omega^{-\frac{1}{2}}(x, y) \Pi(y)\right] \tag{2.1a}
\end{equation*}
$$

$$
\begin{equation*}
\langle\phi| a(x)\left|\Psi_{\alpha}\right\rangle=\alpha(x) \Psi_{\alpha}[\phi] \tag{2.1b}
\end{equation*}
$$

where $\alpha(x)$ is in general complex. The solution to Eq. (2.1b) is

$$
\begin{equation*}
\Psi_{\alpha}[\phi]=N \exp \left[-\frac{1}{2} \iint \phi \Omega \phi+\sqrt{2} \iint \alpha \Omega^{1 / 2} \phi\right] . \tag{2.2}
\end{equation*}
$$

For our calculation it is useful to write $\Psi_{\alpha}$ in terms of the real and imaginary parts of its (functional) eigenvalue $\alpha \equiv \alpha_{R}+i \alpha_{I}$ :

$$
\begin{equation*}
\Psi_{\alpha}[\phi]=N e^{i \int \bar{\pi} \phi} e^{-\frac{1}{2} \iint(\phi-\bar{\phi}) \Omega(\phi-\bar{\phi})} \tag{2.3}
\end{equation*}
$$

where

$$
\begin{align*}
& \bar{\phi}(x)=\sqrt{2} \int_{-\infty}^{\infty} d y \alpha_{R}(y) \Omega^{-\frac{1}{2}}(y, x)  \tag{2.4a}\\
& \bar{\pi}(x)=\sqrt{2} \int_{-\infty}^{\infty} d y \alpha_{I}(y) \Omega^{\frac{1}{2}}(y, x) \tag{2.4~b}
\end{align*}
$$

and a factor $\exp \left(-\frac{1}{2} \int \bar{\phi} \Omega \bar{\phi}\right)$ has been absorbed into the normalization. By tracing over $\phi_{-}$we obtain the reduced density matrix for this state:

$$
\begin{align*}
\rho_{\alpha}\left(\phi_{+}^{1}, \phi_{+}^{2}\right) & =\int \mathcal{D} \phi_{-} \Psi_{\alpha}\left(\phi_{+}^{1}, \phi_{-}\right) \Psi_{\alpha}^{*}\left(\phi_{+}^{2}, \phi_{-}\right) \\
& =e^{i \int \bar{\pi}_{+}\left(\phi_{+}^{1}-\phi_{+}^{2}\right)} \rho_{0}\left(\phi_{+}^{1}-\bar{\phi}_{+}, \phi_{+}^{2}-\bar{\phi}_{+}\right) \tag{2.5}
\end{align*}
$$

where $\rho_{0}$ is the reduced density matrix for the vacuum state given in Eq. (1.3), with the argument $\phi_{+}^{1,2}$ translated by $\bar{\phi}_{+}$. One can show that the phase appearing in $\rho_{\alpha}$ disappears in the functional integration involved in the entropy calculation. It is straightforward to see that the eigenvalues of $\rho_{\alpha}$ are the same as those of $\rho_{0}$, and one concludes that the entanglement entropy arising from the coherent state is given by the vacuum state entanglement entropy:

$$
\begin{align*}
S_{\alpha} & =-\operatorname{Tr} \rho_{\alpha} \ln \rho_{\alpha} \\
& =-\operatorname{Tr} \rho_{0} \ln \rho_{0} \\
& =S_{0} . \tag{2.6}
\end{align*}
$$

We can understand this result with the following argument. The relation between $\rho_{\alpha}$ and $\rho_{0}$ given in Eq. (2.5) reflects the fact that the coherent state labelled by $\alpha$ is related to the vacuum state by a unitary transformation. This becomes clear if we rewrite Eq. (2.3) in terms of the unitary operator constructed from $\Pi$ and $\Phi$,

$$
\begin{equation*}
\Psi_{\alpha}[\phi]=N\langle\phi| e^{i \int \bar{\pi} \Phi} e^{i \int \bar{\phi} \Pi}|0\rangle_{M} . \tag{2.7}
\end{equation*}
$$

Moreover, when we factor the basis state into $|\phi\rangle=\left|\phi_{+}\right\rangle \otimes\left|\phi_{-}\right\rangle$, the unitary operator factors into two pieces, one acting on $\left|\phi_{+}\right\rangle$alone and the other acting on $\left|\phi_{-}\right\rangle$alone, due to the commutation relation $\left[\Pi_{ \pm}, \Phi_{\mp}\right]=0$ :

$$
\begin{equation*}
e^{i \int \bar{\pi} \Phi} e^{i \int \bar{\phi} \Pi}=\left(e^{i \int \bar{\pi}_{+} \Phi_{+}} e^{i \int \bar{\phi}_{+} \Pi_{+}}\right)\left(e^{i \int \bar{\pi}_{-} \Phi_{-}} e^{i \int \bar{\phi}_{-} \Pi_{-}}\right) . \tag{2.8}
\end{equation*}
$$

Eqs. (2.7) and (2.8) then lead to Eq. (2.6).
Our result is rather surprising, since the entropy we calculate is determined by the counting of quantum states, and there is no reason to expect that the entanglement entropy arising from the two different pure states should be the same. This result may be easily generalized to a spacetime of arbitrary dimension and to an unobserved region of any shape.

## 3 Entanglement Entropy of One-Particle States

In this section we study the structure of the divergences of the entanglement entropy arising from one-particle states. Calculation of the entanglement entropy associated with one-particle states is in general extremely complicated due to the difficulty in diagonalizing the reduced density matrix. We find that the calculation is much simpler if we use an alternative quantization of Minkowski space, employing the Rindler space mode functions [14]. Therefore, we shall first describe this formalism.

### 3.1 Rindler Space Description of the Minkowski Vacuum

We introduce the familar two-wedge Rindler coordinates $\xi$ and $\eta$, which are related to the Minkowski coordinates $x$ and $t$ through the relations

$$
\begin{align*}
x & = \pm a^{-1} e^{a \xi} \cosh (a \eta)  \tag{3.1a}\\
t & = \pm a^{-1} e^{a \xi} \sinh (a \eta) \tag{3.1b}
\end{align*}
$$

where the parameter $a$ is a positive constant, and the sign is taken to be positive and negative in the right-hand and left-hand wedges, respectively. The Rindler coordinates take all real values, $-\infty<\xi, \eta<\infty$, and cover the two quadrants of Minkowski space given by $|x|>|t|$. Due to the fact that the Rindler metric is conformal to all of Minkowski space and the massless Klein-Gordon equation is conformally invariant in $(1+1)$-dimensions, there exist mode solutions of the form

$$
\begin{align*}
& { }^{R} u_{k}= \begin{cases}\frac{1}{\sqrt{4 \pi \omega}} e^{i(k \xi-\omega \eta)} & \text { in right wedge } \\
0 & \text { in left wedge }\end{cases}  \tag{3.2a}\\
& { }^{L} u_{k}= \begin{cases}0 & \text { in right wedge } \\
\frac{1}{\sqrt{4 \pi \omega}} e^{i(k \xi+\omega \eta)} & \text { in left wedge }\end{cases} \tag{3.2b}
\end{align*}
$$

where $\omega=|k|$. These mode functions can be analytically continued to the region $|x|<|t|$, and together, ${ }^{L} u_{k}$ and ${ }^{R} u_{k}$ are complete in all of Minkowski space (12]. The field operator $\Phi$, therefore, may be expanded in terms of these solutions, resulting in an alternative Fock space;

$$
\begin{equation*}
\Phi=\int d k\left({ }^{L} b_{k}{ }^{L} u_{k}+{ }^{R} b_{k}{ }^{R} u_{k}+\text { h.c. }\right) \tag{3.3}
\end{equation*}
$$

The operators ${ }^{R} b_{k}$ and ${ }^{L} b_{k}$ annihilate the Rindler vacuum, $|0\rangle_{\text {Rind }} \equiv|0\rangle_{R} \otimes|0\rangle_{L}$,

$$
\begin{equation*}
{ }^{L} b_{k}|0\rangle_{\text {Rind }}={ }^{R} b_{k}|0\rangle_{\text {Rind }}=0 \tag{3.4}
\end{equation*}
$$

With this alternative formulation one can characterize the states of the quantum field either as Minkowski particle states according to the usual quantization, in which the field operator is expanded as

$$
\begin{equation*}
\Phi=\int \frac{d p}{\sqrt{4 \pi \omega}}\left[a_{p} e^{i(p x-\omega t)}+a_{p}^{\dagger} e^{-i(p x-\omega t)}\right] \tag{3.5}
\end{equation*}
$$

with $\omega=|p|$, or as Rindler particle states according to Eq. (3.3).
We note that at a fixed time $t=\eta=0$ the Rindler mode functions in Eq. (3.2a, (3.2b), with $a=1$, are precisely the eigenfunctions in (1.7a) that diagonalize the vacuum reduced density matrix given in Eq. (1.3). When we write the instantaneous field configuration as

$$
\begin{equation*}
\phi(\xi(x))=\theta(x) \int_{-\infty}^{\infty} \frac{d k}{2 \pi} \phi_{R}(k) e^{i k \xi(x)}+\theta(-x) \int_{-\infty}^{\infty} \frac{d k}{2 \pi} \phi_{L}(k) e^{i k \xi(x)} \tag{3.6}
\end{equation*}
$$

the Minkowski vacuum wave functional [Eq. (1.2)] has the following form in the basis $\left|\phi_{L} \phi_{R}\right\rangle$ (13]:

$$
\begin{align*}
\Psi_{0}\left(\phi_{R}, \phi_{L}\right) & =\left\langle\phi_{L} \phi_{R} \mid 0\right\rangle_{M} \\
& \propto \exp \left\{-\frac{1}{2} \int_{-\infty}^{\infty} \frac{d k}{2 \pi}\left[k \operatorname{coth}(\pi k / a)\left(\left|\phi_{R}\right|^{2}+\left|\phi_{L}\right|^{2}\right)-\frac{2 k \operatorname{Re} \phi_{R} \phi_{L}^{*}}{\sinh (\pi k / a)}\right]\right\} . \tag{3.7}
\end{align*}
$$

By constructing the density matrix for this state and integrating over the field configuration for $x<0$, we obtain the reduced density matrix

$$
\begin{align*}
\rho_{0}\left(\phi_{R}^{1}, \phi_{R}^{2}\right) & =\int \mathcal{D} \phi_{L} \mathcal{D} \phi_{L}^{*} \Psi_{0}\left(\phi_{R}^{1}, \phi_{L}\right) \Psi_{0}^{*}\left(\phi_{R}^{2}, \phi_{L}\right) \\
& \propto \exp \left\{-\frac{1}{2} \int_{-\infty}^{\infty} \frac{d k}{2 \pi}\left[k \operatorname{coth}(2 \pi k / a)\left(\left|\phi_{R}^{1}\right|^{2}+\left|\phi_{R}^{2}\right|^{2}\right)-\frac{2 k \operatorname{Re} \phi_{R} \phi_{L}^{*}}{\sinh (2 \pi k / a)}\right]\right\} . \tag{3.8}
\end{align*}
$$

This is precisely the density matrix of Eq. (1.3) when it is diagonalized using Eq. (1.7a, 1.7b). The momentum variable $k$, which we shall call the Rindler momentum,
labels the eigenmode with eigenvalue $\lambda(k)$. Eq. (3.8) has the form of a thermal density matrix at a temperature $T_{R}=a / 2 \pi$.

It is clear from the above discussion that the entanglement entropy for the Minkowski space geometry we are considering may be analyzed in the language of Rindler space. In order to discuss the one-particle Minkowski states we introduce the elegant formulation given in [14], where one constructs the Minkowski positive frequency mode functions

$$
\begin{align*}
& f_{1}=\frac{1}{\sqrt{2 \sinh \pi \omega / a}}\left[e^{\pi \omega / 2 a R} u_{k}+e^{-\pi \omega / 2 a L} u_{-k}^{*}\right]  \tag{3.9a}\\
& f_{2}=\frac{1}{\sqrt{2 \sinh \pi \omega / a}}\left[e^{-\pi \omega / 2 a R} u_{-k}^{*}+e^{\pi \omega / 2 a L} u_{k}\right] \tag{3.9b}
\end{align*}
$$

and expands the field as

$$
\begin{equation*}
\Phi=\int d k\left(d_{k}^{(1)} f_{1}+d_{k}^{(2)} f_{2}+\text { h.c. }\right) \tag{3.10}
\end{equation*}
$$

The operators $d_{k}^{(1,2)}$ are then given by

$$
\begin{align*}
d_{k}^{(1)} & =\frac{1}{\sqrt{2 \sinh \pi \omega / a}}\left[e^{\pi \omega / 2 a R} b_{k}-e^{-\pi \omega / 2 a L} b_{-k}^{\dagger}\right]  \tag{3.11a}\\
d_{k}^{(2)} & =\frac{1}{\sqrt{2 \sinh \pi \omega / a}}\left[e^{\pi \omega / 2 a L} b_{k}-e^{-\pi \omega / 2 a R} b_{-k}^{\dagger}\right] \tag{3.11b}
\end{align*}
$$

and they annihilate the Minkowski vacuum,

$$
\begin{equation*}
d_{k}^{(1,2)}|0\rangle_{M}=0 \tag{3.12}
\end{equation*}
$$

This can be explicitly verified in the functional Schrödinger representation, where the vacuum wave functional is given by Eq. (3.7), and ${ }^{L, R} b_{k}$ and ${ }^{L, R} b_{k}^{\dagger}$ have the following form:

$$
\begin{align*}
{ }^{L, R} b_{k} & =\frac{1}{\sqrt{2}}\left[\sqrt{\omega} \phi_{L, R}(k)+\frac{1}{\sqrt{\omega}} \frac{\delta}{\delta \phi_{L, R}^{*}(k)}\right]  \tag{3.13a}\\
{ }^{L, R} b_{k}^{\dagger} & =\frac{1}{\sqrt{2}}\left[\sqrt{\omega} \phi_{L, R}^{*}(k)-\frac{1}{\sqrt{\omega}} \frac{\delta}{\delta \phi_{L, R}(k)}\right] \tag{3.13b}
\end{align*}
$$

Eq. (3.12) implies that $d_{k}^{(1,2)}$ can be expressed as a superposition of the conventional Minkowski space annihilation operators $a_{p}$,

$$
\begin{equation*}
d_{k}^{(1,2)}=\int_{-\infty}^{\infty} \frac{d p}{2 \pi} D^{(1,2)}(k, p) a_{p} \tag{3.14}
\end{equation*}
$$

We find the expansion coefficients to be

$$
D^{(n)}(k, p)=\left[\frac{2 k}{|p|} \sinh (\pi k / a)\right]^{\frac{1}{2}} \Gamma(k / i a)|p|^{i k / a}\left\{\begin{array}{cc}
\theta(k p), & n=1  \tag{3.15}\\
\theta(-k p), & n=2
\end{array}\right.
$$

where $\Gamma(x)$ and $\theta(x)$ are the gamma and step functions.
To calculate the entanglement entropy for a one-particle state, we find it convenient to use an alternative representation of the Minkowski vacuum, constructed in Ref [14]: from Eqs. (3.11a), (3.11b), and (3.12), we have

$$
\begin{align*}
|0\rangle_{M} & =\prod_{k,-k} N_{k} \sum_{n} e^{-\pi n \omega / a}\left|n_{k}\right\rangle_{R} \otimes\left|n_{-k}\right\rangle_{L} \\
& \equiv \prod_{k,-k}|0\rangle_{M}^{k} \tag{3.16}
\end{align*}
$$

The normalization is $N_{k}=\left(1-e^{-2 \pi \omega / a}\right)^{1 / 2}$. The product $\prod_{k,-k}$ is taken over a complete set of Rindler modes, and $\left|n_{k}\right\rangle_{R, L}$ denotes right and left Rindler states with $n$ particles of Rindler momentum $k$. By tracing over the degrees of freedom in the left region, we obtain the reduced density operator, $\hat{\rho}_{0}$; since it is diagonalized by the Rindler mode functions, it may be written as a product of density operators, one for each Rindler mode:

$$
\begin{equation*}
\hat{\rho}_{0} \equiv \prod_{k,-k} \hat{\rho}_{0}^{k} \tag{3.17}
\end{equation*}
$$

where the contribution from each mode is given by

$$
\begin{align*}
\hat{\rho}_{0}^{k} & =\sum_{m}{ }_{L}\left\langle m_{-k} \mid 0\right\rangle_{M M}^{k}\left\langle 0 \mid m_{-k}\right\rangle_{L} \\
& =N_{k}^{2} \sum_{m} e^{-2 \pi m \omega / a}|m\rangle_{R R}\langle m| . \tag{3.18}
\end{align*}
$$

This is an alternative form of the density matrix given in Eq. (3.8). The entanglement entropy can be calculated simply, and, upon discretizing the spectrum by demanding that ${ }^{R} u_{k}$ vanish at $\xi(L)$ and $\xi(\epsilon)$, we find that it has the same form as Eq. (1.9). The advantage of using this formalism is that it automatically gives the diagonal form of the reduced density matrix for the one-particle state that we are going to consider.

### 3.2 Entanglement Entropy of a One-Particle State

We shall now compute the entanglement entropy of a one-particle Minkowski state with a definite Rindler momentum $k$, by exciting the positive-frequency mode $f_{1}$. This is a particular superposition of Minkowski momentum eigenstates:

$$
\begin{equation*}
d_{k}^{(1) \dagger}|0\rangle_{M}=\int_{-\infty}^{\infty} \frac{d p}{2 \pi} D^{(1) *}(k, p) a_{p}^{\dagger}|0\rangle_{M} \tag{3.19}
\end{equation*}
$$

Using the notation of Eq. (3.16), we write this state as

$$
\begin{equation*}
d_{k}^{(1) \dagger}|0\rangle_{M}=|1\rangle_{M}^{k} \prod_{\ell \neq k}|0\rangle_{M}^{\ell} \tag{3.20}
\end{equation*}
$$

where

$$
\begin{align*}
|1\rangle_{M}^{k}= & \mathcal{N} \sum_{n} e^{-\pi n \omega / a}\left\{\sqrt{n+1}\left|(n+1)_{k}\right\rangle_{R} \otimes\left|n_{-k}\right\rangle_{L}\right. \\
& \left.\quad-e^{-\pi \omega / a} \sqrt{n}\left|n_{k}\right\rangle_{R} \otimes\left|(n-1)_{-k}\right\rangle_{L}\right\} \\
= & \mathcal{N} 2 \sinh (\pi \omega / a) \sum_{n} e^{-\pi n \omega / a} \sqrt{n}\left|n_{k}\right\rangle_{R} \otimes\left|(n-1)_{-k}\right\rangle_{L} \tag{3.21}
\end{align*}
$$

The normalization factor is singular:

$$
\begin{equation*}
\mathcal{N}^{2}={ }_{M}\langle 0| d_{k}^{(1)} d_{k}^{(1) \dagger}|0\rangle_{M}=2 \pi \delta(0) . \tag{3.22}
\end{equation*}
$$

The reduced density matrix, normalized to have unit trace, is then

$$
\begin{equation*}
\hat{\rho}(k)=\frac{1}{2 \pi \delta(0)} \operatorname{Tr}_{L}\left[d_{k}^{(1) \dagger}|0\rangle_{M M}\langle 0| d_{k}^{(1)}\right]=\hat{\rho}_{1}^{k} \prod_{\ell \neq k} \hat{\rho}_{0}^{\ell} \tag{3.23}
\end{equation*}
$$

where $\operatorname{Tr}_{L}$ represents a trace over $\left\{\bigotimes_{k,-k}\left|n_{k}\right\rangle_{L}\right\}$, $\hat{\rho}_{0}^{\ell}$ is given by Eq. (3.18), and

$$
\begin{equation*}
\hat{\rho}_{1}^{k}=4 \sinh ^{2}(\pi \omega / a) \sum_{n} n e^{-2 \pi n \omega / a}\left|n_{k}\right\rangle_{R R}\left\langle n_{k}\right| . \tag{3.24}
\end{equation*}
$$

The density operator is again diagonal, and the entanglement entropy is readily calculated:

$$
\begin{equation*}
S=\mathcal{S}_{1}(k)+\sum_{\ell \neq k} \mathcal{S}_{0}(\ell) \tag{3.25}
\end{equation*}
$$

(The summation reflects the fact that the spectrum must be discretized for this expression to be well-defined.) When we subtract the entanglement entropy of the vacuum state, $S_{0}$, from $S, \Delta S \equiv S-S_{0}$ reduces to

$$
\begin{align*}
\Delta S & =\mathcal{S}_{1}(k)-\mathcal{S}_{0}(k) \\
& =-\ln (1-\mu)-\frac{\mu}{1-\mu} \ln \mu-\frac{(1-\mu)^{2}}{\mu} \sum_{m}(m \ln m) \mu^{m} \tag{3.26}
\end{align*}
$$

where $\mu=e^{-2 \pi \omega / a}$ as before. We have evaluated the last term in this expression numerically: as shown in Fig $1, \Delta S$ is finite for all values of $\omega$. It can also be evaluated analytically in the limits $\omega \rightarrow 0$ and $\omega \rightarrow \infty$ (15]:

$$
\begin{align*}
& \lim _{\omega \rightarrow 0} \Delta S=\gamma_{E} \approx 0.5772  \tag{3.27a}\\
& \lim _{\omega \rightarrow \infty} \Delta S=0 \tag{3.27b}
\end{align*}
$$

We emphasize that the density of states factor, which made the vacuum state entanglement entropy [Eq. (1.9)] diverge, does not enter into $\Delta S$. Thus we have shown that the entanglement entropy for a particular class of one-particle states can be made finite by subtracting from it the vacuum entanglement entropy.

### 3.3 More General One-Particle States

We have computed the entanglement entropy associated with the one-particle state given in Eq. (3.19) in closed form using the fact that the Rindler space formalism
leads to a diagonal reduced density matrix arising from that state. More general oneparticle Minkowski states are created by operators corresponding to both the $f_{1}$ and $f_{2}$ modes [see Eqs. (3.9a, 3.9b)]:

$$
\begin{equation*}
|\psi\rangle=\int_{-\infty}^{\infty} \frac{d k}{\sqrt{2} \pi}\left[\psi_{1}(k) d_{k}^{(1) \dagger}+\psi_{2}(k) d_{-k}^{(2) \dagger}\right]|0\rangle_{M} \tag{3.28}
\end{equation*}
$$

Here $\psi_{1,2}(k)$ are smearing functions determining a superposition of momentum states. With such states the problem of diagonalizing the reduced density matrix remains. Already, for the simple choice $\psi_{1}(k)=\psi_{2}(k)=\sqrt{\pi} \delta(k-\ell)$, i.e.

$$
\begin{equation*}
|\psi\rangle=\frac{1}{\sqrt{2}}\left(d_{\ell}^{(1) \dagger}+d_{-\ell}^{(2) \dagger}\right)|0\rangle_{M} \tag{3.29}
\end{equation*}
$$

the reduced density matrix is not diagonal. However, we have computed the entanglement entropy numerically: when the vacuum contribution is subtracted off, the entanglement entropy $\Delta S \equiv S-S_{0}$ is found to be finite (see Fig 2). Therefore, it is reasonable to expect that, for well-behaved smearing functions that fall off sufficiently fast at large $k$, the entanglement entropy arising from the state $|\psi\rangle$ is finite once the vacuum contribution is subtracted.

In the appendix we provide a formal, but concrete, calculation that supports the expectation that the entanglement entropy $\Delta S$ of one-particle states is finite for states defined by well-behaved smearing functions.

Our calculation presented in this paper suggests that the difference between the entanglement entropy arising from different states, but for the same unobserved region, is finite for a given theory of matter fields. This is complimentary to another finite quantity that has recently been studied in [8], where the excited state was produced by a moving mirror.

## Appendix

We shall consider a one-particle state defined by

$$
\begin{equation*}
|\psi\rangle=\int_{-\infty}^{\infty} \frac{d k}{\sqrt{2 \pi}} \psi(k) d_{k}^{(1) \dagger}|0\rangle_{M} \tag{A.1}
\end{equation*}
$$

and discuss possible divergences in the entanglement entropy $S(\psi)$ associated with $|\psi\rangle$. First, using a variational principle, we show that $S(\psi)$ is bounded by an entropy defined by

$$
\begin{align*}
S_{b} & =\operatorname{Tr} \hat{\rho}_{b} \ln \hat{\rho}_{b}  \tag{A.2a}\\
\hat{\rho}_{b} & =\int_{-\infty}^{\infty} d k|\psi(k)|^{2} \hat{\rho}(k) \tag{A.2b}
\end{align*}
$$

where $\hat{\rho}(k)$ is the reduced density matrix associated with the state $d_{k}^{(1) \dagger}|0\rangle_{M}$ given in Eq. (3.20). We can obtain an explicit expression for $S_{b}$ due to the fact that $\hat{\rho}_{b}$ is diagonal in the basis of $n$-particle Rindler states. Then we shall study the divergence structure of $S_{b}$, and show that $\Delta S_{b} \equiv S_{b}-S_{0}$ is finite for well-behaved smearing functions $\psi(k)$. Although the bound $S(\psi) \leq S_{b}$ (which we shall prove below) is a formal one, in that both $S_{b}$ and $S_{0}$ are infinite, our calculation is interesting since we show that the difference, $\Delta S_{b}$, is finite for well-behaved smearing functions. The finiteness of the difference supports the expectation that $\Delta S(\psi) \equiv S(\psi)-S_{0}$ is finite.

We first present a general extremum principle.

## Lemma:

Let $\hat{\rho}(q)$ be a family of density operators labelled by a continuous parameter $q$, and consider operators of the form

$$
\begin{equation*}
\hat{\rho}[h]=\int d q h(q) \hat{\rho}(q) \tag{A.3}
\end{equation*}
$$

We guarantee that $\hat{\rho}[h]$ will be a density operator by demanding that $h(q)>0$ and $\int d q h(q)=1$. Subject to these constraints we extremize the functional

$$
\begin{equation*}
F[h]=-\operatorname{Tr} \hat{\rho}[h] \ln \hat{\rho}[h]+\lambda\left(\int d q h(q)-1\right) \tag{A.4}
\end{equation*}
$$

Upon setting $\delta F[h] / \delta h(q)=0$ we find the extremum condition

$$
\begin{equation*}
-\operatorname{Tr} \hat{\rho}(q) \ln \hat{\rho}[h]=1-\lambda . \tag{A.5}
\end{equation*}
$$

The right hand side of this expression is independent of $q$, so Eq. (A.4) is extremized by the $h(q)$ that makes the left hand side constant as well.

Next we construct a family of density operators $\hat{\rho}_{\psi}(q)$ in the following way: define a set of states generalizing Eq. (A.1),

$$
\begin{equation*}
|\psi, q\rangle=\int_{-\infty}^{\infty} \frac{d k}{\sqrt{2 \pi}} q^{-i k} \psi(k) d_{k}^{(1) \dagger}|0\rangle_{M} \tag{A.6}
\end{equation*}
$$

and compute the reduced density operator

$$
\begin{equation*}
\hat{\rho}_{\psi}(q)=\operatorname{Tr}_{L}|\psi, q\rangle\langle\psi, q| . \tag{A.7}
\end{equation*}
$$

In the calculations that follow we shall use a scaling relation for the states $|\psi, q\rangle$ that is manifest in the functional Schrödinger representation described in section (3.1). Under the transformation $\phi_{R, L} \rightarrow q^{i k} \phi_{R, L}$, the one-particle state

$$
\begin{equation*}
\left\langle\phi_{L} \phi_{R}\right| d_{k}^{(1) \dagger}|0\rangle_{M}=N_{k}\left[\phi_{R}^{*}(k)-e^{-\pi \omega / a} \phi_{L}^{*}(k)\right] \Psi_{0}\left(\phi_{R}, \phi_{L}\right) \tag{A.8}
\end{equation*}
$$

(where $N_{k}$ is a normalization constant) becomes

$$
\begin{equation*}
\left\langle q^{i k} \phi_{L}, q^{i k} \phi_{R}\right| d_{k}^{(1) \dagger}|0\rangle_{M}=q^{-i k}\left\langle\phi_{L} \phi_{R}\right| d_{k}^{(1) \dagger}|0\rangle_{M} \tag{A.9}
\end{equation*}
$$

Combining Eqs. (A.9) and (A.6), we find the scaling relation that we seek,

$$
\begin{equation*}
\left\langle\phi_{L} \phi_{R} \mid \psi, q\right\rangle=\left\langle q^{i k} \phi_{L}, q^{i k} \phi_{R} \mid \psi, 1\right\rangle . \tag{A.10}
\end{equation*}
$$

Furthermore, the functional measures $\mathcal{D} \phi_{R} \mathcal{D} \phi_{R}^{*}$ and $\mathcal{D} \phi_{L} \mathcal{D} \phi_{L}^{*}$ are invariant under this transformation.

Now we use the lemma: consider the extremum condition Eq. (A.5), which we write as

$$
\begin{equation*}
-\operatorname{Tr} \hat{\rho}_{\psi}(q) \ln \left[\int_{0}^{\infty} d q^{\prime} h\left(q^{\prime}\right) \hat{\rho}_{\psi}\left(q^{\prime}\right)\right]=\text { constant } \tag{A.11}
\end{equation*}
$$

Using Eq. ( $\widehat{\mathrm{A} .10}$ ), and the invariance of the functional measure, we have

$$
\begin{equation*}
-\operatorname{Tr} \hat{\rho}_{\psi}(q) \ln \left[\int_{0}^{\infty} d q^{\prime} h\left(q^{\prime}\right) \hat{\rho}_{\psi}\left(q^{\prime}\right)\right]=-\operatorname{Tr} \hat{\rho}_{\psi}(1) \ln \left[\int_{0}^{\infty} d q^{\prime} h\left(q^{\prime}\right) \hat{\rho}_{\psi}\left(q^{\prime} q^{-1}\right)\right] \tag{A.12}
\end{equation*}
$$

which is $q$-independent if $d q^{\prime} h\left(q^{\prime}\right)=d\left(q q^{\prime}\right) h\left(q q^{\prime}\right)$. The normalized $h(q)$ with this property is

$$
\begin{equation*}
h(q)=\frac{1}{2 \pi \delta(0)} \frac{1}{q} \tag{A.13}
\end{equation*}
$$

where we have written the singular normalization $\left(\int_{0}^{\infty} d q q^{-1}\right)^{-1}$ as $(2 \pi \delta(0))^{-1}$. The corresponding density operator $\hat{\rho}[h]$ is

$$
\begin{align*}
\hat{\rho}[h] & =\frac{1}{2 \pi \delta(0)} \int_{-\infty}^{\infty} d k|\psi(k)|^{2} \operatorname{Tr}_{L}\left[d_{k}^{(1) \dagger}|0\rangle_{M M}\langle 0| d_{k}^{(1)}\right] \\
& =\int_{-\infty}^{\infty} d k|\psi(k)|^{2} \hat{\rho}(k) \tag{A.14}
\end{align*}
$$

where $\hat{\rho}(k)$ is given by Eq. (3.23). This density operator is our $\hat{\rho}_{b}$ in Eq. (A.2b). It is diagonal in the basis $\left\{\bigotimes_{k,-k}\left|n_{k}\right\rangle_{R}\right\}$.

We must show that the extreme value is a maximum for $F[h]$, if it is to provide a bound. We take the second functional derivative, evaluated at the extremum,

$$
\begin{equation*}
\frac{\delta^{2} F[h]}{\delta h(q) \delta h\left(q^{\prime}\right)}=-\operatorname{Tr} \frac{\hat{\rho}(q) \hat{\rho}\left(q^{\prime}\right)}{\hat{\rho}_{b}} \tag{A.15}
\end{equation*}
$$

(We take the trace with respect to the basis in which $\hat{\rho}_{b}$ is diagonal.) The functional Hessian matrix, Eq. (A.15), is diagonalized by the set of basis functions $q^{i k-1}$ :

$$
\begin{equation*}
-\int_{0}^{\infty} d q q^{-i k-1} \int_{0}^{\infty} d q^{\prime} q^{i k^{\prime}-1} \operatorname{Tr} \frac{\hat{\rho}(q) \hat{\rho}\left(q^{\prime}\right)}{\hat{\rho}_{b}}=-2 \pi \delta\left(k-k^{\prime}\right) \int_{0}^{\infty} d \xi \xi^{i k^{\prime}-1} \operatorname{Tr} \frac{\hat{\rho}(1) \hat{\rho}(\xi)}{\hat{\rho}_{b}} \tag{A.16}
\end{equation*}
$$

We clarify this expression by defining $\hat{\mathcal{O}}(k)=\int_{0}^{\infty} d q q^{i k-1} \hat{\rho}(q)$, and writing the diagonal elements as $-\operatorname{Tr} \hat{\mathcal{O}}^{\dagger}(k) \hat{\mathcal{O}}(k) / \hat{\rho}_{b}$. They are seen to be the negative trace of the ratio of two positive operators, and are therefore negative. The eigenvalues of the Hessian matrix are thus all negative, and the extremum is a maximum. We have therefore established the formal bound $S(\psi) \leq S_{b}$.

We shall now calculate $S_{b}$. We first discretize the spectrum by introducing infrared and ultraviolet cutoffs as given in Eq. (1.8b). This leads to the following expression for $\hat{\rho}_{b}$ :

$$
\begin{equation*}
\hat{\rho}_{b}=\lim _{\Delta k \rightarrow 0} \sum_{n} \Delta k\left|\psi\left(k_{n}\right)\right|^{2} \hat{\rho}\left(k_{n}\right) \tag{A.17}
\end{equation*}
$$

where $\Delta k=\pi / \ln (L / \epsilon), k_{n}=n \Delta k$, and $\hat{\rho}\left(k_{n}\right)$ is given by (using Eqs. (3.18) and (3.24)),

$$
\begin{equation*}
\hat{\rho}\left(k_{n}\right)=\lim _{\Delta k \rightarrow 0} \frac{1-\mu_{n}}{\mu_{n}} \prod_{i}\left[\left(1-\mu_{i}\right) \sum_{m_{i}=0}^{\infty} m_{n} \mu_{i}^{m_{i}}\left|m_{i}\right\rangle_{R R}\left\langle m_{i}\right|\right] \tag{A.18}
\end{equation*}
$$

with $\mu_{n}=e^{-2 \pi\left|k_{n}\right| / a}$. With the expression above for $\hat{\rho}_{b}$, the entropy $S_{b}$, when the vacuum entanglement entropy is subtracted, reduces to

$$
\begin{align*}
\Delta S_{b} & \equiv S_{b}-S_{0} \\
& =-\sum_{n} \frac{f_{n} \ln \mu_{n}}{1-\mu_{n}}-\prod_{i}\left(1-\mu_{i}\right) \sum_{m_{1}, m_{2}, \ldots=0}^{\infty} \mu_{n}^{m_{n}} F(f) \ln F(f) \tag{A.19}
\end{align*}
$$

where we have defined $f_{n} \equiv\left|\psi_{n}\right|^{2} \Delta k$, and

$$
\begin{equation*}
F(f) \equiv \sum_{n} \frac{1-\mu_{n}}{\mu_{n}} f_{n} m_{n} \tag{A.20}
\end{equation*}
$$

This gives an upper bound for $\Delta S(\psi)$ :

$$
\begin{equation*}
\Delta S(\psi)=S(\psi)-S_{0} \leq \Delta S_{b} \tag{A.21}
\end{equation*}
$$

Although $\Delta S_{b}$ cannot be evaluated exactly for arbitrary $\psi(k)$, it provides the following information: due to the normalization of $\hat{\rho}_{b}$, which implies the normalization of the
smearing function $\sum_{n} \Delta k\left|\psi\left(k_{n}\right)\right|^{2}=1$, the density of states does not appear in $\Delta S$ as an overall factor. Thus the only possible source of divergence is the smearing function $\psi(k)$. However, one can evaluate Eq. (A.19) for smearing functions that satisfy $\psi(k)=0$ for $|k|>k_{0}$, where $k_{0} \ll a$. In this limit, $\Delta S_{b}$ is well approximated by a multidimensional integral: if we set $y_{n}=-m_{n} \ln \mu_{n}$, and $d y_{n} \sim-\ln \mu_{n}$, then

$$
\begin{align*}
\Delta S & \approx 1-\int_{0}^{\infty} \cdots \int_{0}^{\infty} d y_{1} d y_{2} \ldots e^{-\sum_{n} y_{n}} \sum_{n} y_{n} f_{n} \ln \sum_{n} y_{n} f_{n} \\
& =1-\|f\| \ln \|f\|-\|f\|\left(1-\gamma_{E}\right) \tag{A.22}
\end{align*}
$$

where $\|f\|^{2} \equiv \sum_{n}\left|f_{n}\right|^{2} \leq 1$. For any of the allowed values of $\|f\|$ this expression is finite and nonzero. It is reasonable to suppose that $\Delta S_{b}$ will remain finite for smearing functions that drop off sufficiently fast at large $k$ : this supports the expectation that, for resonable values of the smearing function, the difference between the entanglement entropy of the state given by Eq. (A.1) and that of the vacuum is finite.

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[15] We obtain the $\omega \rightarrow 0$ limit by writing Eq. (3.26) as

$$
\begin{equation*}
\Delta S=-\frac{\mu}{1-\mu} \ln \mu-\frac{(1-\mu)^{2}}{\mu} \sum_{m} m \ln [m(1-\mu)] \mu^{m} \tag{0.23}
\end{equation*}
$$

Defining $x=2 \pi \omega / a$, so that $\mu=e^{-x}$, and taking the limit $x \rightarrow 0$,

$$
\begin{equation*}
\lim _{x \rightarrow 0} \Delta S=\lim _{x \rightarrow 0} 1-x^{2} \sum_{m} m \ln (x m) e^{-x m}=1-\int_{0}^{\infty} d y y \ln y e^{-y}=\gamma_{E} \tag{0.24}
\end{equation*}
$$

## Figure Captions

Fig. 1: $\Delta S$ for the state in Eq. (3.19) as a function of $k / a$. The dashed line is $\gamma_{E}$.

Fig. 2: $\Delta S$ for the state in Eq. (3.29) as a function of $\ell / a$.




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