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FREE VIBRATION OF AXISYMMETRIC AND BEAM-LIKE CYLINDRICAL SHELLS, PARTIALLY FILLED WITH LIQUID

by

Aouni A. Lakis and Marwan Sinno

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Department of Mechanical Engineering
Ecole Polytechnique de Montréal
Campus de l'Université de Montréal
C.P. 6079, Succ. A, Montreal
Quebec, Canada, H3C 3A7

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FREE VIBRATION OF AXISYMMETRIC AND BEAM-LIKE CYLINDRICAL SHELLS' PARTIALLY FILLED WITH LIQUID

ABSTRACT

This report presents a theory for the determination of free vibration characteristics of anisotropic thin cylindrical shells, partially or completely filled with liquid for two circumferential wave numbers, n = 0 breathing and n = 1 beam-like. The method used was a hybrid one, based on the finite element method and supported by classical shell theories. The shell was subdivided into cylindrical finite elements and the displacement functions were obtained using the shell's equations. Expressions for the mass and stiffness matrices for a fnite element and for the whole structure were obtained. Similarly, a finite element was developed for the liquid in cases of potential flow. The natural frequencies of the shell in vacuo and partially filled were obtained and compared with existing experiments and other theories.

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LIST OF SYMBOLS

a	:	internal radius of the shell
a_i, b_i, c_i, d_i	:	constants determined in (5.7)
$A_1; A_2$	:	Lamé parameter
A_j,	:	constant in equation of axial displacement, U
B_j, C_j	:	constants in equations of tangential & radical displacements, V and W
b/L	:	ratio of liquid level
c	:	speed of sound in fluid
e	:	exponential function
E	:	Young's modulus
h	:	constant determined in (5.10)
g	:	acceleration constant ($g = 980$ cm/sec^2)
i	:	$i^2 = -1$
J_n	:	Bessel function of the first kind and of order n
ℓ	:	Length of a finite element
L	:	Length of the shell

m	:	half-number of axial mode
$M_1, M_2, M_{12}, M_{21}, \bar{M}_{12}$	:	resultant moments
$M_x, M_\theta, M_{x\theta}$	:	resultant moments for a cylindrical shell
$\dot{M}_1$	:	boundary couple value
n	:	number of circumferential modes
N	:	number of finite elements
NDF	:	number of degrees of freedom for each node
N_1, N_2, N_{12}	:	resultant constraints
$N_x, N_\theta, \bar{N}_{x\theta}$		resultant constraints for a cylindrical shell
P_{ij}	:	matrix of elasticity element $(i = 1,\ldots 6)$; $(j = 1,\ldots 6)$
P_x, P_θ, P_r	:	loads applied
P_t	:	lateral pressure exerted on the structure
P_i, P_e	:	internal and external pressure
Q_1, Q_2	:	resultant shear constraints for a cylindrical shell
Q_x, Q_θ	:	resultant shear constraints for a cylindrical shell
r	:	radius of the shell
r_1	:	radius of the first finite element
R_1, R_2	:	radius fo curvature for the surface of reference

r_q, s_q	:	expressions determined by (6.8)
t	:	thickness of the shell
t_1	:	thickness of first finite element
U, V, W	:	axial, tangential and radial displacement.
$\bar{\bar{V}}$	:	resultant shear constraint for a boundary
V_x, V_r, V_θ	:	axial raidal and tangential compoenents, of the fluid velocity (6.3)
U_o, $\bar{U}_e$, $\bar{U}_i$	:	parameters determined by equation (6.11)
x	:	coordinate of the shell generator
Y_n	:	Bessel function of the second kind and of order n.
α_i, β_i	:	determined by (4.7) and given by (4.8) and (4.10)
γ_i, γ_e	:	determined by (6.11)
δ_i, δ_e	:	determined by (6.11)
$\varepsilon_x, \varepsilon_\theta, \varepsilon_{x\theta}$	:	deformations of the surface of reference
ζ_1, ζ_2	:	coordinates for the surface of reference
θ	:	circumferential coordinate
$\kappa_x, \kappa_\theta, \bar{\kappa}_{x\theta}$	:	rotations of the surface fo reference
Λ	:	natural value
λ_j	:	complex roots of a characteristic equation
ν	:	Poisson's ratio
ρ	:	density of the shell material

ρ_ℓ : liquid density

ω : natural frequency of system (rd/sec)

ω_o : determined by (6.14)

Ω : vibration parameter

$(\Omega = \omega r \sqrt{\rho (1 - \nu^2)/E})$

LIST OF MATRICES

$[A]$, $[A_o]$, $[A'_o]$	:	determined respectively by equations (4.11), (4.20) and (4.24)
$[B]$	:	determined by equation (5.1)
$\{C\}$	:	vector determined by equation (4.1)
$[C_f]$, $[C_o]$	:	determined by equation (6.1)
$[D]$, $[D_o]$	:	determined respectively by equations (4.4) and (4.16)
$[D_f]$	:	determined by equation (6.10)
$[DD]$	:	determined by equation (6.15)
$\{F\}$	:	external force vector
$[G]$, $[G_o]$, $[G_f]$	:	determined respectively by equations (5.6), (5.8) and (6.10)
$[I]$	:	identity matrix
$[k_f]$	:	stiffness matrix of a finite fluid element
$[k]$	:	stiffness matrix of a finite element of the shell
$[K]$	:	stiffness matrix of the system
$[K_o]$	:	stiffness matrix of the shell
$[K_f]$	:	stiffness matrix of the fluid column

$[K_f']$: determined by equation (6.1)

$[m]$, $[m_f]$: respectively, matrices of a finite element of the shell and of the fluid

$[M_f]$: mass matrix of the fluid column

$[M_o]$: mass matrix of the shell

$[M_f']$: determined by equation (6.1)

$[M]$: mass matrix of the system

$[N]$, $[N_o]$, $[N_o']$: respectively determined by (4.12), (4.22) and (4.24)

$[P]$: matrix of elasticity

$[R]$, $[R_o]$, $[R_o']$: respectively determined by equations (4.9), (4.18) and (4.24)

$[Q]$, $[Q_o]$, $[Q_o']$: respectively determined by equations (5.1), (A-3.2) and (A3.13)

$[S]$, $[S_o]$, $[S_f]$: respectively determined by equations (5.6), (5.8) and (6.10)

$[T]$: determined by equation (4.2)

$[Z]$, $[Z_o]$: respectively determined by equations (5.7) and (5.9)

$[\Gamma]$, $[\Gamma_o]$: respectively determined by equations (5.7) and (5.9)

$\{\delta_i\}$, $\{\delta_j\}$: nodal displacement vectors

$\{\Delta\}$: vector determined by equation (4.20)

$\{\varepsilon\}$, $\{\varepsilon_o\}$, $\{\varepsilon_o\}$: deformation vectors

$\{\sigma\}$, $\{\sigma_o\}$, $\{\sigma_o'\}$: stress vectors

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CHAPTER 1

INTRODUCTION

1.1 General

Thin shells have been and are still very important elements employed virtually throughout modern chemical, nuclear, aeronautical and space industries. There is a plethora of published static and dynamic shell studies. Many theories have been put forth and possible applications of these theories have been investigated.

Aron, in 1874, was one of the first to attempt to formulate a theory for thin curved shells, starting with the general elasticity equation. He was followed, in 1888, by Love [1] with his approximation theory. Since that time and up to the present, linear elastic shell theory has been examined and re-examined by scientific researchers throughout the literature ([2] to [7]).

More specifically relating to cylindrical shells, Arnold and Waburton derived the dynamic equations for a uniform cylindrical shell by using the energy method, the Timoshenko stress-strain relationships and the Lagrange equations[8]. Baron and Bleich [9] based their theory on an energy method, treating the shell as a membrane and introducing a correction factor that took the curvature into consideration. Galletly [10] reapplied Arnold and Waburton's study to the reinforced cylindrical arched shell. The free vibration of non-uniform cylindrical shells was studied theoretically by Al-Najafi and Waburton [11], and Falkiewicz [12] investigated them experimentally.

Dynamic studies, based on simple equations of motion for particular vibration problems, were undertaken by Seide [13]. Reissner was interested in shells with sandwich-type arch [14]; but the essence of these studies appeared in Ambartsumyan's paper [15], which involved several cases of anisotropic shells.

The effects of a fluid on the dynamic state of a shell can manifest itself in many ways. If the shell contains a low-pressure gas, then the vibration system (shell-gas) differs very little from that of the empty shell. However, this is not the case when the shell contains high-pressure gas. furthermore, if the fluid is compressible, the compressibility can affect the effective stiffness of the system. In addition, if the density of the fluid is relatively high, as is the case for the liquid, what will occur is a considerable intertial load upon the shell, which will result in a reduction in the systems natural frequency.

There have been several studies conducted on cylindrical shells partially filled with liquid. Niordson [16] was the first to present a study concerning the effect of the liquid on the natural frequency of the shell. Berry and Reissner [17] studied the case of a simply supported cylindrical containing pressurized gas. However the natural vibration of a shell completely or partially filled with non-pressurized liquid were theoretically and experimentally investigated by Lindholm and Kana [18].

1.2 Research objectives

This report is an attempt to determine the natural frequencies of anisotropic cylindrical shells and shells partially filled with liquid for the following two cases of circumferential modes: axisymetric (n = 0) and beam-like (n = 1).

A hybrid method was used, based on the finite element method and classical thin shell theory. The finite element chosen for a cylindrical element had two nodes with four degrees of freedom for each node for n = 1, and one to two degrees, for n = 0. This therefore made it possible to employ the thin shell equations so that we could determine the displacement functions and then the mass and stiffness matrices of the element. This is not feasible however, if a triangular or rectangular element is employed [19] and [20]. Similarly, the method in reference [22] was used to obtain the fluid's potential flow for a finite element. The natural frequencies of the empty and the partially liquid-filled shells were determined and compared to values found in the literature and existing experiments.

1.3 Outline of the report

The present study is divided into nine chapters. A description of the contents will now briefly be given.

Chapter 2 is a review of basic thin shell theory and an outline of the numerical method used.

Chapter 3 presents formulations of three equations of motion for this study in conjunction with the displacement of the shells of reference and components of the matrix of elasticity, as derived from the general equations for shells of revolution and their elasticity relationships.

In Chapter 4, the displacement functions are chosen for a finite element from the exact solutions of the three equations presented.

The mass and stiffness matrices of each element, as well as for the system as a whole are determined in Chapter 5.

The liquid's apparent mass and stiffness as well as absorption matrices are developed in Chapter 6. The study of free vibrations in a cylindrical shell partially filled with liquid is then dealt with.

In Chapter 7, the method of computing natural frequencies and eigen vectors of the (liquid shell) system are described.

The synthesis obtained from numerical calculations and from our research, compared with the findings of other authors, are given in Chapter 8.

Finally, Chapter 9 presents the general conclusions.

CHAPTER 2

BASIC THEORY AND METHOD

2.1 Classical thin elastic shell theory

The classical thin elastic shell theory is derived from the approximation of the tridimensional theory of elasticity. It originates out of the first approximation of Love [1], which is based on the following hypothesis:

a) thickness (t) is small compared to the minimal radius of curvature (R_{min});

b) the wall displacements of the shell are small compared to shell thickness;

c) the constraints which follow a normal axis to the surface of reference are insignificant;

d) the normals at the surface of reference remain normal and are not subjected to any elongation;

Hypothesis (a) represents the definition of thin shells ($r/t > 10$). The higher-order terms for displacement are negligible when compared to the frist-order terms with assurance of linearity of the differential equations. Hypothesis (c) and (d) assume that the constraints normal to the surface and the transverse shear deformation are negligible.

In this report Sander's theory is used, based on the first Love approximation and the fact that unit deformations are cancelled for all rigid-body movement. The equations of equilibrium, the strain-displacement and stress-strain relationships are given in Appendix A-1.

2.2 Method

As previously mentioned (in section 1.2), the method used was one, developed in references [21], [22] and [23] specifically for cylindrical shells with a circumferential mode equal to or greater than 2 ($n \geqslant 2$). This same method was applied to cylindrical shells for two cases: axisymetric ($n = 0$) and beam-like behaviours ($n = 1$). The principle points of this method are as follows:

a) The shell is subdivided into several cylindrical elements (Fig. 2). The cylindrical element is determined by two nodes i, j and the boundaries of the nodal surface (Fig. 3). The displacement functions can be determined by:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$$

Where δ_i represent the nodal displacements and [N] is a matrix of general position functions.

b) The displacement functions chosen must adequately represent the real displacements of the shell. Accordingly, thin shell equations were used to determine the displacement functions. These yield more precise results than if the displacement functions are expressed in polynomial form. The fundamental equations from Sanders in A-1 were reduced to three differential equations as a function of axial, radial and circumferential displacements of the surface of reference. The solution of these equations gives us the displacement functions.

CHAPTER 3

EQUATIONS OF MOTION FOR ANISOTROPIC CYLINDRICAL SHELLS

3.1 Equations of equilibrium, strain-displacement and stress-strain relationships

Sander's theory of thin shells [4], was used to describe the behaviour of cylindrical curved shells (Appendix 1).

By eliminating the shear forces Q_x and Q_θ by means of equations A-1.5 d,e, we obtain the three equilibrium equations:

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial \overline{N}_{x\theta}}{\partial \theta} - \frac{1}{2r^2}\frac{\partial \overline{M}_{x\theta}}{\partial \theta} = 0$$

$$\frac{\partial \overline{N}_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{3}{2r}\frac{\partial \overline{M}_{x\theta}}{\partial x} + \frac{1}{r^2}\frac{\partial M_\theta}{\partial \theta} = 0 \qquad (3.1)$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{2}{r}\frac{\partial^2 \overline{M}_{x\theta}}{\partial x \partial \theta} - \frac{1}{r} N_\theta = 0$$

The deformation vector $\{\varepsilon\}$ is given by:

$$\{\varepsilon\} = \left\{\begin{matrix} \varepsilon_x \\ \varepsilon_\theta \\ 2\bar{\varepsilon}_{x\theta} \\ \kappa_x \\ \kappa_\theta \\ 2\bar{\kappa}_{x\theta} \end{matrix}\right\} = \left\{\begin{matrix} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} + \dfrac{W}{r} \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{r}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{r^2}\left(\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial V}{\partial \theta}\right) \\ -\dfrac{2}{r}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2r}\dfrac{\partial V}{\partial x} - \dfrac{1}{2r^2}\dfrac{\partial U}{\partial \theta} \end{matrix}\right\} \tag{3.2}$$

Where U, V and W are, respectively, displacement that are axial tangential and radial to the shell's surface of reference.

The relationships between the stress and strain vectors and the reference surface for an anisotropic shell are given as follows:

$$\{\sigma\} = \left\{\begin{matrix} N_x \\ N_\theta \\ \bar{N}_{x\theta} \\ M_x \\ M_\theta \\ \bar{M}_{x\theta} \end{matrix}\right\} = [P]\ \{\varepsilon\} \tag{3.3}$$

Where [P] is the matrix of elasticity [15].

Elements P_{ij} in [P] characterize the mechanical properties of the shell material, and so in general we assume that:

$$[P]=\begin{bmatrix} P_{11} & P_{12} & 0 & P_{14} & P_{15} & 0 \\ P_{21} & P_{22} & 0 & P_{24} & P_{25} & 0 \\ 0 & 0 & P_{33} & 0 & 0 & P_{36} \\ P_{41} & P_{42} & 0 & P_{44} & P_{45} & 0 \\ P_{51} & P_{52} & 0 & P_{54} & P_{55} & 0 \\ 0 & 0 & P_{63} & 0 & 0 & P_{66} \end{bmatrix} \qquad (3.4)$$

By substituting (3.2) to (3.4) in the equations of equilibrium (3.1), new equations (3.5) are obtained in conjunction with [P] elements P_{ij} and with the axial tangential and radial displacements, U, V and W of the reference surface of the shell:

$$\begin{cases} L_1(U,V,W,P_{ij}) = 0 \\ L_2(U,V,W,P_{ij}) = 0 \\ L_3(U,V,W,P_{ij}) = 0 \end{cases} \qquad (3.5)$$

(these equations are given in Appendix A-2).

Similar to Donnell [24], Sander's (3.5) equations of motion have been simplified for the n = 1 case and become:

$$\begin{cases} S_1(U,V,W,P_{ij}) = 0 \\ S_2(U,V,W,P_{ij}) = 0 \\ S_3(U,V,W,P_{ij}) = 0 \end{cases} \tag{3.6}$$

(these equations are given in Appendix A-2).

3.2 Matrix of elasticity

The matrix fo elasticity [P] is generally given by equation (3.4); thus this theory can be applied to:

(i) shells composed of only one layer or an arbitrary number of isotropic or orthotropic layers;

(ii) double-walled shells, with slabs or ribs;

(iii) ring-stiffered shells with grooves of known characteristics;

(iv) shells where [P] can be experimentally evaluated.

We will now restrict ourselves to shells made of only one layer or of an arbitrary number of symmetrical isotropic or orthotropic layers relatively placed at the surface coordinate.

For the case of an arbitrary number of orthotropic layers [15], we assume that there is no sliding between them and that the principle direction of elasticity at each point or the shell coincides with the line direction coordinates.

(i) For a number of layers equal to 2V, P_{ij} of [P] can be written as follows:

$$P_{ij} = 2 \sum_{S=1}^{V} B_{ij}^{S} \, (t_S - t_{S+1}); \; i = 1 \text{ to } 3 \text{ and } j = 1 \text{ to } 3$$

$$P_{ij} = (2/3) \sum_{S=1}^{V} B_{i-3,j-3}^{S} \, (t_S^3 - T_{S+1}^3); \; i = 4 \text{ to } 6 \text{ and}$$

$$j = 4 \text{ to } 6$$

$$P_{ij} = 0; \; i = 1 \text{ to } 3 \text{ and } j = 4 \text{ to } 6; \; i = 4 \text{ to } 6 \text{ and } j = 1 \text{ to } 3$$

(3.7)

(ii) For an odd number, 2V + 1, we have:

$$P_{ij} = 2 \left[B_{ij}^{V+1} \, t_{V+1} + \sum_{S=1}^{V} B_{ij}^{S} \, (t_S - t_{S+1}) \right]; \; i = 1 \text{ to } 3$$

$$\text{and } j = 1 \text{ to } 3$$

$$P_{ij} = (2/3) \left[B_{i-3,j-3}^{V+1} \, t_{V+1} + \sum_{S=1}^{V} B_{i-3,j-3}^{S} \, (t_S - t_{S+1}) \right];$$

$$i = 4 \text{ to } 6 \text{ and } j = 4 \text{ to } 6 \qquad (3.8)$$

$$P_{ij} = 0; \; i = 1 \text{ to } 3 \text{ and } j = 4 \text{ to } 6; \; i = 4 \text{ to } 6 \text{ and } j = 1 \text{ to } 3$$

where

$$B_{11}^{S} = \left[E_1^S/(1 - \nu_1^S \nu_2^S)\right]; \quad B_{22}^{S} = \left[E_2^S/(1 - \nu_1^S \nu_2^S)\right]$$

$$B_{12}^{S} = B_{21}^{S} = \left[\nu_2^S \, E_1^S/(1 - \nu_1^S \nu_2^S)\right]; \quad B_{33}^{S} = 0.5 \, G_{12}^{S}$$

$$B_{ij}^{S} = 0 \quad \text{elsewhere}$$

t_s is the coordinate of the 6 th layer using the surface of reference as a reference point as illustrated in Fig. 4; $(E_1^S \nu_1^S)$ and $(E_2^S \nu_2^S)$, which are, respectively, the elastic modulus and Poisson ratio in relation to directions X and θ; and G_{12}^S is the shear modulus of elasticity.

CHAPTER 4

DISPLACEMENT FUNCTIONS

4.1 Choice of displacement functions

In accordance with the hybrid method mentioned in 2.2, the shell is subdivided into several uniform cylindrical elements (Figures 2-3). The cylindrical element is defined by two nodes i, j, and the three components U, V and W respectively represent the axial, tangential and radial displacements, from a point located on the shell's surface of reference.

4.1.1 Arbitraty loads (n = 1)

The equations of motion (A-2) are given by:

$$\begin{aligned} S_1(U,V,W,P_{ij}) &= 0 \\ S_2(U,V,W,P_{ij}) &= 0 \\ S_3(U,V,W,P_{ij}) &= 0 \end{aligned} \qquad (4.1)$$

The displacement functions can be written as:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T] \begin{Bmatrix} u(x) \\ w(x) \\ v(x) \end{Bmatrix} \qquad (4.2)$$

[T] is a (3x3) matrix θ given in Appendix A-4 and u(x), v(x) and w(x) are function only of x.

Setting:

$$u(x) = Ae^{\lambda x/r} \qquad v(x) = Be^{\lambda x/r} \qquad w(x) = Ce^{\lambda x/r} \tag{4.3}$$

and

Substituting (4.3) and (4.2) in (4.1), we obtain three homogeneous equations which are functions of constants A, B and C.

$$[D]\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0 \tag{4.4}$$

In order to arrive at a non-trivial solution, the determinant of matrix [D] must be equal to zero. From this the following characteristic equation is derived:

$$|D| = h_8\lambda^8 + h_6\lambda^6 + h_4\lambda^4 + h_2\lambda^2 + h_0 = 0 \tag{4.5}$$

The values of coefficient h_j in the polynomial are given in Appendix A-4.

Each root λ_j of this equation yields a solution for the equations of motion (4.1). The complete solution is obtained by totalling the sum of the eight solutions independently from the constants A_j, B_j and $C_j (j = 1,2...8)$.

$$u(x) = \sum_{j=1}^{8} A_j e^{\lambda_j x/r} \qquad v(x) = \sum_{j=1}^{8} B_j e^{\lambda_j x/r}$$

$$w(x) = \sum_{j=1}^{8} C_j e^{\lambda_j x/r} \tag{4.6}$$

As constants A_j, B_j and C_j are not independent, we will express A_j and B_j in conjunction with C_j

$$\begin{aligned} A_j &= \alpha_j C_j \\ B_j &= \beta_j C_j \end{aligned} \qquad j = 1,\ 2,\ \ldots\ 8 \tag{4.7}$$

The values α_j and β_j $(j = 1,2...8)$ can be obtained by nears of following relationship:

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{Bmatrix} \alpha_j \\ \beta_j \end{Bmatrix} = \begin{Bmatrix} -d_{13} \\ -d_{23} \end{Bmatrix} \tag{4.8}$$

where $d_{k\ell}$ are the coefficients of matrix [D] given in Appendix A-4.

By introducing expressions (4.6) and (4.7) into (4.2), displacements $U(x,\theta)$, $V(x,\theta)$ and $W(x,\theta)$ can only be expressed in conjunction with the eight C_j constants and may be written as:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T]\ [R]\ \{C\} \tag{4.9}$$

where [R] is a (3x8) matrix given in Appendix A-4 and {C} is an eight orders vector of constants C_j

$$\{C\} = \{C_1 \quad C_2 \quad \ldots C_8\}^T$$

To determine the eight C_j constants, eight boundary conditions for the finite element must be formulated.

Hence, displacements at boundaries $i(x=0)$ and $j(x=\ell)$ (Fig. 3) are expressed by:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{u_i \quad w_i \quad \left.\frac{\partial w}{\partial x}\right)_i \quad v_i \quad u_j \quad w_j \quad \left.\frac{\partial w}{\partial x}\right)_j \quad v_j\}^T \tag{4.10}$$

and $\{C\}$ becomes:

$$\{C\} = [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.11}$$

$[A]$ is an (8x8) matrix given Appendix A-4 and derived from matrix $[R]$ by successively setting $x = 0$ and $x = \ell$.

Substituting (4.11) into (4.9) we obtain

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T]\ [R]\ [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.12}$$

representing the displacement functions.

4.1.2 Symmetric loads ($n = 0$)

The equations of motion (A-2) are given by:

$$\begin{aligned} L_1(U,V,W,P_{ij}) &= 0 \\ L_2(U,V,W,P_{ij}) &= 0 \\ L_3(U,V,W,P_{ij}) &= 0 \end{aligned} \tag{4.13}$$

In the particular case of axisymmetric motion, the displacements are only a function of x. All those derived in relation to θ then become zero and equations (4.13) are written as:

$$\begin{aligned} L_1'(U,W,P_{ij}) &= 0 \\ L_3'(U,W,P_{ij}) &= 0 \\ L_2'(V,P_{ij}) &= 0 \end{aligned} \tag{4.14a-c}$$

(these equations are given in Appendix A-2).

Two systems of equations have been obtained: one, the so-called torsional system, is represented by equation (4.14c) and is a function of V only; the other is a non-torsional method, represented by the two equations (4.14a-b) which are functions of U and W.

a) Non-torsional system

Setting:

$$u(x) = Ae^{\lambda x/r} \qquad w(x) = Ce^{\lambda x/r} \tag{4.15}$$

and substituting (4.15) in (4.14a-b), we obtain two homogeneous equations which are functions of constants A and C:

$$[D_0]\begin{Bmatrix} A \\ C \end{Bmatrix} = 0 \tag{4.16}$$

Proceeding analagously in this fashion, the case of an arbitrary load becomes the characteristic equation:

$$|D_0| = h_4'\lambda_4^4 - h_2'\lambda^2 + h_0' = 0 \tag{4.17}$$

The values of coefficient h_j' in this polynomial are given in Appendix A-4.

Setting $A_j = \alpha_j C_j$ and substituting them in (4.15) we obtain:

$$\begin{Bmatrix} U(x) \\ W(x) \end{Bmatrix} = [R_0]\,\{C\} \tag{4.18}$$

where $[R_0]$ is a (2x4) matrix given in Appendix A-4; $\{C\}$ is a fourth order vector for constant C_j

$$\{C\} = \{C_1 \quad C_2 \ldots C_4\}^T$$

The values for α_j are given by:

$$\alpha_j = \left(\frac{P_{14}}{r}\lambda_j - \frac{P_{12}}{\lambda_j}\right)/P_{11} \qquad (j = 1,2..4) \tag{4.19}$$

To determine the four C_j constants, four boundary conditions for each element must be formulated. To this end, displacements for boundaries i $(x = 0)$ and $j(x = \ell)$ are expressed by:

$$\{\Delta\} = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [A_0]\,\{C\} \tag{4.20}$$

where $\{\Delta\}$ can be determines by one of the two models below:

Model I

$$\{\Delta\} = \left\{ W_i \left.\frac{\partial W}{\partial x}\right)_i W_j \left.\frac{\partial W}{\partial x}\right)_j \right\}^T$$

Model II

$$\{\Delta\} = \left\{ u_i \left.\frac{\partial W}{\partial x}\right)_i u_i \left.\frac{\partial W}{\partial x}\right)_j \right\}^T$$

The $[A_0]$ matrices are given in Appendix A-4 and vector $\{C\}$ is given by:

$$\{C\} = [A_0^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.21}$$

Substituting (4.21) into (4.18) we obtain:

$$\begin{Bmatrix} U(x) \\ W(x) \end{Bmatrix} = [R_0]\ [A_0^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N_0] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.22}$$

which represents the displacement functions.

b) Torsional system

The displacement function can be represented by:

$$V(x) = B_0 + B_1 x, \text{ such that:} \tag{4.23}$$

$$\{V(x)\} = [R_0'] [A_0^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N_0'] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.24}$$

where

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \begin{Bmatrix} V_i \\ V_j \end{Bmatrix} = [A_0'] \{B\} \tag{4.25}$$

and matrices $[A_0']$ and $[R_0']$ are given in Appendix A-4.

CHAPTER 5

MATRIX CONSTRUCTION

This chapter will deal with determination of the mass and stiffness matrices and the method of constructing global matrices.

5.1 Arbitrary load in beam like conditions (n = 1)

5.1.1 Strain-displacement relationships

By using equations (3.2) and (4.12), vector $\{\varepsilon\}$ takes the form of:

$$\{\varepsilon\} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} [Q]\,[A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{5.1}$$

where matrices $[T]$, $[A]$ and $[Q]$ are given in Appendix A-4.

5.1.2 Stress-strain relationships

By using equations (3.3) and (5.1), the stress is given by:

$$\{\sigma\} = \begin{Bmatrix} N_x \\ N_\theta \\ \bar{N}_{x\theta} \\ M_x \\ M_\theta \\ \bar{M}_{x\theta} \end{Bmatrix} = [P]\,\{\varepsilon\} = [P]\,[B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [ST] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{5.2}$$

5.1.3 Mass and stiffness matrices

The mass and stiffness matrices can be expressed as

$$[k] = \iint [B]^T \; [P] \; [B] \; dA$$

$$[m] = \rho t \iint [N]^T \; [N] \; dA \qquad (5.3)$$

where $dA = r\,d\theta dx$

Matrices $[P]$, $[N]$ and $[B]$ are determined in (3.4), (4.12) and (5.1); substituting them in (5.3) and integrating them with respect to θ, we obtain:

$$k = [A^{-1}]^T \; [\pi r \int_0^{\ell} [Q]^T \; [P] \; [Q] \; dx] \; [A^{-1}] = [A^{-1}]^T \; [G] \; [A^{-1}] \qquad (5.4)$$

$$[m] = \rho t \; [A^{-1}]^T \; [\pi r \int_0^{\ell} [R]^T \; [R] \; dx] \; [A^{-1}] = \rho t \; [A^{-1}]^T \; [S] \; [A^{-1}] \qquad (5.5)$$

The elements of matrices $[G]$ and $[S]$ are determined in conjunction with elements $[P]$, λ and α.

For $\lambda_j + \lambda_i \neq 0$

$$G(i,j) = \frac{\pi r^2}{\lambda_i + \lambda_j} [e^{(\lambda_i + \lambda_j)\ell/r} - 1] \, Z(i,j)$$

$$S(i,j) = \frac{\pi r^2}{\lambda_i + \lambda_j} [e^{(\lambda_i + \lambda_j)\ell/r} - 1] \, \Gamma(i,j) \tag{5.6a-b}$$

and $\lambda_i + \lambda_j = 0$

$G(i,j) = \pi r \ell \times Z(i,j)$

$S(i,j) = \pi r \ell \times \Gamma(i,j)$

where

$$\begin{aligned} Z(i,j) = {} & P_{11} a_i a_j + (P_{12} + P_{15}/r)\,(a_i b_j + a_j b_i) - P_{14}(\lambda_i^2 a_j \\ & + \lambda_j^2 a_i)/r^2 + b_i b_j\,(P_{22} + 2P_{25}/r + P_{55}/r^2) - (P_{24} \\ & + P_{45}/r)\,(\lambda_i^2 b_j + \lambda_j^2 b_i)/r^2 + P_{36}(c_i d_j + c_j d_i) \\ & + P_{33} c_i c_j + P_{66} d_i d_j + P_{44} \lambda_i^2 \lambda_j^2 / r^4 \end{aligned}$$

$$\Gamma(i,j) = (\alpha_i \alpha_j + \beta_i \beta_j + 1) \tag{5.7a-b}$$

where

$$a_i = \alpha_i \lambda_i / r \qquad b_i = (\beta_i + 1)/r \qquad i,j = 1,2..8$$

$$d_i = (2\lambda_i + 3\beta_i \lambda_i /2 + \alpha_i /2)/r^2$$

$$c_i = (\beta_i \lambda_i - \alpha_i)/r$$

5.2 Symmetric load (n = 0)

Proceeding analogously to the case involving arbitrary loads, we set:

5.2.1 Non-torsional

for $\lambda_i + \lambda_j \neq 0$

$$G_0(i,j) = \frac{2\pi r^2}{\lambda_i + \lambda_j} [e^{(\lambda_i + \lambda_j)\ell/r} - 1] \, Z_0(i,j)$$

$$S_0(i,j) = \frac{2\pi r^2}{\lambda_i + \lambda_j} [e^{(\lambda_i + \lambda_j)\ell/r} - 1] \, \Gamma_0(i,j) \qquad (5.8a\text{-}b)$$

and $\lambda_i + \lambda_i = 0$

$$G_0(i,j) = 2\pi r\ell Z_0(i,j)$$

$$S_0(i,j) = 2\pi r\ell \Gamma_0(i,j)$$

where

$$Z_0(i,j) = P_{11}a_i a_j + P_{12}(a_i + a_j)/r - P_{14}(\lambda_i^2 a_j + a_i\lambda_j^2)/r^2$$
$$+ P_{22}/r^2 - P_{24}(\lambda_j^2 + \lambda_i^2)/r^3 + P_{44}\lambda_i^2\lambda_j^2/r^4$$

and

$$\Gamma_0(i,j) = (\alpha_i\alpha_j + 1)$$

with

$$\alpha_i = \alpha_i\lambda_i/r \qquad i,j = 1,2..4$$

5.2.2 Torisonal

$$k(1,1) = k(2,2) = -k(1,2) = -k(2,1) = 2\pi rh/\ell$$

$$m(1,1) = m(2,2) = 2m(1,2) = 2m(2,1) = 2\pi r\ell\rho t/3 \qquad (5.10a\text{-}b)$$

where

$$h = (P_{33} + 9P_{66}/4r^2 + 3P_{63}/r)$$

Comments:

The complete expansion of n = 0 is given in Appendix A-3.

5.3 Global mass and stiffness matrices for the shell

The shell is subdivided into several cylindrical elements. The mass and stiffness matrices of each element are assembled in such a way that equilibrium forces and continuity of the displacements at each node must be satisfied. The vectors $\{F_i\}$ and $\{F_j\}$ represent the internal forces active on nodes (i, j) and $\{\delta_i\}$ and $\{\delta_j\}$ are the displacements associated with F_i and F_j. The sum of the forces and the moments of a node must be equal, respectively, to the sum of the external forces and the sum of the moments applied at this node.

$$\{F\}^e = F_j + F_{i+1}$$

and $\delta_j = \delta_{i+1}$

Through these relationships, the mass and stiffness of each element can be superimposed to yield the global mass and stiffness matrices of the shell. $\{K_0\}$ and $[M_0\}$ are square matrices of the order NDF (N + 1), where N is the number of finite elements and NDF is the number of degrees of freedom at each node. This is schematically represented in Figure 5.

CHAPTER 6

FREE VIBRATION OF CYLINDRICAL SHELLS PARTIALLY FILLED WITH LIQUID

This chapter presents the final formulations of the equations of motion for cylindrical shells partially filled with liquid, by using the relationships established in this report together with the ones formulated in reference [22]

6.1 Equations of motion

The dynamic behaviour of a shell subjected to a pressure field can be represented by the following system:

$$\{[M_0] - [M'_f]\}\{\ddot{\delta}\} + \{[C_0] - [C_f]\}\{\dot{\delta}\} + \{[K_0] - [K'_f]\}\{\delta\} = \{F\} \tag{6.1}$$

where

$\{\delta\}$ is the displacement vector; $[M_0]$ and $[K_0]$ are, respectively, the mass and stiffness matrices of "in vacuo" systems, $[M'_f]$, $[C'_f]$ and $[K'_f]$ represent the inertial, Coriolis and centrifugal forces of the liquid flow; $[C_0]$ is the system damping matrix and the external forces $\{F\}$ represent the random pressure field induced by the boundary layer [22].

6.2 Inertial, Coriolis and centrifugal forces of the flow

It is considered here that the shell is subjected to only a potential flow which induces inertial, Coriolis and centrifugal forces participating in the vibration pattern. These forces are coupled with the elastic deformation of the shell.

The mathematical model used is based on the following hypothesis [22]:

a) The liquid flow is potential;

b) Vibration is linear (small deformation);

c) Pressure on the wall is purely lateral;

d) The velocity distribution of the liquid is assumed constant through out the shell section and finally;

e) The fluid is incompressible.

6.2.1 Determination of the apparent mass, stiffness and damping matrices of the liquid flow

The potential flow is governed by the equation:

$$\nabla^2 \Phi = \frac{1}{c^2}\left[\frac{\partial^2 \Phi}{\partial t^2} + 2U_x \frac{\partial^2 \Phi}{\partial x \partial t} + U_x^2 \frac{\partial^2 \Phi}{\partial x^2}\right] \tag{6.2}$$

where

c is the speed of sound in the fluid; U_x is the velocity of the liquid through the shell section and Φ is the potential function that represents potential velocity.

$$V_x = U_x + \frac{\partial\Phi}{\partial x} \; ; \; V_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} \; ; \; V_r = \frac{\partial\Phi}{\partial r} \tag{6.3}$$

where

V_x, v_θ and V_r are the axial, tangential and radial components of the fluid velocity.

The pressures exerted upon the wall are given by:

$$P_i = -\rho_i \left(\frac{\partial\Phi_i}{\partial t} + U_{xi}\frac{\partial\Phi_i}{\partial x}\right)_{r=a}$$

$$P_e = -\rho_e \left(\frac{\partial\Phi_e}{\partial t} + U_{xe}\frac{\partial\Phi_e}{\partial x}\right)_{r=a+t} \qquad (6.4a\text{-}b)$$

where

(a) and (t) are, respectively, the radius and thickness of the shell, and subscripts i and e indicate the internal and external locations of the structure.

Finally, the condition

$$(V_r)_{r=a} = \left(\frac{\partial \Phi}{\partial r}\right)_{r=a} = \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x}\right)_{r=a} \tag{6.5}$$

must be satisfied in order to obtain the contact between the shell's surface and the peripherial layer of the fluid.

Assuming that the form of the displacement functions is given by equation (4.12) and setting:

$$\Phi(x,\theta,r,t) = \sum_{q=1}^{8} R_q(r)\ S_q(x,\theta,t) \tag{6.6}$$

we obtain the pressure exerted on the wall as follows (for more detailed information, the reader is refered to references [22,23] or to Appendix A-5):

$$P_t = \sum_{q=1}^{8} \{[-a_i\rho_i r_q + a_e\rho_e s_q]\ \ddot{W}_q + 2\ [-\rho_i a_i U_{xi} r_q + \rho_e a_e U_{xe} s_q]\ \dot{W}'_q$$
$$+ [-a_i\rho_i U_{xi}^2 r_q + a_e\rho_e U_{xe}^2 s_q]\ W''_q\} \tag{6.7}$$

where

$W_q(x,\theta,t) = C_q\, e^{(\lambda_q x/a + i\omega t)} \cos n\theta$ is given by (4.12); $\rho_{i,e}$ is the density of the fluids; ω is the natural frequency and

$$r_q = 1/[n - m_q a \left(J_{n+1}(m_q a)/J_n(m_q a)\right)]$$

$$z_q = 1/[n - m_q a \left(Y_{n+1}(m_q a)/Y_n(m_q a)\right)] \tag{6.8}$$

$$m_q a = \left[\lambda_q^2 - (a^2/c^2)\left(\frac{\lambda_q U_x}{a} + i\omega\right)^2\right]^{\frac{1}{2}}$$

$J_n(m_q a)$ and $Y_n(m_q a)$ are, respectively, Bessel functions of the first and second kind and of order n.

Substituting (4.1) into equation (6.7), using the finite element method and then integrating them for x and θ, we obtain the inertial, Coriolis and centrifugal forces as follows [22]:

$$\underset{8\times 8}{[m_f]} = [A^{-1}]^T [S_f] [A^{-1}]$$

$$\underset{8\times 8}{[c_f]} = [A^{-1}]^T [D_f] [A^{-1}] \tag{6.9}$$

$$\underset{8\times 8}{[k_f]} = [A^{-1}]^T [G_f] [A^{-1}]$$

$$S_f(k,q) = -\pi\ \delta_i\ \gamma_i^2\ r_q\ I_{kq}\ (\ell,a_i) + \pi\ \delta_e\ \gamma_e^2\ s_q\ (\ell,a_e)$$

$$D_f(k,q) = 2\lambda_q\pi\ [-\delta_i\ \overline{U}_i\ \gamma_i\ r_q\ I_{kq}\ (\ell,a_i) + \delta_e\ \overline{U}_e\ \gamma_e\ s_q\ (\ell,a_e)] \qquad (6.10)$$

$$G_f(k,q) = \lambda_q^2\pi\ [-\delta_i\ \overline{U}_i^2\ r_q\ I_{kq}\ (\ell,a_i) + \delta_e\ \overline{U}_e^2\ s_q\ I_{kq}\ (\ell,a_e)]$$

The non-dimensional terms in equations (6.10) are determined by:

$$\delta_i = (a_i/t_1)\ (\rho_i/\rho_1) \quad ; \quad \delta_e = (a_e/t_1)\ (\rho_e/\rho_1)$$

$$\overline{U}_0^2 = P(1,1,1)/\rho_1 t_1 \quad ; \quad \overline{U}_i = U_{xi}/U_0 \quad ; \quad \omega_0 = U_0/r_1 \qquad (6.11)$$

$$\overline{U}_e = U_{xe}/U_0 \quad ; \quad \gamma_i = a_i/r_1 \quad ; \quad \gamma_e = a_e/r_1$$

p_1, t_1 and r_1 are respectively, the radius, thickness and density of the first element of the shell. The element p (1,1,1,) is the first term of elasticity matrix [P].

$$I_{kq}(\ell,a_{i,e}) = \frac{1}{(\lambda_q + \lambda_k)}\ [e^{(\lambda_q + \lambda_k)\ell/a_{i,e}} - 1] \quad \text{for} \quad \lambda_q + \lambda_k \neq 0$$

$$I_{kq}(\ell,a_{i,e}) = \ell/a_{i,e} \quad \text{for} \quad \lambda_q + \lambda_k = 0$$

Matrices $[M_f]$, $[C_f]$ and $[K_f]$ are, respectively, the global mass, stiffness and damping matrices of the fluid column, which are obtained by overlaying the mass $[m_f]$, stiffness $[k_f]$ and damping $[c_f]$ matrices for each element.

Note: The foregoing development given for the n = 1 case is still valid for n = 0, with the following changes:

The matrices $[S_f]$, $[D_f]$ will be remultiplied by two and k,q = 1,2..4. Matrix [A] will be given by [A].

6.3 Eigen value and eigen vector problems

For free vibration, equation (6.1) may be rewritten as follows:

$$\begin{bmatrix} 0 & \frac{1}{\omega_0} M \\ \frac{1}{\omega_0^2} M & \frac{1}{\omega_0} C \end{bmatrix} \begin{Bmatrix} \ddot{\delta} \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{\omega_0} M & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} \dot{\delta} \\ \delta \end{Bmatrix} = \{0\} \tag{6.13}$$

where

$$\begin{aligned} &[M] = [M_t] - [M_f] \;;\; [K] = [K_t] - [K_f] \\ &[C] = [C_f] \\ &[M_t] = [M_0]/\rho_1 t_1 r_1^2 \;;\; [K_t] = [K_0]/P(1,1,1) \;; \\ &\omega_0^2 = P(1,1,1)/\rho_1 t_1 r_1^2 . \end{aligned} \tag{6.14}$$

and the problem for eigen values will be given by:

$$\left| [DD] - \Lambda[I] \right| = 0$$

where

$$[DD] = \begin{bmatrix} 0 & I \\ -\frac{1}{\omega_0^2} K^{-1} M & -\frac{1}{\omega_0} K^{-1} C \end{bmatrix} \tag{6.15}$$

$\Lambda = \frac{1}{1\omega}$ (ω is the natural frequency of the system).

<u>Particular cases</u>

If the shell is partially or completely filled with liquid ($U_X = 0$), the eigen value and eigen vector problem is restated as:

$$\left| \frac{1}{\omega_0^2} K^{-1} M - \Lambda[I] \right| = 0 \tag{6.16}$$

and $\omega(rd/sec) = \sqrt{1/\Lambda}$

Matrices [K], [M] and [C] are square matrices of order NDF (N + 1), where NDF is the number of degrees of freedom at each node and N is the number of finite elements in the shell.

CHAPTER 7

CALCULATION METHOD

7.1 Calculation method and the computer program

The non-uniform cylindrical shell is subdivided into a sufficient number of finite elements. Calculation of eigenvalues and eigenvectors is done with the help of a computer program, which determines the mass and stiffness matrices of an element, assembles the global matrices of the system and calculates the free vibrations and corresponding principle modes.

The program is written in Fortran IV and is performed on a CDC (Syber Model 173). The flow chart for the principal program is given in Figure 6.

a) The program is composed of finite elements with the radius, thickness and length of each element defined, as are the mechanical properties and the harmonic number in order to calculate ($n = 0$ or 1).

b) The program proceeds as follows for each finite element:

(i) The roots of the characteristic equation of λ_j ($j = 1,2...8$) for $n = 1$ are calculated by the Newton-Raphson iterative method and then α_j and β_j are obtained.

(ii) The intermediate matrices $[A]$, $[G]$ and $[S]$ are calculated as, given respectively, by equations (A-4), (5.6a-b).

(iii) Mass $[m]$ and stiffness $[k]$ matrices are determined by equations (5.4) and (5.5), respectively.

c) The mass and stiffness matrices for the entire system are assembled as described in section 5.3.

d) The boundary conditions are applied; $[K_0]$ and $[M_0]$ are now reduced to square matrices of order NDF (N + 1) - j, where j is the number of constraint equations imposed. The geometric boundary conditions are simply specified: hence, for a shell free at the ends, j = 0; for a shell simply supported (V = W = 0), j = 4, and for a shell clamped at both ends, j = 8.

e) The natural frequencies ωi and the corresponding modes of a real, non symmetric square matrix of the form $[M_0^{-1}]_{red}$ $[K_0]$ are obtained. Where i = 1,2.. NDF (N + 1) - j, $[K_0]_{red}$ $[M_0]$ are real symmetric matrices. The calculations are done with the help of an EIGZE sub routine from the IMSL catalog. The frequencies and corresponding modes are real.

f) For the liquid component:

(i) Matrices $[S_f]$, $[G_f]$ and $[D_f]$, which are given by system equations (6.10), are calculated.

(ii) Matrices $[m_f]$, $[k_f]$ and $[c_f]$ are then determined, as given by (6.9), for each element of the fluid column.

(iii) These matrices are superimposed on to the mass and stiffness matrices of the empty shell.

(iv) The frequencies and principle modes are obtained by solving the system equation (6.15) where [K], [M] and [C] are square matrices of order NDF (N + 1) - j. Two sub routines, HSVEC and HESSEIN from the IMSL catalog to the calculation.

CHAPTER 8

CALCULATIONS AND DISCUSSION

The main purpose of this study was to compare results obtained with the methods proposed with results from other classical numerical and experimental methods for different boundary conditions.

The first section was devoted for determining the eigen values and eigen vectors of the empty uniform cylindrical shells. The second section consisted of a study of the natural vibration of cylindrical shells, partially or completely filled with liquid.

8.1 Natural vibration of a cylindrical shell in vacuo

The first example of calculations to determine the natural frequencies and corresponding modes for a cylindrical shell simply supported at both ends was the analysis done by Michalopoulos and Muster [28] and also by Baron and Bleich [9]. This shell had the following proprieties:

$L = 18.65$ in., $t = 0.047$ in., $r = 4.08$ in., $E = 29.5 \times 10^6$

lb/in^2; $\rho = 0.734 \times 10^{-3}$ lb - sec^2/in^4 et $\nu = 0.3$

we calculated the natural frequencies of this shell using our method and compared results with what Michalopoulos and Must [28] obtained for $n = 0,1$ and $m = 1$ (Table 1).

Baron and Bleich [9] based their theory on the energy method, treating the shell as a membrane, and introducing a correction factor to take the bending into consideration. By using the energy method, and Lagrange equations, and by expressing the displacements in Fourier series, Michalopoulos and Muster [28] derived the equations of motion in matrix. The natural frequencies were obtained by a Jacobi integration method form.

In our case, the shell was subdivided into ten equal finite elements. The results from this method were in strong agreement with results obtained by the other theories, in particular in Michalopoulos and Muster [28].

In this particular case, the first natural frequencies associated with $n = 0$ and $n = 1$ were, respectively, 8 and 4 times higher than the lowest frequency (436 Hz) which was associated with ($n \geqslant 2$), which was due to a very high level of dissipation energy deformation for $n = 0$ and $n = 1$. This phenomenon has already been discussed in detail in references [8,33 and 34].

The normalized eigenvectors were calculated and are presented in Figure 7. In this case, radial motion was dominant for $n \geqslant 2$ [28] and circumferential motion dominant for $n \leqslant 2$ (see table in Figure 7).

The second of example calculation determine the number of finite elements required to correspond to the appropriate natural frequencies.

The shell under study here was the same as the one in the first example for $n = 0,1$, and the number of finite elements varied from 2 to 10. the results obtained are presented in Figures 8a ($m = 1$) and 8b ($m = 2$ and 3). For $m = 2$ and $m = 3$, the natural frequencies converged more rapidly.

The third example consisted of determining the natural frequencies of a cylindrical clamped-free shell whose proprieties were as follows:

$r/t = 100$, $r/L = 0.448$, $r = 10.16$ cm, $\nu = 0.3$, $g = 980$ cm/sec^2, $\rho = 7.84 \times 10^{-3}$ kg/cm^3 et $E = 2.11 \times 10^6$ kg/cm^2.

the results obtained are given in Table 2 where they are compared with results from other dynamic discretization methods used by Tottenham and Shimizu [29] and Sankran [30].

Following Donnell's equations and expressing the displacements in series form, Sakran [30] resolved the equations of motion by using the numerical integration technique. Tottenham and Shimizu [29] used the progression method and Flügge's theory.

In this case, the shell was subdivided into ten equal finite elements and the results obtained with our method were in agreement with other numerical methods ([29] and [30]), especially for low modes, but increasing differences showed up with higher modes, which were dependent upon the degree of "discretization" and the number of elements chosen. In Sankran's article [30], seven points of integration were used.

The fourth example of calculations consisted of determining the frequency parameters (Ω) for different values of r/t and L/r for different boundary conditions of a cylindrical shell.

The results obtained with our method (10 elements) were compared with Baron and Bleich's [9] results for a simply supported shell, and with Sharma and John's [31] findings for a clamped-free shell. The Sharma and Johns method was based on Flügge's theory, and Rayleigh-Ritz's method. The frequency parameters (Ω) are indicated in Tables 3 and 4 for $n = 1$ and the first five axial modes. Table 5 describes the results for $n = 0$ and $m = 1$.

For $n = 1$, the natural frequencies increased with increasing axial mode (m) for all the parameter values of r/t and L/r and for different boundary conditions (see Tables 3 and 4 and Figure 9, respectively).

The natural frequencies for $n = 0$ and $n = 1$ essentially depended upon ratio L/r. The r/t ratio was only weakly affected, especially for $L/r \geqslant 4$ and $n = 1$ (Figure 10) and $L/r \geqslant 2.5$ for $n = 0$ (Table 5). This can be accounted for by the fact that the deformation energy of the membrane was more relevant than the energy due to the curvature [8]. Membrane theory was applied in this case [9].

For $n = 1$ and $m = 1$, the motion's radial and circumferential amplitudes were almost identical and greater than longitudinal amplitudes for $L/r \geqslant 3$. In the case of $L/r \leqslant 3$, radial motion was dominant (Figure 11). For $n = 0$ and $m = 1$, circumferential motion dominated when $L/r \geqslant 2$ and radial motion dominated when $L/r \leqslant 2$.

8.2 Free vibration of cylindrical shells partially or completely filled with liquid

For the first set of calculations, it was necessary to determine the frequency parameters (Ω) for different values of r/t and L/r and the different boundary conditions for a shell completely filled with liquid.

The results obtained (10 elements) for $n = 1$ and the first five axial modes are indicated in Table 3 in the case of a simply supported shell, and in Table 4 for a clamped-free shell.

We concluded that the frequency parameters (Ω) depended both on L/r and on r/t, (contrary to the empty shell case where the r/t ratio was only weakly supported (Figure 13)), as a result of the lateral pressure of the liquid being exerted on the structure.

The particular case of a cylindrical shell completely filled with liquid and simply supported at both ends was analyzed by Niordson [16]. The shell had the following properties:

$r/t = 60$, $L/r = 24.98$, $r = 35.43$in., $\nu = 0.3$, $E = 29.5 \times 10^6$ lb/in^2, $\rho = 0.734 \times 10^{-3}$ lb-sec^2/in^4 et $\rho_\ell = 0.935 \times 10^{-4}$ lb-sec^2/in^4.

The results obtained (10 elements) with our method for $n = 1$ and the first four modes were in accordance with Niordsons [16] results. (Table 6).

Experimental results were obtained by lindoholm and Kana [32] for a clamped-free cylindrical shell partially filled with liquid having the properties:

$L = 14.95$in., $r = 1.485$in., $t = 0.01$in., $E = 29.5 \times 10^6$ lb/in^2, $\rho = 0.734 \times 10^{-3}$ lb-sec^2/in^4 et $\rho_\ell = 0.935 \times 10^4$ lb-sec^2/in^4.

Our results (10 elements) as shown in Figure 14, were in agreement with the experimental results above.

Finally, a calculation example was studied in the case of $n = 0$ and $n = 1$ for a cylindrical shell simply supported at both ends and partially filled with liquid. The shell proprieties were:

$L/r = 2$, $r/t = 100$, $t = 0.01$in., $E = 29.5 \times 10^6$ lb/in^2

$\nu = 0.3$, $\rho = 0.734 \times 10^{-3}$ lb-sec^2/in^2 et $\rho_\ell = 0.935 \times 10^{-4}$

lb-sec^2/in^4.

The results obtained for $b/L = 0.0$, 1/5, 2/5, 1/2, 4/5 and 1 are presented in Figure 15. For $n = 1$ and $m = 1,2,3$, the natural frequencies decreased considerably with the variation in liquid level. In the case where $n = 0$ and $m = 1,2,3$, the natural frequencies decreased rapidly for $b/L \leqslant 0.4$.

The normalized eigen vectors are presented in Figure 16 for $n = 1$ and $m = 1$. For the empty and full shell, the eigen vectors were identical to a half-period of "sin" and "cos". With the dropping level of liquid, however, longitudinal displacement tended towards zero at the base of the shell, while at the same time, the peaks of the radial and circumferential displacement curves were tending towards the shell base.

Above the liquid level, the radial and longitudinal displacements tended to move rapidly towards zero for $n = 0$.

CHAPTER 9

CONCLUSION

The present report developed a particular method for determining the natural frequencies and corresponding modes of anisotropic cylindrical shells, partially or completely filled with liquid. It was a hybrid method based upon classical thin shell theory and the finite element method.

A cylindrical finite element was used, contrary to the earlier practice of using triangular or rectangular elements. Consequently the derivations of the displacement functions for classical shell theory were formulated and hence the mass and stiffness matrices of each element were obtained.

Only the axisymmetric ($n = 0$) and beam-like cases ($n = 1$) were dealt with in this paper. In the axisymmetric case ($n = 0$), two systems of displacement functions were derived for the torsional and non-torsional motions. The equations of motion derived from classical shell theory were solved to obtain functions which adequately represented real displacements. Then, we used the finite element method to obtain the mass and stiffness matrices of an element in vacuo and then partially filled with liquid. This method was developed in references [21], [22] and [23] for cylindrical shells with a circumferential mode equal to or greater than 2 ($n \geq 2$).

The eigen values and eigen vectors of a shell were determined with the help of a computer program. The results obtained with this numerical method were in several cases in accordance with those from other classical numerical or experimental theories. The method used in this research may be applied to the static and dynamic analysis of axisymmetric non-uniform shells and more precisely to free or forced vibration problems [22].

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APPENDIX A

APPENDIX A-1

SANDERS SHELL THEORY

a) General equations of equilibrium

The development of equations for the static equilibrium of thin shells has been the focus of several books and publications [4,6]. For our purposes here, we shall limit ourselves to the final five equations of motion only, as given by Sanders [4] in the form (Figure 1):

$$\frac{\partial A_2 N_1}{\partial \zeta_1} + \frac{\partial A_1 \bar{N}_{12}}{\partial \zeta_2} + \bar{N}_{12} \frac{\partial A_1}{\partial \zeta_2} - N_2 \frac{\partial A_2}{\partial \zeta_1} + \frac{A_1 A_2 Q_1}{R_1} + \frac{A_1}{2} \frac{\partial}{\partial \zeta_2} \left[\left(\frac{1}{R_1} - \frac{1}{R_2} \right) \bar{M}_{12} \right] = 0 \tag{a}$$

$$\frac{\partial A_2 \bar{N}_{12}}{\partial \zeta_1} + \frac{\partial A_1 N_2}{\partial \zeta_2} + \frac{\partial A_2}{\partial \zeta_1} \bar{N}_{12} - \frac{\partial A_1}{\partial \zeta_2} N_1 + \frac{A_1 A_2}{R_2} Q_2 + \frac{A_2}{2} \frac{\partial}{\partial \zeta_1} \left[\left(\frac{1}{R_2} - \frac{1}{R_1} \right) \bar{M}_{12} \right] = 0 \tag{b}$$

$$\frac{\partial A_2 Q_1}{\partial \zeta_1} + \frac{\partial A_1 Q_2}{\partial \zeta_2} - \left(\frac{N_1}{R_1} + \frac{N_2}{R_2} \right) A_1 A_2 = 0 \qquad \text{(A-1.1)} \quad \text{(c)}$$

$$\frac{\partial A_2 M_1}{\partial \zeta_1} + \frac{\partial A_1 \overline{M}_{12}}{\partial \zeta_2} + \overline{M}_{12} \frac{\partial A_1}{\partial \zeta_2} - M_2 \frac{\partial A_2}{\partial \zeta_1} - A_1 A_2 Q_1 = 0 \qquad \text{(d)}$$

$$\frac{\partial A_2 \overline{M}_{12}}{\partial \zeta_1} + \frac{\partial A_1 M_2}{\partial \zeta_2} + \overline{M}_{12} \frac{\partial A_2}{\partial \zeta_1} - M_1 \frac{\partial A_1}{\partial \zeta_2} - A_1 A_2 Q_2 = 0 \qquad \text{(e)}$$

$$\text{with} \begin{cases} \overline{N}_{12} = \frac{1}{2} (N_{12} + N_{21}) \\ \overline{M}_{12} = \frac{1}{2} (M_{12} + M_{21}) \end{cases}$$

b) <u>Deformation vectors</u>

Deformation vector $\{\varepsilon\}$ is given by:

$$\varepsilon_1 = \frac{1}{A_1} \frac{\partial U_1}{\partial \zeta_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \zeta_2} U_2 + \frac{W}{R_1}$$

$$\varepsilon_2 = \frac{1}{A_2} \frac{\partial U_2}{\partial \zeta_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \zeta_1} U_1 + \frac{W}{R_2} \qquad \text{(A-1.2)}$$

$$\overline{\varepsilon}_{12} = \frac{1}{2A_1 A_2} \left(A_2 \frac{\partial U_2}{\partial \zeta_1} + A_1 \frac{\partial U_1}{\partial \zeta_2} - \frac{\partial A_1}{\partial \zeta_2} U_1 - \frac{\partial A_2}{\partial \zeta_1} U_2 \right)$$

$$\kappa_1 = \frac{1}{A_1} \frac{\partial \beta_1}{\partial \zeta_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \zeta_2} \beta_2$$

$$\kappa_2 = \frac{1}{A_2} \frac{\partial \beta_2}{\partial \zeta_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \zeta_1} \beta_1$$

with
$$\begin{cases} \beta_1 = \dfrac{U_1}{R_1} - \dfrac{1}{A_1}\dfrac{\partial W}{\partial \zeta_1} \\[2ex] \beta_2 = \dfrac{U_2}{R_2} - \dfrac{1}{A_2}\dfrac{\partial W}{\partial \zeta_2} \end{cases}$$

c) Boundary conditions

The boundary conditions are given by:

$$\begin{aligned}
&\frac{M_1}{R_1} + N_1 = \overline{\overline{N}}_1 && \text{or} \quad U_1 = \overline{\overline{U}}_1 \\
&\overline{N}_{12} + \left(\frac{3}{R_2} - \frac{1}{R_1}\right)\frac{\overline{M}_{12}}{2} = \overline{\overline{T}}_{12} && \text{or} \quad U_2 = \overline{\overline{U}}_2 \\
&Q_1 + \frac{1}{A_2}\frac{\partial M_{12}}{\partial \zeta_2} = \overline{\overline{V}}_1 && \text{or} \quad W = \overline{\overline{W}} \\
&M_1 = \overline{\overline{M}}_1 && \text{or} \quad \frac{W}{\partial \zeta_1} = \frac{\overline{\overline{W}}}{\partial \zeta_1}
\end{aligned} \tag{A-1.3}$$

For a constant boundary $a\zeta_1$, where the double barred terms correspond to boundary values.

d) <u>The parameters for a cylindrical shell of revolution (Fig. 2 and 3) give us:</u>

$$\zeta_1 = x \qquad U_1 = U \qquad R_1 = \infty \qquad A_1 = 1$$
$$\zeta_2 = \theta \qquad U_2 = V \qquad R_2 = r \qquad A_2 = r \tag{A-1.4}$$

Substituting these parameters into the five equations of equilibrium (A-1.1), we obtain:

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial \overline{N}_{x\theta}}{\partial \theta} - \frac{1}{2r^2}\frac{\partial \overline{M}_{x\theta}}{\partial \theta} = 0 \qquad \text{(a)}$$

$$\frac{\partial \overline{N}_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{1}{2r}\frac{\partial \overline{M}_{x\theta}}{\partial x} + \frac{1}{r}Q_\theta = 0 \qquad \text{(b)}$$

$$\frac{\partial Q_x}{\partial x} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} - \frac{1}{r}N_\theta = 0 \qquad \text{(A-1.5)} \quad \text{(c)}$$

$$\frac{\partial M_x}{\partial x} + \frac{1}{r}\frac{\partial \overline{M}_{x\theta}}{\partial \theta} - Q_x = 0 \qquad \text{(d)}$$

$$\frac{\partial \overline{M}_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} - Q_\theta = 0 \qquad \text{(e)}$$

APPENDIX A-2

EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin cylindrical shell, which were referenced in the various chapters of this report.

a) Sander's equations of motion (2.5)

$$L_1 \ (U,V,W) = 0$$

$$P_{11} \frac{\partial^2 U}{\partial x^2} + \frac{P_{12}}{r} \frac{\partial W}{\partial x} - P_{14} \frac{\partial^3 W}{\partial x^3} + \left(\frac{1}{r} (P_{12} + P_{33}) + \frac{1}{r^2} (P_{15} + P_{36}) - \frac{3}{4r^3} P_{66} \right) \frac{\partial^2 V}{\partial \theta \partial x} + \frac{1}{r^2} \left(P_{33} - \frac{P_{36}}{r} + \frac{P_{66}}{4r^2} \right) \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{r^2} \left(P_{15} + 2P_{36} - \frac{P_{66}}{r} \right) \frac{\partial^3 W}{\partial x \partial \theta^2} = 0$$

$$L_2 \ (U,V,W) = 0$$

$$\frac{1}{r} \left(P_{33} + P_{21} + \frac{P_{36}}{r} + \frac{P_{51}}{r} - \frac{3}{r^2} P_{66} \right) \frac{\partial^2 U}{\partial x \partial \theta} + \frac{1}{r^2} \left(P_{22} + \frac{P_{55}}{r^2} + \frac{2}{r} P_{25} \right) \frac{\partial^2 V}{\partial \theta^2} + \left(P_{33} + \frac{3}{r} P_{36} + \frac{9}{4r^2} P_{66} \right) \frac{\partial^2 V}{\partial x^2} + \frac{1}{r^2} \left(P_{22} + \frac{P_{52}}{r} \right) \frac{\partial W}{\partial \theta} - \frac{1}{r^3} \left(P_{25} + \frac{P_{55}}{r} \right) \frac{\partial^3 W}{\partial \theta^3} - \frac{1}{r} \left(2P_{36} + P_{24} + \frac{3}{r} P_{66} + \frac{P_{54}}{r} \right) \frac{\partial^3 W}{\partial \theta \partial x^2} = 0$$

$$L_3\ (U,V,W) = 0$$

$$-\frac{P_{21}}{r}\frac{\partial U}{\partial x} - \frac{1}{r^2}\left(P_{22} + \frac{P_{25}}{r}\right)\frac{\partial V}{\partial \theta} - \frac{P_{22}}{r^2} W + P_{41}\frac{\partial^3 U}{\partial x^3} + \frac{1}{r^2}\left(P_{51}\right.$$

$$\left. + 2P_{63} - \frac{P_{66}}{r}\right)\frac{\partial^3 U}{\partial x \partial \theta^2} + \frac{1}{r^3}\left(P_{52} + \frac{P_{55}}{r}\right)\frac{\partial^3 V}{\partial \theta^3} + \frac{1}{r}\left(P_{42} + 2P_{63}\right.$$

$$\left. + \frac{P_{45}}{r} + \frac{3}{r}P_{66}\right)\frac{\partial^3 V}{\partial \theta \partial x^2} + \frac{2P_{25}}{r^3}\frac{\partial^2 W}{\partial \theta^2} - \frac{P_{55}}{r^4}\frac{\partial^4 W}{\partial \theta^4} + \frac{2}{r}P_{24}\frac{\partial^2 W}{\partial x^2} - P_{44}$$

$$\frac{\partial^4 W}{\partial x^4} - \frac{1}{r^2}(2P_{45} + 4P_{66})\frac{\partial^4 W}{\partial x^2 \partial \theta} = 0$$

b) Sander's simplified equations of motion (2.6)

$$S_1\ (U,V,W) = 0$$

$$P_{11}\frac{\partial^2 U}{\partial x^2} + \frac{P_{12}}{r}\frac{\partial W}{\partial x} + \frac{1}{r}\left(P_{12} + P_{33}\right)\frac{\partial^2 V}{\partial x \partial \theta} - P_{14}\frac{\partial^3 W}{\partial x^3} + \frac{P_{33}}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

$$-\frac{1}{r^2}\left(P_{15} + 2P_{36}\right)\frac{\partial^3 W}{\partial x \partial \theta^2} = 0$$

$$S_2\ (U,V,W) = 0$$

$$\frac{1}{r}\left(P_{21} + P_{33}\right)\frac{\partial^2 U}{\partial x \partial \theta} + \frac{P_{22}}{r^2}\frac{\partial^2 V}{\partial \theta^2} + P_{33}\frac{\partial^2 V}{\partial x^2} + \frac{P_{22}}{r^2}\frac{\partial W}{\partial \theta} - \frac{P_{25}}{r^3}\frac{\partial^3 W}{\partial \theta^3}$$

$$-\frac{1}{r}\left(P_{24} + 2P_{36}\right)\frac{\partial^3 W}{\partial x^2 \partial \theta} = 0$$

$$S_3 \quad (U,V,W) = 0$$

$$-\frac{P_{21}}{r}\frac{\partial U}{\partial x} - \frac{P_{22}}{r^2}\frac{\partial V}{\partial \theta} - \frac{P_{22}}{r^2} W + P_{41}\frac{\partial^3 U}{\partial x^3} + \frac{1}{r^2}\left(P_{51} + 2P_{63}\right)\frac{\partial^3 U}{\partial x \partial \theta^2}$$

$$+ \frac{P_{52}}{r^3}\frac{\partial^3 V}{\partial \theta^3} + \frac{1}{r}\left(P_{42} + 2P_{63}\right)\frac{\partial^3 V}{\partial x^2 \partial \theta} + \frac{2P_{52}}{r^3}\frac{\partial^2 W}{\partial \theta^2} - \frac{P_{55}}{r^4}\frac{\partial^4 W}{\partial \theta^4}$$

$$+ \frac{2}{r} P_{42}\frac{\partial^2 W}{\partial x^2} - P_{44}\frac{\partial^4 W}{\partial x^4} - \frac{2}{r^2}\left(P_{45} + 2P_{66}\right)\frac{\partial^4 W}{\partial x^2 \partial \theta^2} = 0$$

c) <u>Equations of motion (4.14) (n = 0)</u>

$$L_1' \quad (U,W) = 0$$

$$P_{11}\frac{\partial^2 U}{\partial x^2} + \frac{P_{12}}{r}\frac{\partial W}{\partial x} - P_{14}\frac{\partial^3 W}{\partial x^3} = 0$$

$$L_2' \quad (V) = 0$$

$$\left(P_{33} + \frac{2}{r} P_{36} + \frac{9}{4r^2} P_{66}\right)\frac{\partial^2 V}{\partial x^2} = 0$$

$$L_3' \quad (U,W) = 0$$

$$-\frac{P_{21}}{r}\frac{\partial U}{\partial x} - \frac{P_{22}}{r^2} W + P_{41}\frac{\partial^3 U}{\partial x^3} + \frac{2}{r} P_{24}\frac{\partial^2 W}{\partial x^2} - P_{44}\frac{\partial^4 W}{\partial x^4} = 0$$

APPENDIX A-3

MATRIX CONSTRUCTION (n = 0)

In this appendix, the mass, stiffness and intermediate matrices are determines for cases of asixymmetric motion (n = 0).

A-3.1 Non-torsional

a) Strain-displacement relationships

The deformation vector $\{\varepsilon\}$ is given by:

$$\{\varepsilon_0\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \kappa_x \end{Bmatrix} = \begin{Bmatrix} \partial U/\partial x \\ W/r \\ -\partial^2 W/\partial x^2 \end{Bmatrix} \qquad \text{(A-3.1)}$$

Substituting equations (4.22) and (A-3.1), vector $\{\varepsilon\}$ takes the form

$$\{\varepsilon_0\} = [Q_0]\ [A_0^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B_0] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad \text{(A-3.2)}$$

Matrices $[Q_0]$ and $[A_0]$ are given in Appendix A-4.

b) Stress-strain relationships

The resultant stress vector takes the form:

$$\{\sigma_0\} = [P_0]\ \{\varepsilon_0\} \tag{A-3.3}$$

with

$$[P_0] = \begin{bmatrix} P_{11} & P_{12} & P_{14} \\ P_{21} & P_{22} & P_{24} \\ P_{41} & P_{42} & P_{44} \end{bmatrix} \tag{A-3.4}$$

By using equations (A-3.2 and A-3.3), the stress vector is given by:

$$\{\sigma_0\} = \begin{Bmatrix} N_x \\ N_\theta \\ M_x \end{Bmatrix} = [P_0]\ [B_0] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{A-3.5}$$

c) The mass and stiffness matrices

The mass and stiffness matrices can be expressed in the same way as equations (5.3).

$$[k] = \iint [B_0]^T \, [P_0] \, [B_0] \; dA$$
$$[m] = \rho t \iint [N_0]^T \, [N_0] \; dA \qquad \text{(A-3.6)}$$

where $dA = r d\theta \; dx$

Matrices $[N_0]$ $[P_0]$ and $[B_0]$ are determined in (4.22), (A-3.4) and (A-3.5). Substituting them in (A-3.6) and integrating them on θ, we obtain:

$$[k] = [A_0^{-1}]^T \left[2\pi r \int_0^{\ell} [Q_0]^T \, [P_0] \, [Q_0] \; dx \right] [A_0^{-1}] = [A_0^{-1}]^T [G_0] \, [A_0^{-1}] \qquad \text{(A-3.7)}$$

$$[m] = \rho t \, [A_0^{-1}]^T \left[2\pi r \int_0^{\ell} [R_0]^T \, [R_0] \; dx \right] [A^{-1}] = \rho t \, [A_0^{-1}]^T [S_0] \, [A_0^{-1}]$$

The elements of matrices $[G_0]$ and $[S_0]$ are determined as a function of elements of matrix $[P_0]$ λ and α

$$\lambda_i + \lambda_j \neq 0$$

$$G_0\ (i,j) = \frac{2\pi r^2}{(\lambda_i + \lambda_j)} \left(e^{(\lambda_i + \lambda_j)\ell/r} - 1\right) Z_0\ (i,j) \tag{A-3.8}$$

$$S_0\ (i,j) = \frac{2\pi r^2}{(\lambda_i + \lambda_j)} \left(e^{(\lambda_i + \lambda_j)\ell/r} - 1\right) \Gamma_0\ (i,j) \tag{A-3.9}$$

and

$$\lambda_i + \lambda_j = 0$$

$$G_0\ (i,j) = 2\pi r\ell\ Z_0\ (i,j) \tag{A-3.10}$$

$$S_0\ (i,j) = 2\pi r\ell\ \Gamma_0\ (i,j)$$

where

$$Z_0\ (i,j) = P_{11}\ a_i\ a_j + \frac{P_{12}}{r}(a_i + a_j) - \frac{P_{14}}{r^2}(a_i\ \lambda_j^2 + a_j\ \lambda_i^2) + \frac{P_{22}}{r^2} - \frac{P_{24}}{r^3}(\lambda_i^2 + \lambda_j^2) + \frac{P_{44}}{r^4}\lambda_i^2\ \lambda_j^2 \tag{A-3.11}$$

and $\Gamma_0\ (i,j) = (\alpha_i\ \alpha_j + 1)$

with $a_i = \dfrac{\alpha_i\ \alpha_j}{r}$ $\qquad i,j = 1,2\ldots4$

A-3.2 Torsional

a) Strain-displacement relationships

Deformation vector $\{\varepsilon_0'\}$ is given by:

$$\{\varepsilon_0'\} = \begin{Bmatrix} 2\bar{\varepsilon}_{x\theta} \\ 2\bar{\kappa}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \partial V/\partial x \\ \frac{3}{2r}\ \partial V/\partial x \end{Bmatrix} \qquad \text{(A-3.12)}$$

Using equations (2.2), and (4.23) and (A.12), vector $\{\varepsilon\}$ takes the form:

$$\{\varepsilon_0\} = [Q_0'] \, [A_0']^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B_0'] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad \text{(A-3.13)}$$

Matrices $[Q_0']$ and $[A_0']$ are given in Appendix A-4.

b) Stress-strain relationships

The resultant stress vector takes the form:

$$\{\sigma_0\} = [P_0'] \{\varepsilon_0'\} \qquad \text{(A-3.14)}$$

with

$$[P_0'] = \begin{bmatrix} P_{33} & P_{36} \\ P_{63} & P_{66} \end{bmatrix} \tag{A-3.15}$$

Using equations (2.3), (A-3.13) and (A-3.14), the stress vector is given by:

$$\{\sigma_0\} = \begin{Bmatrix} \overline{N}_{x\theta} \\ \overline{M}_{x\theta} \end{Bmatrix} = [P_0'] \, [B_0'] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{A-3.16}$$

c) Mass and stiffness matrices

Analogously to the non-torsional case, the mass and stiffness matrices here can be written as:

$$[k] = [A_0'^{-1}]^T \left[2\pi r \int_0^{\ell} [Q_0']^T \, [P_0'] \, [Q_0'] \, dx \right] [A_0'^{-1}]$$

$$[m] = \rho t \, [A_0'^{-1}]^T \left[2\pi r \int_0^{\ell} [R_a']^T \, [R_0] \, dx \right] [A_0'^{-1}]$$

The elements of matrices [k] and [m] are given by:

$$k(1,1) = \frac{2\pi r h}{\ell}$$

$$k(2,2) = \frac{2\pi r h}{\ell}$$

$$k(1,2) = -\frac{2\pi r h}{\ell}$$

$$k(2,1) = -\frac{2\pi r h}{\ell}$$

and

$$m(1,1) = \frac{2}{3}\pi r \ell \rho t$$

$$m(2,2) = m(1,1)$$

$$m(1,2) = m(1,1)/2$$

$$m(2,1) = m(1,2)$$

where

$$h = \left(P_{33} + \frac{9}{4r^2} P_{66} + \frac{3}{r} P_{63}\right)$$

Quantities ℓ, r, and t are, respectively, the length, radius and thickness of the element and ρ is the density of the material.

APPENDIX A-4

This appendix contains the matrices referenced in the bibliography during the course of out analytic developments.

These matrices are classified as follows:

$[D]$; $[D_0]$	(Table 1)
$[A]$; $[A_0]$; $[A_0']$	(Table 2)
$[T]$; $[R]$; $[R_0]$; $[R_0']$	(Table 3)
$[Q]$ $[Q_0]$; $[Q_0']$	(Table 4)

The subscript $(_0)$ corresponds to cases of symmetric loads. The subscripts i,j assigned to the matrice correspond to: the roots of the characteristic equations; coordinate $x = \ell_i = 0$; coordinate $x = \ell_j = \ell$.

Quantities ℓ,r are, respectively, the length and radius of each element. The roots of the characteristic equation are represented by λ_κ, where κ = I,..8 for n = 1 and κ = 1,..4 for n = 0. The values of α_κ and β_κ are determined by equations (4.8) for n = 1 and (4.19) for n = 0.

TABLE 1

Matrix $\underset{3x3}{[D]}$

$$\underset{3x3}{[D]} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0 \; ; \qquad [D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

with

$$d_{11} = P_{11} \lambda^2 - P_{33}$$

$$d_{12} = h_1 \lambda$$

$$d_{13} = \lambda (h_3 - \frac{P_{14}\lambda^2}{r})$$

$$d_{21} = h_1 \lambda$$

$$d_{22} = P_{22} - P_{33} \lambda^2$$

$$d_{23} = h_5 - h_7 \lambda^2$$

$$d_{31} = \lambda (h_3 - \frac{P_{14}\lambda^2}{r})$$

$$d_{32} = h_5 - h_7 \lambda^2$$

$$d_{33} = P_{44} \frac{\lambda^4}{r^2} - h_9 \lambda^2 + h_{11}$$

and

$$h_1 = P_{12} + P_{33}$$

$$h_2 = P_{12} + \frac{1}{r}(P_{15} + 2P_{36})$$

$$h_5 = P_{22} + \frac{P_{25}}{r}$$

$$h_7 = \frac{1}{r}(P_{24} + 2P_{36})$$

$$h_9 = \frac{2}{r}\left[\frac{1}{r}(P_{45} + 2P_{66}) + P_{24}\right]$$

$$h_{11} = \frac{1}{r}\left(2P_{25} + \frac{P_{55}}{r}\right) + P_{22}$$

The characteristic equation (4.5) is of the form:

$$h_8\lambda^8 + h_6\lambda^6 + h_4\lambda^4 + h_2\lambda^2 + h_0 = 0$$

where

$$h_8 = P_{33}(P_{14}^2 - P_{11}P_{44})$$

$$h_6 = P_{11}\left[P_{22}P_{44} + r^2(P_{33}h_9 - h_7^2)\right] + P_{33}^2P_{44} + h_1(rP_{14}h_7 - P_{44}h_1) - P_{14}(2rP_{33}h_3 + P_{14}P_{22} - rh_1h_7)$$

$$h_4 = r^2P_{11}\ (2h_5h_7 - P_{22}h_9 - P_{33}h_{11}) - P_{33}\Big[P_{22}P_{44}$$

$$+ r^2\ (P_{33}h_9 - h_7^2)\Big] + r^2h_1\ (h_1h_9 - h_3h_7) + rh_3\Big[P_{14}P_{22}$$

$$+ r\ (P_{33}h_3 - h_1h_7)\Big] - rP_{14}\ (2h_1h_5 - P_{22}h_3)$$

$$h_2 = r^2\Big[P_{11}(P_{22}h_{11} - h_5^2) - P_{33}(2h_5h_7 - P_{22}h_9 - P_{33}h_{11})$$

$$+ h_1(2h_3h_5 - h_1h_{11}) - P_{22}h_3^2)\Big]$$

$$h_0 = P_{33}(P_{25}^2 - P_{22}P_{55})$$

Matrix $\underset{2x2}{[D]}$

$$\underset{2x2}{[D_0]} \begin{Bmatrix} A \\ C \end{Bmatrix} = 0 \; ; \qquad [D_0] = \begin{bmatrix} d'_{11} & d'_{12} \\ d'_{21} & d'_{22} \end{bmatrix}$$

with

$$d'_{11} = P_{11}\lambda$$

$$d'_{12} = P_{12} - \frac{P_{14}}{r} \lambda^2$$

$$d'_{21} = P_{21}\lambda - \frac{P_{41}}{r} \lambda^3$$

$$d'_{22} = P_{22} - \frac{2P_{24}}{r} \lambda^2 + \frac{P_{44}}{r^2} \lambda^4$$

The characteristic equation (4.17) is of the form:

$$h'_4\lambda^4 - h'_2\lambda^2 + h'_0 = 0$$

where

$$h'_4 = \frac{1}{r^2}\left(P_{11}P_{44} - P^2_{14}\right)$$

$$h'_2 = \frac{2}{r}\left(P_{11}P_{24} - P_{12}P_{14}\right)$$

$$h'_0 = P_{11}P_{22} - P^2_{12}$$

TABLE 2

Matrix $\underset{8\times 8}{[A]}$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \underset{8\times 8}{[A]}\ \underset{8\times 1}{\{C\}}; \qquad \{C\} = \{C_1 \quad C_2 \quad \ldots . C_8\}^T$$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{u_i \ w_i \ \partial w/\partial x)_i \ v_i \ u_j \ w_j \ \partial w/\partial x)_j \ v_j\}^T$$

$$A(1,\kappa) = \alpha_\kappa$$

$$A(2,\kappa) = 1$$

$$A(3,\kappa) = \frac{\lambda_\kappa}{r}$$

$$A(4,\kappa) = \beta_\kappa$$

$$A(5,\kappa) = A(1,\kappa) \times a_\kappa$$

$$A(6,\kappa) = A(2,\kappa) \times a_\kappa$$

$$A(7,\kappa) = A(3,\kappa) \times a_\kappa$$

$$A(8,\kappa) = A(4,\kappa) \times a_\kappa$$

$$\kappa = 1,2\ldots 8$$

$$a_\kappa = e^{(\lambda_\kappa \ell/r)}$$

Matrix $\underset{4\text{x}4}{[A_0]}$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \underset{4\text{x}4}{[A_0]} \; \underset{4\text{x}1}{\{C\}} \; ; \qquad \{C\} = \{C_1 \quad C_2 \quad ..C_4\}^T$$

Model I

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{w_i \;\; \partial w/\partial x)_i \quad w_j \;\; \partial w/\partial x)_j\}^T$$

$$A_0(1,\kappa) = 1$$

$$A_0(2,\kappa) = \frac{\lambda_\kappa}{r}$$

$$A_0(3,\kappa) = A_0(1,\kappa) \text{ x } a_\kappa$$

$$A_0(4,\kappa) = A_0(2,\kappa) \text{ x } a_\kappa$$

Model II

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{u_i \;\; \partial w/\partial x)_i \quad u_j \;\; \partial w/\partial x)_j\}^T$$

$$A_0(1,\kappa) = \alpha_\kappa$$

$$A_0(2,\kappa) = \frac{\lambda_\kappa}{r}$$

$$A_0(3,\kappa) = A_0(1,\kappa) \times a_\kappa$$

$$A_0(4,\kappa) = A_0(2,\kappa) \times a_\kappa$$

where

$$a_\kappa = e^{(\lambda_\kappa \ell/r)} \qquad \kappa = (1,\ 2,\ ..4)$$

Matrix $\underset{2\times 2}{[A_0']}$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \underset{2\times 2}{[A_0']} \underset{2\times 1}{\{B\}} \qquad \{B\} = \begin{Bmatrix} B_0 & B_1 \end{Bmatrix}^T$$

with

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{V_i \;\; V_j\}^T$$

$A_0'(1,1) = 1$

$A_0'(1,2) = 0$

$A_0'(2,1) = 1$

$A_0'(2,2) = \ell$

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = \underset{3x3}{[T]}\ \underset{3x8}{[R]}\ \underset{8x1}{\{C\}}$$

with

$$\{C\} = \{C_1 \quad C_2 \quad \ldots C_8\}^T$$

$$[T] = \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \sin\theta \end{bmatrix}$$

$$R(1,\kappa) = \alpha_\kappa\, e^{(\lambda_\kappa x/r)}$$

$$R(2,\kappa) = e^{(\lambda_\kappa x/r)} \qquad\qquad \kappa = (1,\ 2,\ ..8)$$

$$R(3,\kappa) = \beta_\kappa\, e^{(\lambda_\kappa x/r)}$$

Matrices $[R_0]$; $[R_0']$

$$\begin{Bmatrix} U(x) \\ W(x) \end{Bmatrix} = \underset{2x4}{[R]} \; \underset{4x1}{\{C\}} \; ; \qquad V = \underset{1x2}{[R_0']} \; \underset{2x1}{\{B\}}$$

with

$$\underset{4x1}{\{C\}} = \{C_1 \; ..C_4\}^T$$

$$R_0(1,\kappa) = \alpha_\kappa \, e^{(\lambda_\kappa \, x/r)}$$

$$R_0(2,\kappa) = e^{(\lambda_\kappa \, x/r)}$$

$$R_0'(1,1) = 1$$

$$R_0'(1,2) = x$$

$$\{B\} = \{B_0 \; B_1\}^T$$

TABLE 4

Matrix [Q]

$$\{\varepsilon\} = \underset{6\mathrm{x}6}{\begin{bmatrix} [T] & 0 \\ 0 & [T] \end{bmatrix}} \quad \underset{6\mathrm{x}8}{[Q]} \; \underset{8\mathrm{x}8}{[A^{-1}]} \; \underset{8\mathrm{x}1}{\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}}$$

with

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ 2\bar{\varepsilon}_{x\theta} \\ \kappa_x \\ \kappa_\theta \\ 2\bar{\kappa}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \partial U/\partial x \\ (1/r)\ \partial V/\partial\theta + W/r \\ \partial V/\partial x + (1/r)\ \partial U/\partial\theta \\ -\partial^2 W/\partial x^2 \\ (-1/r^2)\left(\partial^2 W/\partial\theta^2 - \dfrac{\partial V}{\partial\theta}\right) \\ (-2/r)\ \partial^2 W/\partial x\partial\theta + \dfrac{3}{2r}\dfrac{\partial V}{\partial x} - \dfrac{1}{2r^2}\dfrac{\partial U}{\partial\theta} \end{Bmatrix}$$

$$[T] = \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \sin\theta \end{bmatrix}$$

$$Q(1,\kappa) = a_\kappa\ e^{b_\kappa}$$

$$Q(2,\kappa) = \frac{1}{r}\ (\beta_\kappa + 1)\ e^{b_\kappa}$$

$$Q(3,\kappa) = \frac{1}{r}\ (\beta_\kappa\ \lambda_\kappa - \alpha_\kappa)\ e^{b_\kappa}$$

$$Q(4,\kappa) = -\left(\frac{\lambda_\kappa}{r}\right)^2 e^{b_\kappa} \qquad \kappa = (1,2, \, ..8)$$

$$Q(5,\kappa) = \frac{1}{r^2}(1 + \beta_\kappa)\, e^{b_\kappa}$$

$$Q(6,\kappa) = \frac{1}{r^2}\left(2\lambda_\kappa + \frac{3}{2}\beta_\kappa\lambda_\kappa + \frac{1}{2}\alpha_\kappa\right) e^{b_\kappa}$$

where

$$b_\kappa = \frac{\lambda_\kappa x}{r}$$

$$a_\kappa = \frac{\alpha_\kappa \lambda_\kappa}{r}$$

Matrices $[Q_0]_{3x4}$; $[Q_0']_{2x2}$

a) $\{\varepsilon\} = \underset{3x4}{[Q_0]}\ \underset{4x4}{[A_0^{-1}]}\ \underset{4x1}{\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}}$

with

$$\{\varepsilon_0\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \kappa_x \end{Bmatrix} = \begin{Bmatrix} \partial U/\partial x \\ W/r \\ -\partial^2 W/\partial x^2 \end{Bmatrix}$$

$$Q_0(1,\kappa) = a_\kappa\ e^{b_\kappa}$$

$$Q_0(2,\kappa) = \frac{e^{b_\kappa}}{r}$$

$$Q_0(3,\kappa) = -\left(\frac{\lambda_\kappa}{r}\right)^2\ e^{b_\kappa}$$

b) $\{\varepsilon_0'\} = [Q_0]\ [A_0']^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$

$$Q_0'(1,1) = 0$$

$$Q_0'(1,2) = 1$$

$$Q_0'(2,1) = 0$$

$$Q_0'(2,2) = \frac{3}{2r}$$

where

$$b_\kappa = \frac{\lambda_\kappa x}{r}$$

$$a_\kappa = \frac{\alpha_\kappa \lambda_\kappa}{r}$$

APPENDIX A-5

FREE VIBRATION OF CYLINDRICAL SHELLS PARTIALLY FILLED WITH LIQUID

A-5.1 Equations of motion

The dynamic behaviour of a shell subjected to pressures fields can be represented by the following equation system.

$$\Big[[M_0] - [M_f']\Big]\{\ddot{\delta}\} + \Big[[C_0] - [C_f']\Big]\{\dot{\delta}\} + \Big[[K_0] - [K_f']\Big]\{\delta\} = \{F\} \tag{5.1}$$

where

$\{\delta\}$ is the displacement vector; $[M_0]$ and $[K_0]$ are, respectively, global matrices for the mass and stiffness of the in-vacuo system, and $[M_f']$; $[C_f']$ and $[K_f']$ represent the intertial Coriolis and centrifugal forces of the liquid flow; $[C_0]$ is the system damping matrix and external forces $\{F\}$ represent the field pressure hazardous induced by the boundary layer [22].

A-5.2 Inertial, Coriolis and centrifugal forces of the liquid flow

The shell is subject to intertial, Coriolis and centrifugal forces because of the liquid flow, which participates within the vibration of the structure. These forces are coupled with the elastic deformation of the shell.

The mathematic model used is based on the following hypotheses [22]:

a) The liquid flow is potential;

b) Vibration is linear (small deformation);

c) Pressure on the wall is purely lateral;

d) The speed distribution of the fluid is assumed constant throughout the shell section;

e) The fluid is incompressible.

A-5.2.1 Determination of the apparent mass, stiffness and damping matrices if the liquid flow

The potential flow is governed by the following equations:

$$\nabla^2\Phi = \frac{1}{C^2}\left[\frac{\partial^2\Phi}{\partial t^2} + 2U_x\frac{\partial^2\Phi}{\partial x\partial t} + U_x^2\frac{\partial^2\Phi}{\partial x^2}\right] \qquad \text{(a)}$$

and (5.2)

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial x^2} \qquad \text{(b)}$$

where

C is the speed of sound in the fluid; U_x is the velocity of the liquid through the shell section and Φ is the potential function that represents potential velocity:

$$V_x = U_x + \frac{\partial\Phi}{\partial x};\ V_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta};\ V_r = \frac{\partial\Phi}{\partial r} \qquad (5.3)$$

and

V_x, V_θ and V_r are the axial, tangential and radial components of the fluid velocity.

The pressures exerted on the wall are given by:

$$P_i = -\rho_i \left(\frac{\partial \Phi_i}{\partial t} + U_{xi} \frac{\partial \Phi_i}{\partial x} \right)_{r=a}$$

$$P_e = -\rho_e \left(\frac{\partial \Phi_e}{\partial t} + U_{xe} \frac{\partial \Phi_e}{\partial x} \right)_{r=a+t} \tag{5.4}$$

where

a and t are, respectively, the radius and thickness of the shell, and subscripts i and e indicate the internal and external locations of the shell.

Finally, the condition

$$(V_r)_{r=a} = \left(\frac{\partial \Phi}{\partial r} \right)_{r=a} = \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x} \right)_{r=a} \tag{5.5}$$

must be satisfied in order to obtain the contact between the shell's surface and the peripheral layer of the fluid.

From equation (4.12) we have [22]:

$$W(x,\theta,t) = \sum_{q=1}^{8} C_q \, e^{(\lambda_q x/a + iwt)} \cos n\,\theta \tag{5.6}$$

and setting:

$$\Phi(x,\theta,r,t) = \sum_{q=1}^{8} R_q(r)\; S_q(x,\theta,t) \tag{5.7}$$

Substituting (5.6) and (5.7) in (5.5), we obtain:

$$S_q(x,\theta,t) = \frac{1}{R_q(a)} \left[\dot{W}_q + U_x W_q'\right]_{r=a} \tag{5.8}$$

and equation (5.7) becomes:

$$\Phi(x,\theta,r,t) = \sum_{q=1}^{8} \frac{R_q(r)}{R_q(a)} \left[\dot{W}_q + U_x W_q'\right]_{r=a} \tag{5.9}$$

Substituting equations (5.6) and (5.9) in equation (5.2), we obtain for each value of q:

$$R_q''(r) + \frac{1}{r} R_q'(r) - R_q(r) \left[\frac{n^2}{r^2} - m_q^2\right] \tag{5.10}$$

where

$$m_q^2 = \frac{\lambda_q^2}{a^2} - \frac{1}{C^2}\left(\frac{\lambda_q}{a} U_x + iw\right)^2$$

The solution of equation (5.10) is in the form:

$$R(r) = D_1 J_n(m_q r) + D_2 Y_n(m_q r) \tag{5.11}$$

such that

$J_n(m_q r)$ and $Y_n(m_q r)$ are, respectively, Bessel functions of the first and second kind and of order n; the potential functions will be given by:

$$\Phi_i = \sum_{q=1}^{8} \frac{J_n(m_q r)}{J'_n(m_q a_i)} [\dot{W}_q + U_{x_i} W'_q]_{r=a} \tag{5.12}$$

and

$$\Phi_e = \sum_{\Phi=1}^{8} \frac{Y_n(m_q r)}{Y'_n(m_q a_e)} [\dot{W}_q + U_{x_e} W'_q]_{r=a} \tag{5.13}$$

and the pressure exerted on the wall

$$P_t = P_i - P_e$$

$$P_t = \sum_{q=1}^{8} \{[-a_i \rho_i r_q + a_e \rho_e s_q] \ddot{W}_q + 2 [-\rho_i a_i U_{xi} r_q + \rho_e a_e U_{xe} s_q]$$
$$\dot{\dot{W}}_q + [-\rho_i a_i U_{xi}^2 r_q + \rho_e a_e U_{xe}^2 s_q] W''_q\} \tag{5.14}$$

where

$$W(x,\theta,t) = \sum_{q=1}^{8} C_q e^{(\lambda_q x/a + i\omega t)} \cos n\theta, \ \rho_i \quad \rho_e$$

are, respectively, internal and external fluid densities, and ω is the free vibration of the system.

$$r_q = 1 \Big/ \left(n - m_q a \, J_{n+1} (m_q a_i) \Big/ J_n (m_q a_i)\right) \tag{5.15}$$

$$s_q = 1 \Big/ \left(n - m_q a \, Y_{n+1} (m_q a_e) \Big/ Y_n (m_q a_e)\right) \tag{5.16}$$

Using the finite element method, vector $\{F\}$ will be given:

$$\{F\} = \iint [N]^T \{P\} \, a \, dx d\theta \tag{5.17}$$

where the pressure vector is determined by:

$$\{P\} = \begin{Bmatrix} 0 \\ P_t \\ 0 \end{Bmatrix} \tag{5.18}$$

Finally, substituting (5.14) and (5.18) in (5.17) and integrating on x and θ, we obtain the intertial Coriolis and centrifugal forces of the liquid how as follows:

$$\begin{aligned} [m_f] &= [A^{-1}]^T [S_f] [A^{-1}] \\ [c_f] &= [A^{-1}]^T [D_f] [A^{-1}] \end{aligned} \tag{5.19}$$

and

$$[k_f] = [A^{-1}]^T [G_f] [A^{-1}]$$

where

$$S_f(k,q) = -\pi\delta_i\gamma_i^2 r_q I_{kq}\ (\ell, a_i) + \pi\delta_e\gamma_e^2 s_q\ (\ell, a_e)$$

$$D_f(k,q) = 2\lambda_q\pi\ [-\delta_i\bar{U}_i\gamma_i r_q I_{kq}\ (\ell, a_i) + \delta_e\bar{U}_e\gamma_e s_q\ (\ell, a_e)] \tag{5.20}$$

$$G_f(k,q) = \lambda_q^2\pi\ [-\delta_i\overline{U}_i^2 r_q I_{kq}\ (\ell, a_i) + \delta_e\overline{U}_e^2 s_q I_{kq}\ (\ell, a_e)]$$

In the equations, the non-dimensional terms are determined by:

$$\delta_i = \left(a_i/t_1\right)\left(\rho_i/\rho_1\right), \quad \delta_e = \left(a_e/t_1\right)\left(\rho_e/\rho_1\right)$$

$$U_0^2 = P(1,1,1)/\rho_1 t_1, \quad \overline{U}_i = \frac{U_{xi}}{U_0}, \quad \omega_0 = \frac{U_0}{r_1} \qquad (5.21)$$

$$\overline{U}_e = U_{xe}/U_0, \quad \gamma_i = a_i/r_1, \quad \gamma_e = a_e/r_1$$

where

r_1, t_1 and ρ_1 are, respectively, radius, thickness density of the first element of the shell. The element P(1,1,1) is the first term in the matrix of elasticity [P].

$$I_{kq}(\ell,a) = \frac{1}{(\lambda_q + \lambda_k)}\left[e^{(\lambda_k + \lambda_q)\ell/a} - 1\right] \text{for} \quad \lambda_k + \lambda_q \neq 0$$

$$I_{kq}(\ell,a) = \ell/a \quad \text{for} \quad \lambda_k + \lambda_q = 0 \qquad k,q = 1,2,\ldots 8$$

and ℓ is the element length.

Matrices $[M_f]$, $[C_f]$ and $[K_f]$ are, respectively, the global matrices mass, stiffness and for the apparent damping of the entire fluid column, which are obtained by superposition of the mass $[m_f]$, stiffness $[k_f]$ and damping $[C_f]$ matrices for each element in the fluid column.

Note: The preceding developement, given for the case of $n = 1$, is still applicable for $n = 0$ with the two following changes:

Matrices $[S_f]$, $[D_f]$ and $[G_f]$ will be re-multiplied by two and $k,q = 1,2,..4$. Matrix $[A]$ will be given by $[A_0]$.

APPENDIX B

TABLES

n	BARON & BLEICH	MICHALOPOULOS & MUSTER	PRESENT METHOD
0	3540	3384	3398
1	1920	1775	1790

TABLE 1 - Natural frequencies (Hz) of a uniform cylindrical shell simply supported at both ends, calculated by various theories (m = 1)

<u>Shell</u>

Length	:	18.54 in.
Radius (average)	:	4.08 in.
Thickness	:	0.047 in.
Young's modulus	:	3×10^7 lb/in^2
Poisson's ratio	:	0.3
Density of material	:	0.7324×10^{-3} lb - sec^2/Po^4
Boundary conditions	:	simply supported shell
		$V = W = 0$ to $x = 0$ and $x = L$

n = 1; clamped-free

m	TOTTENHAM* & SHIMIZU [29]	SANKARAN** [30]	PRESENT METHOD
1	2033	2080	2043
2	5387	5450	5557
3	6957	7000	7289
4	7532	7740	8163

* Progression method
** Integral method

TABLE 2 - Natural frequencies (Hz) of a clamped-free cylindrical shell calculated by various numerical methods (n = 1).

<u>Shell</u>

$r/t = 100$: $\nu = 0.3$ $g = 980$ cm/sec^2

$r/L = 0.448$: $\rho = 7.84 \times 10^{-3}$ kg/cm^3 $r = 10.16$ cm

	r/t L/r	20	50	100	200	300	Baron & Bleich all values of r/t
empty		0.8581	0.8482	0.8501	0.8626	0.8808	0.8448
full	1.0	0.6915	0.5647	0.4621	0.3596	0.3018	—
empty		0.7151	0.7153	0.7229	0.7446	0.7635	0.7117
full	1.5	0.5358	0.4214	0.3320	0.2485	0.2050	—
empty		0.5746	0.5770	0.5849	0.5985	0.6055	0.5728
full	2.0	0.4180	0.3219	0.2491	0.1840	0.1519	—
empty		0.4605	0.4632	0.4688	0.4748	0.4771	0.4591
full	2.5	0.3314	0.2525	0.1936	0.1426	0.1178	—
empty		0.3738	0.3761	0.3794	0.3819	0.3828	0.3727
full	3.0	0.2674	0.2025	0.1546	0.1138	0.0941	—
empty		0.3082	0.3100	0.3117	0.3127	0.3132	0.3072
full	3.5	0.2195	0.1665	0.1260	0.0927	0.0767	—
empty		0.2577	0.2590	0.2598	0.2604	0.2606	0.2569
full	4.0	0.1828	0.1373	0.1044	0.0767	0.0635	—
empty		0.1870	0.1875	0.1877	0.1879	0.1879	0.1864
full	5.0	0.1315	0.0983	0.0745	0.0548	0.0464	—
empty		0.1411	0.1412	0.1413	0.1414	0.1414	0.1406
full	6.0	0.0986	0.0734	0.0556	0.0408	0.0337	—
empty		0.1098	0.1098	0.1099	0.1099	0.1099	0.1096
full	7.0	0.0762	0.0566	0.0428	0.0314	0.0260	—
empty		0.0876	0.0876	0.0876	0.0877	0.0877	0.0874
full	8.0	0.0605	0.0448	0.0339	0.0248	0.0205	—
empty		0.0714	0.0714	0.0714	0.0714	0.0714	0.0713
full	9.0	0.0491	0.0363	0.0274	0.0201	0.0166	—
empty		0.0592	0.0592	0.0592	0.0592	0.0592	0.0592
full	10.0	0.0406	0.0300	0.0226	0.0166	0.0137	—

TABLE 3 Vibration parameter (Ω) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$), $m = 1$, $\nu = 03$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P_0^4 and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	r/t L/r	20	50	100	200
empty		0.2796	0.2796	0.2796	0.2796
full	15	0.1901	0.1398	0.1052	0.0770
empty		0.1610	0.1610	0.1610	0.1610
full	20	0.1090	0.0801	0.0602	0.0440
empty		0.1043	0.1043	0.1043	0.1043
full	25	0.0704	0.0516	0.0338	0.0284
empty		0.0729	0.0729	0.0729	0.0729
full	30	0.0492	0.0360	0.0270	0.0198
empty		0.0538	0.0537	0.0537	0.0537
full	35	0.0363	0.0265	0.0199	0.0146
empty		0.0414	0.0413	0.0413	0.0413
full	40	0.0279	0.0203	0.0153	0.0112
empty		0.0329	0.0327	0.0327	0.0327
full	45	0.0222	0.0161	0.0121	0.0088
empty		0.0268	0.0265	0.0265	0.0265
full	50	0.0180	0.0130	0.0098	0.0072

TABLE 3 Vibration parameter ($\Omega \times 10$) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m = 1$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec$^2/P_0^4$ and $\Omega = \omega r \sqrt{\rho(1-\nu^2)/E}$)

	r/t L/r	20	50	100	200	300
empty full	1.0	0.0956 0.9972	0.9585 0.7719	0.9419 0.6631	0.9543 0.5470	0.9790 0.5420
empty full	1.5	0.9331 0.7938	0.9075 0.6573	0.9223 0.5497	0.9763 0.4243	0.0219 0.3479
empty full	2.0	0.8611 0.6934	0.8647 0.5688	0.9018 0.0480	0.9636 0.3417	0.9848 0.2802
empty full	2.5	0.7937 0.6113	0.8120 0.4898	0.8548 0.3864	0.8928 0.2855	0.9040 0.2351
empty full	3.0	0.7226 0.5394	0.7464 0.4252	0.7788 0.3305	0.8000 0.2436	0.8091 0.2011
empty full	3.5	0.6512 0.4766	0.6735 0.3712	0.6936 0.2860	0.7061 0.2106	0.7123 0.1705
empty full	4.0	0.5838 0.4224	0.6015 0.3260	0.6133 0.2499	0.6209 0.1840	0.6246 0.1520
empty full	5.0	0.4687 0.3356	0.4778 0.2557	0.4821 0.1951	0.4850 0.1435	0.4864 0.1187
	6.0	0.3802 0.2709	0.3845 0.2049	0.3863 0.1559	0.3876 0.1147	0.3881 0.0949
	7.0	0.3128 0.2221	0.3148 0.1672	0.3157 0.1270	0.3163 0.0934	0.3165 0.0774
	8.0	0.2610 0.1847	0.2620 0.1385	0.2625 0.1052	0.2628 0.0774	0.2629 0.0641
	9.0	0.2206 0.1556	0.2211 0.1164	0.2214 0.0883	0.2216 0.0649	0.2216 0.0537
	10.0	0.1886 0.1326	0.1889 0.0990	0.1890 0.0751	0.1891 0.0552	0.1892 0.0456

TABLE 3a Vibration parameter (Ω) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m = 2$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P$_0^4$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	r/t L/r	20	50	100	200
empty full	15	0.9819 0.6806	0.9821 0.5047	0.9823 0.3814	0.9824 0.2798
empty full	20	0.5934 0.4075	0.5934 0.3008	0.5934 0.2269	0.5934 0.1662
empty full	25	0.3945 0.2692	0.3944 0.1982	0.3944 0.1493	0.3944 0.1093
empty full	30	0.2801 0.1904	0.2800 0.1400	0.2800 0.1053	0.2800 0.0771
empty full	35	0.2087 0.1415	0.2086 0.1039	0.2086 0.0782	0.2086 0.0572
empty full	40	0.1614 0.1092	0.1612 0.0801	0.1612 0.0602	0.1612 0.0440
empty full	45	0.1284 0.0868	0.1282 0.0636	0.1282 0.0476	0.1282 0.0349
empty full	50	0.1046 0.0706	0.1043 0.0517	0.1043 0.0388	0.1043 0.0284

TABLE 3a Vibration parameter ($\Omega \times 10$) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m = 2$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec$^2/P_0^4$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	L/r \ r/t	20	50	100	200	300
empty full	1.0	1.6030 1.5829	1.0781 0.9433	0.9850 0.7663	0.9850 0.5917	1.0000 0.5817
empty full	1.5	1.0973 0.9986	0.9655 0.7732	0.9658 0.6584	1.0216 0.5169	1.0685 0.4216
empty full	2.0	0.9701 0.8415	0.9396 0.6942	0.9818 0.5786	1.0532 0.4359	1.0636 0.3556
empty full	2.5	0.9139 0.7608	0.9281 0.6310	0.9916 0.5122	1.0320 0.3806	1.0346 0.3125
empty full	3.0	0.8731 0.6996	0.9137 0.5745	0.9713 0.4570	0.9926 0.3381	1.0008 0.2785
empty full	3.5	0.8338 0.6458	0.8856 0.5233	0.9270 0.4104	0.9464 0.3030	0.9591 0.2499
empty full	4.0	0.7917 0.5964	0.8420 0.4768	0.8709 0.3703	0.8984 0.2730	0.9033 0.2253
empty full	5.0	0.6985 0.5082	0.7311 0.3971	0.7468 0.3050	0.7609 0.2245	0.7668 0.1855
empty full	6.0	0.6040 0.4333	0.6218 0.3333	0.6304 0.2547	0.6370 0.1874	0.6401 0.1550
empty full	7.0	0.5192 0.3707	0.5289 0.2823	0.5335 0.2153	0.5368 0.1584	0.5383 0.1311
empty full	8.0	0.4476 0.3190	0.4530 0.2416	0.4556 0.1840	0.4574 0.1354	0.4582 0.1121
empty full	9.0	0.3883 0.2764	0.3915 0.2086	0.3931 0.1587	0.3941 0.1168	0.3945 0.0967
empty full	10.0	0.3393 0.2412	0.3413 0.1815	0.3422 0.1381	0.3429 0.1016	0.3431 0.0841

TABLE 3b Vibration parameter (Ω) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m = 3$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P_0^4 and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

L/r \ r/t		20	50	100	200
empty full	15	1.9058 1.3411	1.9085 1.0017	1.9048 0.7597	1.9106 0.5585
empty full	20	1.2048 0.8390	1.2052 0.6234	1.2055 0.4717	1.2057 0.3463
empty full	25	0.8240 0.5691	0.8239 0.4213	0.8240 0.3182	0.8240 0.2333
empty full	30	0.5961 0.4093	0.5959 0.3021	0.5959 0.2279	0.5959 0.1670
empty full	35	0.4499 0.3076	0.4496 0.2266	0.4496 0.1707	0.4496 0.1250
empty full	40	0.3509 0.2391	0.3506 0.1759	0.3506 0.1324	0.3505 0.0969
empty full	45	0.2810 0.1910	0.2807 0.1403	0.2806 0.1056	0.2806 0.0773
empty full	50	0.2299 0.1560	0.2295 0.1145	0.2295 0.0861	0.2295 0.0630

TABLE 3b Vibration parameter (Ω x 10) of a cylindrical shell simply supported at both ends and filled with liquid (n = 1, m = 3, ν = 0.3, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P_0^4 and $\Omega r \sqrt{\rho(1-\nu^2)/E}$)

	r/t L/r	20	50	100	200	300
empty full	1.0	2.4855 2.7245	1.3232 1.2442	1.0620 0.8805	1.0020 0.6339	1.0133 0.5919
empty full	1.5	1.3952 1.3392	1.0358 0.8791	0.9942 0.7278	1.0387 0.5706	1.0802 0.4628
empty full	2.0	1.1018 1.0020	0.9825 0.7768	1.0125 0.6519	1.0662 0.4932	1.0670 0.4006
empty full	2.5	0.9986 0.8740	0.9761 0.7184	1.0376 0.5934	1.0624 0.4431	1.0676 0.3627
empty full	3.0	0.9513 0.8046	0.9808 0.6717	1.0374 0.7402	1.0490 0.4051	1.0533 0.3329
empty full	3.5	0.9226 0.7551	0.9714 0.6293	0.9797 0.5029	0.9819 0.3728	1.0386 0.3068
empty full	4.0	0.8994 0.7135	0.9116 0.5890	0.9165 0.4651	0.9196 0.3440	1.0194 0.2835
empty full	5.0	0.8480 0.6389	0.8996 0.5140	0.9222 0.3994	0.9450 0.2947	0.9576 0.2433
empty full	6.0	0.7793 0.5692	0.8146 0.4478	0.8341 0.3449	0.8521 0.2542	0.8615 0.2161
empty full	7.0	0.7011 0.5047	0.7245 0.3910	0.7349 0.2998	0.7360 0.2208	0.7544 0.1827
empty full	8.0	0.6246 0.4470	0.6400 0.3429	0.6483 0.2622	0.6547 0.1931	0.6576 0.1598
empty full	9.0	0.5552 0.3965	0.5623 0.3023	0.5705 0.2308	0.5731 0.1700	0.5759 0.1407
empty full	10.0	0.4946 0.3529	0.5012 0.2679	0.5045 0.2043	0.5068 0.1505	0.5078 0.1246

TABLE 3c Vibration parameter (Ω) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m = 4$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P_0^4 and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	L/r \ r/t	20	50	100	200
empty full	15	2.9489 2.0945	2.9602 1.5747	2.9654 1.1977	2.9686 0.8817
empty full	20	1.9324 1.3611	1.9351 1.0175	1.9364 0.7721	1.9372 0.5678
empty full	25	1.3545 0.9461	1.3551 0.7042	1.3555 0.5332	1.3558 0.3917
empty full	30	0.9972 0.6916	0.9971 0.5130	0.9973 0.3878	0.9974 0.28463
empty full	35	0.7622 0.5257	0.7619 0.3888	0.7619 0.2936	0.7619 0.2153
empty full	40	0.6000 0.4120	0.5996 0.3041	0.5996 0.2294	0.5996 0.1681
empty full	45	0.4838 0.3311	0.4834 0.2440	0.4833 0.1839	0.4833 0.1347
empty full	50	0.3980 0.2716	0.3975 0.1998	0.3974 0.1505	0.3974 0.1102

TABLE 3c Vibration parameter (Ω) x 10) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m = 4$, $\nu = 0.3$, $\rho_{\ell} = 0.935 \times 10^{-4}$ lb - sec$^2/P_0^4$ and $\Omega = \omega r \rho \sqrt{\rho(1 - \nu^2)/E}$)

	r/t L/r	20	50	100	200	300
empty full	1.0	3.7146 4.7684	1.7265 1.7700	1.7264 1.0570	1.2250 0.7577	1.1233 0.6250
empty full	1.5	1.8645 1.8981	1.1544 1.0264	1.0928 0.8707	1.0676 0.6087	1.0815 0.4895
empty full	2.0	1.3124 1.2415	1.1101 0.8547	1.0583 0.7048	1.0739 0.5314	1.0729 0.4292
empty full	2.5	1.1107 1.0077	1.0115 0.7841	1.0567 0.6500	1.0654 0.4860	1.0786 0.3963
empty full	3.0	1.0261 0.9012	1.0166 0.7402	1.0559 0.6081	1.0617 0.4534	1.0606 0.3715
empty full	3.5	0.9864 0.8408	1.0222 0.7049	1.0480 0.5723	1.0600 0.4258	1.0654 0.3496
empty full	4.0	0.9650 0.7993	1.0169 0.6719	1.0354 0.5396	1.0494 0.4006	1.0567 0.3295
empty full	5.0	0.9355 0.7355	0.9817 0.6078	1.0033 0.4796	1.0220 0.3551	1.0321 0.2929
empty full	6.0	0.8950 0.6768	0.9339 0.5465	0.9573 0.4262	0.9795 0.3152	0.9916 0.2605
empty full	7.0	0.8386 0.6184	0.8722 0.4899	0.8948 0.3793	0.9157 0.2802	0.9269 0.2319
empty full	8.0	0.7739 0.5620	0.8015 0.4393	0.8196 0.3385	0.0853 0.2499	0.8433 0.2068
empty full	9.0	0.7076 0.5096	0.7287 0.3945	0.7416 0.3030	0.7519 0.2236	0.7569 0.1851
empty full	10.0	0.6440 0.4619	0.6595 0.3552	0.6683 0.2722	0.6749 0.2000	0.6780 0.1663

TABLE 3d Vibration parameter (Ω) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m = 5$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P_0^4 and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$

	r/t L/r	20	50	100	200
empty full	15	4.0712 2.9075	4.1033 2.2016	4.1181 1.6793	4.1279 1.2381
empty full	20	2.7485 1.9523	2.7573 1.4683	2.7612 1.1173	2.7636 0.8230
empty full	25	1.9661 1.3863	1.9689 1.0375	1.9702 0.7878	1.9710 0.5796
empty full	30	1.4692 1.0286	1.4700 0.7666	1.4705 0.5811	1.4709 0.4270
empty full	35	1.1356 0.7900	1.1357 0.5870	1.1359 0.4442	1.1360 0.3261
empty full	40	0.9018 0.6241	0.9014 0.4624	0.9015 0.3495	0.9016 0.2564
empty full	45	0.7321 0.5045	0.7315 0.3730	0.7315 0.2816	0.7315 0.2065
empty full	50	0.6053 0.4157	0.6047 0.3068	0.6046 0.2314	0.6046 0.1696

TABLE 3d Vibration parameter ($\Omega \times 10$) of a cylindrical shell simply supported at both ends and filled with liquid ($n = 1$, $m =$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P_0^4 and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	r/t	10	20	50		100		200		300	
	L/r	THIS METHOD	THIS METHOD	THIS METHOD	METHOD IN REF. [31]	THIS METHOD	METHOD IN REF. [31]	THIS METHOD	METHOD IN REF. [31]	THIS METHOD	METHOD IN REF. [31]
empty	10	2.2290	2.2240	2.2206	2.2042	2.2193	2.2042	2.2183	2.2042	2.2179	2.2179
full		1.7839	1.5032	1.1003	—	0.8262	—	0.6038	—	0.4987	—
empty	15	1.0275	1.0242	1.0229	0.9984	1.0224	0.9983	1.0222	0.9983	1.0220	0.9983
full		0.8213	0.6911	0.5056	—	0.3796	—	0.2774	—	0.2290	—
empty	20	0.5889	0.5838	0.5830	0.5655	0.5828	0.5654	0.5826	0.5654	0.5826	0.5654
full		0.4705	0.3937	0.2879	—	0.216	—	0.1579	—	0.1304	—
empty	25	0.3859	0.3764	0.3754	0.3631	0.3753	0.3630	0.3752	0.3630	0.3752	0.3630
full		0.3082	0.2537	0.1853	—	0.1391	—	0.1016	—	0.0839	—
empty	30	0.2801	0.26308	0.2616	0.2526	0.2615	0.2526	0.2615	0.2526	0.2615	0.2526
full		0.2237	0.1772	0.1291	—	0.0968	—	0.0707	—	0.0584	—
empty	35	0.2234	0.1949	0.1926	0.1858	0.1925	0.1858	0.1925	0.1858	0.1925	0.1858
full		0.1784	0.1313	0.0950	—	0.0713	—	0.0521	—	0.0430	—
empty	40	0.1950	0.1514	0.1477	0.1423	0.1476	0.1423	0.1476	0.1423	0.1476	0.1423
full		0.1557	0.1019	0.0728	—	0.0546	—	0.0400	—	0.0329	—
empty	45	0.1843	0.1224	0.1169	0.1125	0.1168	0.1125	0.1167	0.1125	0.1167	0.1125
full		0.1471	0.0824	0.0576	—	0.0432	—	0.0316	—	0.0261	—
empty	50	0.1845	0.1030	0.0949	0.0912	0.0946	0.0912	0.0946	0.0912	0.0946	0.0912
full		0.1473	0.0694	0.0467	—	0.0350	—	0.0256	—	0.0211	—

TABLE 4 Vibration parameter ($\Omega \times 10^2$) of a clamped-free cylindrical shell filled with liquid ($n = 1$, $m = 1$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec$^2/P_0^4$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	r/t L/r	10	20	50	100	200	300
empty full	10	10.705 8.7322	10.6850 7.4361	10.675 5.5102	10.673 4.1637	10.671 3.0542	10.670 2.5259
empty full	15	5.5648 4.4988	5.5536 3.8127	5.5478 2.8097	5.5457 2.1171	5.5445 1.5505	5.5440 1.2815
empty full	20	3.3543 2.7000	3.3470 2.2821	3.3436 1.6773	3.3425 1.2622	3.3420 0.9236	3.3415 0.7632
empty full	25	2.2226 1.7868	2.2196 1.5076	2.2174 1.1063	2.2167 0.8319	2.2163 0.6084	2.2162 0.5027
empty full	30	1.5801 1.2665	1.5735 1.0662	1.5717 0.7816	1.5712 0.5874	1.5710 0.4295	1.5721 0.3548
empty full	35	1.1795 0.9444	1.1712 0.7923	1.1696 0.5803	1.1692 0.4361	1.1691 0.3188	1.1690 0.2633
empty full	40	0.9161 0.7330	0.9047 0.6115	0.9031 0.4475	0.9030 0.3361	0.9027 0.2457	0.9027 0.2029
empty full	45	0.7355 0.5882	0.7197 0.4860	0.7178 0.3553	0.7175 0.2668	0.7175 0.1950	0.7174 0.1611
empty full	50	0.6086 0.4865	0.5862 0.3957	0.5839 0.2881	0.5837 0.2169	0.5836 0.1585	0.5836 0.1309

TABLE 4a Vibration parameter ($\Omega \times 10^2$) of a clamped-free cylindrical shell filled with liquid ($n = 1$, $m = 2$, $\nu = 0.3$ $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P$_0^4$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	r/t L/r	10	20	50	100	200	300
empty full	10	24.049 19.849	24.081 17.072	24.118 12.780	24.140 9.701	24.156 7.1349	24.163 5.9061
empty full	15	13.399 10.945	13.383 9.3376	13.378 6.9333	13.378 5.2439	13.378 3.8490	13.378 3.1840
empty full	20	8.4535 6.8613	8.4387 5.8301	8.4327 4.3097	8.4310 3.2525	8.4302 2.3843	8.4298 1.9714
empty full	25	5.7750 4.6673	5.7628 3.9552	5.7577 2.9154	5.7565 2.1972	5.7556 1.6093	5.7553 1.3303
empty full	30	4.1752 3.3643	4.1642 2.8450	4.1603 2.0931	4.1592 1.5760	4.1587 1.1536	4.1584 0.9534
empty full	35	3.1506 2.5333	3.1398 2.1383	3.1364 1.5709	3.1355 1.1820	3.1351 0.8649	3.1350 0.7147
empty full	40	2.4584 1.9738	2.4470 1.6627	2.4440 1.2201	2.4433 0.9176	2.4430 0.6713	2.4430 0.5546
empty full	45	1.9711 1.5807	1.9583 1.3283	1.9533 0.9739	1.9550 0.7322	1.9545 0.5355	1.9544 0.4424
empty full	50	1.6165 1.2952	1.6015 1.0849	1.6080 0.7947	1.5980 0.5973	1.5977 0.4368	1.5976 0.3608

TABLE 4b Vibration parameter ($\Omega \times 10^2$) of a clamped-free cylindrical shell filled with liquid ($n = 1$, $m = 3$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P$_0^4$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	r/t L/r	10	20	50	100	200	300
empty full	10	38.212 31.903	38.475 27.776	38.711 21.025	38.831 16.030	38.912 11.811	38.948 9.7823
empty full	15	22.626 18.660	22.644 16.042	32.673 12.008	22.689 9.1170	22.700 6.7066	22.704 5.5519
mpty full	20	14.819 12.129	14.807 10.366	14.807 7.7127	14.808 5.8390	14.810 4.2890	14.811 3.5489
empty full	25	10.395 8.4619	10.378 7.2044	10.374 5.3388	10.373 4.0344	10.372 2.9600	10.372 2.4480
empty full	30	7.6590 6.2092	7.6420 5.2718	7.6401 3.8954	7.6353 2.9400	7.6347 2.1547	7.6345 1.7816
empty full	35	5.8580 4.7347	5.8413 4.0107	5.8359 2.9572	5.8347 2.2292	5.8342 1.6330	5.8340 1.3500
empty full	40	4.6152 3.7217	4.5986 3.1462	4.5936 2.3159	4.5923 1.7445	4.5918 1.2773	4.5918 1.0558
empty full	45	3.7253 2.9986	3.7080 2.5299	3.7030 1.8598	3.7021 1.4000	3.7016 1.0248	3.7015 0.8468
empty full	50	3.0680 2.4660	3.0496 2.0762	3.0447 1.5245	3.0438 1.1471	3.0435 0.8394	3.0433 0.6930

TABLE 4c Vibration parameter ($\Omega \times 10^2$) of a clamped-free cylindrical shell filled with liquid ($n = 1$, $m = 4$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - $\sec^2/P_0^4$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

	r/t L/r	10	20	50	100	200	300
empty full	10	52.313 44.032	53.012 38.809	53.669 29.731	54.012 22.764	54.252 16.800	54.355 13.916
empty full	15	32.801 27.235	32.936 23.600	33.068 17.807	33.131 13.567	33.174 9.9978	33.192 8.2817
mpty full	20	22.118 18.230	22.137 15.677	22.166 11.7417	22.180 8.9187	22.190 6.5828	22.194 5.4338
empty full	25	15.843 12.981	15.833 11.108	15.837 8.2784	15.840 6.2734	15.843 4.6102	15.843 3.8152
empty full	30	11.860 9.6719	11.840 8.2470	11.838 6.1230	11.838 4.6319	11.838 3.4003	11.838 2.8130
empty full	35	9.1822 7.4607	9.1007 6.3431	9.1547 4.6964	9.1541 3.5477	9.1541 2.6022	9.1541 2.1521
empty full	40	7.3024 5.9155	7.2795 5.0174	7.2728 3.7066	7.2714 2.0503	7.2714 2.0503	7.2714 1.6953
empty full	45	5.9364 4.7976	5.9127 4.0603	5.9059 2.2942	5.9043 1.6540	5.9043 1.6540	5.9083 1.3674
empty full	50	4.9157 3.9651	4.8910 3.3488	4.8840 2.4657	4.8824 1.3606	4.8824 1.3606	4.8822 1.1246

TABLE 4d Vibration parameter ($\Omega \times 10^2$) of a clamped-free cylindrical shell filled with liquid ($n = 1$, $m = 5$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/P_0^4 and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

r/t L/r	20	200	Baron and Bleich all values of r/t
1	0.9647	0.9614	0.9489
2	0.9333	0.9331	0.9294
4	0.4666	0.4665	0.4646
6	0.3111	0.3110	0.3097
8	0.2333	0.2332	0.2323
9	0.2074	0.2074	0.2065
10	0.1866	0.1866	0.1859

TABLE 5 Vibration parameter (Ω) of a cylindrical shell simply supported at both ends ($n = 0$, $m = 1$, $\nu = 0.3$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

n = 1; simply supported - supported

m	NIORDSON 7	PRESENT METHOD	CONDITION
1	9.956 4.504	9.861 4.549	empty shell shell filled with liquid
2	37.504 17.257	37.290 17.46	empty shell shell filled with liquid
3	77.271 36.361	77.900 37.137	empty shell shell filled with liquid
4	123.693 59.594	128.120 62.115	empty shell shell filled with liquid

TABLE 6 Free vibration (Hz) of a cylindrical shell simply supported at both ends and filled with liquid [in list of Tables].

<u>Shell</u>

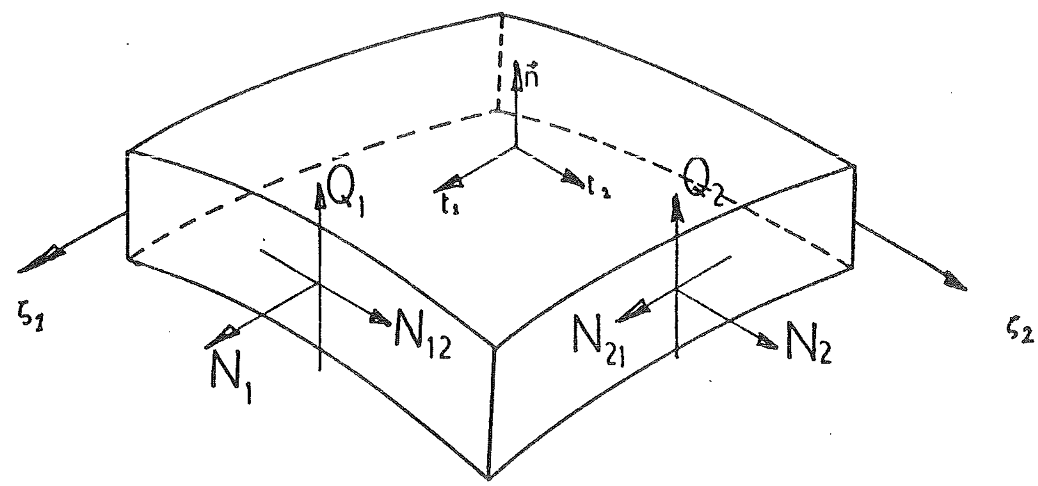
$r/t = 60$ $\quad V = 0.3$ $\quad E = 29.5 \times 10^{6}$ lb/in^2 $\quad r = 35.43$ in.

$L/r = 24.98$ $\quad \rho = 0.734 \times 10^{-3}$ lb - sec^2/in^4 $\quad \rho_{\ell} = 0.935 \times 10^{-4}$

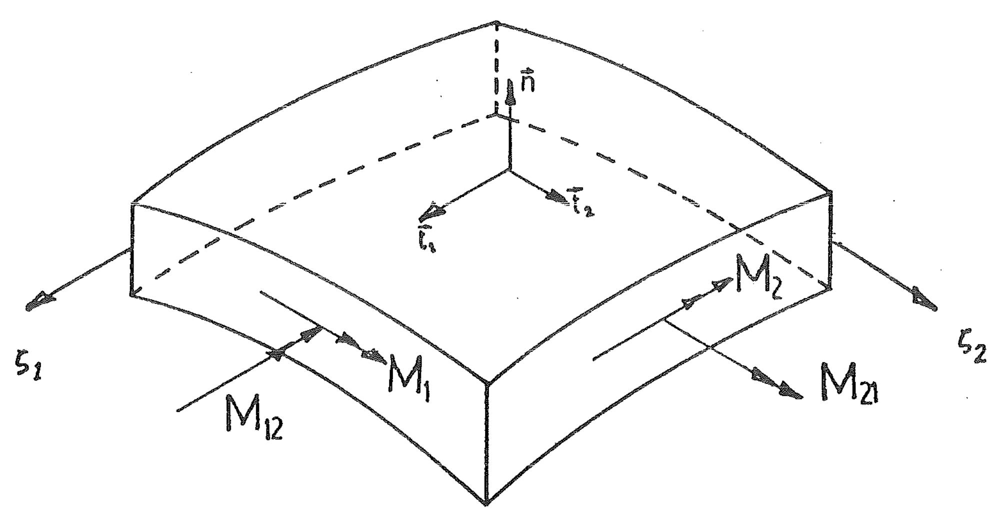
lb - sec^2/in^4

APPENDIX C

FIGURES



Direction of resultant constraints



Direction of resultant moments

FIGURE 1 - DIFFERENTIAL ELEMENTS FOR A THIN SHELL

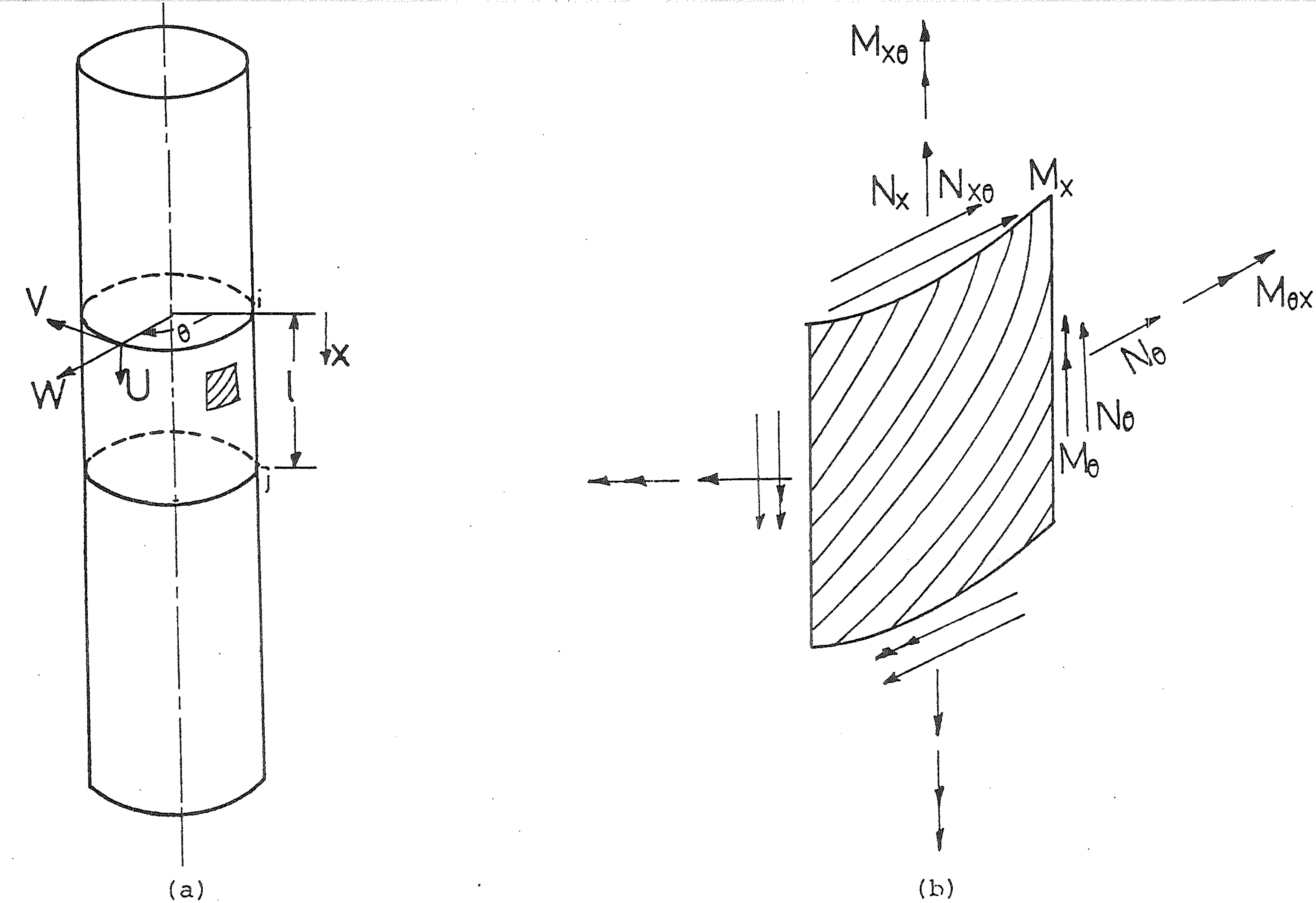


FIGURE 2 - (a) GEOMETRY OF THE SURFACE OF REFERENCE FOR A CYLINDRICAL SHELL AND A CYLINDRICAL ELEMENT DETERMINED BY NODES i AND j

(b) RESULTANT CONSTRAINTS FOR A FINITE ELEMENT (FOR CLARIFICATION Q_x AND Q_θ ARE OMITTED)

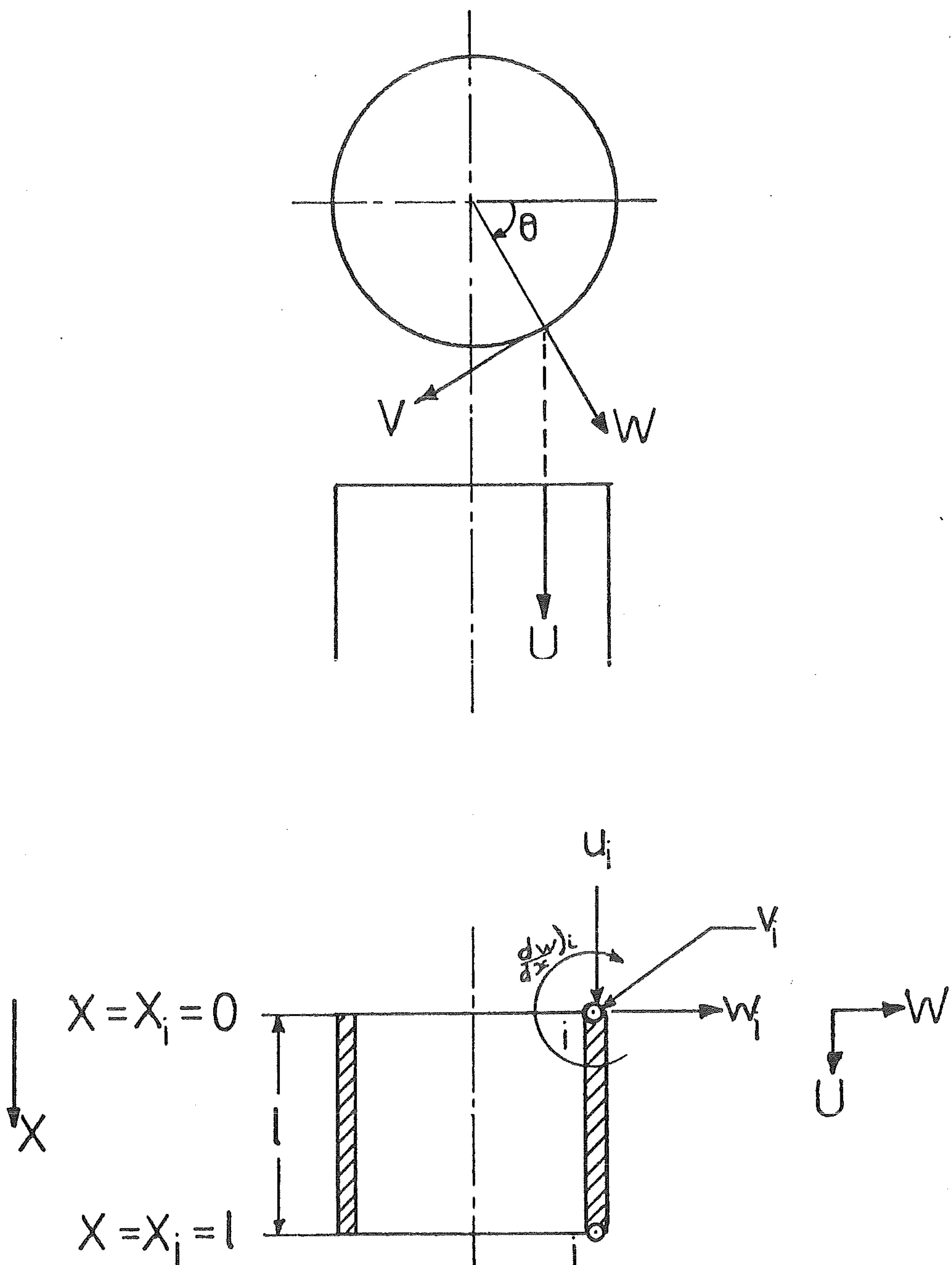


FIGURE 3 - NODAL DISPLACEMENTS AT JOINTS i AND j

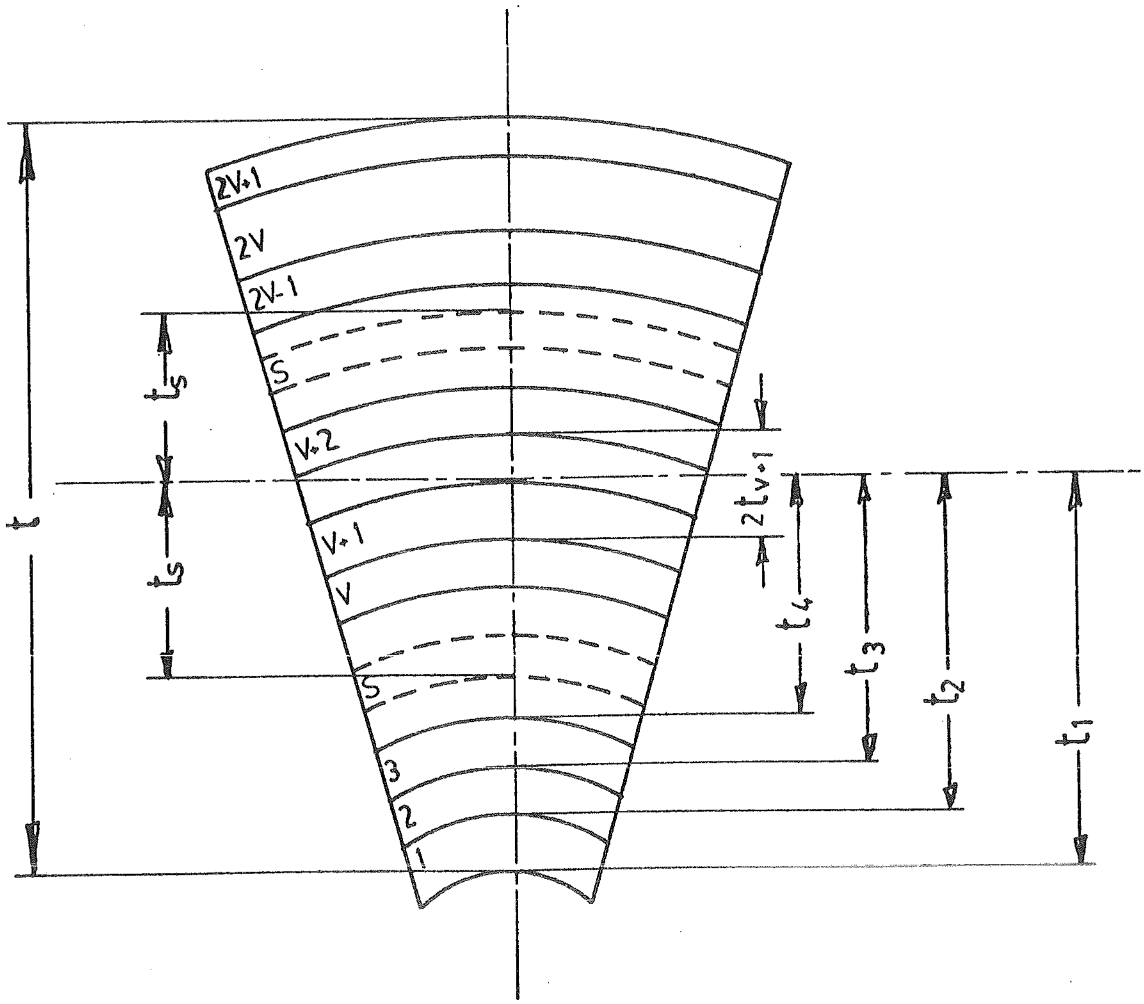


FIGURE 4 - SHELL COMPOSED OF BY AN ODD NUMBER (2V + 1)
OF ANISOTROPIC LAYERS

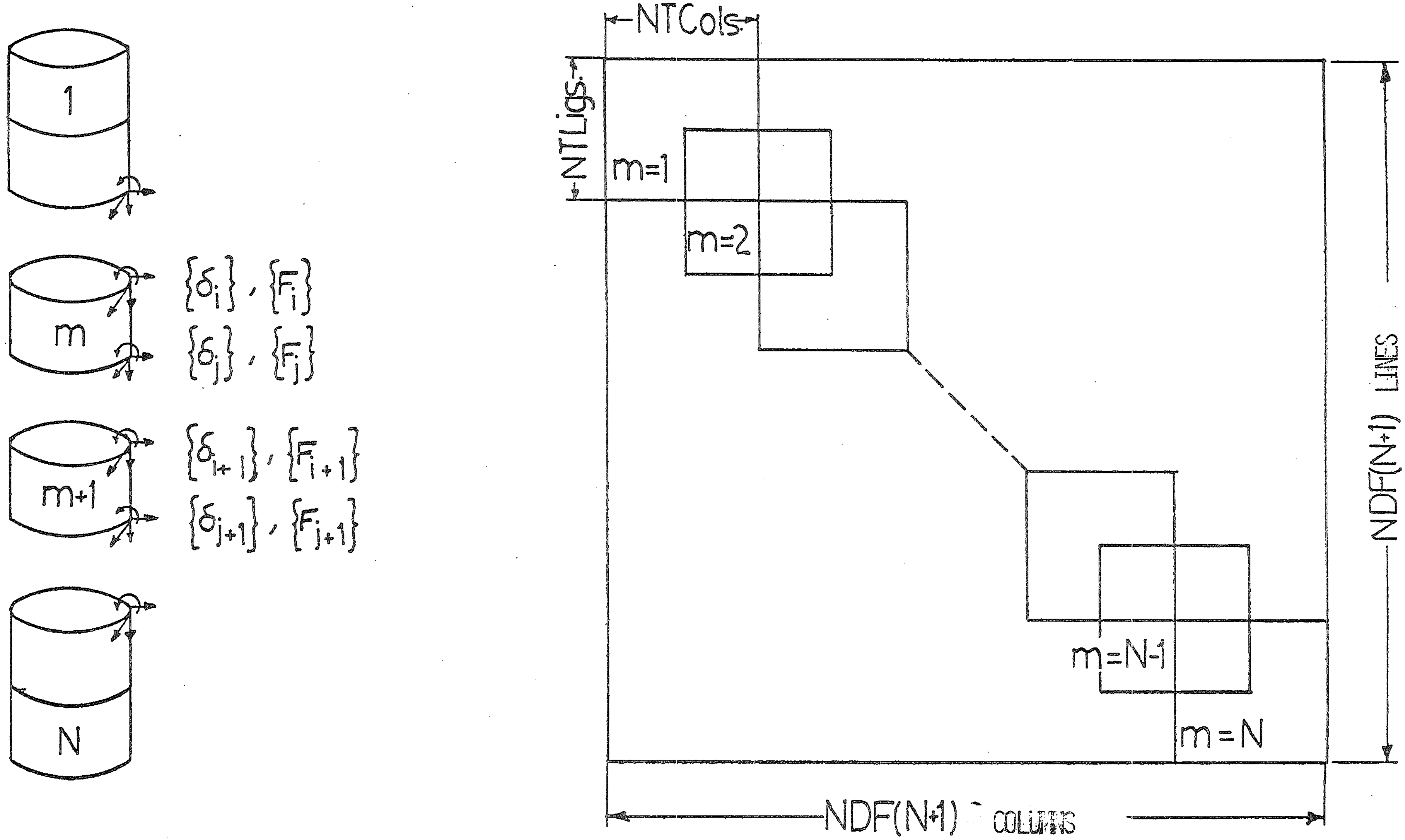


FIGURE 5 - ASSEMBLY DIAGRAM OF MASS AND STIFFNESS MATRICES FOR COMPLETE SYSTEM (N = NUMBER OF FINITE ELEMENTS, NDF = NUMBER OF DEGREES OF FREEDOM AND NT = 2NDF)

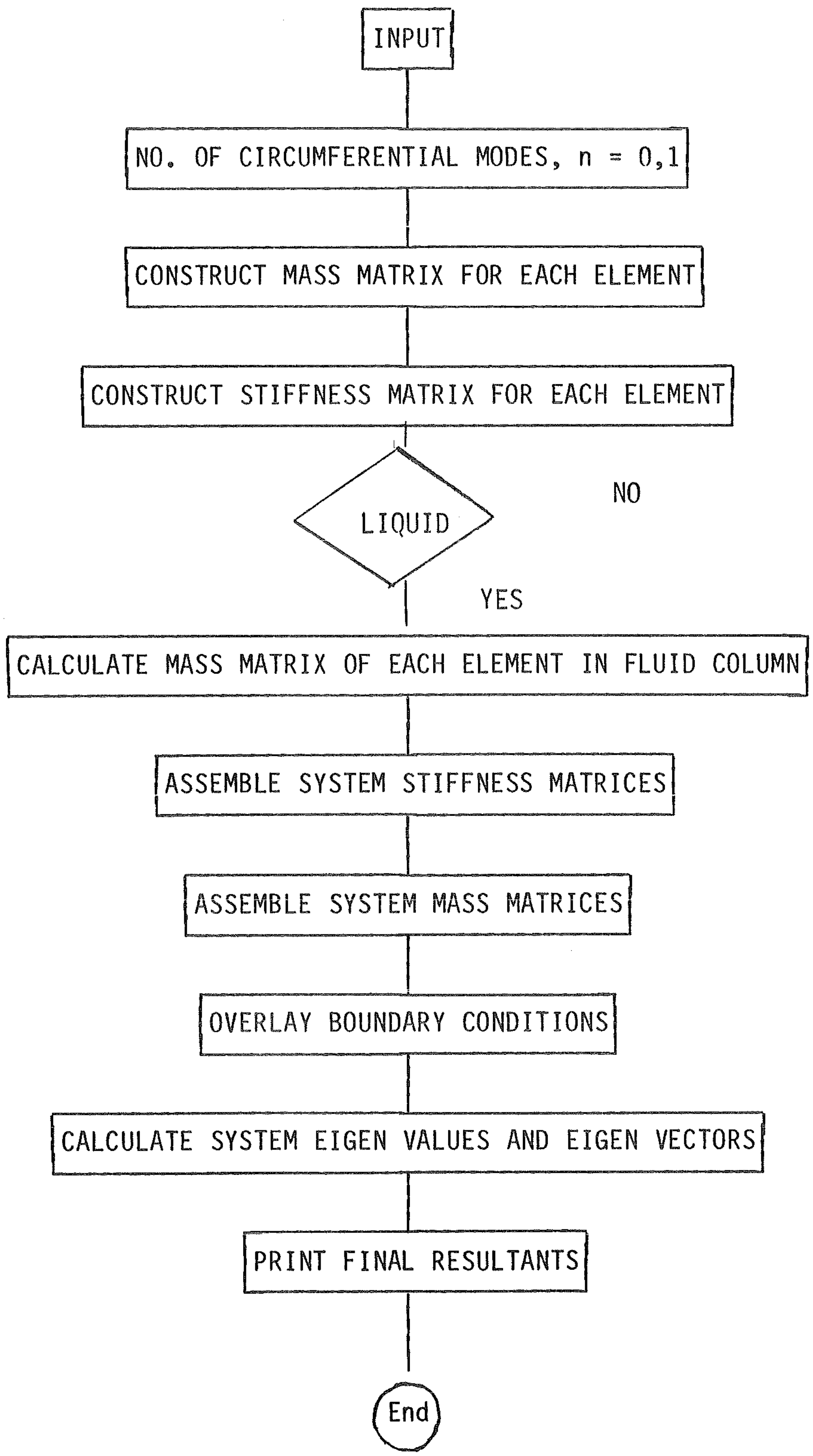
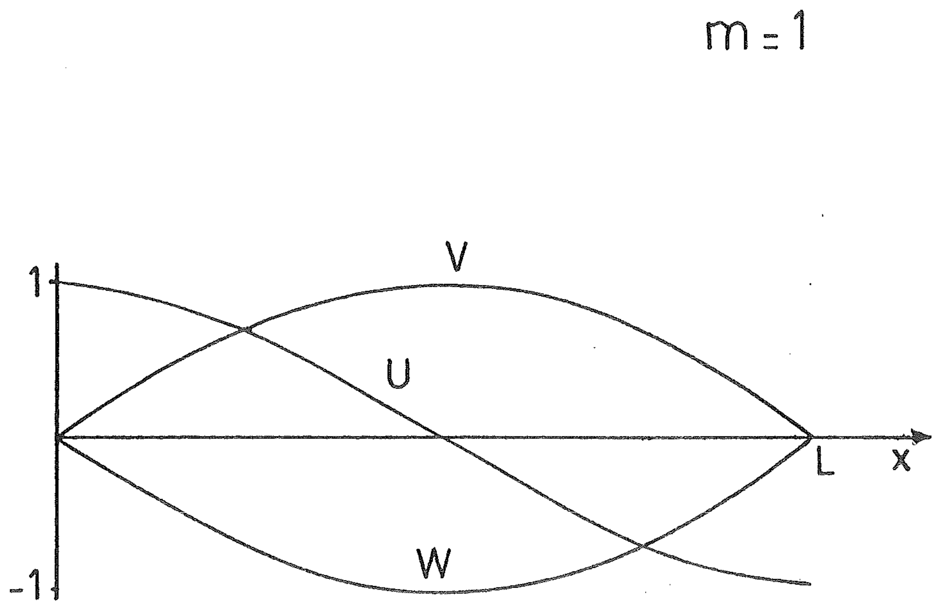


FIGURE 6 - Slow chart of the principle program



n	U_{max}/V_{max}	W_{max}/V_{max}	V_{max}
0	—	—	1
1	0.3153	0.9799	—
3	0.2024	2.9638	—

* see ref. [21]

FIGURE 7 - NORMALIZED NATURAL EIGEN VECTORS OF A CYLINDRICAL SHELL SIMPLY SUPPORTED AT BOTH ENDS

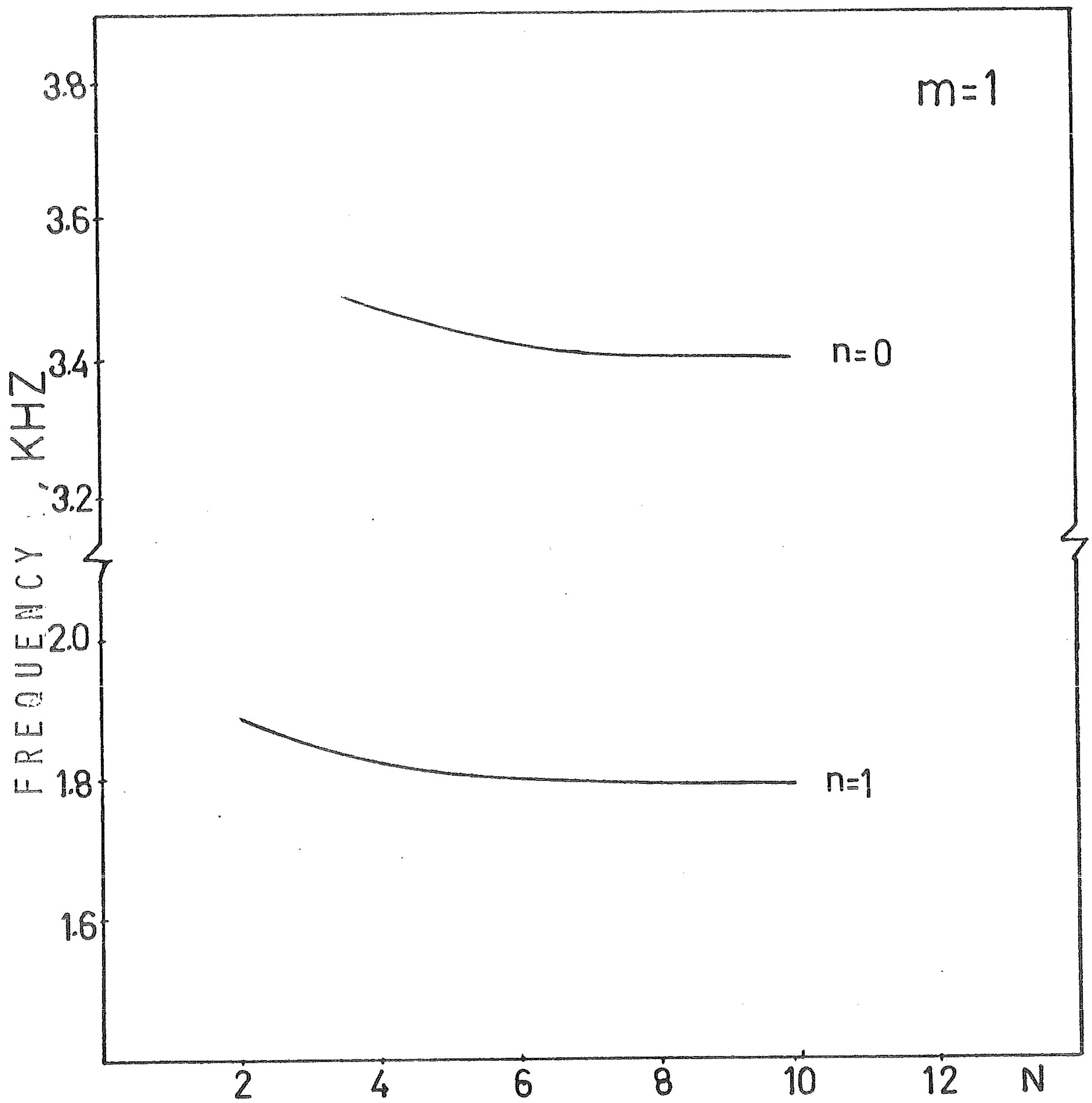


FIGURE 8a - CONVERGENCE TEST OF A CYLINDRICAL SHELL SIMPLY SUPPORTED AT BOTH ENDS FOR $n = 0$ and $n = 1$ (N = number of finite elements)

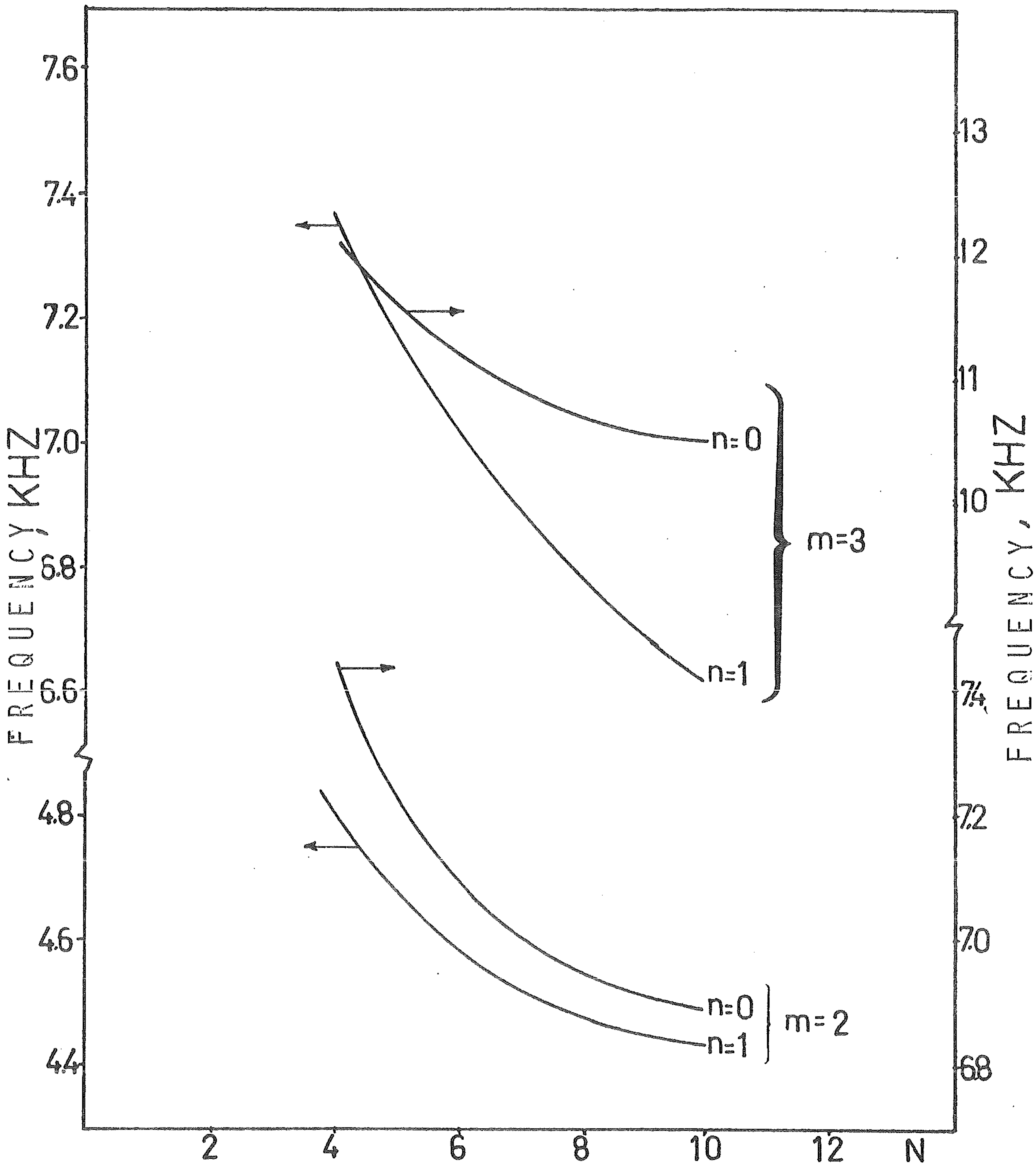


FIGURE 8b - CONVERGENCE TEST OF A CYLINDRICAL SHELL SIMPLY SUPPORTED AT BOTH ENDS FOR n = 0 and n = 1 (N = number of finite elements)

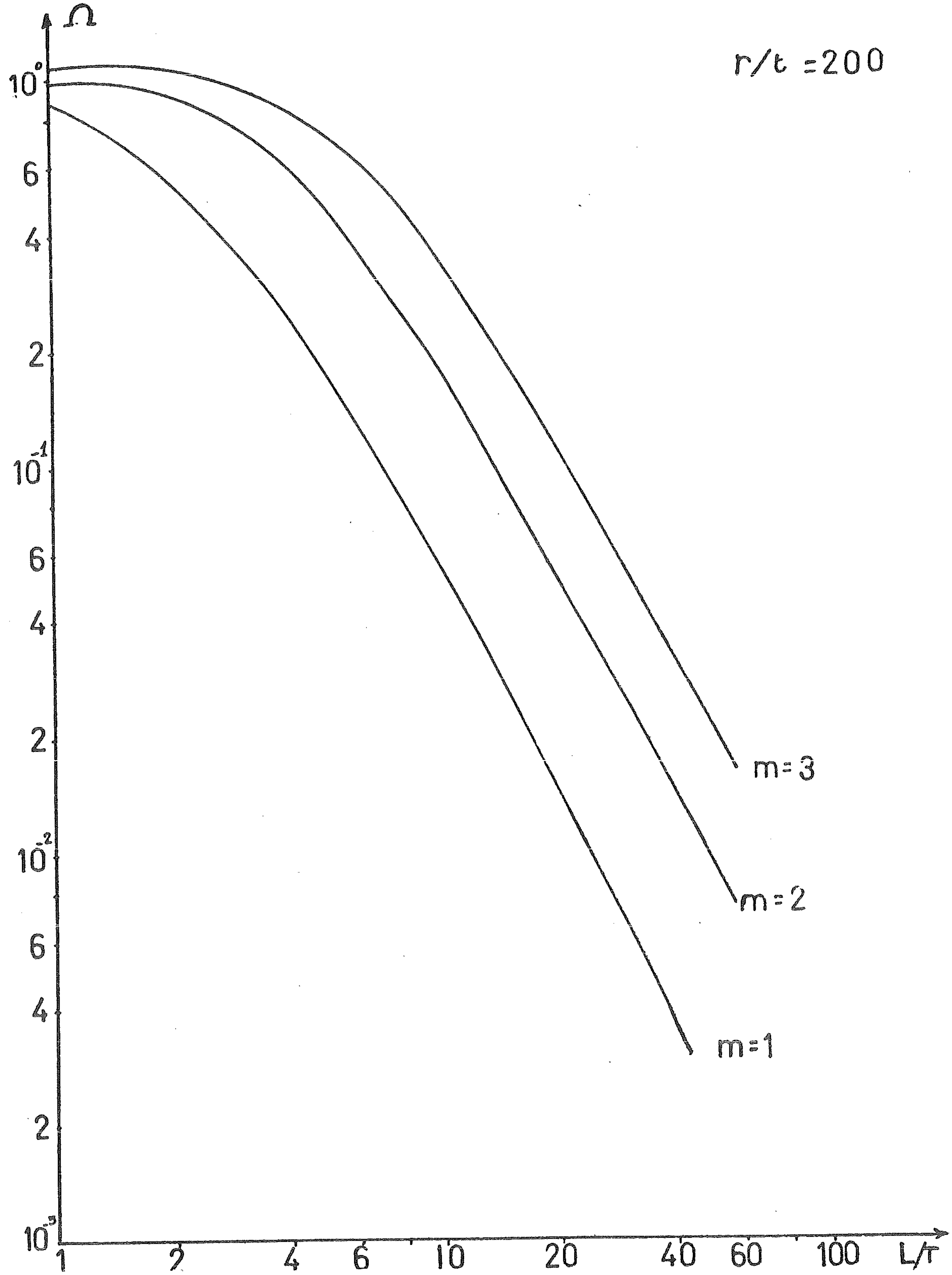


FIGURE 9 - VARIATION OF VIBRATION PARAMETER (Ω) IN CONJECTION WITH L/r OF A CYLINDRICAL SHELL SIMPLY SUPPORTED AT BOTH ENDS (n = 1), $\nu = 0.3$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$

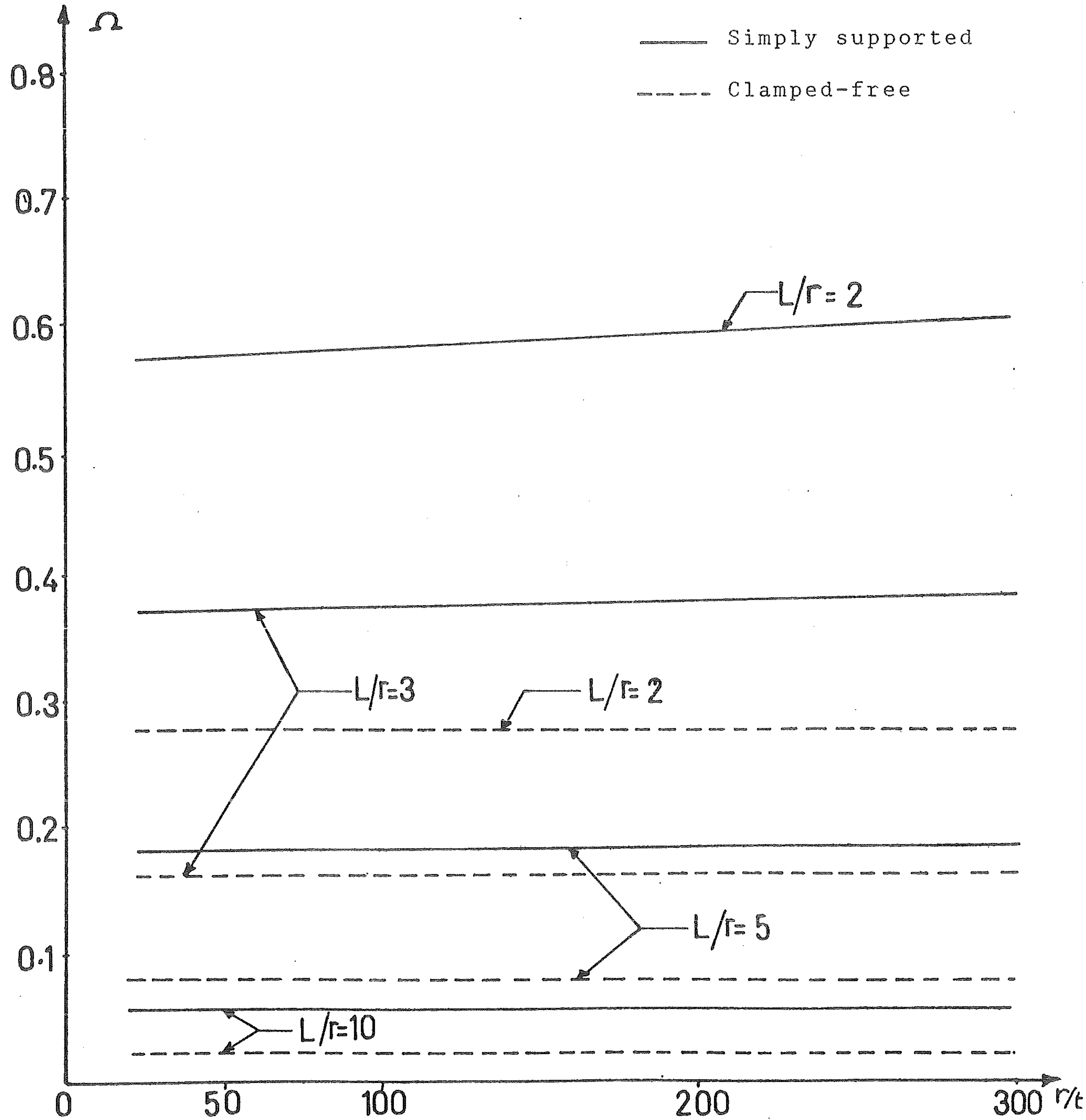


FIGURE 10 - VARIATION OF VIBRATION PARAMETER (Ω) IN CONJUNCTION WITH r/t OF A CYLINDRICAL SHELL ($n = 1$, $m = 1$, $\nu = 0.3$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

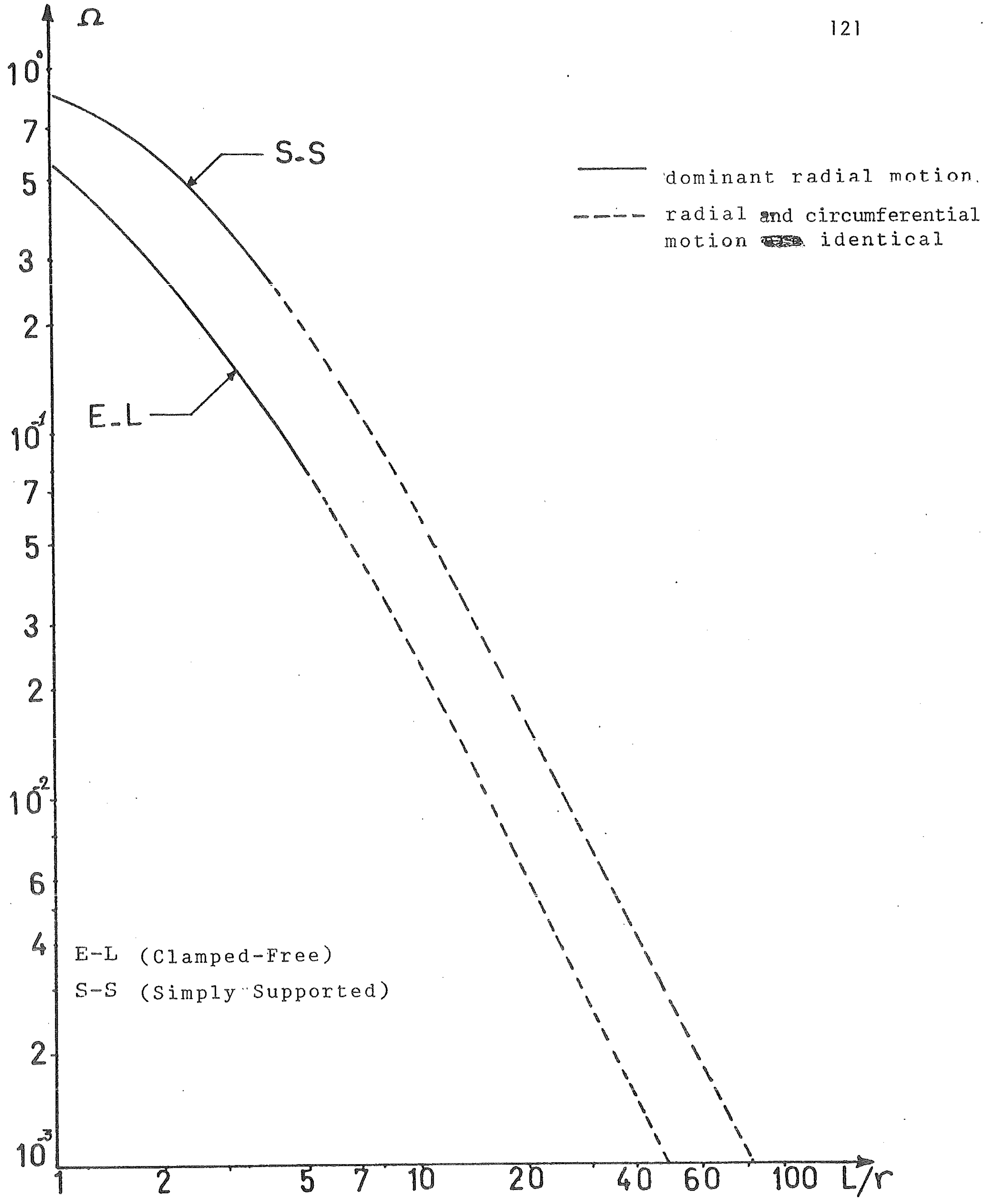


FIGURE 11 - EFFECT OF L/r ON EIGEN VALUES AND EIGEN VECTORS FOR DIFFERENT BOUNDARY CONDITION ($n = 1$, $m = 1$, $r/t = 100$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

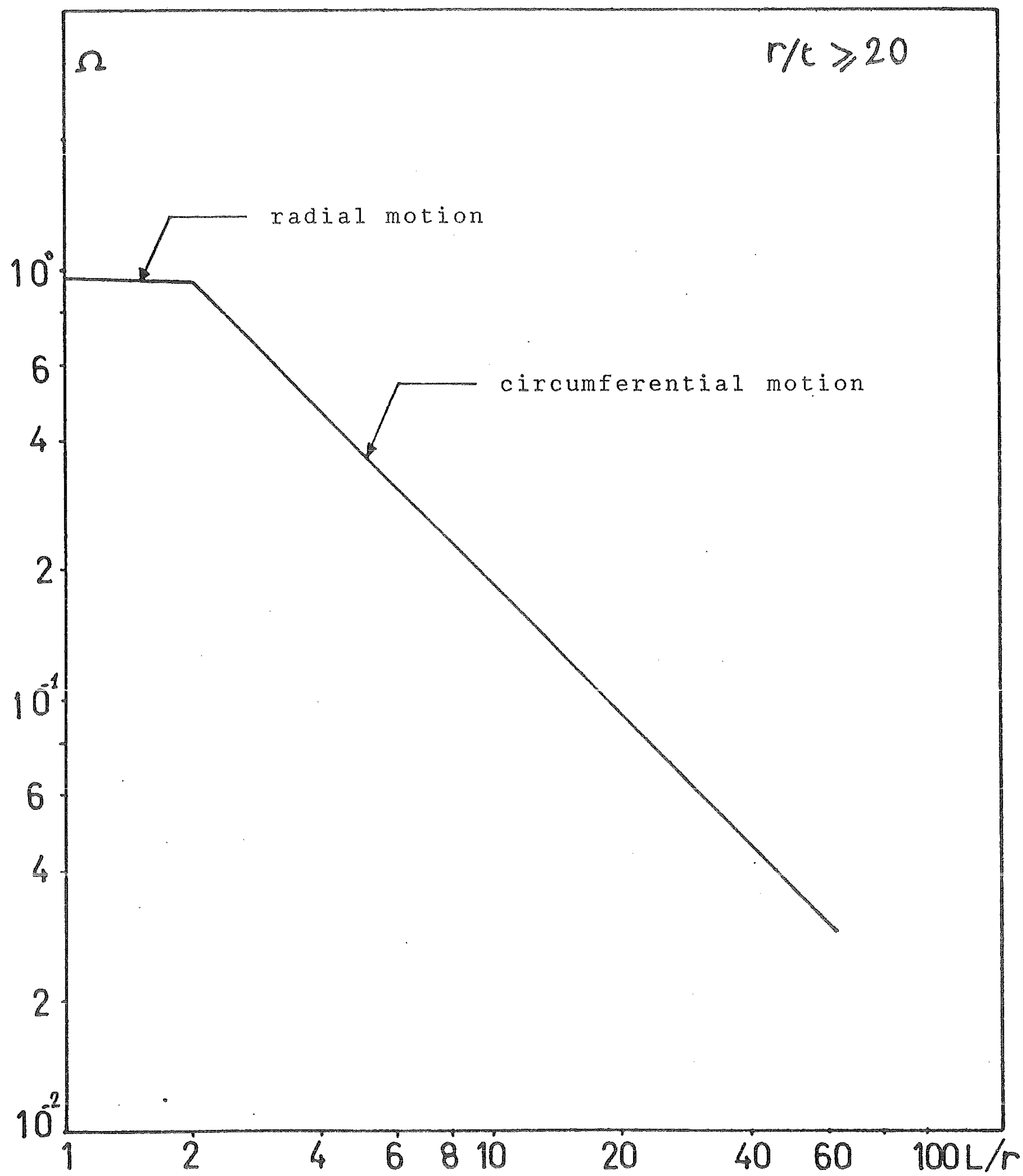


FIGURE 12 - VARIATION OF VIBRATION PARAMETER (Ω) IN CONJUNCTION WITH L/r OF A CYLINDRICAL SHELL SIMPLY SUPPORTED AT BOTH ENDS

($n = 0$, $m = 1$, $\nu = 0.3$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

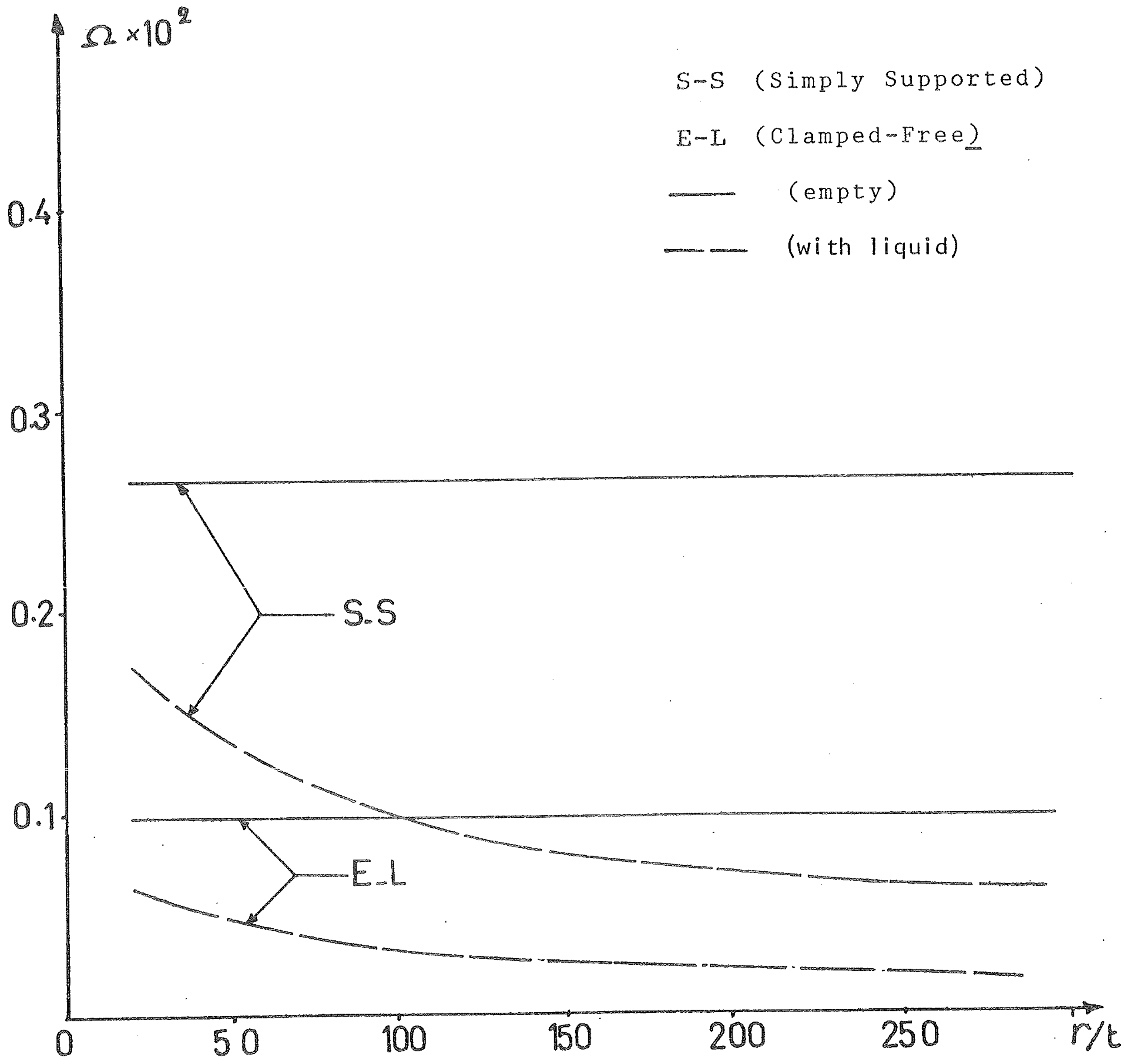


FIGURE 13 - EFFECT OF THE LIQUID ON VIBRATION PARAMETER (Ω)
($n = 1$, $m = 1$, $\nu = 0.3$, $\rho_\ell = 0.935 \times 10^{-4}$ lb - sec^2/in^4
$L/r = 50$ and $\Omega = \omega r \sqrt{\rho(1 - \nu^2)/E}$)

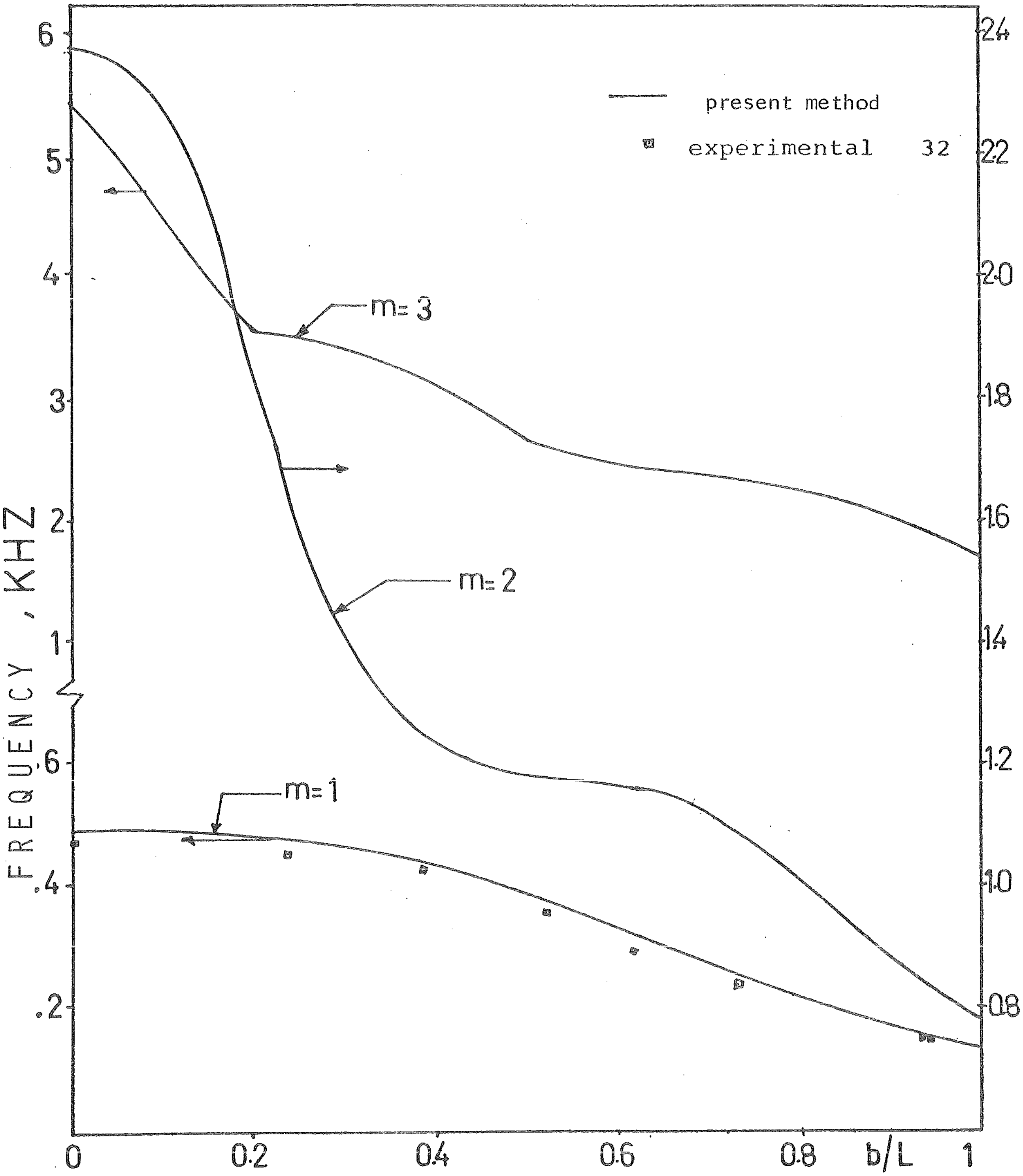


FIGURE 14 - COMPARISON OF PRESENT METHOD WITH EXPERIMENTAL VASLUES [23] FOR A CYLINDRICAL SHELL PARTIALLY FILLED WITH LIQUID ($n = 1$, $\nu = 0.3$ and $\rho_{\ell} = 0.935 \times 10^{-4}$ lb - sec^2/in^4

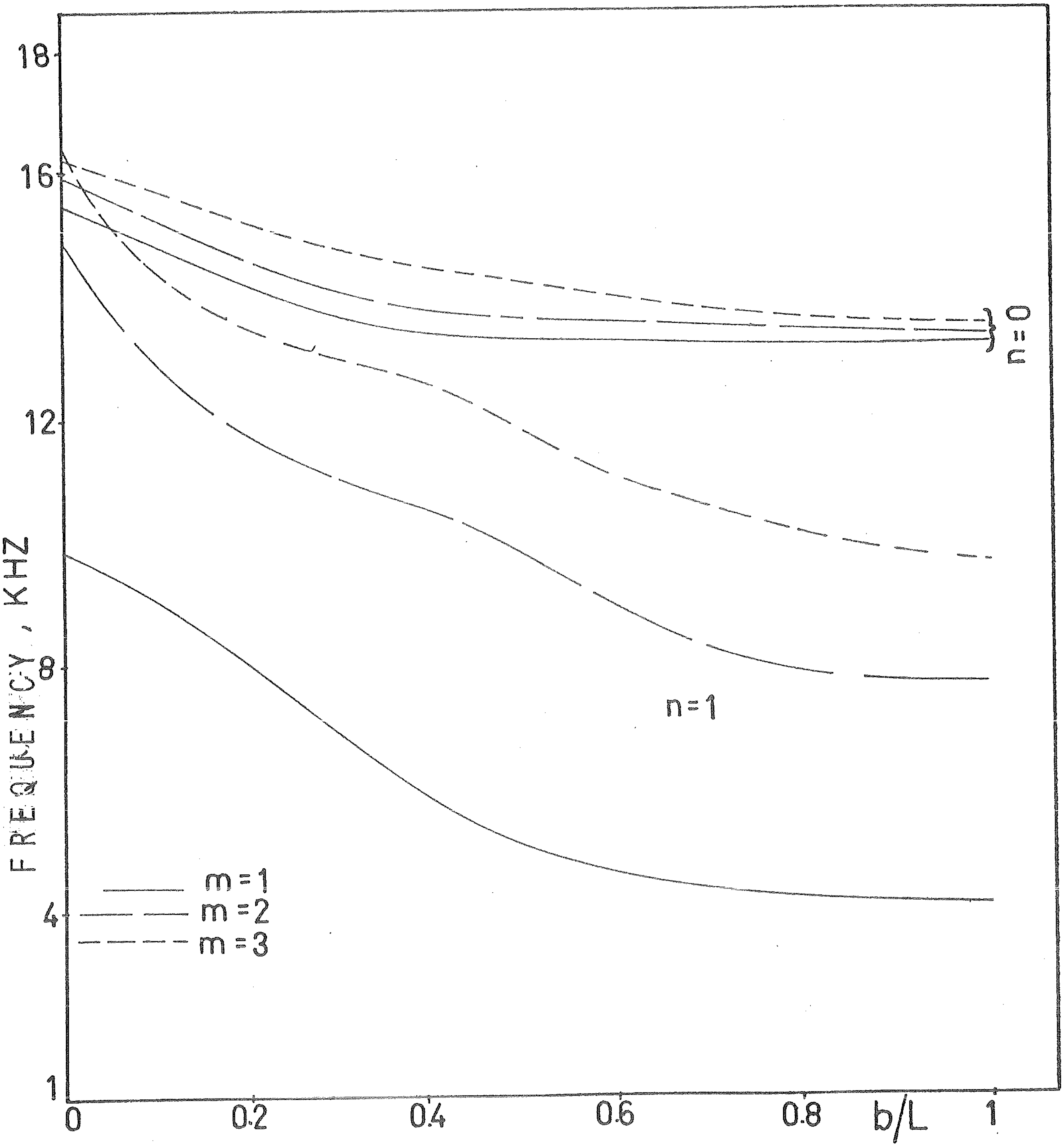


FIGURE 15 - VARIATION IN FREE VIBRATION OF A CYLINDRICAL SHELL PARTIALLY FILLED WITH LIQUID AND SIMPLY SUPPORTED AT BOTH ENDS (ρ_ℓ = 0.935 x 10^{-4} lb - sec^2/in^4

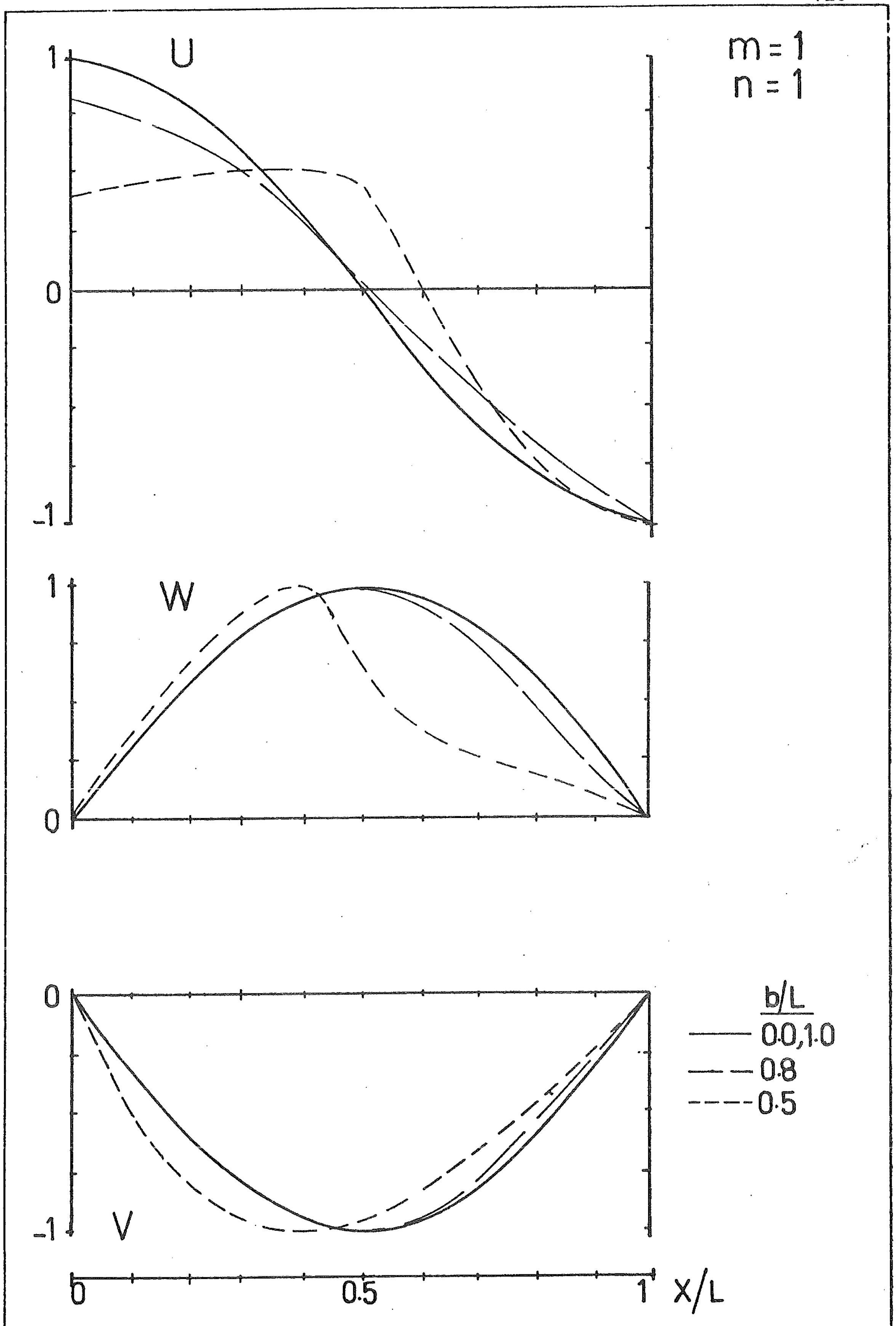


FIGURE 16 - NORMALIZED EIGEN VECTORS OF A SHELL SIMPLY SUPPORTED AT BOTH ENDS AND PARTIALLY FILLED WITH LIQUID