Transversals of Longest Cycles in Partial k-Trees and Chordal Graphs

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Abstract

Let $\operatorname{lct}(G)$ be the minimum cardinality of a set of vertices that intersects every longest cycle of a 2-connected graph G. We show that $\operatorname{lct}(G) \leq k-1$ if G is a partial k-tree and that $\operatorname{lct}(G) \leq \max\{1, \omega(G)-3\}$ if G is chordal, where $\omega(G)$ is the cardinality of a maximum clique in G. Those results imply that all longest cycles intersect in 2-connected series parallel graphs and in 3-trees.

1 Introduction

It is known that, in a 2-connected graph, every pair of longest cycles intersect each other. A natural question is whether all longest cycles have a vertex in common. This has in general a negative answer, as the Petersen's graph shows. Thus, it is interesting to look for a set of vertices that intersects every longest cycle of the graph. Such a set is called a *longest cycle transversal*, or just a transversal. The minimum cardinality of a transversal in a graph G is denoted by lct(G). It is interesting to search for good upper bounds for lct(G). Note that lct(G) = 1 if and only if all longest cycles have a common vertex.

Consider a 2-connected graph G with n vertices. Thomassen [28] showed that $\operatorname{lct}(G) \leq \lceil n/3 \rceil$. This bound was improved by Rautenbach and Sereni [25], who proved that $\operatorname{lct}(G) \leq \lceil \frac{n}{3} - \frac{n^{2/3}}{36} \rceil$. Jobson et al. [23] showed that $\operatorname{lct}(G) = 1$ if G is a dually chordal graph, a class of graphs that includes doubly chordal, strongly chordal, and interval graphs. They also mention that their proof can be applied to show that $\operatorname{lct}(G) = 1$ if G is a 2-connected split graph. Fernandes and the author [10] showed that $\operatorname{lct}(G) = 1$ if G is a 3-tree, and that $\operatorname{lct}(G) \leq 2$ if G is a partial 3-tree. In this paper, we give results for $\operatorname{lct}(G)$ when G is a partial k-tree and when G is chordal. A previous extended abstract containing these results was presented at LATIN 2018 [17].

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This paper is organized as follows. In Section 2, we establish basic concepts on paths and cycles, which includes the very important concept of attractor. In Section 3, we give definitions on tree decompositions and branches. In Section 4, we define the classes of partial k-trees and chordal graphs. In Section 5, we state a central lemma (Lemma 5.3) that will be used in the next two sections. In Section 6, we show that $lct(G) \leq k-1$ for every 2-connected partial k-tree G (Theorem 6.2) and, in Section 7, we show that $lct(G) \leq max\{1, \omega(G)-3\}$ for every 2-connected chordal graph G (Theorem 7.2). Finally, in Section 8, we present some concluding remarks. In this paper, all graphs considered are simple and the notation used is standard [3, 9].

2 Paths, Cycles, and Attractors

Given two paths C' and C'', if $C' \cup C''$ is a path or a cycle, it is denoted by $C' \cdot C''$. For a pair of vertices $\{a,b\}$ in a cycle C, let C' and C'' be the paths such that $C = C' \cdot C''$ and $V(C') \cap V(C'') = \{a,b\}$. We refer to these paths as the ab-parts of C. Moreover, we can extend this notation and define, for a triple of vertices $\{a,b,c\}$ in a cycle C, the abc-parts of C; and, when the context is clear, we denote by C_{ab} , C_{bc} , and C_{ac} the corresponding abc-parts of C.

In what follows, let G be a graph and let $S \subseteq V(G)$. We say that S separates vertices u and v if u and v are in different components of G-S. Let $X \subseteq V(G)$. We say that S separates X if S separates at least two vertices in X. We say that a path or cycle C' k-intersects S if $|V(C') \cap S| = k$. Moreover, we also say that C' k-intersects S at $V(C') \cap S$. A path or cycle C' crosses S if S separates S in S separates S in S separates S in S separates S in S in S separates S in S in S separates S in S

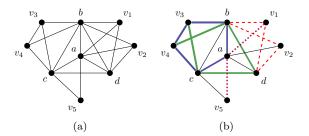


Figure 1: (a) A graph G with $S = \{a, b, c, d\}$. (b) Consider, in G, paths $P_1 = v_1 a v_5$ and $P_2 = v_3 c d b v_4$, and cycles $C_1 = v_1 b v_2 d v_1$ and $C_2 = v_3 v_4 c a b v_3$. Then P_1 and C_1 cross S, and P_2 and C_2 are fenced by S. Moreover, P_1 1-crosses S, P_2 is 3-fenced by S, C_1 2-crosses S and C_2 is 3-fenced by S. (Also note that path c d and cycle a b d a are fenced by S.) Cycles C_2 and $v_1 b c v_5 a v_1$ are S-equivalent.

The length of a path or a cycle C' in G is the number of edges of C' and it is

denoted by |C'|. A cycle in G is called a *longest cycle* if it has maximum length over all cycles in G. Two cycles are S-equivalent if they intersect S at the same set of vertices (Figure 1). Let C be a longest cycle in G. We say that C is an attractor for S if C is fenced by S and all S-equivalent longest cycles are also fenced by S. We say that C is a k-attractor for S if C k-intersects S. In this case, we also say that S has a K-attractor.

The next proposition is well-known, but, to our knowledge, no simple proof of it has been written. We use it several times through the text without making any reference to it.

Proposition 2.1. Let C and D be a pair of longest cycles in a 2-connected graph. Then $|V(C) \cap V(D)| \geq 2$.

Proof. Suppose by contradiction that $|V(C) \cap V(D)| \leq 1$. As G is 2-connected, there exist two disjoint paths R and S, both of them with one extreme in C, the other in D, and internally disjoint from both C and D [3, Proposition 9.4]. Note that, when $|V(C) \cap V(D)| = 1$, it can be the case that exactly one of $\{R, S\}$ has zero length. Let $\{x_1\} = V(C) \cap V(R)$ and $\{x_2\} = V(C) \cap V(S)$. Let $\{y_1\} = V(D) \cap V(R)$ and $\{y_2\} = V(D) \cap V(S)$. Let C' and C'' be the two x_1x_2 -parts of C. Let D' and D'' be the two y_1y_2 -parts of D. Then $C' \cdot R \cdot D' \cdot S$ and $C'' \cdot R \cdot D'' \cdot S$ are both cycles, one of them longer than |C|, a contradiction. \square

3 Tree Decomposition and Branches

A tree decomposition [9, p. 337] of a graph G is a pair (T, \mathcal{V}) , consisting of a tree T and a collection $\mathcal{V} = \{V_t : t \in V(T)\}$ of (different) bags $V_t \subseteq V(G)$, that satisfies the following three conditions:

- $\bigcup_{t \in V(T)} V_t = V(G);$
- for every $uv \in E(G)$, there exists a bag V_t such that $u, v \in V_t$;
- if $v \in V(G)$ is in two different bags V_{t_1} and V_{t_2} , then v is also in any bag V_t such that t is on the path from t_1 to t_2 in T.

The treewidth tw(G) is the number $\min\{\max\{|V_t|-1:t\in V(T)\}:(T,\mathcal{V})\}$ is a tree decomposition of $G\}$. We refer to the vertices of T as nodes.

If G is a graph with treewidth k, then we say that (T, \mathcal{V}) is a full tree decomposition of G if $|V_t| = k + 1$ for every $t \in V(T)$, and $|V_t \cap V_{t'}| = k$ for every $tt' \in E(T)$ (Figure 2).

Proposition 3.1 ([2, Lemma 8][15, Theorem 2.6]). Every graph has a full tree decomposition.

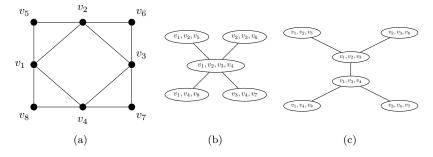


Figure 2: (a) A graph G with treewidth two. (b) A tree decomposition of G that is not full. (c) A full tree decomposition of G.

Let G be a graph and (T, \mathcal{V}) be a tree decomposition of G. Given two different nodes $t, t' \in V(T)$, we denote by $\operatorname{Br}_t(t')$ the component of T-t where t' lies. We say that such component is a branch of T at t, and that the components of T-t are the branches of T at t [18]. Similarly, for a vertex $v \notin V_t$, it is denoted by $\operatorname{Br}_t(v)$ the branch $\operatorname{Br}_t(t')$ of T at t such that $v \in V_{t'}$. In that case, we also say that $v \in \operatorname{Br}_t(t')$.

Let $t \in V(T)$. Let C' be a path or cycle in G fenced by V_t . It is easy to see that, for every $u, v \in V(C') \setminus V_t$, we have $\operatorname{Br}_t(u) = \operatorname{Br}_t(v)$. Hence, when $V(C') \not\subseteq V_t$, we say that $\operatorname{Br}_t(C') = \operatorname{Br}_t(v)$, where v is an arbitrary vertex of $V(C') \setminus V_t$. The next proposition relates the concepts of separation and branches.

Proposition 3.2 ([9, Lemma 12.3.1]). Let G be a graph and (T, \mathcal{V}) be a tree decomposition of G. Let $tt' \in E(T)$. Let $u, v \in V(G)$ be such that $u \notin V_t$ and $v \notin V_{t'}$. If $u \in \operatorname{Br}_t(t')$ and $v \in \operatorname{Br}_{t'}(t)$, then $V_t \cap V_{t'}$ separates u and v.

4 Partial k-trees and Chordal Graphs

A clique in a graph is a set of pairwise adjacent vertices. A k-clique is a clique of cardinality k. The cardinality of a maximum clique in G is denoted by $\omega(G)$. A k-tree is defined recursively as follows. The complete graph on k vertices is a k-tree. Any graph obtained from a k-tree by adding a new vertex and making it adjacent to exactly all the vertices of an existing k-clique is also a k-tree. A graph G is a partial k-tree if and only if G is the subgraph of a k-tree. Partial k-trees are closely related to the definition of tree decomposition. In fact, a graph G is a partial k-tree if and only if $tw(G) \leq k$ [2, Theorem 35] (Figure 3).

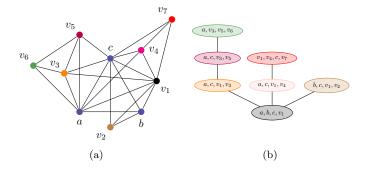


Figure 3: (a) A 3-tree G. To construct G, we begin with triangle abc and add the following sequence of vertices: v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 . (b) We can obtain a tree decomposition of G in the following way: each time we add a new vertex, say v_i , to an already existing triangle, say xyz, we also add a new node, with corresponding bag $\{x, y, z, v_i\}$, to the tree decomposition and we make it adjacent to an already existing node whose corresponding bag contains x, y and z. Moreover, the tree decomposition obtained by this procedure is a full tree decomposition of G.

A graph is called *chordal* if every induced cycle has length three. A tree decomposition (T, \mathcal{V}) of a graph G is called a *clique tree* if \mathcal{V} is the set of all maximal cliques in G (Figure 4).

Proposition 4.1 ([12, Theorem 2, Theorem 3]). Every chordal graph has a clique tree.

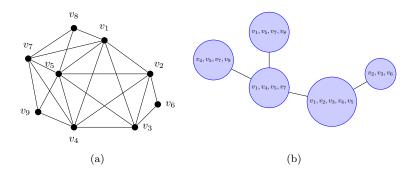


Figure 4: (a) A chordal graph G with $\omega(G) = 5$. (b) A clique tree of G.

5 Our main technique

In this section we introduce the technique for proving our results on partial k-trees and chordal graphs. A similar technique and notation was introduced

in [8]. We begin by showing a new proof for the well-known Helly Property on trees (see [19] and [21]). Given a tree T, a partial orientation of T is a digraph T' such that V(T') = V(T) and, if $uv \in E(T')$, then $uv \in E(T)$. Note that not all edges of T are present in T' as arcs.

Lemma 5.1 ([21, Theorem 4.1]). Let T be a tree. Let \mathscr{C} be a set of pairwise vertex-intersecting subtrees of T. There exists a vertex $t \in V(T)$ such that every tree in \mathscr{C} contains t.

Proof. We define a partial orientation T' of T as follows: $tt' \in E(T')$ if and only if there exists a tree $P \in \mathscr{C}$, that does not contain t, such that V(P) and t' are in the same component of T-t. Suppose by contradiction that the lemma is false for T. Then every node in T' has outdegree at least one. Let tt' be the last arc of a maximal directed path in T'. As T is a tree, t't is also an arc in T', which implies that there exist two trees P and Q in $\mathscr C$ such that V(P) and t' are in the same component of T-t, and V(Q) and t are in the same component of T-t'. But then $V(P) \cap V(Q) = \emptyset$, a contradiction (Figure 5).

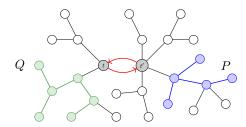


Figure 5: The subtrees P and Q and the last arc tt' of a maximal directed path in the proof of Lemma 5.1.

Our main technique for partial k-trees and chordal graphs is inspired in the proof of Lemma 5.1, but adapted to the tree decomposition of the graph. This is shown in Lemma 5.3. Before it, we show a useful property.

Proposition 5.2. Let (T, V) be a tree decomposition of a graph G. If C is a path or cycle in G fenced by V_t , for some $t \in V(T)$, then either $V(C) \subseteq V_t$ or there exists an edge $tt' \in E(T)$ such that $Br_t(C) = Br_t(t')$.

Proof. If $V(C) \subseteq V_t$, then there is nothing to prove. Otherwise, let $u \in V(C) \setminus V_t$. As $u \notin V_t$, there exists a bag $V_{t''}$ that contains u. Let t' be the neighbor of t in T such that t' is in the path from t to t'' in T. Then $\operatorname{Br}_t(C) = \operatorname{Br}_t(u) = \operatorname{Br}_t(t'') = \operatorname{Br}_t(t')$.

The next lemma is crucial for our results on partial k-trees and chordal graphs.

Lemma 5.3. Let (T, V) be a tree decomposition of a graph G. For every node t, let $\mathcal{C}(t)$ be a set of cycles in G fenced by V_t but not contained in $G[V_t]$. If $\mathcal{C}(t) \neq \emptyset$

for every node $t \in V(T)$, then there exists an edge $tt' \in E(T)$ and two cycles $C \in \mathcal{C}(t)$ and $D \in \mathcal{C}(t')$ such that $\operatorname{Br}_t(C) = \operatorname{Br}_t(t')$ and $\operatorname{Br}_{t'}(D) = \operatorname{Br}_{t'}(t)$.

Proof. We define a partial orientation T' of T as follows: $tt' \in E(T')$ if and only if $tt' \in E(T)$ and there exists a cycle $C \in \mathcal{C}(t)$ such that $\operatorname{Br}_t(C) = \operatorname{Br}_t(t')$. For every $t \in V(T)$, as $\mathcal{C}(t) \neq \emptyset$, there exists a cycle C fenced by V_t with $V(C) \nsubseteq V_t$. Thus, by Proposition 5.2, there exists a neighbor t' of t in T such that $\operatorname{Br}_t(C) = \operatorname{Br}_t(t')$. Hence every node in T' has outdegree at least one. Let tt' be the last arc of a maximal directed path in T'. As T is a tree, t't is also an arc in T', which implies that there exist two cycles $C \in \mathcal{C}(t)$ and $D \in \mathcal{C}(t')$ such that $\operatorname{Br}_t(C) = \operatorname{Br}_t(t')$ and $\operatorname{Br}_{t'}(D) = \operatorname{Br}_{t'}(t)$.

Immediate results are obtained for partial k-trees and chordal graphs using Lemma 5.3 (see also [9, Theorem 12.3.9] and [25, Proposition 2.6]). Recall that $\omega(G)$ is the maximum cardinality of a clique in G.

Corollary 5.4. Let G be a 2-connected graph. Then $lct(G) \le tw(G) + 1$. And, if G is chordal, then $lct(G) \le \omega(G)$.

Proof. It suffices to prove the first part, as the second part follows directly by [9, Corollary 12.3.12]. Suppose by contradiction that lct(G) > tw(G) + 1 and let (T, \mathcal{V}) be a tree decomposition for G of width tw(G). Then, for every $t \in V(T)$, there exists a longest cycle that does not intersect V_t . Thus, by Lemma 5.3, there exists an edge $tt' \in E(T)$ and two longest cycles C and D such that C is fenced by V_t and does not intersect V_t , D is fenced by $V_{t'}$ and does not intersect $V_{t'}$, $\operatorname{Br}_t(C) = \operatorname{Br}_t(t')$ and $\operatorname{Br}_{t'}(D) = \operatorname{Br}_{t'}(t)$. But then $V(C) \cap V(D) = \emptyset$, a contradiction.

The main task in this paper is to improve the bounds given by Corollary 5.4. So we have to find a longest cycle fenced by V_t that satisfies a particular property, which will make our set $\mathcal{C}(t)$ nonempty for every $t \in V(T)$, to finally apply Lemma 5.3. The main difficulty is that, when the bounds are diminished, the corresponding cycles can intersect several times the corresponding bag.

6 Result for Partial k-Trees

By Corollary 5.4, we have that $lct(G) \le k+1$ when G is a 2-connected partial k-tree. In this section we improve this result and show that, in fact, $lct(G) \le k-1$ (Theorem 6.2). We begin by showing a useful lemma.

Lemma 6.1. Let G be a 2-connected graph. Let (T, V) be a full tree decomposition of G. Let $t \in V(T)$. If $lct(G) > |V_t| - 2$, then V_t has an ℓ -attractor with $\ell \leq 2$.

Proof. As $\operatorname{lct}(G) > |V_t| - 2$, for every subset of V_t with cardinality $|V_t| - 2$, there exists a longest cycle that does not contain any vertex of it. If any of these cycles intersects V_t at most once, then there is an ℓ -attractor for V_t with $\ell \leq 1$ and we are done. Hence, every such cycle 2-intersects V_t . So, for every pair of vertices

in V_t , there exists a longest cycle that 2-intersects V_t at such pair. Suppose by contradiction that V_t has no ℓ -attractor, with $\ell \leq 2$. Then, for every pair of vertices in V_t , there exists a longest cycle that 2-crosses V_t at such pair. Observe that it cannot be the case that all such cycles contain an edge of V_t . Hence, there exists a longest cycle C that 2-crosses V_t , say at $\{a,b\}$, such that $ab \notin E(C)$.

Let C' and C'' be the two ab-parts of C. As both C' and C'' are fenced by V_t and are not contained in V_t , by Proposition 5.2, there exists two nodes t' and t'', neighbors of t in T, such that $\operatorname{Br}_t(C'') = \operatorname{Br}_t(t')$ and $\operatorname{Br}_t(C'') = \operatorname{Br}_t(t'')$, where possibly t' = t''. As (T, \mathcal{V}) is a full tree decomposition, we have $|V_t \cap V_{t'}| = |V_t \cap V_{t''}| = |V_t| - 1$, so $V_t \setminus V_{t'}$ consists on one vertex, say x. If $V_t \cap V_{t'} \neq V_t \cap V_{t''}$, let y be the vertex in $V_t \setminus V_{t''}$. If $V_t \cap V_{t'} = V_t \cap V_{t''}$, let y be an arbitrary vertex in V_t different from x. Let D be a longest cycle that 2-crosses V_t at $\{x,y\}$ and let D' and D'' be the two xy-parts of D. Note that $\operatorname{Br}_t(D')$ and $\operatorname{Br}_t(D'')$ are different from both $\operatorname{Br}_t(t')$ and $\operatorname{Br}_t(t'')$. Then, by Proposition 3.2, C and D intersect each other in at most one vertex, a contradiction to the fact that G is 2-connected (Proposition 2.1).

Finally, we prove our main theorem.

Theorem 6.2. For every 2-connected partial k-tree G, we have $lct(G) \le k-1$.

Proof. Let (T, \mathcal{V}) be a full tree decomposition of G. For every $t \in V(T)$, let $\mathscr{C}(t)$ be the set of longest cycles in G such that, for every $C \in \mathscr{C}(t)$, C is an ℓ -attractor for V_t with $\ell \leq 2$. Suppose by contradiction that $\operatorname{lct}(G) > k - 1$. Then, as $|V_t| = k + 1$, by Lemma 6.1, $\mathscr{C}(t) \neq \emptyset$ for every $t \in V(T)$. Thus, by Lemma 5.3, there exists an edge $tt' \in E(T)$ and two longest cycles C and D in G such that $\operatorname{Br}_t(C) = \operatorname{Br}_t(t')$, $\operatorname{Br}_{t'}(D) = \operatorname{Br}_{t'}(t)$, C is an ℓ -attractor for V_t with $\ell \leq 2$, and D is an ℓ -attractor for $V_{t'}$ with $\ell' \leq 2$.

It is easy to see, by Proposition 3.2, that $u \notin V(C)$. Analogously, we can conclude that $w \notin V(D)$ and therefore $V(C) \cap V(D) \subseteq V_t \cap V_{t'}$. Hence, as C and D are given by Lemma 6.1 and G is 2-connected, we may assume that $V(C) \cap V_t = V(D) \cap V_t = \{a,b\}$. Let C' and C'' be the two ab-parts of C. Let D' and D'' be the two ab-parts of D. As $|V(C) \cap V(D)| = 2$, we can conclude that |C'| = |C''| = |D''| = |D''|. We may assume that $w \notin V(D')$. Hence, $C' \cdot D'$ is a longest cycle that 2-crosses V_t at $\{a,b\} = V(C) \cap V_t$, a contradiction to the fact that C is a 2-attractor for V_t .

The previous theorem implies the following result.

Corollary 6.3. All longest cycles intersect in 2-connected partial 2-trees, also known as series-parallel graphs.

Also, we have the following corollary due to results of Fomin and Thilikos [11], and Alon, Seymour, and Thomas [1].

Corollary 6.4. For every 2-connected planar graph G on n vertices, we have $lct(G) < 3.182\sqrt{n}$, and for every 2-connected K_r -minor free graph G, we have $lct(G) < r^{1.5}\sqrt{n}$.

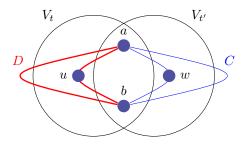


Figure 6: Situation in the proof of Theorem 6.2.

7 Result for Chordal Graphs

In this section, we prove that $lct(G) \leq max\{1, \omega(G) - 3\}$ for every 2-connected chordal graph G (Theorem 7.2). Throughout this section, we denote by L := L(G) the length of a longest cycle in G. Recall that $\omega(G)$ is the cardinality of a maximum clique in G.

7.1 Proof of the main theorem

The next lemma conceals the heart of the proof of Theorem 7.2. The proof of that lemma is presented in Subsection 7.2.

Lemma 7.1. Let G be a 2-connected chordal graph such that $lct(G) > max\{1, \omega(G) - 3\}$. Let k be an integer with $k \geq 2$. For each maximal k-clique in G, there exists an ℓ -attractor with $\ell \leq min\{3, k - 1\}$.

Using this lemma, we derive the main result of this section.

Theorem 7.2. For every 2-connected chordal graph G, $lct(G) \le max\{1, \omega(G) - 3\}$.

Proof. Let (T, \mathcal{V}) be a clique tree of G, which exists by Proposition 4.1. If some clique in \mathcal{V} has cardinality one, then |V(G)|=1 and we are done. Thus, every clique in \mathcal{V} has cardinality at least two. For every $t \in V(T)$, let $\mathscr{C}(t)$ be the set of longest cycles in G which are ℓ -attractors for V_t , with $\ell \leq \min\{3, |V_t|-1\}$. Suppose by contradiction that $\mathrm{lct}(G) > \max\{1, \omega(G)-3\}$. Then, by Lemma 7.1, $\mathscr{C}(t) \neq \emptyset$ for every $t \in V(T)$. Observe that, as V_t is a clique, any cycle in $\mathscr{C}(t)$ has no edges in $G[V_t]$; indeed, otherwise, such cycle will contain all vertices of V_t , a contradiction to the fact that $\ell \leq \min\{3, |V_t|-1\}$. This implies that, for any $t \in V(T)$, no cycle in $\mathscr{C}(t)$ is contained in $G[V_t]$. Thus, by Lemma 5.3, there exists an edge $tt' \in E(T)$ and two cycles $C \in \mathscr{C}(t)$ and $D \in \mathscr{C}(t')$ such that $\mathrm{Br}_t(C) = \mathrm{Br}_t(t')$, $\mathrm{Br}_{t'}(D) = \mathrm{Br}_{t'}(t)$, C is an ℓ -attractor for V_t with $\ell \leq \min\{3, |V_t|-1\}$, and D is an ℓ -attractor for $V_{t'}$ with $\ell \leq \min\{3, |V_{t'}|-1\}$.

Suppose for a moment that $|V_t \cap V_{t'}| \le \omega(G) - 2$. As $lct(G) > max\{1, \omega(G) - 3\}$, there exists a longest cycle that contains at most one vertex of $V_t \cap V_{t'}$. As such cycle must intersect both C and D twice, this is a

contradiction to Proposition 3.2. Hence $|V_t \cap V_{t'}| \ge \omega(G) - 1$. Moreover, as both V_t and $V_{t'}$ are maximal and different, we conclude that $|V_t| = |V_{t'}| = \omega(G)$. Let $\{u\} = V_t \setminus V_{t'}$ and $\{w\} = V_{t'} \setminus V_t$. It is easy to see that $u \notin V(C)$ and $w \notin V(D)$. As G is 2-connected, $|V(C) \cap V(D)| \ge 2$, so we have the following cases.

Case 1: Both C and D 3-intersect V_t and $V_{t'}$, respectively.

Let $V(C) \cap V_t = \{a, b, c\}$. Consider the case when $V(D) \cap V_{t'} = \{a, b, c\}$. We may assume, without loss of generality, that $w \notin C_{ab}$ and $u \notin D_{ab}$. As $(C - C_{ab}) \cdot D_{ab}$ and $(D - D_{ab}) \cdot C_{ab}$ are cycles, $|C_{ab}| = |D_{ab}|$ and both are longest cycles. Hence, $(C - C_{ab}) \cdot D_{ab}$ is a longest cycle that 3-crosses V_t at $V(C) \cap V_t$, a contradiction to the fact that C is an attractor for V_t (Figure 7(a)). Now suppose that $V(D) \cap V_{t'} = \{b, c, d\}$, with $d \neq a$. Then, $C_{bc} \cdot C_{ca} \cdot ad \cdot D_{db}$ and $D_{bc} \cdot D_{cd} \cdot da \cdot C_{ab}$ are cycles, one of them longer than L, a contradiction.

Case 2: Both C and D 2-intersect V_t and $V_{t'}$, respectively.

Let $\{a,b\} = V(C) \cap V_t = V(D) \cap V_{t'}$. Let C' and C'' be the two ab-parts of C. Let D' and D'' be the two ab-parts of D. As $C' \cdot D'$, $C' \cdot D''$, $C'' \cdot D'$ and $C'' \cdot D''$ are cycles, |C'| = |C''| = |D'| = |D''| = L/2. Without loss of generality, we may assume that $u \notin V(D')$. Hence, $D' \cdot C'$ is a longest cycle that 2-crosses V_t at $V(C) \cap V_t$, a contradiction to the fact that C is an attractor for V_t (Figure 7(b)).

Case 3: C 3-intersects V_t and D 2-intersects $V_{t'}$.

We may assume that $V(C) \cap V_t = \{a, b, c\}$ and that $V(D) \cap V_{t'} = \{a, b\}$. Let D' and D'' be the two ab-parts of D. Without loss of generality, we may assume that $u \notin V(D')$. As $(C - C_{ab}) \cdot D'$ and $(D - D') \cdot C_{ab}$ are cycles, $|C_{ab}| = |D'|$ and both are longest cycles. Hence, $(C - C_{ab}) \cdot D'$ is a longest cycle that 3-crosses V_t at $V(C) \cap V_t$, a contradiction to the fact that C is an attractor for V_t (Figure 7(c)).

This concludes the proof.

Note that k-trees are chordal [14, Theorem 4.1] and their maximum cliques have cardinality at most k + 1. Also, every planar graph is K_5 -free. Hence, we have the following corollary.

Corollary 7.3. If G is a k-tree, with k > 2, then $lct(G) \le k - 2$. Moreover, all longest cycles intersect in 2-trees, 3-trees, and in 2-connected chordal planar graphs.

7.2 Proof of the main lemma

We next show the proof of Lemma 7.1. Before that, we present new useful definitions. If C' and D' are paths fenced by a set of vertices K in a graph G, we write $C' \sim_K D'$ if there exist vertices $u \in V(C')$ and $v \in V(D')$ such that u and v are in the same component of G - K. Otherwise, we write $C' \sim_K D'$. If the context is clear, we write $C' \sim D'$ and $C' \nsim D'$. Given a cycle C that

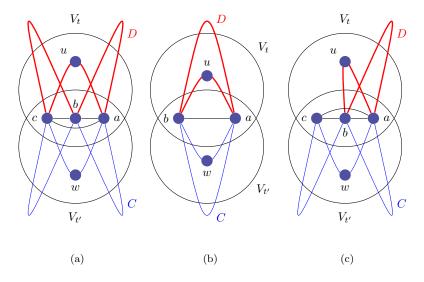


Figure 7: Cases in the proof of Theorem 7.2.

3-crosses K at $\{a,b,c\}$, we say that a breaks C if $C_{ab} \sim_K C_{ac}$. If the context is clear, we also say that a is a C-breaking vertex or that a is a breaking vertex (Figure 8). Recall that two paths or cycles C' and C'' are K-equivalent if $V(C') \cap K = V(C'') \cap K$.

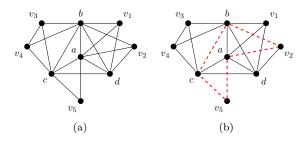


Figure 8: (a) A graph with $K = \{a, b, c, d\}$. (b) Consider cycle $C = av_2bcv_5a$. Then $C_{ab} \nsim_K C_{ac}$, $C_{ab} \nsim_K C_{bc}$ and $C_{bc} \nsim_K C_{ac}$. Hence, a breaks C.

Proof of Lemma 7.1. Let K be a maximal clique in G and $k \geq 2$ be its cardinality. If there is a longest cycle that intersects K at most once, then we are done. Indeed, such cycle would be an ℓ -attractor for K with $\ell \in \{0,1\}$. Hence, we may assume that every longest cycle intersects K at least twice. Note that this implies, as lct(G) > 1, that $k \geq 3$. Let (T, \mathcal{V}) be a clique tree of G. We have the next two cases.

Case 1: There exists a longest cycle that 2-intersects K.

Let $\{a,b\}\subseteq K$ be such that there exists a longest cycle that 2-intersect K at $\{a,b\}$. If all cycles that 2-intersect K at $\{a,b\}$ are fenced by K, then we have an ℓ -attractor with $\ell \leq 2$ and we are done. Hence, there exists a longest cycle C that 2-crosses K at $\{a,b\}$. As K is a maximal clique in G, there exists $t \in V(T)$ such that $V_t = K$. It is straightforward to see that, as $k \geq 3$, no ab-part of C is an edge. Let C' and C'' be the two ab-parts of C. By Proposition 5.2, there exists two edges $tt_1, tt_2 \in E(G)$ such that $Br_t(C') = Br_t(t')$ and $Br_t(C'') = Br_t(t'')$. As V_t , $V_{t'}$ and $V_{t''}$ are maximal cliques in G, there exist two vertices $c \in V_t \setminus V_{t'}$ and $d \in V_t \setminus V_{t''}$. Note that $c, d \notin \{a, b\}$. Note also that, although C' and C'' are in different components of G-K, it can be the case that t'=t'', implying that c = d. Let us assume that $c \neq d$ (so $k \geq 4$), as the proof when c = d is very similar. As $lct(G) > max\{1, \omega(G) - 3\} \ge k - 3$, there exists a longest cycle D that does not contain any vertex of $V_t \setminus \{b, c, d\}$. If D and any cycle equivalent to D, is fenced by K, then we are done. Indeed, as D will be a ℓ -attractor with $\ell \leq 3 \leq \min\{3, k-1\}$. So, we may assume, without loss of generality that D crosses K.

Suppose for a moment that D 2-intersects K at $\{x,y\}$ (note that it can be the case that $\{x,y\} \cap \{b\} \neq \emptyset$). Let D' and D'' be the two xy-parts of D. As both C and D cross K, we may assume, without loss of generality, that C' is internally disjoint from D' and that C'' is internally disjoint from D''. But then, $C' \cdot bx \cdot D' \cdot ya$ and $C'' \cdot bx \cdot D'' \cdot ya$ are both cycles, one of them longer than L, a contradiction (Figure 9(a)). Hence, we may assume that D 3-intersects K. If $V(D) \cap K \cap \{a,b\} = \emptyset$ then the proof is very similar to the previous case. So, let us assume that D intersects K at $\{b,c,d\}$. Recall that $c \in V_t \setminus V_{t'}$ and $d \in V_t \setminus V_{t''}$. Thus, by Proposition 3.2, we have $C' \nsim_K D_{bc}$ and $C' \nsim_K D_{cd}$. Analogously, $C'' \nsim_K D_{bd}$. Hence, $C' \cdot D_{bc} \cdot D_{cd} \cdot da$ and $C'' \cdot D_{bd} \cdot da$ are both cycles, one of them longer than L, a contradiction (Figure 9(b)).

Case 2: Every longest cycle intersects K at least three times.

As $\operatorname{lct}(G) > \max\{1, \omega(G) - 3\} \ge k - 3$, for every triangle $\Delta \subseteq K$, there exists a longest cycle that 3-intersects K at Δ . Suppose by contradiction that none of these cycles is a 3-attractor for K. Then, for every triangle $\Delta \subseteq K$, there exists a longest cycle C_{Δ} that 3-crosses K at Δ . Let $\Delta \subseteq K$. As C_{Δ} 3-crosses K at Δ , Δ has at least two C_{Δ} -breaking vertices. As there are $\binom{|K|}{3}$ triangles in K, by pigeonhole principle, there exists a vertex $x \in K$ such that x is a breaking vertex for at least $\frac{(|K|-1)(|K|-2)}{3}$ of the triangles incident to x.

Suppose for a moment that $|K| \geq 5$. Then, there exists two edge-disjoint triangles incident to x such that x is a breaking vertex for both of them. Let xab and xcd be such triangles, and let C and D be the corresponding cycles respectively. As x breaks both C and D, without loss of generality we may assume that $C_{xa} \nsim_K D_{xc}$ and that $C_{xb} \nsim_K D_{xd}$. Also, there exists a part $P \in \{D_{xc}, D_{xd}\}$ such that $C_{ab} \nsim_K P$ and a part $Q \in \{C_{xa}, C_{xb}\}$ such that $D_{cd} \nsim_K Q$. Without loss of generality, we may suppose that $C_{ab} \nsim_K D_{xd}$. If $D_{cd} \nsim_K C_{xa}$ then $D_{xc} \cdot D_{cd} \cdot da \cdot C_{ax}$ and $D_{dx} \cdot C_{xb} \cdot C_{ba} \cdot ad$ are cycles, a contradiction (Figure 10(a)). So $D_{cd} \sim_K C_{xa}$, implying that $D_{cd} \nsim_K C_{xb}$. If $C_{ab} \nsim_K D_{cd}$ then

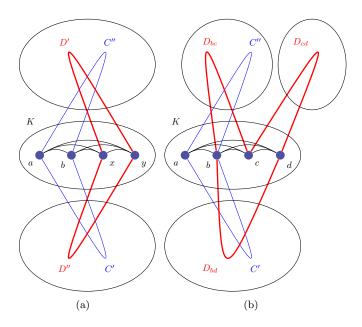


Figure 9: Situations in the proof of Lemma 7.1.

 $C_{xa} \cdot ac \cdot D_{cx}$ and $C_{xb} \cdot C_{ba} \cdot ac \cdot D_{cd} \cdot D_{dx}$ are cycles, a contradiction (Figure 10(b)). So, $C_{ab} \sim_K D_{cd}$. As $D_{cd} \sim_K C_{xa}$, we conclude that $C_{ab} \sim_K C_{xa}$. As $C_{xa} \nsim_K D_{xc}$, we conclude that $C_{ab} \nsim_K D_{xc}$. Then, $C_{xa} \cdot C_{ab} \cdot bc \cdot D_{cx}$ and $C_{xb} \cdot bc \cdot D_{cd} \cdot D_{dx}$ are both cycles, again a contradiction (Figure 10(c)).

Now suppose that |K| = 4. Then x is a breaking vertex for two triangles incident to x. Let xbd and xcd be these two triangles. Let C and D be the corresponding longest cycles respectively. Hence,

$$C_{xb} \nsim_K C_{xd} \text{ and } D_{xc} \nsim_K D_{xd}.$$
 (1)

Also, note that

$$C_{xb} \nsim_K D_{cd} \text{ and } C_{bd} \nsim_K D_{xc}.$$
 (2)

Is the proof of (2): by Proposition 5.2, there exists an edge $tt' \in E(T)$ such that $\operatorname{Br}_t(C_{xb}) = \operatorname{Br}_t(t')$. As both V_t and $V_{t'}$ are maximal cliques in G, there exists a vertex in $V_t \setminus V_{t'} \subseteq \{c, d\}$. Thus, by Proposition 3.2, $V_t \cap V_{t'}$ separates C_{xb} from D_{cd} ; hence $C_{xb} \nsim D_{cd}$. Analogously, $C_{bd} \nsim D_{xc}$.

By (1), either $C_{xb} \nsim D_{xd}$ and $C_{xd} \nsim D_{xc}$, or $C_{xb} \nsim D_{xc}$ and $C_{xd} \nsim D_{xd}$. In the first case, by (2), $C_{bx} \cdot D_{xd} \cdot D_{dc} \cdot cb$ and $D_{xc} \cdot cb \cdot C_{bd} \cdot C_{dx}$ are cycles, one of them longer than L, a contradiction (Figure 10(d)).

In the second case, $C_{bd} \sim D_{cd}$. Indeed, suppose for a moment that $C_{bd} \nsim D_{cd}$. Thus, $C_{xd} \cdot D_{xd}$ and $C_{xb} \cdot C_{bd} \cdot D_{dc} \cdot D_{cx}$ are cycles (Figure 10(e)). But then, $C_{xd} \cdot D_{xd}$ is a longest cycle that 2-intersects V_t , a contradiction. Hence $C_{bd} \sim D_{cd}$. If $C_{bd} \nsim D_{xd}$ and $C_{xd} \nsim D_{cd}$, then $C_{bx} \cdot C_{xd} \cdot dc \cdot D_{cd}$ and $D_{cx} \cdot D_{xd} \cdot C_{db} \cdot bc$ are cycles, a contradiction (Figure 10(f)). So we

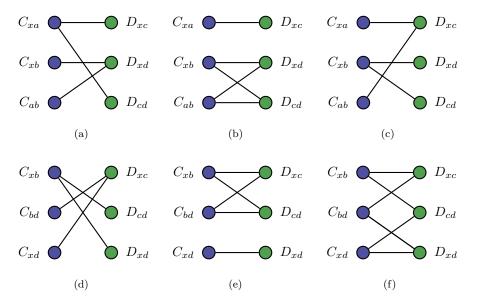


Figure 10: Each bipartite graph represents the situation of the cycles C and D in the proof of Lemma 7.1. Each side of the bipartition has three vertices that represent the parts of each cycle. There is a straight edge in the graph if the corresponding parts, say P and Q, are such that $P \sim_K Q$.

may assume, without loss of generality, that $C_{xd} \sim D_{cd}$. Thus, we have, $C_{bd} \sim C_{xd} \sim D_{cd}$. By Proposition 3.2, there exists an edge $tt' \in E(G)$ such that $\operatorname{Br}_t(t') = \operatorname{Br}_t(C_{xd})$. As both V_t and $V_{t'}$ are maximal cliques in G, there exists a vertex in $V_t \setminus V_{t'} \subseteq \{b,c\}$. If $V_t \setminus V_{t'} = \{b\}$, then by Proposition 3.2, $V_t \cap V_{t'}$ separates C_{xd} from C_{bd} , a contradiction. If $V_t \setminus V_{t'} = \{c\}$, then by Proposition 3.2, $V_t \cap V_{t'}$ separates C_{xd} from C_{cd} , again a contradiction.

This concludes the proof of the lemma.

8 Concluding remarks

In this paper, we showed upper bounds for the minimum cardinality of a set of vertices that intersects all longest cycles in a 2-connected partial k-tree and in a 2-connected chordal graph. We showed that, in partial k-trees, there is a set of at most k-1 vertices that intersects all longest cycles of the graph, and that in chordal graphs there is such a set with cardinality at most $\max\{1, \omega-3\}$, where ω is the cardinality of a maximum clique of the graph. This implies that all longest cycles intersect in partial 2-trees and in 3-trees.

The question of whether lct(G) = 1 when G is a 2-connected chordal graph is still open, we conjecture a positive answer to that question. As any graph is a

partial k-tree for some k, we have that lct(G) > 1 when G is a 2-connected partial k-tree. However, for partial 3-trees, it has been proved that all longest cycles intersect [17, 16]. For partial 4-trees, there exists a 2-connected graph G given by Thomassen on 15 vertices [26, Figure 16], with tw(G) = 4 and lct(G) = 2. Hence, by Theorem 6.2, we conclude the following corollary and conjecture that $\ell = 2$.

Corollary 8.1. Let ℓ be the minimum integer such that $lct(G) \leq \ell$ for every 2-connected partial 4-tree G. Then, $\ell \in \{2,3\}$.

Transversals of longest paths has been also studied [4, 5, 6, 8, 13, 24, 25]. Also, other questions about intersection of longest cycles have been rised by several authors [7, 20, 22, 27].

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